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Universal R-Matrix of the Quantum Superalgebra osp(2 | 1)

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Abstract. A quantum analogue of the simplest superalgebra osp(2 | 1) and its finite-dimensional, irreducible representations are found. The corresponding constant solution to the Yang–Baxter equation is constructed and is used to formulate the Hopf superalgebra of functions on the quantum supergroup OSp(2 | 1).

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The development of the quantum inverse problem method [1] gives rise to the notions of the quantum group and quantum algebra [2–5]. This field is under active study and papers containing general [6, 7] and concrete results [8–10] on quantum groups have appeared.

Taking into account the recent interest in 'supermathematics', in particular, in the theory of Lie superalgebras [11], in this Letter, we construct a quantum analogue [12] of the Lie superalgebra osp(2 | 1) [13, 14].

The Letter is organized as follows. In Section 1 we recall the main properties of the Lie superalgebra osp(2 | 1) and the corresponding group. Section 2 contains the definition of the quantum analogue of osp(2 | 1), the universal R-matix and the finite-dimensional representations. The algebra of functions $A(R)$ on the quantum formal group is given in Section 3.

1. The rank-one Lie superalgebra osp(2 | 1) plays a special role in the classification theory of Lie superalgebras [11]. This is due to the fact that many of its properties are analogous to those of sl(2) ⊂ osp(2 | 1). It contains the three even generators $X_\pm, H$ of sl(2) and two odd generators $v_\pm$ which satisfy the (anti)commutation relations

$$\begin{aligned} &[H, X_\pm] = \pm X_\pm, \qquad [X_+, X_-] = 2H, \\ &[H, v_\pm] = \pm\tfrac{1}{2}v_\pm, \qquad [X_\pm, v_\mp] = v_\pm, \qquad [X_\pm, v_\pm] = 0, \\ &[v_+, v_-]_+ = -\tfrac{1}{2}H, \qquad [v_\pm, v_\pm]_+ = \pm\tfrac{1}{2}X_\pm, \end{aligned} \tag{1}$$

where $[\cdot\,,\cdot]_+$ denotes the anticommutator.

The centre of the corresponding universal enveloping superalgebra $\mathscr{U}(\text{osp}(2\,|\,1))$ is generated by the Casimir operator

$$C_2 = H^2 + \tfrac{1}{2}[X_+, X_-]_+ + [v_+, v_-]. \tag{2}$$

There is a graded anti-involution $*$,

$$(X_\pm)^* = X_\mp, \qquad H^* = H, \qquad (v_\pm)^* = \pm(-1)^\varepsilon v_\mp, \quad \varepsilon = 0, 1,$$
$$(XY)^* = (-1)^{p(X)p(Y)} Y^* X^*, \qquad (X^*)^* = (-1)^{p(X)} X, \tag{3}$$

where $p(X) = 0, 1$ is the parity of a homogeneous element, $X \in \mathrm{osp}(2 \mid 1)$. We shall use $\varepsilon = 0$ later on.

The irreducible, finite-dimensional representations V_l with the highest weight vector are parametrized by the half-integer $s = l/2$ or by the integer $l \in \mathbb{N}$. Their dimension is $\dim V_l = 2l + 1$ and the corresponding value of C_2 defined by formula (2) is $s(s + 1/2) = l(l + 1)/4$ [13, 14]. With respect to the even subalgebra, sl(2), every representation is decomposed into the direct sum of two irreducible representations,

$$V_l = D_l \oplus D_{l-1},$$

which correspond to the sl(2)-spin, $s = l/2$ and $s = (l - 1)/2$. Let us point out that, for any l, the matrix H has the eigenvalue zero and, in a basis of eigenvectors, is

$$H = \tfrac{1}{2} \operatorname{diag}(l, l-1, \ldots, 0, -1, \ldots, -l). \tag{4}$$

The decomposition of the tensor product of two irreducible representations is multiplicity free,

$$V_l \otimes V_p = \sum_{m=|l-p|}^{l+p} V_m. \tag{5}$$

The corresponding Clebsch–Gordan coefficients are known [13, 14]. The action of the generators (1) in the tensor product (5) is defined by the coproduct Δ,

$$\Delta(a) = a \otimes I + I \otimes a, \quad a \in \mathrm{osp}(2 \mid 1). \tag{6}$$

As a result $\mathscr{U}(\mathrm{osp}(2 \mid 1))$ is a Hopf superalgebra [2].

