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Nelson Algebras Through Heyting Ones[1]: I

Abstract. The main aim of the present paper is to explain a nature of relationships exist between Nelson and Heyting algebras. In the realization, a topological duality theory of Heyting and Nelson algebras based on the topological duality theory of Priestley ([15], [16]) for bounded distributive lattices are applied. The general method of construction of spaces dual to Nelson algebras from a given dual space to Heyting algebra is described (Thm 2.3). The algebraic counterpart of this construction being a generalization of the Fidel-Vakarelov construction ([6], [25]) is also given (Thm 3.6). These results are applied to compare the equational category $\boldsymbol{N}$ of Nelson algebras and some its subcategories (and their duals) with the equational category $\boldsymbol{H}$ of Heyting algebras (and its dual). It is proved (Thm 4.1) that the category N is topological over the category $\boldsymbol{H}$.

A Nelson algebra is an algebra $(A, \vee, \wedge, \rightarrow, \neg, \sim, 0, 1)$ of the type (2, 2, 2, 1, 1, 0, 0) satisfying some appriopriate axioms (for details see Section 0). Nelson algebras under the name N-lattices have been introduced by H. Rasiowa [19] as an algebraic semantic of the constructive logic with strong negation of Nelson [14] and Markov [11]. This logic being a conservative extension of the intuitionistic logic arises naturally as a research to omitting the non-constructivity of the intuitionistic negation. So, to study this logic and its extensions the detail explanation of relationships appearing between Nelson algebras and Heyting ones being an algebraic semantic of the intuitionistic logic seems to be indispensable. The great number of solved problems concerning the theory of Heyting algebras supplies an additional motivation for such investigations.

In this paper, the starting point to explain mentioned relationships stands the following fact: for any Nelson algebra $\mathfrak{A} = (A, \vee, \wedge, \rightarrow, \neg, \sim, 0, 1)$ the relation $\approx$ on A defined by the rule $a \approx b$ iff $a \rightarrow b = 1$ and $b \rightarrow a = 1$ is a congruence relation on the algebra $(A, \vee, \wedge, \rightarrow, \neg, 0, 1)$ and the quotient algebra, denote it by $\mathfrak{A}^*$, is a Heyting algebra. It is known that each Heyting algebra $\mathfrak{B}$ can be represented up to isomorphism as $\mathfrak{A}^*$ for some Nelson algebra $\mathfrak{A}$. The Fidel-Vakarelov construction of Nelson algebra $N(\mathfrak{B})$ (see e.g. [25]) yields an example of such an algebra. This also follows from the topological representation of Nelson algebras developed in [19]. On the other hand, it is

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easy to find non-isomorphic Nelson algebras such that their "star" algebras are isomorphic. Thus to explain in detail relationships between Nelson and Heyting algebras it is natural to pointed out the following

PROBLEM 1. Find a usefull and possible simple description (construction) of all up to isomorphism Nelson algebras $\mathfrak{A}$ whose Heyting algebras $\mathfrak{A}^*$ are isomorphic to a given Heyting algebra $\mathfrak{B}$.

With this problem arises the following one which is very important from algebraic and logical points of view

PROBLEM 2. Is it possible to apply the obtained description to state connections between varieties (quasivarieties) of Nelson and Heyting algebras?

In this paper we solve positively these two problems. The presented part I deals with the first problem (for the second problem we refer to part II).

To obtain a solution we apply the known topological duality theory for Heyting algebras and a stated (in Section 1) one for Nelson algebras, both based on the topological duality theory of H. A. Priestley ([15], [16]) for distributive lattices. In Section 2 the construction of all up to isomorphism dual spaces of Nelson algebras $\mathfrak{A}$ from a given dual space of Heyting algebra $\mathfrak{B}$ such that $\mathfrak{A}^*$ are isomorphic to $\mathfrak{B}$ is given. Hence, up to duality, the all Nelson algebras $\mathfrak{A}$ with $\mathfrak{A}^*$ isomorphic to a given $\mathfrak{B}$ are described. From this description follows that these algebras are determined by $\mathfrak{B}$ and only one parameter associated with $\mathfrak{B}$, namely a Boolean congruence on $\mathfrak{B}$. In Section 3 we presente the construction of all Nelson algebras $\mathfrak{A}$ with $\mathfrak{A}^*$ isomorphic to $\mathfrak{B}$ being an algebraic realization of this observation. This construction is a generalization of the Fidel-Vakarelov one.

These results lead us to state in Section 4 that the investigated connection has a topological nature. More precisely, we prove that the functor * from the category $\boldsymbol{N}$ of Nelson algebras to the category $\boldsymbol{H}$ of Heyting algebras is topological, and hence the category $\boldsymbol{N}$ is topological over the category $\boldsymbol{H}$. This means that every Nelson algebra $\mathfrak{A}$ with $\mathfrak{A}^*$ isomorphic to a given $\mathfrak{B}$ can be treated as $\mathfrak{B}$ equipped with some additional structure, and among these structures there are distinguished two, namely "discrete" and "indiscrete" ones. So we get functors $\overleftarrow{N}$ and $\overrightarrow{N}$ from $\boldsymbol{H}$ to $\boldsymbol{N}$ being left and right adjoint to the the functor *, respectively. These functors allow us to presente the interplaies between categories $\boldsymbol{N}$ (and some its subcategories), $\boldsymbol{H}$ and categories of spaces dually equivalent to them in a elegant mathematical fashion.

0. Preliminaries

An algebra $(A, \vee, \wedge, \rightarrow, \neg, \sim, 0, 1)$ of type (2, 2, 2, 1, 1, 0, 0) is said to be a *Nelson algebra* if the following hold:

(I) $(A, \vee, \wedge, \sim, 0, 1)$ is a Kleene algebra i.e. it is a De Morgan algebra satisfying for all $a, b \in A$ the inequality $a \wedge \sim a \leqslant b \vee \sim b$ ($\leqslant$ denotes the lattice order),

(II) The relation $\prec$ on A defined by the rule $a \prec b$ iff $a \to b = 1$ is reflexive and transitive,

(III) The equivalence relation $\approx$ on A induced by $\prec$ is a congruence relation on $(A, \vee, \wedge, \to, \neg, 0, 1)$ and the quotient algebra is a Heyting algebra.

(IV) For all $a, b \in A$ there hold: $\sim(a \to b) \approx a \wedge \sim b$, $a \wedge \sim a \prec 0$, $\neg a = a \to 0$, and $a \leqslant b$ iff $a \prec b$ and $\sim b \prec \sim a$,

where by Heyting algebra we mean any algebra $(B, \vee, \wedge, \Rightarrow, -, 0, 1)$ of type $(2, 2, 2, 1, 0, 0)$ such that $(B, \vee, \wedge, 0, 1)$ is bounded lattice, $a \Rightarrow b$ is the greatest element in the set $\{c \in A;\ a \wedge c \leqslant b\}$ and $-a = a \Rightarrow 0$

The above definition of Nelson algebra is adopted from the one presented in [19].

The operations $\Rightarrow$, $\to$, $-$ and $\neg$ are called *relative pseudocomplementation, weak relative pseudocomplementation, pseudocomplementation* and *weak pseudocomplementation*, respectively. Both, the class $\boldsymbol{H}$ of Heyting algebras and the class $\boldsymbol{N}$ of Nelson algebras (in presented types) are varieties of algebras. For more information on Heyting and Nelson algebras, as well as another bounded lattices with additional operations we refer to [20] and [21].

An *ordered topological space* (for short *ordered space*) $(X, \mathscr{T}, \leqslant)$ is a poset (i.e. partially ordered set) $(X, \leqslant)$ endowed with a topology $\mathscr{T}$. A poset $(X, \leqslant)$ as well as an ordered space $(X, \mathscr{T}, \leqslant)$ will be denoted by the same symbol $\mathfrak{X}$. A subset U of a poset or an ordered space $\mathfrak{X}$ is *increasing* (*decreasing*) if $x \leqslant y$, $x \in U$ $(y \in U)$ imply $y \in U$ $(x \in U)$. The smallest increasing (decreasing) set containing a given subset U of X we denote by $\uparrow U$ $(\downarrow U)$; instead of $\uparrow\{x\}$ $(\downarrow\{x\})$ we write $\uparrow x$ $(\downarrow x)$. An ordered space $\mathfrak{X}$ is *totally order disconnected* if for $x, y \in X$ with $x \nleqslant y$ there exists a clopen increasing set U such that $x \in U$ and $y \notin U$.

