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Self-Dual Yang–Mills Fields in $d = 7, 8$, Octonions and Ward Equations

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Abstract. The Yang–Mills theories in $d = 7$ and $d = 8$ with the arbitrary gauge group G are considered. Generalized self-duality-type relations for gauge fields are reduced to systems of nonlinear differential equations on functions of one variable (Ward equations). Ward equations may be reduced to equations which follow from Yang–Baxter equations. This permits us to obtain new classes of explicit solutions of the Yang–Mills equations in $d = 7$ and $d = 8$.

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1. Here, we shall aim at finding the solutions of the classical equations for a pure Yang–Mills (YM) theory in the Euclidean spaces $\mathbf{R}^{7,0}$ and $\mathbf{R}^{8,0}$ with an arbitrary semisimple gauge Lie group $\mathbb{G}$. We start with the potentials A_μ in $\mathbf{R}^{d,0}$ with values in the Lie algebra $\mathscr{G}$ of the Lie group $\mathbb{G}$. The field tensor $F_{\mu\nu}$ is defined as

$$F_{\mu\nu} = [\mathbb{D}_\mu, \mathbb{D}_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \tag{1}$$

where $\mathbb{D}_\mu = \partial_\mu + [A_\mu, \ ]$, $\mu, \nu, \ldots = 1, \ldots, d$.

The Yang–Mills equations for the gauge potentials A_μ have the form

$$\mathbb{D}_\mu F_{\mu\nu} = 0. \tag{2}$$

In a series of papers [1–4], it was shown that Equation (2) is satisfied by virtue of the Bianchi identity $\mathbb{D}_{[\mu} F_{\nu\sigma]} = 0$ if the tensor $F_{\mu\nu}$ satisfies the generalized self-duality equation

$$T_{\mu\nu\sigma\lambda} F_{\sigma\lambda} = \gamma F_{\mu\nu}, \tag{3}$$

where the numerical tensor T is completely antisymmetric and $\gamma = \text{const}$ is a nonzero eigenvalue. Equation (3) generalizes the usual self-duality equations in $d = 4$. In this Letter, we shall describe the classes of self-dual solutions of the YM equations in $d = 7$ and $d = 8$.

2. Let **Ca** be the alternative nonassociative algebra of the octonions. Its defining relations are

$$e_a e_b = -\delta_{ab} e_8 + f_{abc} e_c, \qquad e_a e_8 = e_8 e_a = e_a, \qquad e_8^2 = e_8,$$

where $a, b, \ldots = 1, \ldots, 7$, e_a are the basic octonionic units, and e_8 is the unit element in **Ca**. The Cayley structure constants f_{abc} are totally antisymmetric in (abc) and equal to unity for the seven combinations (or cycles): (123), (246), (435), (367), (651), (572), (714).

In $\mathbf{R}^{7,0}$, one may introduce the antisymmetric four-index tensor h_{abcd} which is dual to f_{abc}:

$$h_{abcd} = -\frac{1}{3!}\varepsilon_{abcdmnk}f_{mnk},$$

where $\varepsilon_{abcdmnk}$ is a completely antisymmetric tensor in $d = 7$. The tensors f_{abc} and h_{abcd} satisfy the following seven-dimensional identities [4, 5]:

$$\begin{aligned} &h_{abcd}h_{ijkd} \\ &\quad =(\delta_{ai}\,\delta_{bj} - \delta_{aj}\,\delta_{bi})\delta_{ck} + (\delta_{bi}\,\delta_{cj} - \delta_{bj}\,\delta_{ci})\delta_{ak} + \\ &\qquad +(\delta_{ci}\,\delta_{aj} - \delta_{cj}\,\delta_{ai})\delta_{bk} - f_{abc}f_{ijk} + h_{abij}\,\delta_{ck} + \\ &\qquad + h_{bcij}\,\delta_{ak} + h_{caij}\,\delta_{bk} + h_{abjk}\,\delta_{ci} + \\ &\qquad + h_{bcjk}\,\delta_{ai} + h_{cajk}\,\delta_{bi} + h_{abki}\,\delta_{cj} + h_{bcki}\,\delta_{aj} + h_{caki}\,\delta_{bj}, \end{aligned} \tag{4a}$$

$$\begin{aligned} &f_{abk}h_{cdek} \\ &\quad = f_{acd}\,\delta_{be} - f_{bcd}\,\delta_{ae} + f_{ade}\,\delta_{bc} - \\ &\qquad - f_{bde}\,\delta_{ac} + f_{aec}\,\delta_{bd} - f_{bec}\,\delta_{ad}. \end{aligned} \tag{4b}$$