To construct the supergroup associated with (1), one needs a linear space of parameters. In our case, it will be an infinite-dimensional complex Grassmann algebra $\Lambda = \Lambda_0 \oplus \Lambda_1$. Multiplying the generators of $\mathrm{osp}(2 \mid 1)$ by the parameters of the corresponding parity (0 or 1) one obtains even elements and, after exponentiating them, group elements. In the representation V_1 these elements of the supergroup $\mathrm{OSp}(2 \mid 1)$ are 3×3 matrices with five even and four odd entries which satisfy the relations

$$g = \begin{pmatrix} a & \alpha & b \\ \gamma & e & \beta \\ c & \delta & d \end{pmatrix} \quad \begin{array}{l} e = 1 + \alpha\delta, \\ \gamma = -a\delta + c\alpha, \quad ad - bc + \alpha\delta = 1, \\ \beta = d\alpha - b\delta. \end{array} \tag{7}$$

The latter relations follow from the equation

$$g^{\mathrm{st}} J g = J, \quad J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \tag{8}$$

where st means supertransposition of the even matrix

$$(g^{st})_{ab} = \exp(i\pi p(a)(p(a) + p(b)))g_{ba}.$$

2. The quantum superalgebra $\mathcal{U}_q(\mathrm{osp}(2 \mid 1))$ is generated as in the sl(2) case [15] by three elements, H and $v_\pm$, with the commutation relations

$$[H, v_\pm] = \pm\tfrac{1}{2}v_\pm, \quad [v_+, v_-] = \frac{-\sinh \eta H}{\sinh 2\eta}. \tag{9}$$

The following formulae for co-multiplication, antipode and co-unit define on $\mathcal{U}_q$ the structure of a Hopf superalgebra

$$\Delta(v_\pm) = v_\pm \otimes e^{\eta H/2} + e^{-\eta H/2} \otimes v_\pm, \tag{10}$$

$$\Delta(H) = H \otimes I + I \otimes H,$$
$$S(H) = -H, \qquad S(v_\pm) = -\exp(\pm\eta/4)v_\pm, \tag{11}$$

$$\varepsilon(1) = 1, \qquad \varepsilon(H) = \varepsilon(v_\pm) = 0. \tag{12}$$

The co-multiplication Δ is not cocommutative, it does not coincide with $\Delta' = \sigma \circ \Delta$, where σ is the permutation map

$$\sigma(a \otimes b) = (-1)^{p(a)p(b)} b \otimes a.$$

Due to this fact, the maps Δ' and $S' = S^{-1}$ define on $\mathcal{U}_q$ another Hopf–superalgebra structure. It follows from the theory of the quantum double [2, 9] that there exists a canonical element $R \in \mathcal{U}_q \otimes \mathcal{U}_q$ which defines a similarity transformation of Δ into Δ',

$$R\Delta'(\,\cdot\,) = \Delta(\,\cdot\,)R. \tag{13}$$

In terms of

$$e = x\, e^{-\eta H/2} v_+, \qquad f = x\, e^{\eta H/2} v_-, \quad x = \left(4 \sinh \frac{\eta}{2} \sinh 2\eta\right)^{1/2}$$

it has the following form, similar to the sl(2) case [2, 9, 10]

$$R = \exp(-2\eta H \otimes H) \sum_{k=0}^{\infty} a_k e^k \otimes f^k,$$
$$a_k = \frac{q^{k(k+1)/4}}{(q^2 - 1)^k} \frac{1}{[k]_+!}, \quad [k]_+ = ((-1)^{k-1} q^{k/2} + q^{-k/2})/(q^{1/2} + q^{-1/2}), \tag{14}$$

where $q = \exp(-\eta/2)$.

The canonical element R is also called the universal R-matrix. It satisfies the relations

$$(\Delta \otimes \mathrm{id})R = R_{13}R_{23}, \tag{15}$$

$$(\mathrm{id} \otimes \Delta)R = R_{13}R_{12}, \tag{16}$$

$$(S \otimes \mathrm{id})R = R^{-1}, \tag{17}$$

where the indices show the embedding of R into the graded tensor product, $\mathcal{U}_q \otimes \mathcal{U}_q \otimes \mathcal{U}_q$. Formulae (15, 16) imply the Yang–Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \tag{18}$$

where the graded tensor product law must be taken into account,

$$(a \otimes b)(c \otimes d) = (-1)^{p(b)p(c)}(ac \otimes bd). \tag{19}$$

The R-matrix (14) commutes with $\exp(\lambda H) \otimes \exp(\lambda H)$. This property is used in the theory of link invariants [9, 10].