Denote the category of compact totally order disconnected spaces (Priestley spaces) with continuous and order-preserving maps by $\boldsymbol{P}$; and the category of bounded distributive lattices with bounded lattice homomorphisms by $\boldsymbol{D}_{01}$. The set $\mathscr{O}(X)$ of all clopen increasing subsets of an object $\mathfrak{X}$ of $\boldsymbol{P}$ is closed under the set union and intersection, so $\mathscr{O}(\mathfrak{X}) = (\mathscr{O}(X), \cup, \cap, \emptyset, X)$ is an object of $\boldsymbol{D}_{01}$; the set $\mathscr{P}(A)$ of all prime filters of an object $\mathfrak{A}$ of $\boldsymbol{D}_{01}$ ordered by inclusion and topologised by taking the family of sets $\mu(a) = \{F \in \mathscr{P}(A);\ a \in F\}$, for $a \in A$, and their complements as a subbase, denote it by $\mathscr{P}(\mathfrak{A})$, is an object of $\boldsymbol{P}$. Furthermore, setting $\mathscr{P}(h) = h^{-1}$: $\mathscr{P}(B) \to \mathscr{P}(A)$, for every morphism h: $\mathfrak{A} \to \mathfrak{B}$ of $\boldsymbol{D}_{01}$, and $\mathscr{O}(f) = f^{-1}$: $\mathscr{O}(Y) \to \mathscr{O}(X)$, for every morphism f: $\mathfrak{X} \to \mathfrak{Y}$ of $\boldsymbol{P}$ we obtain contravariant functors $\mathscr{P}$: $\boldsymbol{D}_{01} \to \boldsymbol{P}$ and $\mathscr{O}$: $\boldsymbol{P} \to \boldsymbol{D}_{01}$. Priestley in [15] and [16] proved that these functors establish a dual equivalence between categories $\boldsymbol{D}_{01}$ and $\boldsymbol{P}$.

Priestley's duality has been applied to obtain a description of dually equivalent categories to various equational subcategories of $\boldsymbol{D}_{01}$ (the list of examples may be found in [18]). To our considerations will be indispensable descriptions two of them, namely, the category dually equivalent to the category $\boldsymbol{H}$ of Heyting algebras and to $\boldsymbol{Kl}$ of Kleene algebras.

The specialization of the Priestley duality to the case of Heyting algebras is essentially a part of the folklore of Duality Theory (see [18]). A H-space is an object $\mathfrak{X}$ of $\boldsymbol{P}$ such that for every open subset Y of X, $\downarrow Y$ is open (or equivalently, for all $U, V \in \mathcal{O}(X)$, $\downarrow(U \setminus V)$ is open). Let $\boldsymbol{P}$-$\boldsymbol{H}$ be a category whose objects are H-spaces and whose morphisms from the H-space $\mathfrak{X}$ to the H-space $\mathfrak{Y}$ are all morphisms $f\colon \mathfrak{X} \to \mathfrak{Y}$ in the category $\boldsymbol{P}$ such that $f(\uparrow x) = \uparrow f(x)$, for all $x \in X$. For any H-space $\mathfrak{X}$ and $U, V \in \mathcal{O}(X)$ define:

$$U \Rightarrow V = X \setminus \downarrow(U \setminus V) \quad \text{and} \quad -U = X \setminus \downarrow U.$$

Then the algebra $\mathcal{O}_{\mathrm{H}}(\mathfrak{X}) = (\mathcal{O}(X), \cup, \cap, \Rightarrow, -, \emptyset, X)$ is a Heyting algebra. Furthermore, for any morphism $f\colon \mathfrak{X} \to \mathfrak{Y}$ in $\boldsymbol{P}$-$\boldsymbol{H}$, $\mathcal{O}_{\mathrm{H}}(f) = \mathcal{O}(f)$ is a Heyting algebra homomorphism from $\mathcal{O}_{\mathrm{H}}(\mathfrak{Y})$ into $\mathcal{O}_{\mathrm{H}}(\mathfrak{X})$. On the other hand, for each Heyting algebra $\mathfrak{B}$, $\mathcal{P}_{\mathrm{H}}(\mathfrak{B}) = \mathcal{P}(\mathfrak{B})$ is a H-space; and for each Heyting algebra homomorphism $h\colon \mathfrak{B} \to \mathfrak{D}$, $\mathcal{P}_{\mathrm{H}}(h) = \mathcal{P}(h)$ is a morphism in $\boldsymbol{P}$-$\boldsymbol{H}$ from $\mathcal{P}_{\mathrm{H}}(\mathfrak{D})$ to $\mathcal{P}_{\mathrm{H}}(\mathfrak{B})$. Therefore we have two contravariant functors $\mathcal{P}_{\mathrm{H}}\colon \boldsymbol{H} \to \boldsymbol{P}$-$\boldsymbol{H}$ and $\mathcal{O}_{\mathrm{H}}\colon \boldsymbol{P}$-$\boldsymbol{H} \to \boldsymbol{H}$. These functors establish a dual equivalence between the categories $\boldsymbol{H}$ and $\boldsymbol{P}$-$\boldsymbol{H}$.

The extension of the Priestley duality to the case of Kleene algebras is due to Cornish and Fowler [5]. A Kl-space is an object $\mathfrak{X}$ of $\boldsymbol{P}$ with a homeomorphism g on $\mathfrak{X}$, that is, a pair $(\mathfrak{X}, g)$ such that for any $x, y \in X$:

(k_1) if $x \leqslant y$ then $g(y) \leqslant g(x)$,
(k_2) $g(g(x)) = x$,
(k_3) $x \leqslant g(x)$ or $g(x) \leqslant x$.

Let $\boldsymbol{P}$-$\boldsymbol{Kl}$ be a category whose objects are Kl-spaces and whose morphisms from the Kl-space $(\mathfrak{X}, g_{\mathrm{X}})$ to the Kl-space $(\mathfrak{Y}, g_{\mathrm{Y}})$ are all morphisms $f\colon \mathfrak{X} \to \mathfrak{Y}$ in the category $\boldsymbol{P}$ such that $fg_{\mathrm{X}} = g_{\mathrm{Y}}f$. For any Kl-space $(\mathfrak{X}, g)$ and $U \in \mathcal{O}(X)$ define

$$\sim U = X \setminus g(U).$$

Then the algebra $\mathcal{O}_{\mathrm{Kl}}(\mathfrak{X}, g) = (\mathcal{O}(X), \cup, \cap, \sim, \emptyset, X)$ is a Kleene algebra. Moreover, for any morphism $f\colon (\mathfrak{X}, g_{\mathrm{X}}) \to (\mathfrak{Y}, g_{\mathrm{Y}})$ in $\boldsymbol{P}$-$\boldsymbol{Kl}$, $\mathcal{O}_{\mathrm{Kl}}(f) = \mathcal{O}(f)$ is a Kleene algebra homomorphism from $\mathcal{O}_{\mathrm{Kl}}(\mathfrak{Y}, g_{\mathrm{Y}})$ to $\mathcal{O}_{\mathrm{Kl}}(\mathfrak{X}, g_{\mathrm{X}})$. On the other hand, $\mathcal{P}_{\mathrm{Kl}}(\mathfrak{A}) = (\mathcal{P}(\mathfrak{A}), g)$, where the map $g\colon \mathcal{P}(A) \to \mathcal{P}(A)$ is defined by $g(F) = \mathcal{P}(A) \setminus \{\sim a;\ a \in F\}$, is a Kl-space; and for each Kleene algebra homomorphism $h\colon \mathfrak{A} \to \mathfrak{C}$, $\mathcal{P}_{\mathrm{Kl}}(h) = \mathcal{P}(h)$ is a morphism in $\boldsymbol{P}$-$\boldsymbol{Kl}$ from $\mathcal{P}_{\mathrm{Kl}}(\mathfrak{C})$ to $\mathcal{P}_{\mathrm{Kl}}(\mathfrak{A})$. Therefore we have two contravariant functors $\mathcal{P}_{\mathrm{Kl}}\colon \boldsymbol{Kl} \to \boldsymbol{P}$-$\boldsymbol{Kl}$ and $\mathcal{O}_{\mathrm{Kl}}\colon \boldsymbol{P}$-$\boldsymbol{Kl} \to \boldsymbol{Kl}$. These functors establish a dual equivalence of the categories $\boldsymbol{Kl}$ and $\boldsymbol{P}$-$\boldsymbol{Kl}$.

