Letters in Mathematical Physics **17** (1989) 149–155.

On Finite-Zone Solutions of Relativistic Toda Lattices

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(Received: 27 July 1988)

Abstract. The finite-zone solutions of relativistic Toda lattices are investigated using the recurrence relations method. As a result, a nonlinear bundle of relativistic Toda lattices is with corresponding stationary and dynamical systems. New Poisson and Hamiltonian structures are introduced. Then the problem of integrating the obtained canonical systems are reduced to the Jacobi problem of inversion.

AMS subject classification (1980). 58F07.

1. Introduction

Between 1975 and 1976 [1,2], the author found a method of recurrence relations for investigation integrable equations and systems. This method was used [3–7] for discrete integrable systems (in particular, for Toda lattices). Also studied were the corresponding stationary and dynamical systems and a new Hamiltonian structure was found. Then the problem of finite-zone solutions was reduced to the oridinary Jacobi problem of inversion.

In March 1988, O. Ragnisco and M. Bruschi acquainted me with their new paper 'Lax representation and complete integrability for the periodic relativistic Toda lattice' [11]. They found a very important symmetric representation for the relativistic Toda lattice and Poisson and Hamiltonian structures for the periodicity problem. They also proved that the periodic relativistic Toda lattice was a completely integrable Hamiltonian system.

Using the results of [3–7] and the Ragnisco–Bruschi representation for the relativistic Toda lattice, in this letter is found a nonlinear bundle of relativistic Toda lattices with corresponding stationary and dynamical systems. Then, new Poisson and Hamiltonian structures are introduced *and the finite-zone problem is reduced to the Jacobi problem of inversion.*

2. Generating Equations

Here, we obtain a complete set of higher-order relativistic Toda lattices and describe the corresponding recurrence chains.

In the case of the relativistic Toda lattices, the corresponding discrete Schrödinger operator L has the form

$$(L\psi)_k = Ed_{k-1}\psi_{k-1} + (v_k - E^2)\psi_k + Ed_k\psi_{k+1}\,, \tag{2.1}$$

where $d_k(t)$, $v_k(t)$, $\psi_k(t)$ are functions of a discrete variable k and the continuous variable t, E is a free complex parameter. (Note that the Ragnisco–Bruschi representation uses the operator L/E.) The operator (2.1) is similar to the operator for the usual Toda lattice [6].

We chose the operators A in the form

$$(A\psi)_k = \mathscr{A}_k(t, E)\psi_k + Ed_k\mathscr{B}_k(t, E)\psi_{k+1}\ . \tag{2.2}$$

THEOREM 1. *The lax system of identities*

$$L\psi = 0\ , \tag{2.3}$$

$$\left(\frac{\partial L}{\partial t} + LA\right)\psi = 0\ . \tag{2.4}$$

is equivalent to the system of generating equations

$$\dot{d}_{k-1} = -d_{k-1}(\mathscr{A}_{k-1} + \mathscr{A}_k - \mathscr{B}_k(v_k - E^2))\ , \tag{2.5}$$

$$\dot{v}_k = -2\mathscr{A}_k(v_k - E^2) + \mathscr{B}_k(v_k - E^2)^2 - E^2(d_{k-1}^2\cdot\mathscr{B}_{k-1} - d_k^2\cdot\mathscr{B}_{k+1})\ . \tag{2.6}$$

It is obvious that the functions $\mathscr{A}_k$ and $\mathscr{B}_k$ can be chosen as the functions on $\lambda = E^2$.

If we consider the system of equations (2.5) and (2.6) for all polynomials $\mathscr{A}_k$, $\mathscr{B}_k$

$$\mathscr{A}_k = a_{0,k}\lambda^{m+1} + \cdots + a_{m,k}\lambda + \hat{a}_{m+1,k}\ , \tag{2.7}$$

$$\mathscr{B}_k = b_{0,k}\lambda^m + \cdots + b_{m,k} \tag{2.8}$$

of the powers $m = 0, 1, 2, \ldots$, we obtain an infinite set of usual and higher-order relativistic Toda lattices.

