EXISTENCE OF STAR-PRODUCTS AND OF FORMAL DEFORMATIONS OF THE POISSON LIE ALGEBRA OF ARBITRARY SYMPLECTIC MANIFOLDS

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ABSTRACT. We prove the existence of star-products and of formal deformations of the Poisson Lie algebra of an arbitrary symplectic manifold. Moreover, all the obstructions encountered in the step-wise construction of formal deformations are vanishing.

0. INTRODUCTION

In the Hamiltonian formulation of classical mechanics, a phase space is nothing else but a symplectic manifold. Passing to quantum theory in the classical way implies a fundamental change in the nature of observables and makes the interpretation of the classical theory as a limit of the new one uneasy in many respects. An important aspect of quantization is its relation to deformations of classical theories. In that spirit, Flato, Lichnerowicz and Sternheimer have proposed building up quantum mechanics on an ordinary phase space in such a way that quantization manifests itself in a deformation of the algebra of observables. The value of the parameter of deformation is closely related to the Planck constant and letting it tend to zero gives back classical mechanics as a limit case. An account of the deformation approach to quantization can be found in [1, 6].

The algebra of observables is the space of smooth functions over a symplectic manifold with its natural structure of associative algebra and the appropriate deformations of this structure are called the star-products. The first star-product appeared as the inverse Weyl transform of the product of operators (Moyal [9]). It was rediscovered by Vey [10] who also proved the existence of nontrivial deformations of the Poisson Lie algebra structure for a symplectic manifold with a vanishing third De Rham cohomology group. The result was extended to associative deformations by Neroslavsky and Vlassov [8] under the same assumption. In the mean time, various classes of manifolds where this assumption is not necessary have been exhibited [4].

The cohomological obstructions appear as follows: a star-product or a deformation of the Poisson bracket are usually constructed step by step.

In passing from step k to step $k+1$, one encounters a Chevalley or a Hochschild cocycle which should be a coboundary to allow the construction to continue. The work of Vey and Neroslavsky and Vlassov consists of confining this cocycle in the De Rham cohomology.

We show in this paper that there exists no obstruction at all: each formal deformation of order

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k of the Poisson bracket extends to a formal deformation and a similar result holds true for star-products.

The basic tools are, first, cohomological properties of the Nijenhuis–Richardson bracket, showing in particular that the bracket of a one-differentiable cocycle with an arbitrary cocycle is always exact. Secondly, if ξ is a conformal nonsymplectic vector field for the symplectic form F of M, homogeneity with respect to ξ allows us to avoid the obstructions. This was first observed in an analytic setting in [2]. An algebraic interpretation led to the proof of the existence of star-products for exact symplectic manifolds [4, 5]. A further refinement combined with gluing allows us to use this type of argument for nonexact F.

1. NOTATIONS AND DEFINITIONS

We will mainly use the notations and definitions of [5]. Some of them have just to be précised.

Let M be a smooth connnected Hausdorff second countable manifold equipped with a symplectic form F. We suppose $\dim M > 2$. We denote by $\Lambda(M)$ the space of smooth forms on M and by $\mathcal{X}(M)$ the space of smooth vector fields on M. As usual, we set $N = \Lambda^0(M)$ and L_X denotes the Lie derivative in the direction of $X \in \mathcal{X}(M)$ acting on $\Lambda(M)$.

If V and V' are vector spaces, $A^p(V, V')$ is the space of $(p+1)$-linear alternating maps from V^{p+1} into V' and $A(V, V')$ is the direct sum of the $A^p(V, V')$'s $(p \geqslant -1)$. For simplicity, we set $A^p(V) = A^p(V, V)$ and $A(V) = A(V, V)$. It is known that $(A(V), [\,,\,])$, where $[\,,\,]$ is the Nijenhuis–Richardson bracket, is a graded Lie algebra, the degree of $C \in A^p(V)$ being p. If $\mathscr{P}$ is a Lie algebra structure on V, the Chevalley coboundary operator of the adjoint representation of $(V, \mathscr{P})$ is up to ± 1 the adjoint action of $\mathscr{P}$ on $A(V)$.