We also introduce the following tensors [5]

$$\begin{aligned} g_{abcd} &\equiv 2(\delta_{ac}\,\delta_{bd} - \delta_{ad}\,\delta_{bc}) - h_{abcd} \\ &= 3(\delta_{ac}\,\delta_{bd} - \delta_{ad}\,\delta_{bc}) - f_{abk}f_{cdk}, \end{aligned} \tag{5a}$$

$$\bar{g}_{abcd} \equiv f_{abk}f_{cdk} = \delta_{ac}\,\delta_{bd} - \delta_{ad}\,\delta_{bc} + h_{abcd}. \tag{5b}$$

The tensors g_{abcd} and $\bar{g}_{abcd}$ project an arbitrary antisymmetric tensor T_{ab} onto the orthogonal 14- and 7-dimensional subspaces of the 21-dimensional vector space of the antisymmetric tensors in $d = 7$:

$$\begin{aligned} T_{ab} &= \tfrac{1}{2}(\delta_{ac}\,\delta_{bd} - \delta_{ad}\,\delta_{bc})T_{cd} \\ &= \tfrac{1}{6}(g_{abcd} + \bar{g}_{abcd})T_{cd} \\ &= \tfrac{1}{6}g_{abcd}T_{cd} + \tfrac{1}{6}\bar{g}_{abcd}T_{cd} \\ &= N_{ab} + \bar{N}_{ab}, \end{aligned}$$

where N_{ab} is the self-dual and $\bar{N}_{ab}$ is the anti-self-dual part of the tensor T_{ab}.

From (4) and (5) one obtains

$$h_{abcd}N_{cd} = -2N_{ab},$$

$$h_{abcd}\bar{N}_{cd} = 4\bar{N}_{ab}.$$

Notice, that these equations are equivalent to the equations [3]:

$$\bar{g}_{cdab}N_{ab} = 0, \tag{6a}$$

$$g_{cdab}\bar{N}_{ab} = 0. \tag{6b}$$

3. Let us consider an ansatz for the fields A_a in $\mathbf{R}^{7,0}$:

$$A_a = \alpha g_{abcd}x_b W_{cd}(u), \tag{7}$$

where $\alpha = \text{const}$ and the antisymmetric tensor $W_{cd} = -W_{dc}$ depends on $u = 1 + x_a x_a$ and takes its values in the Lie algebra $\mathcal{G}$.

Inserting (7) into the definition of F_{ab} and using the identities (4), we obtain the following expression:

$$F_{ab} = -2\alpha g_{abcd}W_{cd} + 2\alpha x_a x_k g_{bkcd}\dot{W}_{cd} - \\ -2\alpha x_b x_k g_{akcd}\dot{W}_{cd} + \alpha^2 x_m x_n g_{amcd}g_{bnek}[W_{cd}, W_{ek}], \tag{8}$$

where

$$\dot{W}_{ce} = \frac{\mathrm{d}W_{ce}}{\mathrm{d}u}.$$

Substitute (8) into the self-duality equations (6a) (i.e. $N_{ab} \to F_{ab}$). After long and tedious calculations using the identities (4) for $\alpha = -\frac{1}{4}$, one obtains the equations:

$$[W_{ab}, W_{cd}] = S_{abcdmn}\dot{W}_{mn}, \tag{9}$$

where

$$S_{abcdmn} = \tfrac{1}{2}(\delta_{ac}\,\delta_{b[m}\,\delta_{n]d} - \delta_{bc}\,\delta_{a[m}\,\delta_{n]d} + \delta_{bd}\,\delta_{a[m}\,\delta_{n]c} - \delta_{ad}\,\delta_{b[m}\,\delta_{n]c})$$

are the SO(7) structure constants.

Equation (9) is a particular case of the equation

$$[W_{m'}, W_{n'}] = C_{m'n'k'}\dot{W}_{k'}, \tag{10}$$

which was considered by Ward [6]. In (10), $C_{m'n'k'}$ are the structure constants of the Lie algebra $\mathcal{H}$ $(m', n', \ldots = 1, \ldots, \dim \mathcal{H})$, and W_m takes its value in the Lie algebra $\mathcal{G}$. We shall call (10) the Ward equations.