EXAMPLE. Let the polynomials $\mathscr{A}_k$ and $\mathscr{B}_k$ be as follows:

$$\mathscr{A}_k = a_0(\lambda - (v_k + d_{k-1}^2 - d_k^2)) \tag{2.9}$$

$$\mathscr{B}_k = -2a_0 \tag{2.10}$$

and let a_0 be equal to $\frac{1}{2}$. Then from (2.5) and (2.6) it follows that the system

$$\dot{d}_k = \tfrac{1}{2}d_k(v_k - v_{k+1} + d_{k-1}^2 - d_{k+1}^2) \tag{2.11}$$

$$\dot{v}_k = v_k(d_{k-1}^2 - d_k^2) \tag{2.12}$$

is the relativistic Toda lattice [11] in the Ragnisco–Bruschi variables.

Note that for $v_k \equiv 0$, system (2.11) and (2.12) turns into the Langmuir lattice [7]

$$\dot{d}_k = \tfrac{1}{2}d_k(d_{k-1}^2 - d_{k+1}^2)\ . \tag{2.13}$$

Now we consider the set of all relativistic Toda lattices for general polynomials (2.7) and (2.8). Equating the coefficients at equal powers of the parameter λ in the identities (2.5) and (2.6), we get two chains of recurrence relations

$$-a_{j,k-1} - a_{j,k} - b_{j,k} + v_k b_{j-1,k} = 0 \ , \tag{2.14}$$

$$2a_{j,k} + b_{j,k} - 2v_k a_{j-1,k} - 2v_k b_{j-1,k} + v_k^2 b_{j-2,k} - \\ - d_{k-1}^2 b_{j-1,k-1} + d_k^2 b_{j-1,k+1} = 0 \ . \tag{2.15}$$

Here $j = 0, 1, \ldots, m$ and $a_{l,k} \equiv 0$, $b_{l,k} \equiv 0$ for all $l < 0$.

The coefficient at λ in (2.6) is equal to

$$2\hat{a}_{m+1,k} - 2v_k a_{m,k} - 2v_k b_{m,k} + v_k^2 b_{m-1,k} - d_{k-1}^2 b_{m,k-1} + \\ + d_k^2 b_{m,k+1} = 0 \ . \tag{2.16}$$

Now we introduce new functions $a_{j,k}$, $b_{j,k}$ for $j > m$ so that they obey the relation

$$2a_{m+1,k} + b_{m+1,k} = \hat{a}_{m+1,k} \tag{2.17}$$

and also satisfy the relations (2.14) and (2.15) for $j > m$.

We get the infinite chains (2.14) and (2.15) of recurrence relations with $j = 0, 1, 2, \ldots$

The dynamical system (2.5) and (2.6) yields the dynamical equations

$$\dot{d}_{k-1} = \frac{d_{k-1}}{2}(b_{m+1,k} - b_{m+1,k-1}) \ ,$$

$$\dot{v}_k = -2v_k a_{m+1,k} - v_k b_{m+1,k} + v_k^2 b_{m,k} \ . \tag{2.18}$$

Thus, the infinite recurrence chains (2.14) and (2.15) are separated from all set of dynamical systems (2.18) ($m = 0, 1, 2, \ldots$).

Infinite chains (2.14) and (2.15) are equivalent to simple identities

$$\hat{\mathscr{F}}_{k-1} + \hat{\mathscr{F}}_k - \hat{\mathscr{G}}_k(v_k - \lambda) = 0 \ , \tag{2.19}$$

$$-2\hat{\mathscr{F}}_k(v_k - \lambda) + \hat{\mathscr{G}}_k(v_k - \lambda)^2 - \lambda(d_{k-1}^2 \hat{\mathscr{G}}_{k-1} - d_k^2 \hat{\mathscr{G}}_{k+1}) = 0 \tag{2.20}$$

for the formal series

$$\hat{\mathscr{F}}_k = \sum_{j=0}^{\infty} a_{j,k}\lambda^{-j} \ , \qquad \hat{\mathscr{G}}_k = \sum_{j=0}^{\infty} b_{j,k}\lambda^{-j-1} \ . \tag{2.21}$$

Multiplying (2.19) by $\hat{\mathscr{G}}_k(v_k - \lambda) - \hat{\mathscr{F}}_k + \hat{\mathscr{F}}_{k-1}$ and (2.20) by $\hat{\mathscr{G}}_k$ and summing both relations, we get the identity

$$-\hat{\mathscr{F}}_k^2 + \lambda d_k^2 \hat{\mathscr{G}}_k \hat{\mathscr{G}}_{k+1} = \hat{c}(\lambda) \ , \tag{2.22}$$

where $\hat{c}(\lambda) = \Sigma_{j=0}^{\infty} c_j \lambda^{-j}$ is a formal series with constant coefficients.