We denote by $A_{\mathrm{loc}}(\mathcal{X}(M), \Lambda^q(M))$ [resp. $A_{\mathrm{loc,\,nc}}(N)$] the space of all $C \in A(\mathcal{X}(M), \Lambda^q(M))$ [resp. $A(N)$] which are local [resp. local and vanishing on the constants]. We set also $A_\nu(N) = E(A(N), \nu]$ and $A_{\nu,\mathrm{loc,\,nc}}(N) = E(A_{\mathrm{loc,\,nc}}(N), \nu)$, where $E(V, \nu)$ denotes the space of formal power series in ν with coefficients in V.

The mapping $\mu: \mathcal{X}(M) \to \Lambda^1(M): X \to -i(X)F$ induces an isomorphism between the spaces of contravariant and covariant tensor fields on M. One sets $\Lambda = \mu^{-1}F$ and, for $u \in N$, $X_u = \mu^{-1}\, du$.

For $C \in A^p(\mathcal{X}(M), \Lambda^q(N))$, set $\mu^*C(u_0, \ldots, u_p) = C(X_{u_0}, \ldots, X_{u_p})$ and, if $q = 2$, $\mu' C = \langle \Lambda, \mu^* C \rangle$.

In particular, $P = \mu^* F$ is the Poisson bracket of M ($\Lambda(M)$ being identified once and for all with a subspace of $A(\mathcal{X}(M), N)$ in the natural way). Denote by $Z^2_{\mathrm{loc,\,nc}}(N)$ the space of Chevalley two-cocycles of (N, P). Then ∂ being the coboundary operator and Γ a symplectic connection of (M, F), each $C \in Z^2_{\mathrm{loc,\,nc}}$ is of the form $C = rS^3_\Gamma + \mu^*\Omega + \partial E$ ($r \in \mathbb{R}$, $\Omega \in \Lambda^2(M)$ closed and $E \in A^0_{\mathrm{loc,\,nc}}(N)$); S^3_Γ is a cocycle of the form $\mu' \Phi_\Gamma$, where Φ_Γ is a cocycle of the Chevalley cohomology of the representation $(\Lambda(M), X \to L_X)$ of $\mathcal{X}(M)$. Further details may be found in [3, 5, 7].

PROPOSITION 1.1. *There exists a linear map* $\tau: A^1_{\mathrm{loc,\,nc}}(N) \to A^1_{\mathrm{loc}}(\mathcal{X}(M), N)$ *such that* $\mu^* \circ \tau = \mathbb{1}$, $\tau \circ \mu^* = \mathbb{1}$ *on* $\Lambda^2(M)$, $\tau(S^3_\Gamma) = \langle \Lambda, \phi_\Gamma \rangle$ *and* $\tau \circ \partial A^0_{\mathrm{loc,\,nc}}(N) \subset \operatorname{im} \partial'$ ($\mathbb{1}$ *denotes the appropriate identical map*; ∂' *is the coboundary operator of the cohomology of the representation* $(\Lambda(M), X \to L_X)$ *of* $\mathcal{X}(M)$.)

Proof. It is easily seen that $\mu^*: A^1_{\mathrm{loc}}(\mathcal{X}(M), N) \to A^1_{\mathrm{loc,\,nc}}(N)$ is surjective. Take then a decomposition

$$A^1_{\mathrm{loc,nc}}(N) = \mathbb{R}S^3_\Gamma \oplus \mu^*\Lambda^2(M) \oplus \partial E_1 \oplus E_2,$$

where

$$\partial A^0_{\mathrm{loc,nc}}(N) = \mu^*\Lambda^2(M) \cap \partial A^0_{\mathrm{loc,nc}}(N) \oplus \partial E_1.$$

Observe that $\mu^*\colon \Lambda^2(M) \to A^1_{\mathrm{loc,nc}}(N)$ is injective. Choose then any right inverse of μ^*, τ_1 on E_1 and τ_2 and E_2 and set $\tau S^3_\Gamma = \langle \Lambda, \Phi_\Gamma \rangle$, $\tau \circ \mu^*|_{\Lambda^2(M)} = \mathbb{1}$, $\tau \circ \partial = \partial' \circ \tau_1$ on E_1 and $\tau = \tau_2$ on E_2. This defines a linear map $\tau\colon A^1_{\mathrm{loc,nc}}(N) \to A^1_{\mathrm{loc}}(\mathcal{H}(M), N)$, which obviously verifies the first three required properties.