When $\alpha = -\frac{1}{4}$ and $W_{ab}(u)$ satisfy the Ward equations (9), it is not difficult to show that the field tensor F_{ab} has the form

$$F_{ab} = \tfrac{1}{16}g_{abcd}\{8W_{cd} + 4(u-1)\dot{W}_{cd} + 4x_d x_k \dot{W}_{ck} - \\ -4x_c x_k \dot{W}_{dk} + x_d x_k \dot{W}_{mn}\bar{g}_{kmnc} - x_c x_k \dot{W}_{mn}\bar{g}_{kmnd}\}. \tag{11}$$

From (11), the self-duality of F_{ab} becomes obvious.

PROPOSITION 1. *For the ansatz* (7), *the self-duality equations* (6a) *of the YM model in* $d = 7$ *with the arbitrary semisimple gauge group* $\mathbb{G}$ *is reduced to the Ward equation* (9). *Conversely, with each solution of the Ward equation* (9) *one may correspond the solution* (7) (*with* $\alpha = -\frac{1}{4}$) *of the self-duality equation* (6a).

Proof. Follows from (8), (6a), and (4) after direct calculations.

4. Consider the eight-dimensional Euclidean space $\mathbf{R}^{8,0}$. In $\mathbf{R}^{8,0}$, let us define the next completely antisymmetric four-index tensor $H_{ABCD}(A, B, \ldots = 1, \ldots, 8)$ [4, 5]:

$$H_{abcd} = h_{abcd}, \qquad H_{abc8} = f_{abc},$$

where $a, b, \ldots = 1, \ldots, 7$, and the tensors h_{abcd} and f_{abc} were introduced in Section 2.

The tensor H_{ABCD} satisfies the identities [5]:

$$\begin{aligned} H_{ABCD}H_{IJKD} &= (\delta_{AI}\,\delta_{BJ} - \delta_{AJ}\,\delta_{BI})\delta_{CK} + (\delta_{BI}\,\delta_{CJ} - \delta_{BJ}\,\delta_{CI})\delta_{AK} + \\ &\quad + (\delta_{CI}\,\delta_{AJ} - \delta_{CJ}\,\delta_{AI})\delta_{BK} + \\ &\quad + H_{ABIJ}\,\delta_{CK} + H_{BCIJ}\,\delta_{AK} + H_{CAIJ}\,\delta_{BK} + \\ &\quad + H_{ABJK}\,\delta_{CI} + H_{BCJK}\,\delta_{AI} + H_{CAJK}\,\delta_{BI} + \\ &\quad + H_{ABKI}\,\delta_{CJ} + H_{BCKI}\,\delta_{AJ} + H_{CAKI}\,\delta_{BJ}. \end{aligned} \tag{12}$$

We also introduce the tensors [4, 5]:

$$G_{ABCD} \equiv 3(\delta_{AC}\,\delta_{BD} - \delta_{AD}\,\delta_{BC}) - H_{ABCD},$$

$$\bar{G}_{ABCD} \equiv \delta_{AC}\,\delta_{BD} - \delta_{AD}\,\delta_{BC} + H_{ABCD}.$$

These tensors project an arbitrary antisymmetric tensor T_{AB} onto the orthogonal 21- and 7-dimensional subspaces of the 28-dimensional vector space of the antisymmetric tensors in $d = 8$:

$$\begin{aligned} T_{AB} &= \tfrac{1}{2}(\delta_{AC}\,\delta_{BD} - \delta_{AD}\,\delta_{BC})T_{CD} = \tfrac{1}{8}(G_{ABCD} + \bar{G}_{ABCD})T_{CD} \\ &= \tfrac{1}{8}G_{ABCD}T_{CD} + \tfrac{1}{8}\bar{G}_{ABCD}T_{CD} = N_{AB} + \bar{N}_{AB}. \end{aligned}$$

Here, N_{AB} is the self-dual and $\bar{N}_{AB}$ is the anti-self-dual parts of the tensor T_{AB}.

From (12), one obtains

$$H_{ABCD}N_{CD} = -2N_{AB}, \qquad H_{ABCD}\bar{N}_{CD} = 6\bar{N}_{AB}.$$

These equations are equivalent to

$$\bar{G}_{CDAB}N_{AB} = 0, \tag{13a}$$

$$G_{CDAB}\bar{N}_{AB} = 0. \tag{13b}$$

5. For the gauge fields A_M in $\mathbf{R}^{8,0}$, we introduce the ansatz

$$A_M = \beta G_{MNCD} x_N W_{CD}(u), \tag{14}$$

where $\beta = \text{const}$ and the antisymmetric tensor $W_{MN} = -W_{NM}$, depending on $u = 1 + x_M x_M$, takes its values in the arbitrary semisimple Lie algebra $\mathscr{G}$.