A new recurrence chain can be obtained from the identity (2.22)

$$d_k^2 \sum_{i=0}^{j-1} b_{i,k} b_{j-1-i,k+1} - \sum_{i=0}^{j} a_{i,k} a_{j-i,k} = c_j \tag{2.23}$$

Comparing (2.19) and (2.20) with (2.5) and (2.6), we are led to the conclusion that system (2.19) and (2.20) is equivalent to the stationary Lax system (2.3) and (2.4) for a formal F-operator

$$(\hat{F}\psi)_k = \hat{\mathscr{F}}_k \psi_k + \lambda d_k \hat{\mathscr{G}}_k \psi_{k+1} . \tag{2.24}$$

Since the functions $a_{j,k}$ and $b_{j,k}$ are connected to $a_{1,k}$ and $b_{1,k}$ by recurrence chains (2.19), (2.20) and (2.22), we can prove the following theorem.

THEOREM 2. *The system* (2.5) *and* (2.6) *is equivalent to the system of dynamical equations*

$$\dot{\hat{\mathscr{F}}}_k = \lambda d_k^2 (\hat{\mathscr{G}}_k \mathscr{B}_{k+1} - \mathscr{B}_k \hat{\mathscr{G}}_{k+1}) , \tag{2.25}$$

$$\dot{\hat{\mathscr{G}}}_k + \hat{\mathscr{G}}_k \frac{\dot{d}_k}{d_k} = (v_{k+1} - \lambda)(\hat{\mathscr{G}}_k \mathscr{B}_{k+1} - \mathscr{B}_k \hat{\mathscr{G}}_{k+1}) -$$

$$- \hat{\mathscr{G}}_k (\mathscr{A}_{k+1} - \mathscr{A}_k) + \mathscr{B}_k (\hat{\mathscr{F}}_{k+1} - \hat{\mathscr{F}}_k) \tag{2.26}$$

and stationary equations (2.19) *and* (2.20).

3. Finite-Zone Dynamical System

Suppose that almost all the coefficients vanish in stationary equations (2.19)–(2.22) and in dynamical equations (2.25) and (2.26)

$$a_{n+1+j,k} \equiv 0 , \qquad b_{n+j,k} \equiv 0 , \quad \text{for all } j > 0 . \tag{3.1}$$

We introduce new functions

$$\mathscr{F}_k = \lambda^{n+1} \hat{\mathscr{F}}_k , \qquad \mathscr{G}_k = \lambda^{n+1} \hat{\mathscr{G}}_k , \qquad C(\lambda) = \lambda^{2n+2} \hat{C}(\lambda) \tag{3.2}$$

and get the following stationary equations

$$\mathscr{F}_k + \mathscr{F}_{k-1} - \mathscr{G}_k (v_k - \lambda) = 0 , \tag{3.3}$$

$$- \mathscr{F}_k^2 + \lambda d_k^2 \mathscr{G}_k \mathscr{G}_{k+1} = C(\lambda) \tag{3.4}$$

and dynamical systems

$$\dot{\mathscr{F}}_k = \lambda d_k^2 (\mathscr{G}_k \mathscr{B}_{k+1} - \mathscr{B}_k \mathscr{G}_{k+1}) , \tag{3.5}$$

$$\dot{\mathscr{G}}_k + \mathscr{G}_k \frac{\dot{d}_k}{d_k} = (v_{k+1} - \lambda)(\mathscr{G}_k \mathscr{B}_{k+1} - \mathscr{B}_k \mathscr{G}_{k+1}) -$$

$$- \mathscr{G}_k (\mathscr{A}_{k+1} - \mathscr{A}_k) + \mathscr{B}_k (\mathscr{F}_{k+1} - \mathscr{F}_k) , \tag{3.6}$$

where $\mathscr{F}_k$, $\mathscr{G}_k$, $\mathscr{A}_k$, $\mathscr{B}_k$, $C(\lambda)$ are polynomials

$$\mathscr{F}_k(\lambda) = a_{0,k} \lambda^{n+1} + \cdots + a_{n+1,k} , \tag{3.7}$$