The notation concerning the theory of universal algebras and the theory of categories is in accordance with [8] and [9], respectively; where all the necessary definitions and results may be found. We only pay the reader attention to one covention, namely, for any class $\boldsymbol{K}$ of algebras of the same similarity type by the same symbol $\boldsymbol{K}$ it is denoted the category whose objects are algebras from this class and whose morphisms are homomorphism between them.

1. Topological duality

In this section we describe a category dually equivalent to the equational category $\boldsymbol{N}$ of Nelson algebras. This category is found as a suitable subcategory of the category $\boldsymbol{P\text{-}Kl}$[2].

If $\mathfrak{X}$ is an ordered space and Y is a subset of X then Y with the restricted order and the induced topology is an order space. This order space, denote it by $\mathfrak{Y}$, we refer to call an order subspace of $\mathfrak{X}$.

For any poset $\mathfrak{X}$ and function $g\colon X \to X$ satysfying conditions (k_1), (k_2) and (k_3) define

$$X^+ = \{x \in X;\ x \leqslant g(x)\}$$

and

$$X^- = \{x \in X;\ g(x) \leqslant x\}.$$

Clearly, $X = X^+ \cup X^-$, $g(X^+) = X^-$ and $g(X^-) = X^+$; X^+ is decreasing and X^- is increasing in $\mathfrak{X}$.

1.1 Lemma. *For any Kl-space* $(\mathfrak{X}, g)$ *the following hold*

(i) *The order subspaces* $\mathfrak{X}^+$ *and* $\mathfrak{X}^-$ *are closed.*

(ii) *Each subset* Y *of* X *is closed (open, clopen) in* $\mathfrak{X}$ *if and only if* $Y \cap X^+$ *and* $Y \cap X^-$ *are closed (open, clopen) in* $\mathfrak{X}^+$ *and* $\mathfrak{X}^-$, *respectively.*

(iii) *Each subset* U_0 *of* X^+ *is closed (open, clopen) increasing in* $\mathfrak{X}^+$ *if and only if* $U_0 = U \cap X^+$, *for some closed (open, clopen) increasing set* U *in* $\mathfrak{X}$.

Proof. (i) is proved in [5], (Lemma 2.1 p. 216). (ii) follows from (i). And (iii) is an easy consequence of the fact that $\mathfrak{X}^+$ is an order subspace in the sense of Priestley [16] (see §3. pp. 509–511).

The following condition for posets $\mathfrak{X}$ with a function g satisfying (k_1), (k_2) and (k_3) is essentially the so-called interpolation condition introduced by Monteiro in [12]

(n) for every $x, y \in X^+$, if $x \leqslant g(y)$ then there exists $z \in X^+$ such that $x \leqslant z \leqslant g(x)$ and $y \leqslant z \leqslant g(y)$.

The role of structures $(\mathfrak{X}, g)$ satisfying (k_1), (k_2), (k_3) and (n) for the theory of Nelson algebras has been investigated by Monteiro in the cited paper [12], nextly by Vakarelov in [25] (under the name M-spaces) and by the author in [22], where the term N-structure has been used. We follow the terminology from [22].

1.2 Lemma. *For any increasing subsets* U, V *and* W *of an* N-*structure* $(\mathfrak{X}, g)$ *there hold*

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[2] The description of the subcategory of $\boldsymbol{P\text{-}Kl}$, dually equivalent to the equational category $\boldsymbol{N}$ of Nelson algebras was announced in abstract form in [23]. An analogous result was also independently obtained by Cignoli in [4].

(i) $U \cap g(U) \cap X^+ = U \cap X^+$,
(ii) $U \cap g(U) \cap X^- = g(U) \cap X^-$,
(iii) $U \cap g(U) \subseteq V \cap g(V)$ iff $U \cap X^+ \subseteq V \cap X^+$,
(iv) $\downarrow(U \cap g(U) \backslash V) \cap X^+ = \downarrow((U \backslash V) \cap X^+) \cap X^+$,
(v) $\downarrow(U \cap g(U) \backslash V) \cap X^- = (g(U) \backslash V) \cap X^-$,
(vi) $U \cap V \cap X^+ \subseteq W \cap X^+$ iff $(U \cap X^+) \cap \downarrow((V \backslash W) \cap X^+) = \emptyset$.

PROOF. By an easy verification.

1.3 PROPOSITION. *Let* $(\mathfrak{X}, g)$ *be a* Kl-*space and let* $(\mathcal{O}(X), \cup, \cap, \sim, \emptyset, X) = \mathcal{O}_{\mathrm{Kl}}(\mathfrak{X}, g)$. *For any* $U, V \in \mathcal{O}(X)$ *define*:

$$U \to V = X \backslash \downarrow(U \cap g(U) \backslash V) \text{ and } \neg U = X \to \emptyset = X \backslash \downarrow(U \cap g(U)).$$

Then $\mathcal{O}_{\mathrm{N}}(\mathfrak{X}, g) = (\mathcal{O}(X), \cup, \cap, \to, \neg, \sim, \emptyset, X)$ *is a Nelson algebra if and only if* $(\mathfrak{X}, g)$ *satisfies the condition* (n) *and for every* $U, V \in \mathcal{O}(X)$, $\downarrow(U \cap g(U) \backslash V)$ *is clopen.*

PROOF. $\Rightarrow$ It is proved in [12] (see also [25] and [22]) that for each Nelson algebra $\mathfrak{A}$, the poset of prime filters of $\mathfrak{A}$ ordered by inclusion with the function g: $\mathcal{P}(A) \to \mathcal{P}(A)$, defined by $g(F) = \mathcal{P}(A) \backslash \{\sim a;\ a \in F\}$ satisfies the condition (n). Therefore, since $\mathcal{P}_{\mathrm{Kl}}(\mathcal{O}_{\mathrm{Kl}}(\mathfrak{X}, g))$ and $(\mathfrak{X}, g)$ are isomorphic objects of the category ***P-Kl***, $(\mathfrak{X}, g)$ has to satisfy the condition (n). To prove the second part of the assertion we use the following property of the weak relative pseudocomplementation operation stated in [12]. Namely, for any elements a, b of the Nelson algebra $\mathfrak{A}$, $a \to b$ is a pseudocomplement of a relative to $\sim a \vee b$, that is, for any $c \in A$

(*) $c \leqslant a \to b$ iff $a \wedge c \leqslant \sim a \vee b$.

Let U and V be clopen increasing in $\mathfrak{X}$, Since $U \to V$ exists to prove $\downarrow(U \cap g(U) \backslash V)$ is clopen it sufficies to show $U \to V = X \backslash \downarrow(U \cap g(U) \backslash V)$. By (*) $U \cap (U \to V) \subseteq \sim U \cup V$ because $U \to V \subseteq U \to V$. This implies $(U \to V) \cap \downarrow(U \cap g(U) \backslash V) = \emptyset$. Hence $U \to V \subseteq X \backslash \downarrow(U \cap g(U) \backslash V)$. For the reverse inclusion assume $x \notin (U \cap g(U) \backslash V)$. So, for all $y \in U \cap g(U) \backslash V$, $x \nleqslant y$. Using total order-disconnectedness and compactness we can select a clopen increasing set W containing x and such that $U \cap g(U) \backslash W \subseteq X \backslash W$. Then $U \cap W \subseteq X \backslash (g(U) \backslash V) = (X \backslash g(U)) \cup V = \sim U \cup V$; and hence, by (*), $x \in W \subseteq U \to V$. This proves $X \backslash \downarrow(U \cap g(U) \backslash V) \subseteq U \to V$.