For the last property, one notes that by [5, prop. 2.2], $\mu^*\Lambda(M) \cap \partial A^0_{\mathrm{loc,nc}}(N) = \partial\mu^*\Lambda^1(M)$ and that on $\Lambda^1(M)$, $\tau \circ \partial \circ \mu = \tau \circ \mu^* \circ \partial' = \partial'$.

In this paper, we fix once and for all a τ such as in Proposition 1.1.

LEMMA 1.2. *Let $R_i \in A_{\mathrm{loc,nc}}(N)$, $S_i \in A^{s_i}_{\mathrm{loc,nc}}(N)$ and a closed $\Omega \in \Lambda^2(M)$ be given. Let also $R'_i, S'_i, T' \in A_{\mathrm{loc}}(\mathcal{H}(M); N)$ be such that $R_i = \mu^* R'_i$, $S_i = \mu^* S'_i$ and $\Sigma_i [R_i, S_i] = \mu^* T'$. There exists $T \in A_{\mathrm{loc,nc}}(N)$ such that*

$$T|_U = \mu^* i(X) T' - \sum_i (-1)^{s_i} [\mu^* i(X) R'_i, S_i] - \sum_i [R_i, \mu^* i(X) S'_i] \tag{1}$$

whenever $\Omega = \mathrm{d}i(X)F$ on an open subset $U \subset M$.

Proof. If $\Omega|_{U'} = \mathrm{d}i(X')F$ and $U \cap U' \neq \phi$, then $X' - X$ is a symplectic vector field on $U \cap U'$. It is, thus, locally of the form X_u. Since $\mu^* i(X_u) = i(u)\mu^*$, the right-hand side of (1) vanishes for $X = X_u$ by the graded Jacobi identity.

2. ONE-DIFFERENTIABLE DEFORMATIONS OF P

Recall that a *formal deformation of order k of P* is an element $\Sigma_{k=0}^\infty \nu^k C_k$ of $A^1_{\nu,\mathrm{loc,nc}}(N)$ such that $C_0 = P$ and such that $[\mathcal{L}_\nu, \mathcal{L}_\nu]$ vanishes at order k (i.e., the components $[\mathcal{L}_\nu, \mathcal{L}_\nu]_l$ of $[\mathcal{L}_\nu, \mathcal{L}_\nu]$ are vanishing for $l \leqslant k$). A *formal deformation of P* is an element $\mathcal{L}_\nu \in A^1_{\nu,\mathrm{loc,nc}}(N)$ such that $C_0 = P$ and that $[\mathcal{L}_\nu, \mathcal{L}_\nu] = 0$.

THEOREM 2.1. *Let $\mathcal{L}_\nu$ be a formal deformation of P and let $\Omega \in \Lambda^2(M)$ be closed.*

(i) *There exists a sequence $\mathbb{L}^k_\nu$ ($k \in \mathbb{N}$) of elements of $A_{\nu,\mathrm{loc,nc}}(N)$ such that $\mathbb{L}^0_\nu = \mathcal{L}_\nu$ and that*

$$l\mathbb{L}^l_\nu|_U = \sum_{p+q=l-1} [\mathbb{L}^p_\nu, \mu^* i(X)\tau(\mathbb{L}^q_\nu)] \tag{2}$$

whenever $\Omega = \mathrm{d}i(X)F$ on the open subset $U \subset M$.

(ii) $\mathbb{L}^1_\nu$ *is a cocycle for $\mathcal{L}_\nu$ and its first term is $\mu^*\Omega$. More generally,*

$$\mathbb{L}_\mu(\mathcal{L}_\nu;\Omega) = \sum_{k=0}^{\infty} \mu^k \mathbb{L}_\nu^k$$

is a formal deformation in μ of $\mathcal{L}_\nu$.

(iii) *In particular, for each $t \geqslant 1$,*

$$\mathcal{L}'_\nu = \sum_{k=0}^{\infty} \nu^{kt} \mathbb{L}_\nu^k$$

is a formal deformation of P such that $\mathcal{L}'_\nu = \mathcal{L}_\nu + \nu^t \mu^* \Omega$ at order t.