Let us insert (14) into the definition of F_{MN} and use the identities (12). We obtain

$$F_{MN} = -2\beta G_{MNCD} W_{CD} + 2\beta x_M G_{NBCD} x_B \dot{W}_{CD} - \\ -2\beta x_N G_{MBCD} x_B \dot{W}_{CD} + \beta^2 x_C x_D G_{MCPQ} G_{NDAB} [W_{PQ}, W_{AB}], \tag{15}$$

where

$$\dot{W}_{CD} = \frac{\mathrm{d} W_{CD}}{\mathrm{d}u}.$$

Substitute (15) into the self-duality equation (13a) (i.e. $N_{AB} \to F_{AB}$). After some algebra using the identities (12), it is not difficult to show that when $\beta = -\frac{1}{6}$, these equations are reduced to

$$[W_{AB}, W_{CD}] = S_{ABCDMN} \dot{W}_{MN}, \tag{16}$$

where

$$S_{ABCDMN} = \tfrac{1}{2}(\delta_{AC}\, \delta_{B[M}\, \delta_{N]D} - \delta_{BC}\, \delta_{A[M}\, \delta_{N]D} + \\ + \delta_{BD}\, \delta_{A[M}\, \delta_{N]C} - \delta_{AD}\, \delta_{B[M}\, \delta_{N]C})$$

are the SO(8) structure constants.

When $\beta = -\frac{1}{6}$ and $W_{AB}(u)$ satisfy (16), the field tensor F_{MN} has the form

$$F_{MN} = \tfrac{1}{36} G_{MNIJ}[12 W_{IJ} + 6(u-1)\dot{W}_{IJ} + \\ + x_I x_B G_{CDBJ} \dot{W}_{CD} - x_J x_B G_{CDBI} \dot{W}_{CD}]. \tag{17}$$

From (17), the self-duality of F_{MN} becomes obvious.

PROPOSITION 2. *For the ansatz* (14), *the self-duality equation* (13a) *of the YM model in $d = 8$ with the arbitrary semisimple gauge group* $\mathbb{G}$, *is reduced to the Ward equation* (16). *Conversely, with each solution of the Ward equation* (16), *one may correspond the solution* (14) (*with* $\beta = -\frac{1}{6}$) *of the self-duality equation* (13a).

Proof. Follows from (12) after substituting (15) into (13a) and direct calculations.

6. Equation (10) was introduced and discussed by Ward [6]. When the gauge algebra $\mathscr{G}$ coincides with $\mathscr{H}$, Equation (10) may be obtained from the well-known classical Yang–Baxter equations. More exactly, Ward has shown [6] (see also [7]), that classical Yang–Baxter equations for the Lie algebra $\mathscr{H}$ may be reduced to Equation (10) (with $W_{m'} \in \mathscr{G} = \mathscr{H}$) if one assumes that $W_{m'} = W_{m'n'}(u) J_{n'}$ has a simple pole at 0 with a residue of the form $\xi g_{m'n'}$ (where $J_{n'}$ are the generators of

$\mathscr{H}$, $\zeta = \text{const}$, and $g_{m'n'}$ is the Killing metric on $\mathscr{H}$) and $W_{m'n'}(-u) = -W_{n'm'}(u)$. For classical Yang–Baxter equations (for the definitions, see [8–10]), solutions with such properties are called nondegenerate [9] and each such solution is, necessarily, also a solution of (10).

Many explicit nondegenerate solutions of classical Yang–Baxter equations are known (see, e.g., [8–10]). All such solutions for the simple Lie algebra $\mathscr{H}$ belong to one of three classes: elliptic, trigonometric, and rational [9, 10]. A detailed description of all nondegenerate elliptic and trigonometric solutions was given in [9] and a number of rational solutions were constructed. This description permits us to obtain new classes of solutions of the YM equations in $d = 7$ and $d = 8$. Namely, by using formula (7) with $\mathscr{G} = \mathscr{H} = \text{so}(7)$ and $\alpha = -\frac{1}{4}$, one may correspond the self-dual solution of the YM equations in $d = 7$ to each nondegenerate solution $W_{ab}(u)$ of the classical Yang–Baxter equations for so(7). Analogously, using formula (14) with $\mathscr{G} = \mathscr{H} = \text{so}(8)$ and $\beta = -\frac{1}{6}$, one may correspond the self-dual solution of the YM equations in $d = 8$ to each nondegenerate solution $W_{AB}(u)$ of the classical Yang–Baxter equations for so(8).