$\Leftarrow$ Note that, if we define $U \to V = X \backslash \downarrow(U \cap g(U) \backslash V)$, for all clopen increasing sets U and V, then

$$U \prec V \text{ iff } U \cap g(U) \subseteq V \cap g(V) \text{ iff } U \cap X^+ \subseteq V \cap X^+$$

the last equivalence by 1.2 (iii) and so

$$U \approx V \text{ iff } U \cap g(U) = V \cap g(V) \text{ iff } U \cap X^+ = V \cap X^+.$$

Therefore $\prec$ is a quasi-order on $\mathcal{O}(X)$ and $\approx$ is an equivalence relation on

$\mathcal{O}(X)$ which satisfies the substitution property with respect to $\cup$ and $\cap$, also with respect to $\rightarrow$ and $\neg$ by 1.2 (iv). Moreover, in the quotient algebra $(\mathcal{O}(X), \cup, \cap, \rightarrow, \neg, \emptyset, X)/_{\approx}$

$$[U]_{\approx} \leqslant [V]_{\approx} \text{ iff } U \cap X^+ \subseteq V \cap X^+.$$

Then by Lemma 1.2 we have

$$\begin{aligned}[U \cap V]_{\approx} \leqslant [W]_{\approx} &\text{ iff } U \cap V \cap X^+ \subseteq W \cap X^+ \\ &\text{ iff } (U \cap X^+) \cap {\downarrow}((V \backslash W) \cap X^+) = \emptyset \\ &\text{ iff } U \cap X^+ \subseteq X \backslash ({\downarrow}((V \backslash W) \cap X^+) \cap X^+) \\ &\text{ iff } U \cap X^+ \subseteq (X \backslash {\downarrow}(V \cap g(V) \backslash W)) \cap X^+ \\ &\text{ iff } U \cap X^+ \subseteq (V \rightarrow W) \cap X^+ \\ &\text{ iff } [U]_{\approx} \leqslant [V \rightarrow W]_{\approx}.\end{aligned}$$

Hence the quotient algebra is a Heyting algebra. Using Lemma 1.2 it is easy to see the remaining conditions of the definition of Nelson algebra are also satisfied.

In the sequal Kl-spaces satisfying the conditions of Proposition 1.3 will be called *N-spaces.*

1.4 PROPOSITION. *Let* $(\mathfrak{X}, g_X)$ *and* $(\mathfrak{Y}, g_Y)$ *be* N-*spaces and let* $f\colon (\mathfrak{X}, g_X) \rightarrow (\mathfrak{Y}, g_Y)$ *be a morphism in* ***P-Kl****. Then the map* $\mathcal{O}(f)\colon \mathcal{O}(Y) \rightarrow \mathcal{O}(X)$ *is a homomorphism from the Nelson algebra* $\mathcal{O}_N(\mathfrak{Y}, g_Y)$ *to the Nelson algebra* $\mathcal{O}_N(\mathfrak{X}, g_X)$ *if and only if for every* $x \in X^+$, $f({\uparrow}x \cap X^+) = {\uparrow}f(x) \cap Y^+$.

PROOF. Obviously, $\mathcal{O}(f)$ is a homomorphism between considered Nelson algebras iff $\mathcal{O}(f)$ is $\rightarrow$-preserving. But, by the definition of $\rightarrow$ and since $f \circ g_X = g_Y \circ f$, it is equivalent to the following

$(*)\quad f^{-1}\big({\downarrow}(U \cap g_Y(U) \backslash V)\big) = {\downarrow}f^{-1}(U \cap g_Y(U) \backslash V)$, for all $U, V \in \mathcal{O}(Y)$.

$\Rightarrow$ Assume $(*)$ is satisfied. Let $x \in X^+$ and $u \in {\uparrow}x \cap X^+$. Then $f(x) \leqslant f(u)$ and $f(u) \leqslant f(g_X(u)) = g_y(f(u))$, which show $f({\uparrow}x \cap X^+) \subseteq {\uparrow}f(x) \cap Y^+$. To the reverse inclusion assume $y \in {\uparrow}f(x) \cap Y^+$. Suppose $y \neq f(z)$ for all $z \in {\uparrow}x \cap X^+$. Since ${\uparrow}x \cap X^+$ is closed in $\mathfrak{X}$ by total order-disconnectedness and compactness there exist clopen increasing sets U and V such that $f({\uparrow}x \cap X^+) \cap (U \cap g_Y(U) \backslash V) = \emptyset$ and $y \in U \cap g_Y(U) \backslash V$. Because $f(x) \leqslant y$, $f(x) \in {\downarrow}(U \cap g_Y(U) \backslash V)$. Then by $(*)$, $x \in {\downarrow}f^{-1}(U \cap g_Y(U) \backslash V)$. Thus, for some u in X, $x \leqslant u$ and $f(u) \in U \cap g_Y(U) \backslash V$. If $u \in X^+$ then $u \in {\uparrow}x \cap X^+$. Hence $f(u) \in f({\uparrow}x \cap X^+)$, which is impossible. If $u \notin X^+$ then by (n) there exists $w \in X^+$ such that $x \leqslant w \leqslant g_X(x)$ and $g_X(u) \leqslant w \leqslant u$. This implies $f(w) \in f({\uparrow}x \cap X^+) \cap (U \cap g_Y(U) \backslash V)$, which is also impossible.

$\Leftarrow$ Since f preserves an order ${\downarrow}f^{-1}(U \cap g_Y(U) \backslash V) \subseteq f^{-1}\big({\downarrow}(U \cap g_Y(U) \backslash V)\big)$. For the inverse inclusion assume $x \in f^{-1}\big({\downarrow}(U \cap g_Y(U) \backslash V)\big)$. So, $f(x) \leqslant y$ for some $y \in U \cap g_Y(U) \backslash V$.

Case 1. $x \in X^-$. Then $g_Y(y) \leqslant g_Y(f(x)) = f(g_X(x)) \leqslant f(x) \leqslant y$. Since both y and $g_Y(y)$ are in the convex set $U \cap g_Y(U)$, $f(x) \in U \cap g_Y(U)$. Moreover, because V is increasing, $f(x) \notin V$. This shows $x \in f^{-1}(U \cap g_Y(U) \backslash V) \subseteq \downarrow f^{-1}(U \cap g_Y(U) \backslash V)$.

Case 2. $x \notin X^-$. If $y \in Y^+$ then $y \in \uparrow f(x) \cap Y^+$. By hypothesis $y = f(u)$, $u \in \uparrow x \cap X^+$. Consequently $x \leqslant u \in f^{-1}(U \cap g_Y(U) \backslash V)$, which proves $x \in \downarrow f^{-1}(U \cap g_Y(U) \backslash V)$. If $y \notin Y^+$ then $f(x) \leqslant f(g_X(x)) = g_Y(f(x))$ and $g_Y(y) \leqslant y$. So, by (n), there exists $z \in Y^+$ such that $f(x) \leqslant z \leqslant g_Y(f(x))$ and $g_Y(y) \leqslant z \leqslant y$. Then $z \in \uparrow f(x) \cap Y^+$ and $z \in U \cap g_Y(U) \backslash V$. Thus as in the case $y \in Y^+$ (here the role of y plaies z) we obtain $x \in \downarrow f^{-1}(U \cap g_Y(U) \backslash V)$.

The composition of morphisms in ***P-Kl*** between N-spaces satisfying the condition of Proposition 1.4 also satisfies this condition. Hence the class of all N-spaces with such morphisms forms a subcategory ***P-N*** of ***P-Kl***. By Proposition 1.3 and 1.4 it follows that the category ***P-N*** is dually equivalent to the equational category ***N*** of Nelson algebras. This dual equivalence is established by the contravariant functors $\mathscr{P}_N$: $\boldsymbol{N} \to \boldsymbol{P\text{-}N}$ and $\mathscr{O}_N$: $\boldsymbol{P\text{-}N} \to \boldsymbol{N}$, where for each Nelson algebra $\mathfrak{A}$, $\mathscr{P}_N(\mathfrak{A}) = \mathscr{P}_{Kl}(\mathfrak{A})$ and for each Nelson algebra homomorphism h, $\mathscr{P}_N(h) = \mathscr{P}(h)$; while, for each N-space $(\mathfrak{X}, g)$, $\mathscr{O}_N(\mathfrak{X}, g)$ is a Nelson algebra defined in Proposition 1.3, and for each morphism f in ***P-N***, $\mathscr{O}_N(f) = \mathscr{O}(f)$.