(iv) *If $\mathcal{L}_\nu$ is one-differentiable at order s (i.e., C_l is one-differentiable for $l \leqslant s$), then $\mathbb{L}_\nu^k$ ($k \in \mathbb{N}$) and $\mathcal{L}'_\nu$ are one-differentiable at order s.*

(If it is needed to recall $\mathcal{L}_\nu$ and Ω, we sometimes will write $\mathbb{L}_\nu^k = \mathbb{L}_\nu^k \ddagger (\mathcal{L}_\nu;\Omega)$).

Proof. Assume the existence of solutions $\mathbb{L}_\nu^l$ ($l < k$) of (2) such that

$$\sum_{p+q=l} [\mathbb{L}_\nu^p, \mathbb{L}_\nu^q] = 0, \quad \forall l < k. \tag{3}$$

From Lemma 1.2, we obtain the existence of a globally defined $\mathbb{L}_\nu^k$ verifying (2) for $l = k$. Then applying (2) and the graded Jacobi identity:

$$k \sum_{p+q=k} [\mathbb{L}_\nu^p, \mathbb{L}_\nu^q]$$

$$= \sum_{p+q+r=k-1} \{[\mathbb{L}_\nu^p, \mu^* i(X)\tau(\mathbb{L}_\nu^r)], \mathbb{L}_\nu^q] + [\mathbb{L}_\nu^p, [\mathbb{L}_\nu^q, \mu^* i(X)\tau(\mathbb{L}_\nu^r)]]\}$$

$$= \sum_{p+q+r=k-1} [[\mathbb{L}_\nu^p, \mathbb{L}_\nu^q], \mu^* i(X)\tau(\mathbb{L}_\nu^r)] = 0$$

in view of (3). Hence, (i), (ii) and (iii). For (iv), one has simply to note that the Nijenhuis--Richardson bracket of one-differentiable cochains is one-differentiable.

Let us say that a formal deformation $\mathcal{L}_\nu$ of order k is a *driver* of a formal deformation $\mathcal{L}'_\nu$ if $\mathcal{L}'_\nu = \mathcal{L}_\nu$ at order k.

THEOREM 2.2. *Every one-differentiable formal deformation of order k of P is a driver of a one-differentiable formal deformation of P. In particular, there exist one-differentiable formal deformations of P with driver $P + \nu\mu^*\Omega$, where Ω is an arbitrary closed two-form of M.*

Proof. Let $\mathcal{L}_\nu$ be a one-dimensional formal deformation of order k of P. Let $0 \leqslant l < k$ be given. Suppose that there exists a one-differentiable formal deformation $\mathcal{L}_\nu^l$ of P such that $C_i^l = C_i$ for $i \leqslant l$. Then $C_{l+1}^l - C_{l+1}$ is a one-differentiable cocycle. It is thus of the form $\mu^*\Omega$ for some closed two-form Ω. By theorem 2.1

$$\mathcal{L}_\nu^{l+1} = \mathbb{L}_{\nu^{l+1}}(\mathcal{L}_\nu^l;\Omega)$$

is a one-differentiable formal deformation of P such that $C_i^{l+1} = C_i$ for $i \leqslant l+1$. Taking $\mathcal{L}_\nu^0 = P$, the result follows by induction on l.

3. FORMAL DEFORMATIONS OF P WITH DRIVER $P + \nu r S_\Gamma^3$

Let U be open and suppose that $F|_U = \mathrm{d}i(\xi)F$ for some $\xi \in \mathcal{K}(U)$. It has been seen in [6] that

LEMMA 3.1.

(i) $L_\xi P = P$,

(ii) $L_\xi \circ \partial = \partial \circ L_\xi - \partial$,

(iii) $L_\xi \circ \partial' = \partial' \circ L_\xi$,

(iv) $L_\xi \circ \mu^* = \mu^* \circ L_\xi - p\mathbb{1}$ *on* $A^{p-1}(\mathcal{K}(M), N)$,

(v) $L_\xi \circ \mu' = \mu' \circ L_\xi - p\mathbb{1}$ *on* $A^p(\mathcal{K}(M), \Lambda^2(M))$.