$$\mathscr{G}_k(\lambda) = b_{0,k} \lambda^n + \cdots + b_{n,k} , \tag{3.8}$$

$$\mathscr{A}_k(\lambda) = a_{0,k}\lambda^{m+1} + \cdots + \hat{a}_{m+1,k} , \tag{3.9}$$

$$\mathscr{B}_k(\lambda) = b_{0,k}\lambda^{m} + \cdots + b_{m,k} , \tag{3.10}$$

$$C(\lambda) = c_0\lambda^{2n+2} + \cdots + c_{2n+2} . \tag{3.11}$$

Now we go over from the variables $b_{j,k}$ to new root variables. Let b_0 be equal to -1. Represent the polynomial $\mathscr{G}_k$ in the form

$$\mathscr{G}_k = \prod_{j=1}^{n} (\lambda - \gamma_{j,k}) . \tag{3.12}$$

If we set $\lambda = \gamma_{j,k}$ in equations (3.4) and (3.6), we obtain the equations in root variables

$$(\mathscr{F}_k|_{\lambda=\gamma_{j,k}})^2 = -C(\gamma_{j,k}) \tag{3.13}$$

and

$$\dot{\mathscr{G}}_k|_{\lambda=\gamma_{j,k}} = -2\mathscr{B}_k\mathscr{F}_k|_{\lambda=\gamma_{j,k}} = \mp 2\mathscr{B}_k(\gamma_{j,k})\sqrt{-C(\gamma_{j,k})} \tag{3.14}$$

From (3.12)–(3.14) follows the dynamical system of equations

$$\dot{\gamma}_{j,k} = \frac{\mp 2\sqrt{-C(\gamma_{j,k})}}{\Pi_{s\neq j}(\gamma_{j,k} - \gamma_{s,k})}\mathscr{B}_k(\gamma_{j,k}) \tag{3.15}$$

The stationary system can be obtained from (3.3) and (3.4) by putting $\lambda = \gamma_{j,k}$. We have

$$\mathscr{F}_k(\gamma_{j,k}) + \mathscr{F}_{k-1}(\gamma_{j,k}) = 0 , \tag{3.16}$$

$$\mathscr{G}_{k+1}(\gamma_{j,k}) = \frac{\dfrac{\partial}{\partial\lambda}(\mathscr{F}_k^2 + C(\lambda))|_{\lambda=\gamma_{j,k}}}{d_k^2\,\gamma_{j,k}\,\Pi_{s\neq j}(\gamma_{j,k} - \gamma_{s,k})} . \tag{3.17}$$

System (3.15) can be directly reduced to the Jacobi system of inversion

$$\frac{1}{2}\sum_j \int_{\gamma_{j,k}^0}^{\gamma_{j,k}} \frac{\gamma_{j,k}^{n-l}}{\sqrt{-C(\gamma_{j,k})}}\, d\gamma_{j,k} = \mp\delta_{m+1}^{l}(t - t_0) \quad l = 1, \ldots, n \tag{3.18}$$

on the Riemann surface

$$W^2 = -C(\lambda) .$$

Here δ is the Kronecker symbol

$$\delta_{m+1}^{l} = \begin{cases} 0, & \text{if } l \neq m+1 , \\ 1, & \text{if } l = m+1 . \end{cases}$$

We can also find a new Hamiltonian structure for the system of equations (3.15)–(3.17).

First of all we introduce impulses $W^{\gamma}_{j,k}$ conjugating to $\gamma_{j,k}$

$$W^{\gamma}_{j,k} = \mathscr{F}_k(\gamma_{j,k}) \ . \tag{3.19}$$

Then we introduce a Poisson structure by a skew-symmetric matrix

$$\begin{aligned} P_{i,n+j} &= \delta^i_j \gamma_{j,k} \ , \quad \text{for } 1 \leqslant i \leqslant n \ , \\ P_{n+i,j} &= -\delta^i_j \gamma_{j,k} \ , \quad \text{for } 1 \leqslant i \leqslant n \ . \end{aligned} \tag{3.20}$$