1.5 PROPOSITION. (i) *Each* Kl-*space* $(\mathfrak{X}, g)$ *satisfying the condition* (n) *is an* N-*space if and only if the order subspace* $\mathfrak{X}^+$ *is a* H-*space.*

(ii) *If* $(\mathfrak{X}, g_X)$ *and* $(\mathfrak{Y}, g_Y)$ *are* N-*spaces, then each morphism* f: $(\mathfrak{X}, g_X) \to (\mathfrak{Y}, g_Y)$ *in* ***P-Kl*** *is a morphism in* ***P-N*** *if and only if* $f^+ = f \upharpoonright X^+ : \mathfrak{X}^+ \to \mathfrak{Y}^+$ *is a morphism in* ***P-H***.

PROOF. Throughout the proof the symbol $\downarrow_+ Z$ ($\uparrow_+ Z$) denotes the smallest decreasing (increasing) set in the order subspace $\mathfrak{X}^+$ containing a given set $Z \subseteq X^+$.

(i) $\Rightarrow$ Clearly, $\mathfrak{X}^+$ is a Priestley space. So, it sufficies to show $\mathfrak{X}^+$ satisfies the defining property to be a H-space. Let U_0 and V_0 be clopen increasing in $\mathfrak{X}^+$. By 1.1 (iii) there exist clopen increasing U and V in $\mathfrak{X}$ such that $U_0 = U \cap X^+$ and $V_0 = V \cap X^+$. Therefore

$$\begin{aligned} \downarrow_+(U_0 \backslash V_0) &= \downarrow(U_0 \backslash V_0) \cap X^+ \\ &= \downarrow((U \cap X^+) \backslash (V \cap X^+)) \cap X^+ \\ &= \downarrow(U \cap g(U) \backslash V) \cap X^+ \end{aligned}$$

(the last equation by 1.2 (iv)). Thus $\downarrow_+(U_0 \backslash V_0)$ is clopen in $\mathfrak{X}^+$ because by hypothesis $\downarrow(U \cap g(U) \backslash V)$ is clopen in $\mathfrak{X}$.

$\Leftarrow$ Let U and V be clopen increasing in $\mathfrak{X}$. Then by 1.2 (iv)

$$\begin{aligned} \downarrow(U \cap g(U) \backslash V) \cap X^+ &= \downarrow((U \cap X^+) \backslash (V \cap X^+)) \cap X^+ \\ &= \downarrow_+((U \cap X^+) \backslash (V \cap X^+)) \end{aligned}$$

and by 1.2 (v)

$$\downarrow(U \cap g(U) \backslash V) \cap X^- = (g(U) \backslash V) \cap X^- .$$

Thus the restriction of $\downarrow(U \cap g(U) \setminus V)$ to X^+ is clopen in $\mathfrak{X}^+$ and to X^- is clopen in $\mathfrak{X}^-$ which, by 1.1 (ii), shows this set is clopen in $\mathfrak{X}$.

(ii) Since the order in $\mathfrak{X}^+$ is a restriction of the order on X to X^+, $\uparrow_+ x = \uparrow x \cap X^+$ for all $x \in X^+$. Then the assertion follows immediately from the definitions.

1.6 COROLLARY. *For each* N*-space* $(\mathfrak{X}, g)$, *Heyting algebras* $(\mathcal{O}(X), \cup, \cap, \rightarrow, \neg, \emptyset, X)/_{\approx}$ *and* $(\mathcal{O}(X^+), \cup, \cap, \Rightarrow, -, \emptyset, X^+)$ *are isomorphic.*

PROOF. As we have seen in the proof of 1.3, for all U, $V \in \mathcal{O}(X)$, $[U]_{\approx} \leqslant [V]_{\approx}$ iff $U \cap X^+ \subseteq V \cap X^+$. Then the map $[U]_{\approx} \mapsto U \cap X^+$ is a lattice isomorphism (onto by 1.1 (iii)). It by 1.2 (iv) also preserves relative pseudocomplements.

2. Topological construction

If $(\mathfrak{Z}, g)$ is a dual N-space of the Nelson algebra $\mathfrak{A}$, then by Corollary 1.6 the order subspace $\mathfrak{Z}^+$ is up to isomorphism in ***P-H*** a dual H-space of the Heyting algebra $\mathfrak{A}^*$. Therefore, to find all Nelson algebras $\mathfrak{A}$ whose $\mathfrak{A}^*$ algebras are isomorphic to a given Heyting algebra $\mathfrak{B}$, it sufficies to find all N-spaces $(\mathfrak{Z}, g)$ such that their order subspaces $\mathfrak{Z}^+$ are isomorphic in ***P-H*** with a dual H-space of an algebra $\mathfrak{B}$. In this section we give a method of construction all N-spaces $(\mathfrak{Z}, g)$ whose order subspaces $\mathfrak{Z}^+$ coincide with a given H-space $\mathfrak{X}$.

2.1. LEMMA. *Let* $(\mathfrak{X}, g)$ *be an* N*-structure such that each* x *in* X^+ *is majorized by at least one* y *maximal in* $\mathfrak{X}^+$. *Then for every* U *increasing in* $\mathfrak{X}^+$ *and* V *increasing in* $\mathfrak{X}^-$, $g(U \cap \mathrm{Max}(X^+)) \subseteq V$ *implies* $U \cup V$ *increasing in* $\mathfrak{X}$.

PROOF. Let $x \leqslant y$ and $x \in U \cup V$.

Case 1. $x \in U$. Since U is increasing in $\mathfrak{X}^+$, if $y \in X^+$ then $y \in U$. So, let $y \in X^-$. Then by (n) there exists $z \in X^+$ such that $x \leqslant z \leqslant g(x)$ and $g(y) \leqslant z \leqslant y$. By hypothesis $z \leqslant u$, for some $u \in \mathrm{Max}(X^+)$. This implies $g(u) \in g(U \cap \mathrm{Max}(X^+)) \subseteq V$. Thus $y \in V$, because $g(u) \leqslant g(z) \leqslant g(g(y)) = y$ and V is increasing in $\mathfrak{X}^-$.

Case 2. $x \notin U$. So $x \in V$. If $y \in X^+$ then $g(x) \leqslant x \leqslant y \leqslant g(y)$; and hence $y = g(g(y)) \leqslant g(y) \leqslant g(x) \leqslant g(x)) = x$. So $y = x \in V$. If $y \in X^-$ then $y \in V$ since V is increasing in $\mathfrak{X}^-$.

2.2 LEMMA. *For any* H*-space* $\mathfrak{X}$ *there hold*

(i) *The set* $\mathrm{Max}(X)$ *of all maximal elements in* $\mathfrak{X}$ *is closed in* $\mathfrak{X}$.

(ii) *For every clopen set* U *in the subspace* $\mathrm{Max}(X)$, $\downarrow U$ *is clopen in* $\mathfrak{X}$.

PROOF. Since every Heyting algebra is a pseudocomplemented distributive lattice, it immediately follows from Lemma 1 p. 216 and Corollary 6 p. 218 of [17].