In addition, one may state

LEMMA 3.2. *For* $p \neq 2, 3$, $L_\xi + p\mathbb{1} - \partial \circ \mu^* \circ i(\xi) \circ \tau$ *is a linear proof bijection from* $Z^2_{\mathrm{loc,nc}}(N)$ *into itself.*

Proof. Recall that $C \in Z^2_{\mathrm{loc,nc}}(N)$ admits a decomposition $C = rS_\Gamma^3 + \mu^*\Omega + \partial E$ where $\Omega \in \Lambda^2(M)$ is closed, $r \in \mathbb{R}$. Moreover, C is exact if and only if $r = 0$ and Ω is exact.

By Proposition 1.1, $C' = \mu^*\Omega + \partial E = \mu^*\tau(C')$ where $\tau(C')$ is a cocycle for ∂'. Using Lemma 3.1,

$$(L_\xi + p\mathbb{1} - \partial \circ \mu \circ i(\xi) \circ \tau)(rS_\Gamma^3 + C') = r(p-3)S_\Gamma^3 + (p-2)C'. \tag{4}$$

Thus $L_\xi + p\mathbb{1} - \partial \circ \mu^* \circ i(\xi) \circ \tau$ is surjective. Moreover, if the right-hand side of (4) is vanishing, $r = 0$ and $C' = 0$. Hence, the injectivity.

Let $\Pi \in A^0(E(N, \nu))$ be defined by

$$\Pi\left(\sum_{k=0}^{\infty} \nu^k u_k\right) = \sum_{k=0}^{\infty} k\nu^k u_k$$

and set $\Theta = 2\Pi - \mathbb{1}$. As can easily be seen, for every $T_\nu \in A_\nu(N)$,

$$T_\nu = \sum_{k=0}^{\infty} \nu^k T_k \Rightarrow [\Pi, T_\nu] = \sum_{k=0}^{\infty} k\nu^k T_k.$$

THEOREM 3.3. *Let* $r \in \mathbb{R}_0$ *be given. There exists a unique formal deformation* $\mathcal{L}_\nu$ *of* P *of driver* $P + \nu r S_\Gamma^3$ *such that*

$$\mathbb{L}_\nu^1(\mathcal{L}_\nu; F) + \partial_\nu \Theta = 0. \tag{5}$$

(∂_ν *is the coboundary operator associated to the adjoint representation of* $(E(N, \nu), \mathcal{L}_\nu)$).

Proof. (i) If U is open and $F|_U = di(\xi)F$ for some $\xi \in \mathcal{H}(U)$, after its definition in Theorem 2.1, $\mathbb{L}_\nu(\mathcal{L}; F)$ reads

$$\sum_{k=0}^{\infty} \nu^k \sum_{p+q=k} [C_p, \mu^* i(\xi)\tau(C_q)].$$

Thus, the kth component of the left-hand side of (5) is

$$\sum_{p+q=k} [C_q, \mu^* i(\xi)\tau(C_q)] - (2k+1)C_k.$$

Observe, moreover, that $L_\xi = \mu^* i(\xi)F = \mu^* i(\xi)\tau(C_0)$. Hence, the above expression can be written, for $k > 0$,

$$-L_\xi C_k - (2k+1)C_k + \partial\mu^* i(\xi)\tau(C_k) + \sum_{\substack{p+q=k \\ p,q>0}} [C_p, \mu^* i(\xi)\tau(C_q)].$$

(ii) Let $\mathcal{L}_\nu$ and $\mathcal{L}'_\nu$ have driver $P + \nu r S^3_\Gamma$ and satisfy (5). Assume that $k > 1$ and that $C'_i = C_i$ for $i < k$. Then

$$(L_\xi + (2k+1)\mathbb{1} - \partial\mu^* i(\xi)\tau)(C'_k - C_k) = 0.$$

It follows from Lemma 3.2 that $C'_k = C_k$ on U. Thus $C'_k = C_k$, hence, the uniqueness by induction on k.

(iii) Observe that (5) means that for every open subset U on which $F = di(\xi)F$, $\mathscr{D}_U = \mu^* i(\xi)\tau(\mathcal{L}_\nu) + \Theta$ is a derivation of $\mathcal{L}_{\nu|U}$. Set $C_0 = P$, $C_1 = rS^3_\Gamma$, $\mathcal{L}^1_\nu = C_0 + \nu C_1$ and $\mathscr{D}^1_U = L_\xi + \mu^* i(\xi)\tau(S^3_\Gamma) + \Theta$. One has

$$[\mathscr{D}^1_U, \mathcal{L}^1_\nu]_0 = L_\xi P + P = 0$$

and

$$[\mathscr{D}^1_U, \mathcal{L}^1_\nu]_1 = r(L_\xi S^3_\Gamma + 3S^3_\Gamma - \partial\mu^* i(\xi)\tau(S^3_\Gamma)) = 0$$

by Lemma 3.1.