The simplest rational solution of the classical Yang–Baxter equations for $\mathscr{H}$ has the form [8–10]:

$$W \equiv g^{m'n'} W_{m'} \otimes J_{n'} = -\frac{1}{u} g^{m'n'} J_{m'} \otimes J_{n'} \Leftrightarrow W_{m'} = -\frac{1}{u} J_{m'}.$$

In particular, for Equations (9) and (16), we obtain the following well-known solutions:

$$W_{ab} = -\frac{1}{(1 + x_c x_c)} J_{ab}, \tag{18a}$$

$$W_{AB} = -\frac{1}{(1 + x_M x_M)} J_{AB}, \tag{18b}$$

where J_{ab} are the generators of the Lie algebra $\mathscr{H} = \text{so}(7)$, and J_{AB} are the generators of the Lie algebra $\mathscr{H} = \text{so}(8)$. The solution (7) with W_{ab} from (18a) was obtained in [11], and solution (14) with W_{AB} from (18b) was obtained in [3, 4].

7. New solutions of the YM equations in $d = 7$ and $d = 8$ may be obtained by the substitution of the solutions of the classical Yang–Baxter equations, which are more general than (18), into (7) and (14). As an example, we write out the trigonometric solution from [10]. It has the form

$$W_{m'} = \frac{1}{2q} \sum_{k=0}^{q-1} \left(\text{cth} \frac{(u - i2\pi k)}{2q} \right) \Theta^k(J_{m'}), \tag{19}$$

where $\Theta: \mathscr{H} \to \mathscr{H}$ is the Coxeter automorphism of the simple Lie algebra $\mathscr{H}$ and q is its order, i.e. $\Theta^q = \text{Id}$. For all simple Lie algebras, the values of the Coxeter numbers q and the description of the automorphism Θ in terms of roots, may be found in [12] (see also [9]).

Let us consider the algebra so(n). Following Belavin and Drinfeld [9], we choose the basis in which the matrices $\mathscr{A} \in$ so(n) satisfy the next relation $\mathscr{A}^t = -S\mathscr{A}S^{-1}$, where S is

$$S = \begin{pmatrix} 0 & & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \\ & \cdot^{\cdot^{\cdot}} & \\ \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & & 0 \end{pmatrix}$$

In this basis, the automorphism Θ has the form:

$$\Theta(\mathscr{A}) = Q\mathscr{A}Q^{-1}. \tag{20}$$

For $\mathscr{H} =$ so(7), the number q and the matrix Q are

$$q = 6, \qquad Q = \text{diag}(1, \omega, \ldots, \omega^5, 1), \tag{21}$$

where $\omega = \exp(i2\pi/6)$. For $\mathscr{H} =$ so(8), one may choose

$$q = 8, \qquad Q = \text{diag}(1, \omega, \ldots, \omega^5, \omega_6, 1) \tag{22a}$$

or

$$q = 8, \qquad Q = \begin{pmatrix} \varepsilon & & & & & & & \\ & \varepsilon^2 & & & & & 0 & \\ & & \varepsilon^3 & & & & & \\ & & & 0 & 1 & & & \\ & & & 1 & 0 & & & \\ & 0 & & & & \varepsilon^5 & & \\ & & & & & & \varepsilon^6 & \\ & & & & & & & \varepsilon^7 \end{pmatrix} \tag{22b}$$

where $\varepsilon = \exp(i2\pi/8)$.

Using the explicit form of the matrices Q from (21) and (22), one may show that solution (19) is real. With the help of (20)–(22), all constant matrices $\Theta^k(J_{m'})$ may be written out for $\mathscr{H} =$ so(7) and $\mathscr{H} =$ so(8). Notice that solution (19) is singular only in the points $u = i2\pi k, k = 0, \pm 1, \pm 2, \ldots$ [9, 10]. But we have $u = 1 + x_a x_a$ for the ansatz (7) and $u = 1 + x_M x_M$ for the ansatz (14). Therefore, $u \geqslant 1$ and the substitution of (19) into (7) and (14) gives the nonsingular solutions of the YM equations in $d = 7$ and $d = 8$.

The explicit form of the general solutions of classical Yang–Baxter equations of the trigonometric and elliptic types is given in [9, 10, 13]. We don't write out these solutions because it would take up too much room.

To sum up, we have shown that Ward equations and classical Yang–Baxter equations arise in the study of YM equations in $d = 7$ and $d = 8$. It is desirable to find more general classes of solutions of Ward equations than the Yang–Baxter equations give.

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