Let $\mathfrak{X} = (X, \mathscr{T}_X, \leqslant_X)$ and $\mathfrak{Y} = (Y, \mathscr{T}_Y, \leqslant_Y)$ be disjoint homeomorphic and dually order isomorphic ordered spaces; and let the map $f\colon X \to Y$ establish the required order-reversing homeomorphism. Furthermore, let S be an arbitrary subset of the set $\mathrm{Max}(X)$ of all maximal elements in $(X, \leqslant_X)$. Define a topology $\mathscr{T}$, a relation $\leqslant$ and a function g_S on $X \cup f(X \setminus S)$ as follows

(1) $Z \in \mathscr{T}$ iff $j^{-1}(Z) \in \mathscr{T}_X$ and $k^{-1}(Z) \in \mathscr{T}_Y$, where the maps $j\colon X \to X \cup f(X\setminus S)$ and $k\colon Y \to X \cup f(X \setminus S)$ are defined by

$$j(x) = x \text{ and } k(y) = \begin{cases} f^{-1}(y); & \text{if } y \in f(S) \\ y; & \text{otherwise} \end{cases}$$

(2) $\leqslant \; = \; \leqslant_X \cup \leqslant_{k(Y)} \cup (\leqslant_X \circ \varrho \circ \leqslant_{k(Y)})$, where $\leqslant_{k(Y)}$ is a partial order on $k(Y)$ induced from Y by k, and $\varrho = \{(x, k(f(x)));\ x \in X\}$

(3) $$g_S(x) = \begin{cases} k(f(x)); & \text{if } x \in X \\ f^{-1}(x); & \text{otherwise.} \end{cases}$$

Clearly, $\mathscr{T}$ is a well-defined topology on the set $X \cup f(X\setminus S)$. This topological space is homeomorphic to a quotient space of the topological sum of $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ by an equivalence relation corresponding to the partition of the set $X \cup Y$ into sets $\{z, f(z)\}$, for $z \in S$ and singletons $\{z\}$, for $z \notin S$. On the order hand, by Proposition 3.1 [22], $(X \cup f(X\setminus S), \leqslant, g_S)$ is an N-structure. Therefore $(X \cup f(X \setminus S), \mathscr{T}, \leqslant)$ is an ordered space. Obviously, for a given ordered space $\mathfrak{X}$, the ordered space $\mathfrak{Y}$ homeomorphic and dually order-isomorphic to $\mathfrak{X}$ with $X \cap Y = \emptyset$, as well as the order-reversing homeomorphism $f\colon \mathfrak{X} \to \mathfrak{Y}$ can be easily constructed. From now on, it will be assumed that they are constructed in some fixed canonical way, uniform for all ordered spaces $\mathfrak{X}$; the set $X \cup f(X\setminus S)$ will be denoted by $X \nearrow S$ and the ordered space $(X \cup f(X \setminus S), \mathscr{T}, \leqslant)$ by $\mathfrak{X} \nearrow S$.

2.3 THEOREM. (i) *If $\mathfrak{X}$ is a* H-*space and the subset S of* $\mathrm{Max}(X)$ *is closed in $\mathfrak{X}$ (or equivalently closed in the subspace* $\mathrm{Max}(X)$) *then $(\mathfrak{X} \nearrow S, g_S)$ is an* N-*space such that the order subspaces $(\mathfrak{X} \nearrow S)^+$ and $(\mathfrak{X} \nearrow S)^+ \cap (\mathfrak{X} \nearrow S)^-$ coincide with $\mathfrak{X}$ and $\mathfrak{S}$, respectively.*

(ii) *For each* N-*space $(\mathfrak{Y}, g_Y)$ there exist a* H-*space $\mathfrak{X}$ and closed subset S of* $\mathrm{Max}(X)$ *such that* N-*spaces $(\mathfrak{Y}, g_Y)$ and $(\mathfrak{X} \nearrow S, g_S)$ are isomorphic as objects of* ***P-N***.

PROOF. (1) Note that $(X \nearrow S)^+ = X$, $(X \nearrow S)^- = k(Y)$, $\leqslant \cap (X \times X) = \leqslant_X$ and $\leqslant \cap (k(Y) \times k(Y)) = \leqslant_{k(Y)}$. The maps j and k are continuous and closed since S is closed in $\mathfrak{X}$. It follows from these observations that the order subspaces $(\mathfrak{X} \nearrow S)^+$ and $(\mathfrak{X} \nearrow S)^+ \cap (\mathfrak{X} \nearrow S)^-$ coincide with $\mathfrak{X}$ and $\mathfrak{S}$, respectively. Therefore by Proposition 1.5 (i) it sufficies to show that $\mathfrak{X} \nearrow S$ is compact totally order disconnected and the map g_S is a homeomorphism on $\mathfrak{X} \nearrow S$.

First observe that for every subset Z of $X \nearrow S$ $j^{-1}(g_S(Z)) = f^{-1}(k^{-1}(Z))$ and $k^{-1}(g_S(Z)) = f(j^{-1}(Z))$. Then g_S is a homeomorphism because it is an involution.

To show $\mathfrak{X} \nearrow S$ is totally order disconnected assume $x, y \in X \nearrow S$ and $x \nleqslant y$.

Case 1. $x, y \in X$. Then $x \nleqslant_X y$. By total order disconnectedness of $\mathfrak{X}$ there exists a clopen increasing subset U of X such that $x \in U$ and $y \notin U$. Then by Lemma 2.2 (ii) $\downarrow(U \cap \mathrm{Max}(X))$ is clopen in $\mathfrak{X}$. Consider the set $Z = U \cup g_S(\downarrow(U \cap \mathrm{Max}(X)))$. Clearly, $x \in Z$, $y \notin Z$ and Z is clopen in $\mathfrak{X} \nearrow S$ because $j^{-1}(Z) = U$ and $k^{-1}(Z) = f(\downarrow(U \cap \mathrm{Max}(X)))$. It is also increasing by Lemma 2.1.

Case 2. $x, y \in k(Y)$. Then $g_S(x), g_S(y) \in X$ and $g_S(y) \nleqslant_X g_S(x)$. Therefore, as in Case 1, there exists Z clopen increasing in $\mathfrak{X} \nearrow S$ such that $g_S(y) \in Z$ and $g_S(x) \notin Z$. Thus the set $(X \nearrow S) \backslash g_S(Z)$ is clopen increasing in $\mathfrak{X} \nearrow S$; it contains x but not y.

Case 3. $x \in X \backslash S$ and $y \in f(X \backslash S)$. Then, for all $z \in X$, $x \nleqslant_X z$ or $g_S(y) \nleqslant_X z$. Compactness and total order disconnectedness of $\mathfrak{X}$ yield clopen increasing $U_1, \ldots, U_n$ in $\mathfrak{X}$ such that $X = (X \backslash U_1) \cup \ldots \cup (X \backslash U_n)$ and for all i, $i = 1, \ldots, n$, $x \in U_i$ or $g_S(y) \in U_i$. Define $U = \bigcap \{U_i;\ x \in U_i,\ 1 \leqslant i \leqslant n\}$ and $V = \bigcap \{U_i;\ x \notin U_i,\ 1 \leqslant i \leqslant n\}$. Obviously, x is in U and $g_S(y)$ in V. Since x and $g_S(y)$ are in X, both families used to define U and V are non-empty. Hence U and V are disjoint. Consider the set $Z = U \cup g_S(\downarrow(U \cap \mathrm{Max}(X)))$. $x \in Z$ and Z is clopen increasing by the same arguments as in Case 1. Since $U \cap V = \emptyset$ and V is increasing in $\mathfrak{X}$, $\downarrow(U \cap \mathrm{Max}(X)) \subseteq \downarrow U \subseteq X \backslash V$. Therefore $\downarrow(U \cap \mathrm{Max}(X)) \cap V = \emptyset$. This implies $g_S(\downarrow(U \cap \mathrm{Max}(X))) \cap g_S(V) = \emptyset$. Hence $y \notin Z$ because $y \in g_S(V)$ and by hypothesis $y \notin U$.

Case 4. $x \in f(X \backslash S)$ and $y \in (X \backslash S)$. Then for all $z \in S$, $z \nleqslant_X g_S(x)$. Since S is closed in $\mathfrak{X}$, by compactness and total order disconnectedness of $\mathfrak{X}$ there exists U clopen increasing in $\mathfrak{X}$ such that $S \subseteq U$ and $g_S(x) \notin U$. Therefore the set $g_S(X \backslash U)$ is clopen increasing in $\mathfrak{X} \nearrow S$; it contains x but not y.

For compactness of $\mathfrak{X} \nearrow S$ observe that the function l defined by

$$l(x) = \begin{cases} j(x); & \text{if } x \in X \\ k(x); & \text{if } x \in Y \end{cases}$$

is a continuous mapping of the topological sum $\mathfrak{X} \oplus \mathfrak{Y}$ onto $\mathfrak{X} \nearrow S$. Hence $\mathfrak{X} \nearrow S$ is compact because $\mathfrak{X} \oplus \mathfrak{Y}$ is compact and $\mathfrak{X} \nearrow S$ as a totally order disconnected space is Hausdorff.