It is easy to notice that the Poisson structure (3.20) can be transformed to the standard constant skew-symmetric matrix by changing the variables

$$v_{j,k} = \ln \gamma_{j,k} \ .$$

System (3.15) in new variables is a first part of the Hamiltonian system

$$\dot{\gamma}_{j,k} = \frac{2W^{\gamma}_{j,k}}{\Pi_{s\neq j}(\gamma_{j,k} - \gamma_{s,k})} \mathscr{B}_k(\gamma_{j,k}) \ . \tag{3.21}$$

From (3.13) and (3.19), the second part of the Hamiltonian system can be obtained:

$$(W^{\gamma}_{j,k})^2 + C(\gamma_{j,k}) = 0 \ . \tag{3.22}$$

It is obvious that system (3.21) and (3.22) is a Hamiltonian system with the Hamiltonian

$$\mathscr{H}_D = \sum_{j=1}^{n} \frac{(W^{\gamma}_{j,k})^2 + C(\gamma_{j,k})}{\gamma_{j,k} \Pi_{s\neq j}(\gamma_{j,k} - \gamma_{s,k})} \mathscr{B}_k(\gamma_{j,k}) \tag{3.23}$$

and with Poisson structure (3.20).

Now we introduce new 'action-angle' variables and then reduce the finite-zone problem to the Jacobi problem of inversion.

Let S be the action function

$$S(\gamma_{1,k}, \ldots, \gamma_{n,k}; H_1, \ldots, H_n) = \sum_{j=1}^{n} \int_{\gamma^0_{j,k}}^{\gamma_{j,k}} \frac{\sqrt{-C(\gamma_{j,k})}}{\gamma_{j,k}} \mathrm{d}\gamma_{j,k} \tag{3.24}$$

where $H_l = C_{n+1+l}$.

We define the impulse conjugate to H_l as derivatives of the function S

$$W^H_{l,k} = -\frac{\partial S}{\partial H_l} = \tfrac{1}{2} \sum_{j=1}^{n} \int_{\gamma^0_{j,k}}^{\gamma_{j,k}} \frac{\gamma^{n-l}_{j,k}}{\sqrt{-C(\gamma_{j,k})}} \mathrm{d}\gamma_{j,k} \tag{3.25}$$

Using the equations of Hamiltonian system (3.21) and (3.22), we have

$$\frac{\mathrm{d}W^H_{l,k}}{\mathrm{d}t} = \tfrac{1}{2} \sum_{j=1}^{n} \frac{\gamma^{n-l}_{j,k}}{\sqrt{-C(\gamma_{j,k})}} \dot{\gamma}_{j,k} = \sum \frac{\gamma^{n-l}_{j,k}}{\sqrt{-C(\gamma_{j,k})}} \mathscr{B}_k(\gamma_{j,k}) = \delta^l_{m+1} \ . \tag{3.26}$$

After integrating this equation, the Jacobi system of inversion on the Riemann surface $W^2 = -C(\gamma)$ has the form

$$\frac{1}{2}\sum_{j=1}^{n}\int_{\gamma_{j,k}^{0}}^{\gamma_{j,k}} \frac{\gamma_{j,k}^{n-l}}{\sqrt{-C(\gamma_{j,k})}}\,d\gamma_{j,k} = \delta_{m+1}^{l}(t-t_0) + W_{l,k}^{\gamma}(t_0)\ . \tag{3.27}$$

Finally, the finite-zone potentials can be found in the form

$$d_k^2 = \frac{1}{b_0^2}\,(c_1 + 2a_0 a_{1,k})\ , \tag{3.28}$$

$$v_k = \frac{1}{b_0}\left(\sum_{j=1}^{n}\gamma_{j,k} - b_0(d_{k-1}^2 - d_k^2) + 2a_{1,k}\right)\ , \tag{3.29}$$

where

$$a_{1,k} = \sum_{j=1}^{n}\frac{\sqrt{-C(\gamma_{j,k})} - \sqrt{-c_0}\,\gamma_{j,k}^{n+1} - \sqrt{-c_{2n+2}}}{\gamma_{j,k}\,\Pi_{s\neq j}(\gamma_{j,k} - \gamma_{s,k})}\ .$$

Acknowledgements

This work was produced at the Institute of Physics of the Rome University 'La Sapienza' and at the Courant Institute of Mathematical Sciences of New York University.

It is a pleasure to thank F. Calogero, A. Degasperis, O. Ragnisco, and M. Bruschi. I would also like to express deep gratitude to P. . Lax.

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