Let now $k > 1$ and suppose that the C_i's $\in A^1_{\mathrm{loc,nc}}(N)$ $(i < k)$ have been constructed such that $\mathcal{L}^{k-1}_\nu = \Sigma_{i<k}\nu^i C_i$ is a formal deformation of P of order $k-1$ and that $\mathscr{D}^{k-1}_U = \Sigma_{i<k}\nu^i\mu^* i(\xi)\tau(C_i) + \Theta$ is a derivation of $\mathcal{L}^{k-1}_\nu$ at order $k-1$. Set

$$J_k = \sum_{\substack{p+q=k \\ p,q>0}} [C_p, C_q] \quad \text{and} \quad A^k_U = \sum_{\substack{p+q=k \\ p,q>0}} [C_p, \mu^* i(\xi)\tau(C_q)].$$

Then J_k is known to be a cocycle. As seen in [5], it has a decomposition. $J_k = \mu'\Phi + \mu^*\Psi$ when Φ and Ψ are three-cocycles of the cohomology of the representation $(\Lambda(M), X \to L_X)$ of $\mathcal{H}(M)$. Set $J'_k = \langle \Lambda, \Phi \rangle + \Psi$. By Lemma 1.2, there exists $A \in A^2_{\mathrm{loc,nc}}(N)$ such that

$$A|_U = 2A^k_U - \mu^* i(\xi) J'_k.$$

On the other hand, the coefficient of ν^k of the identity

$$[\mathcal{D}^{k-1}_U, [\mathcal{L}^{k-1}_\nu, \mathcal{L}^{k-1}_\nu]] = -2[\mathcal{L}^{k-1}_\nu, [\mathcal{L}^{k-1}_\nu, \mathcal{D}^{k-1}_U]]$$

reads

$$(L_\xi + 2k + 2)J_k = 2\,\partial A^k_U \tag{6}$$

so that, applying Lemma 3.1,

$$(2k-2)\mu'\Phi + (2k-1)\mu^*\Psi = \partial(2A^k_U - \mu^* i(\xi) J'_k) = \partial A.$$

It follows that $\mu'\Phi$ and $\mu^*\Psi$ are coboundaries [5], prop. 2.3, and thus $J_k = 2\,\partial C$ for some $C \in A^1_{\mathrm{loc,nc}}(N)$.

Substituting $2\partial C$ to J_k in (6) and observing that $\partial L_\xi = L_\xi \partial + \partial$, we see that $L_\xi C + (2k+1)C - A^k_U$ is a cocycle. It also reads, using Lemma 3.1 (iv),

$$\partial\mu^* i(\xi)\tau(C) + \mu^* i(\xi)\,\partial'\tau(C) - A^k_U + (2k-1)C.$$

Therefore, $B \in A^1_{\mathrm{loc,nc}}(N)$ defined by

$$B|_U = \mu^* i(\xi)\,\partial'\tau(C) - A^k_U + (2k-1)C.$$

is a cocycle (cf. Lemma 1.2) and

$$L_\xi C + (2k+1)C - \partial\mu^* i(\xi)\tau(C) - A^k_U - B = 0. \tag{7}$$

In view of the properties of τ, the cocycle B has a decomposition $B = r'S^3_\Gamma + \mu^* B'$ where $B' = \tau(B - r'S^3_\Gamma)$ is a cocycle. Taking

$$C_k = C + (2k-1)^{-1}\mu^* B' + (2k-2)^{-1} r' S^3_\Gamma,$$

we still have $J_k = 2\partial C_k$ and, moreover, (7) transforms into

$$L_\xi C_k + (2k+1)C_k - \partial\mu^* i(\xi)\tau(C_k) - A^k_U = 0.$$

This means that $\mathcal{L}^k_\nu = \mathcal{L}^{k-1}_\nu + \nu^k C_k$ is a formal deformation of P of order k and that $\mathcal{D}^k_U = \mathcal{D}^{k-1}_U +$

$\nu^k \mu^* i(\xi)\tau(C_k)$ is a derivation of order k of $\mathcal{L}_\nu^k|_U$. Hence, the existence of $\mathcal{L}_\nu$, by induction on k.