The q-analogue of the central element (2) is given by

$$c_2(q) = \cosh 2\eta(H + \tfrac{1}{4}) - q^{-1/2} \cosh \eta(H + \tfrac{1}{2})fe - \\ - (2q)^{-1}\left(\cosh \frac{\eta}{4} \Big/ \sinh \frac{\eta}{2}\right)^2 f^2 e^2. \tag{20}$$

The finite-dimensional representations of $\mathcal{U}_q$ as a superalgebra with commutation relations (9) and highest weight vector, have the same structure as in the classical case of osp$(2 \mid 1)$. In the basis $e_m^{(l)}$, one gets

$$\rho_l(H)e_m^{(l)} = \frac{m}{2} e_m^{(l)}, \qquad \rho_l(v_\pm)e_m^{(l)} = N_\pm(l, m)e_{m\pm 1}^{(l)}, \tag{21}$$

$$N_+^2(l, m) = \left(\sinh 2\eta \cosh \frac{\eta}{4}\right)^{-1} \times \\ \times \begin{cases} \sinh \frac{\eta}{4}(l + m + 1) \cosh \frac{\eta}{4}(l - m), & l - 1 - m = 2n \\ \sinh \frac{\eta}{4}(l - m) \cosh \frac{\eta}{4}(l + m + 1), & l - m = 2n \end{cases}$$

$$m = l - 1, \quad l - 2, \ldots, -l; \qquad N_-(l, m) = (-1)^{l-m-1}N_+(l, m-1)$$

The value of the central element (20) in this representation is

$$c_2(q)\,|_{V_l} = \cosh \eta(l + \tfrac{1}{2}) = \tfrac{1}{2}(q^{2l+1} + q^{-2l-1}). \tag{22}$$

The evaluation of the Clebsch–Gordan coefficients for the decomposition of the tensor product $V_l \otimes V_p$ in this case can be done using the projector P on the highest weight vector:

$$P(H) = \sum_{n=0}^{\infty} C_n(H)v_-^n v_+^n, \qquad Pv_- = v_+P = 0, \\ P^2 = P, \qquad C_0(H) = 1, \qquad C_n(H) = \prod_{k=1}^{n} \left(g_k\left(H + \frac{k}{2}\right)\right)^{-1}, \tag{23}$$

$$g_k(H) = \left(\sinh 2\eta \cosh \frac{\eta}{4}\right)^{-1} \\ \times \sinh k(\eta/2 + i\pi)/2 \cosh(\eta(H + (1 - k)/4) - i\pi k/2). \tag{24}$$

The Clebsch–Gordan coefficients are given by the formula

$$\begin{bmatrix} l_1 l_2 l \\ m_1 m_2 m \end{bmatrix} = N_-^{-1}(l, m)\langle e^{(l_1)}_{m_1} \otimes e^{(l_2)}_{m_2}, \Delta(v_-^{l-m})\Delta(P(l)) e^{(l_1)}_{l_1} \otimes e^{(l_2)}_{m-l_1}\rangle. \tag{25}$$

The universal R-matrix (14) can be represented in the space $V_l \otimes V_p$,

$$R^{l,p} = (\rho_l \otimes \rho_p)R.$$

In particular, for $l = p = 1$, one gets

$$R^{1,1} \equiv R^{(+)} = \left(\begin{array}{ccc|ccc|ccc} q &&&&&&&& \\ & 1 && a &&&&& \\ && q^{-1} && b && e && \\ \hline &&& 1 &&&&& \\ &&&& 1 && c && \\ &&&&& 1 && d & \\ \hline &&&&&& q^{-1} && \\ &&&&&&& 1 & \\ &&&&&&&& q \end{array}\right) \qquad \begin{array}{l} a = d = q - q^{-2}, \\ \\ b = c = -a/q^{1/2}, \\ \\ e = a(1 + q^{-1}). \end{array} \tag{26}$$

The matrix $\check{R} = R^{(+)}\mathscr{P}$, where $\mathscr{P}$ is the permutation operator in $V_1 \otimes V_1$, satisfies the relation

$$(\check{R} - q)(\check{R} + q^{-1})(\check{R} + q^{-2}) = 0 \tag{27}$$

which follows from the general decomposition

$$\check{R}^{l,m} = \sum_{j=|l-m|}^{l+m} (-1)^{\nu(j)} \exp(\eta(c(l) + c(m) - c(j)))P_j, \quad \nu(j) = l + m - j, \tag{28}$$

where $c(l)$ is the value of the Casimir operator (2) in V_l, and $\nu(j)$ is the parity of V_j in the Clebsch–Gordan decomposition. One can define matrices $L^{(\pm)}$, the entries of which belong to $\mathscr{U}_q$,

$$L^{(+)} = (\rho_1 \otimes \mathrm{id})R = \begin{pmatrix} e^{-\eta H} & U & Z \\ 0 & 1 & W_- \\ 0 & 0 & e^{\eta H} \end{pmatrix}, \tag{29}$$