(ii) Take $\mathfrak{X} = \mathfrak{Y}^+$, $S = Y^+ \cap Y^-$ and observe that the map l: $Y \to X \nearrow S$ defined by

$$l(y) = \begin{cases} y; & \text{if } y \in Y^+ \\ g_S(g_Y(Y)); & \text{otherwise} \end{cases}$$

and its inverse are morphisms in ***P-N***.

By dualities, immediately from Theorem 2.3 and Corollary 1.6 we have

2.4 Corollary. *Each Nelson algebra $\mathfrak{A}$ can be represented (up to isomorphism) as an algebra $\mathcal{O}_N(\mathfrak{X} \nearrow S, g_S)$, where $\mathfrak{X}$ is a dual* H-*space of an algebra $\mathfrak{A}^*$ and S is some closed subset of* $\mathrm{Max}(X)$.

2.5 COROLLARY. *For each Heyting algebra* $\mathfrak{B}$ *with a dual* H-*space* $\mathfrak{X}$, *the all (up to isomorphism) Nelson algebras* $\mathfrak{A}$ *whose* $\mathfrak{A}^*$ *algebras are isomorphic to* $\mathfrak{B}$ *there are among algebras* $\mathcal{O}_N(\mathfrak{X} \nearrow S, g_S)$, *where* S *is closed subset of* $\mathrm{Max}(X)$. *If* $\mathfrak{B}$ *is non-trivial then at least two of them are non-isomorphic, namely* $\mathcal{O}_N(\mathfrak{X} \nearrow \emptyset, g_\emptyset)$ *and* $\mathcal{O}_N(\mathfrak{X} \nearrow \mathrm{Max}(X), g_{\mathrm{Max}(X)})$.

3. Algebraic counterpart

If $\mathfrak{X}$ is a H-space of a Heyting algebra $\mathfrak{B}$ then every closed subset S of $\mathrm{Max}(X)$ is increasing in $\mathfrak{X}$. Then the order subspace $\mathfrak{S}$ of $\mathfrak{X}$ is up to isomorphism in ***P-H*** a H-space of the quotient algebra $\mathfrak{B}/\Theta$, for some congruence relation Θ on $\mathfrak{B}$. Moreover, since the order in $\mathfrak{S}$ is discrete ($x \leqslant y$ iff $x = y$) the congruence relation Θ is Boolean, that means $\mathfrak{B}/\Theta$ is a Boolean algebra. Hence, accordance with results of the preceding section every Nelson algebra $\mathfrak{A}$ is determined by its Heyting algebra $\mathfrak{A}^*$ and some its Boolean congruence relation. In this section we presente a method of construction of Nelson algebras being an algebraic realization of this observation. The precise formulation and the proof of the main result need some additional definitions and facts.

Let $\mathfrak{A}$ be a Nelson algebra. Let us denote by a^* the element $\sim \neg a$, for $a \in A$; and the set of all elements of this form i.e. the set $\{a^*;\ a \in A\}$ by A^*. Then almost immediately from definitions we have

3.1 LEMMA. *For all* $a, b \in A$ *there hold*
(i) $a \prec b$ *iff* $a^* \leqslant b^*$,
(ii) $a \approx b$ *iff* $a^* = b^*$,
(iii) a^* *is the least element in the class* $[a]_\approx$ *with respect to the lattice ordering in* $\mathfrak{A}$,
(iv) $a^* = 0$ *and* $1^* = 1$,
(v) $a^* = 1$ *iff* $a = 1$,
(vi) $a^{**} = a^* \leqslant a$,
(vii) $(a^* \vee b^*)^* = (a \vee b)^* = a^* \vee b^*$,
(viii) $(a^* \wedge b^*)^* = (a \wedge b)^* \leqslant a^* \wedge b^*$,
(ix) $(a^* \to b^*)^* = (a \to b)^*$,
(x) $(\neg a^*)^* = (\neg a)^*$.

3.2 LEMMA. *For every* $a \in A$ *the following are equivalent:*
(i) $a \in A^*$,
(ii) $a = a^*$,
(iii) $\neg a = \sim a$,
(iv) $\neg a \prec \sim a$.

Let us define in the set A^* operations $\vee^*$, $\wedge^*$, $\Rightarrow^*$ and $-^*$ by formulas

$$a \vee^* b =_{\mathrm{df}} (a \vee b)^*,$$
$$a \wedge^* b =_{\mathrm{df}} (a \wedge b)^*,$$

$$a \Rightarrow^* b =_{\mathrm{df}} (a \to b)^* \text{ and}$$
$$-^* a =_{\mathrm{df}} (\neg a)^*, \text{ for all } a, b \in A^*.$$

3.3 LEMMA. *Algebra* $(A^*, \vee^*, \wedge^*, \Rightarrow^*, -^*, 0, 1)$ *is isomorphic to the quotient algebra* $(A, \vee, \wedge, \to, \neg, 0, 1)/_{\approx}$; *and hence it is a Heyting algebra. The lattice order in this algebra is a restriction of the lattice order in* $\mathfrak{A}$ *to the subset* A^*.

PROOF. Using Lemma 3.1 one can easily verify that the assignment $[a]_{\approx} \mapsto a^*$, for all $a \in A$, is well-defined and establishes the required isomorphism. The second statement holds since the lattice order and the quasi-order $\prec$ on $\mathfrak{A}$ restricted to the subset A^* coincide.

REMARK. From now on the symbol $\mathfrak{A}^*$ previously used to denote the quotient algebra $(A, \vee, \wedge, \to, \neg, 0, 1)/_{\approx}$ will be reserved to denote the algebra $(A^*, \vee^*, \wedge^*, \Rightarrow^*, -^*, 0, 1)$.

For every Nelson algebra $\mathfrak{A}$ the relation $\approx$ is an equivalence relation on the set A (point (III) of the definition). Then the relation $\approxeq$ on A defined by $a \approxeq b$ iff $\sim a \approx \sim b$ is also equivalence relation. Let $\beta_{\mathfrak{A}}$ be a join of $\approx$ and $\approxeq$ in the lattice of all equivalence relations on the set A.

3.4 LEMMA. $\beta_{\mathfrak{A}}$ *is a congruence relation on the Nelson algebra* $\mathfrak{A}$ *generated by the set* $\{(\neg a, \sim a);\ a \in A\}$.

PROOF. Since $\approx$ and $\approxeq$ have the substitution property with respect to the lattice operations, $\beta_{\mathfrak{A}}$ also has this property with respect to them. The subtitution property of $\beta_{\mathfrak{A}}$ with respect to $\sim$ immediately follows by definitions. Furthermore, if $a\beta_{\mathfrak{A}} b$ and $c\beta_{\mathfrak{A}} d$, then $\sim(a \to c) \approx a \wedge \sim c\beta_{\mathfrak{A}} b \wedge \sim d \approx \sim(b \to d)$. Hence $\sim(a \to c)\,\beta_{\mathfrak{A}} \sim(b \to d)$ which implies $a \to c\beta_{\mathfrak{A}} b \to d$. Therefore $\beta_{\mathfrak{A}}$ is a congruence on $\mathfrak{A}$. To show $\beta_{\mathfrak{A}}$ is generated by $\{(\neg a, \sim a);\ a \in A\}$ notice that $a\beta_{\mathfrak{A}} \sim \neg a$, for all $a \in A$, since $a \approx \sim \neg a$; and hence $\sim a\beta_{\mathfrak{A}} \neg a$. So, to finish the proof it sufficies to show that every congruence relation Θ with this property cointains $\beta_{\mathfrak{A}}$. But it is clear that such Θ cointains $\approx$ and $\approxeq$, so it must cointain $\beta_{\mathfrak{A}}$.