Let $T \in A^0_{\mathrm{loc,nc}}(N)$ or $T = r\mathrm{II}(r \in \mathbb{R})$. Denote by ad T its adjoint action on $A(E(N, \nu))$. It is easily seen that, for $t \geqslant 1$, if $\mathcal{L}_\nu$ is a formal deformation of P, then so is

$$\mathrm{Ad}(\exp \nu^t T)\mathcal{L}_\nu = \sum_{k=0}^{\infty} \frac{\nu^{tk}}{k!} (\mathrm{ad}\, T)^k \mathcal{L}_\nu.$$

THEOREM 3.4. *Every formal deformation of order k of P is a driver of a formal deformation of P.*

Proof. (i) Let $\mathcal{L}_\nu$ be a formal deformation of order k of P. Let us assume that $\mathcal{L}_\nu$ is one-differentiable at order $t-1$, where $t < k$. As

$$2\,\partial C_t = \sum_{\substack{p+q=t \\ p,q>0}} [C_p, C_q]$$

is one-differentiable, we see that C_t is of the form $\theta S^3_\Gamma + \mu^*\eta + \partial T$ for some $\theta \in \mathbb{R}$, some $\eta \in \Lambda^2(M)$ and some $T \in A^0_{\mathrm{loc,nc}}(N)$. If $\theta = 0$, then $\mathrm{Ad}(\exp \nu^t T)\mathcal{L}_\nu$ is a formal deformation of order k of P and is one-differentiable at order t. Moreover, $\mathcal{L}_\nu$ is a driver for $\mathcal{L}'_\nu$ if and only if $\mathrm{Ad}(\exp \nu^t T)\mathcal{L}_\nu$, truncated at order k, is a driver for $\mathrm{Ad}(\exp \nu^t T)\mathcal{L}'_\nu$). Thus, replacing $\mathcal{L}_\nu$ by

$$\mathrm{Ad}(\exp \nu^{k-1} T_{k-1}) \circ \cdots \circ \mathrm{Ad}\,(\exp \nu T_1)\mathcal{L}_\nu$$

for suitable T_i's $\in A^0_{\mathrm{loc,nc}}(N)$ if necessary, we may assume that either $\mathcal{L}_\nu$ is one-differentiable or that there exists $s \leqslant k$ such that C_i is one-differentiable for $i < s$ and that $C_s - rS^3_\Gamma$ is one-differentiable for some $r \in \mathbb{R}_0$.

(ii) In the first case, we may conclude by Theorem 2.2. Let us now deal with the second case. Let $\mathcal{L}^*_\nu$ be a formal deformation of P with driver $P + \nu r S^3_\Gamma$ and define formal deformations $\mathcal{L}^0_\nu, \ldots, \mathcal{L}^s_\nu$ of P inductively by $\mathcal{L}^0_\nu = \mathcal{L}^*_{\nu^s}$ and, for $i > 0$,

$$\mathcal{L}^i_\nu = \mathbb{L}_{\nu^i}(\mathcal{L}^{i-1}_\nu; \tau(C_i - C^{i-1}_i)).$$

It is easily seen that $\mathcal{L}^s_\nu = \mathcal{L}_\nu$ at order s. If $s = k$, the proof is achieved. Suppose then that $s < k$ and that we have found a formal deformation $\mathcal{L}'_\nu$ of P such that $\mathcal{L}'_\nu = \mathcal{L}_\nu$ at order l, with $s < l < k$. Then the cocycle $C'_{l+1} - C_{l+1}$ is of the form $r'S^3_\Gamma + \mu^*\Omega + \partial T$ for some $r' \in \mathbb{R}$, some closed $\Omega \in \Lambda^2(M)$ and some $E \in A^0_{\mathrm{loc,nc}}(N)$. Replacing $\mathcal{L}'_\nu$ by $\mathrm{Ad}(\exp \nu^{l+1} T)\mathcal{L}'_\nu$, we may assume that $T = 0$. On the other hand, set

$$\mathcal{L}''_\nu = \mathrm{Ad}(\exp \nu^{l-s+1}\left(\frac{r'}{sr}\right) \mathrm{II}\ \mathcal{L}_\nu.$$