$$Z_- = (1 + q)b^{-1}U_-^2\, e^{\eta H}, \qquad W_- = -q^{1/2}U_- e^{\eta H}, \qquad U_- = x\, e^{-\eta H/2}v_-,$$

$$L^{(-)} = (\rho_1 \otimes \mathrm{id})\mathscr{P}R^{-1}\mathscr{P} = \begin{pmatrix} e^{\eta H} & 0 & 0 \\ W_+ & 1 & 0 \\ Z_+ & U_+ & e^{-\eta H} \end{pmatrix}, \tag{30}$$

$$Z_+ = (1 + q)b^{-1}\, e^{\eta H}U_+^2, \qquad W_+ = q^{1/2}\, e^{\eta H}U_+, \qquad U_+ = x\, e^{-\eta H/2}v_+.$$

The action of the antipode anti-automorphism on these matrices is as follows

$$S(L^{(\pm)}) = C(L^{(\pm)})^{\mathrm{st}}C^{-1} = (L^{(\pm)})^{-1}, \tag{31}$$

$$C = J\exp(\eta\rho_1(H)/2) = \begin{pmatrix} 0 & 0 & q^{1/2} \\ 0 & 1 & 0 \\ -q^{-1/2} & 0 & 0 \end{pmatrix}. \tag{32}$$

When $q \in \mathbb{R}$ the anti-involution $*$ for $\mathscr{U}_q$ is the same as (3)

$$H^* = H, \qquad (v_+)^* = v_-, \qquad (v_-)^* = -v_+. \tag{33}$$

3. Now we describe $A(R)$, the dual object to $\mathscr{U}_q$, which is interpreted as the algebra of functions on the quantum formal group, using the approach [4, 5]. The generators t_{ij}, $i, j = 1, 2, 3$ of $A(R)$ are defined by the relations

$$\begin{aligned} &(1, T_1 T_2 \cdots T_k) = I^{\otimes k}, \\ &(L^{(\pm)}, T_1 T_2 \cdots T_k) = R_1^{(\pm)} R_2^{(\pm)} \cdots R_k^{(\pm)}, \end{aligned} \tag{34}$$

where, for the 3×3 matrix

$$T = \{t_{ij}\}, \quad T_1 = T \otimes I \otimes \cdots \otimes I, \quad T_2 = I \otimes T \otimes I \cdots \otimes I$$

and so on. It follows from these relations that ($\check{R} = R^{(+)}\mathscr{P}$)

$$\check{R} T_1 T_2 = T_1 T_2 \check{R}, \tag{35}$$

and from the crossing property (31) and (34), that

$$T^{st} C^t T = \gamma C^t, \quad \gamma = t_{11}t_{33} - q t_{31} t_{13} - q^{1/2} t_{21} t_{23}, \tag{36}$$

$$\langle L^{(\pm)} S(T) \rangle = \langle S(L^{(\pm)}), T \rangle = C_1^{-1} R^{(\pm) st_1} C_1$$

$$S(T) = C^{-1} T^{st} C = \begin{pmatrix} t_{33} & z^{-1} t_{23} & -z^{-2} t_{13} \\ -z t_{32} & t_{22} & z^{-1} t_{12} \\ -z^2 t_{31} & -z t_{21} & t_{11} \end{pmatrix}, \quad z = q^{1/2}. \tag{37}$$

The comultiplication Δ: $A(R) \to A(R) \otimes A(R)$

$$\Delta(t_{ij}) = \sum_{k=1}^{3} t_{ik} \otimes t_{kj}. \tag{38}$$

the co-unit ε: $A(R) \to \mathbb{C}$, $\varepsilon(t_{ij}) = \delta_{ij}$ and the antipode map S (37) define the structure of a Hopf superalgebra on $A(R)$. The anti-involution $*$ on the superalgebra $A(R)$ can be obtained by the same method, using (33) for the generators H, $v_\pm$ of $\mathscr{U}_q$, the duality relations (34), and $q \in \mathbb{R}$, ($z = q^{1/2}$)

$$T^* = \begin{pmatrix} t_{33} & z t_{32} & -z^2 t_{31} \\ z^{-1} t_{23} & t_{22} & -z t_{21} \\ -z^{-2} t_{13} & -z^{-1} t_{12} & t_{11} \end{pmatrix}. \tag{39}$$

A left- (or right-) co-module of $A(R)$ gives rise to a corepresentation of $A(R)$, and, due to the duality relation (34), to a representation of the quantum algebra

$\mathcal{U}_q(\mathrm{osp}(2 \mid 1))$. The superalgebra $A(R)$ is useful for the description of global properties, e.g., quantum homogeneous spaces such as the quantum analogue of the supersphere for $\mathrm{OSp}(2 \mid 1)$.

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