Then, $C''_i - C_i$ is one-differentiable for $i \leqslant l+1$ and vanishes for $i \leqslant l-s$. Define formal deformations $\hat{\mathcal{L}}^{l-s+1}_\nu, \ldots, \hat{\mathcal{L}}^{l+1}_\nu$ of P inductively by $\hat{\mathcal{L}}^{l-s+1}_\nu = \mathcal{L}''_\nu$ and for $j \geqslant l-s$,

$$\hat{\mathcal{L}}^j_\nu = \mathbb{L}_{\nu^j}(\hat{\mathcal{L}}^{j-1}_\nu; \tau(\hat{C}_j - C^{j-1}_j)).$$

Then $\hat{\mathcal{L}}_\nu^{l+1}$ is a formal deformation of P which coincides with $\mathcal{L}_\nu$ at order $l + 1$. Hence, the result by induction on $l < k$.

It follows from Theorem 3.4 that $[C, C]$ is exact for each two-cocycle C. Moreover, $H_{1-\mathrm{diff,nc}}(N)$ denoting the cohomology of one-differentiable nc cochains, it can be shown that

THEOREM 3.5.

(i) $[H_{\mathrm{loc,nc}}(N), H_{1-\mathrm{diff,nc}}(N)] = 0.$

(ii) $[\oplus_{i\leqslant 2}H^i_{\mathrm{loc,nc}}(N), \oplus_{i\leqslant 2}H^i_{\mathrm{loc,nc}}(N)] = 0.$

4. STAR-PRODUCTS

Recall that a star-product of (M, F) is a formal deformation $\mathcal{M}_\lambda = \Sigma_{k=0}^\infty \lambda^k C_k$ of the associative algebra $(N, \mathcal{M})$, with driver $\mathcal{M} + \lambda P$, where for $k > 0$, the C_k's are local, vanishing on the constants and such that $C_k(v, u) = (-1)^k C_k(u, v)$ for all $u, v \in N$.

THEOREM 4.1. *Every star-product of order $2k$ is a driver of a star-product. In particular, every symplectic manifold admits a star-product.*

Proof. Let $\mathcal{M}_\lambda = \Sigma_{i\leqslant 2k}\lambda^i C_i$ be a star-product of order $2k$ ($k > 0$). Then

$$\mathcal{L}_\nu = \sum_{i<k} \nu^i C_{2i+1} \tag{8}$$

is a formal deformation of order $k - 1$ of P.

In the proof of the Neroslavsky–Vlassov theorem [8], which asserts the existence of a star-product when the third De Rham cohomology space of M vanishes, the key steps are the following.

(i) There exists $C_{2k+1} \in A^1_{\mathrm{loc,nc}}(N)$ such that $\mathcal{M}'_\lambda = \mathcal{M}_\lambda + \nu^{2k+1}C_{2k+1}$ is a star-product of order $2k + 1$; C_{2k+1} is determined up to an arbitrary one-differentiable element of $A^1_{\mathrm{loc,nc}}(N)$.

(ii) $\mathcal{M}'_\lambda$ extends to a star-product of order $2k + 2$ if and only if $\mathcal{L}_\nu + \nu^k C_{2k+1}$ is a formal deformation of order k of P.

(iii)
$$\sum_{p+q=k} [C_{2p+1}, C_{2q+1}] \tag{9}$$

is a one-differentiable Chevalley cocycle.

By Theorem 3.5, (8) is a driver for a formal deformation of P. If $\nu^k C$ is its kth component, then $\partial(C_{2k+1} - C)$ is one-differentiable. Thus $C_{2k+1} - C - \mu^*\Omega$ is a Chevalley cocycle for a suitable $\Omega \in \Lambda^2(M)$. Then replacing C_{2k+1} by $C_{2k+1} - \mu^*\Omega$, (9) vanishes and $\mathcal{M}_\lambda$ extends to a star-product of order $2k + 2$. Hence, the existence of a star-product with driver $\mathcal{M}_\lambda$ by induction on k.

To conclude, observe now that each symplectic manifold admits a star-product of order 2 [8].

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