3.5 LEMMA. *For any* $a, b \in A^*$ *the following are equivalent*:
(i) $a \wedge b \approx 0$ *and* $a \vee b\beta_{\mathfrak{A}} 1$,
(ii) $a \leqslant \sim b$ *and* $a\beta_{\mathfrak{A}} \sim b$,
(iii) $[a]_{\approx} \cap [\sim b]_{\approxeq} \neq \emptyset$,
(iv) $\mathrm{card}([a]_{\approx} \cap [\sim b]_{\approxeq}) = 1$.

PROOF. (i)$\Rightarrow$(ii). Assume (i). Then by $a \vee b\beta_{\mathfrak{A}} 1$ we have $a \vee (b \wedge \sim b) = (a \vee b) \wedge (a \vee \sim b)\,\beta_{\mathfrak{A}}\, a \vee \sim b$ and $a \vee (b \wedge \sim b) \approx a$ because $b \wedge \sim b \approx 0$. Hence $a \approx a \vee (b \wedge \sim b)\,\beta_{\mathfrak{A}}\, a \vee \sim b$. On the other hand, by $a \wedge b \approx 0$ we have $(a \vee \sim b) \wedge (b \vee \sim b) = (a \wedge b) \vee \sim b \approx \sim b$ and $(a \vee \sim b) \wedge (b \vee \sim b) \approxeq a \vee \sim b$, because $b \vee \sim b \approxeq 1$. Hence $a \vee \sim b \approxeq (a \vee \sim b) \wedge (b \vee \sim b) \approx \sim b$. Therefore $a\beta_{\mathfrak{A}} a \vee \sim b\beta_{\mathfrak{A}} \sim b$, which proves $a\beta_{\mathfrak{A}} \sim b$. Moreover, by $a \wedge b \approx 0$, $a \prec \neg b$

and $b \prec \neg a$. But since $a, b \in A^*$, $\neg a = \sim a$ and $\neg b = \sim b$. So, $a \prec \sim b$ and $b \prec \sim a$; which means $a \leqslant \sim b$.

(ii) $\Rightarrow$ (iii). Recall that $a \beta_{\mathfrak{A}} \sim b$ holds iff there are elements $c_1, \ldots, c_{n-1}$ in A such that

(*) $\quad a\Theta_0 c_1 \Theta_1 c_2 \ldots c_{n-1} \Theta_{n-1} \sim b$

where $\Theta_i = \approx$ for i even and $\Theta_i = \approxeq$ for i odd. The proof is by induction with respect to the number n of relational symbols in the sequence (*).

If $n = 1$, then $a \approx \sim b$; and hence $\sim b \in [a]_{\approx} \cap [\sim b]_{\approxeq}$. If $n = 2$, then $a \approx c_1 \approxeq \sim b$; hence $c_1 \in [a]_{\approx} \cap [\sim b]_{\approxeq}$. Assume the assertion holds for the numbers of relational symbols less than n.

Case 1. n is even. Let $n = 2k$. Then by the definition of $\approxeq$ we have

$$b\Theta_0 \sim c_{n-1} \Theta_1 \sim c_{n-2} \ldots \sim c_k \Theta_k \sim c_{k-1} \ldots \sim c_1 \Theta_{n-1} \sim a.$$

This and (*) imply

$$0 \approx c_k \wedge \sim c_k \Theta_k c_{k+1} \wedge \sim c_{k-1} \ldots c_{n-1} \wedge \sim c_1 \Theta_{n-1} \sim b \wedge \sim a.$$

Hence

$$a \approx a \vee (c_k \wedge \sim c_k) \Theta_k a \vee (c_{k+1} \wedge \sim c_{k-1}) \ldots a \vee (c_{n-1} \wedge \sim c_1) \Theta_{n-1} a \vee (\sim b \wedge \sim a).$$

But $\Theta_{n-1} = \approxeq$ and $\sim b \approxeq \sim b \wedge (a \vee \sim a) = a \vee (\sim b \wedge \sim a)$, since by assumption $a \leqslant \sim b$. So we finally have

$$a \approx a \vee (c_k \wedge \sim c_k) \Theta_k a \vee (c_{k+1} \wedge \sim c_{k-1}) \ldots a \vee (c_{n-1} \wedge \sim c_1) \Theta_{n-1} \sim b.$$

Obviously the number of relational symbols in this sequence is less than n. Thus by the assumption the assertion holds.

Case 2. n is odd. Add to the sequence (*) the word $\approxeq \sim b$ and repead the argumentation as in Case 1.

(iii) $\Rightarrow$ (i). Assume there exists c such that $a \approx c \approxeq \sim b$. Then $a \wedge b \approx c \wedge \sim c \approx 0$, since $b \approx \sim c$. The second condition of (i) also holds because $a \vee b \beta_{\mathfrak{A}} \sim b \vee b \approxeq 1$.

(iii) $\Rightarrow$ (iv). It follows by the fact that the meet of equivalences $\approx$ and $\approxeq$ is a diagonal in $A \times A$.

Now we are ready to formulate and proof the main result of this section.

For any Heyting algebra $\mathfrak{B}$ and Boolean congruence relation Θ on $\mathfrak{B}$ define (comp. [25] and [6])

$$N_{\Theta}(B) =_{\mathrm{df}} \{(a, b) \in B^2;\ a \wedge b = 0 \text{ and } a \vee b\Theta 1\}$$

and for all $(a, b), (c, d) \in N_{\Theta}(B)$

$$(a, b) \vee (c, d) =_{\mathrm{df}} (a \vee c, b \wedge d)$$
$$(a, b) \wedge (c, d) =_{\mathrm{df}} (a \wedge c, b \vee d)$$
$$(a, b) \to (c, d) =_{\mathrm{df}} (a \to c, a \wedge d)$$
$$\neg (a, b) =_{\mathrm{df}} (-a, a)$$
$$\sim (a, b) =_{\mathrm{df}} (b, a).$$

3.6 THEOREM. (i) *For each Heyting algebra* $\mathfrak{B}$ *and Boolean congruence relation* Θ *on* $\mathfrak{B}$, *the set* $N_\Theta(B)$ *is closed under the above defined operations and the algebra* $N_\Theta(\mathfrak{B}) = (N_\Theta(B), \vee, \wedge, \rightarrow, \neg, \sim, (0,1), (1,0))$ *is a Nelson algebra such that* $N_\Theta(\mathfrak{B})^*$ *is isomorphic to* $\mathfrak{B}$.

(ii) *For each Nelson algebra* $\mathfrak{A}$ *there is a Boolean congruence relation* Θ *on* $\mathfrak{A}^*$ *such that* $\mathfrak{A}$ *is isomorphic to* $N_\Theta(\mathfrak{A}^*)$.

PROOF. (i) When $\Theta = B^2$, $N_\Theta(\mathfrak{B})$ is a Nelson algebra (Theorem 1 p. 111 in [25]). So to prove $N_\Theta(\mathfrak{B})$ is a Nelson algebra for all Boolean congruences Θ it sufficies to show that for any elements of $N_\Theta(B)$ results with respect to each operation satisfy the second condition of the definition of $N_\Theta(B)$. Let (a, b) and (c, d) belong to $N_\Theta(B)$. Then by $a \vee b\Theta 1$ and $c\Theta c$ we obtain $a \vee b \vee c\Theta 1$. Analogously, $a \vee c \vee d\Theta 1$. Hence by $(a \vee c) \vee (b \wedge d) = (a \vee b \vee c) \wedge (a \vee c \vee d)$, $(a \vee c) \vee (b \wedge d)\Theta 1$. Similarly $(a \wedge c) \vee (b \vee d)\Theta 1$. For the operation $\rightarrow$ notice that $c \vee d \vee -a\Theta 1$ since $c \vee d\Theta 1$ and $-a\Theta -a$. Furthermore, $a \vee -a \vee c\Theta 1$ because $a \vee -a\Theta 1$ and $c\Theta c$. Hence $(a \vee -a \vee c) \vee (c \vee d \vee -a) = (-a \vee c) \vee (a \wedge d)\Theta 1$. On the other hand, since Θ is Boolean, $a \Rightarrow c\Theta -a \vee c$; which implies $(a \Rightarrow c) \vee (a \wedge d)\Theta(-a \vee c) \vee (a \wedge d)$. So finally $(a \Rightarrow c) \vee (a \wedge d)\Theta 1$. The proofs for the rest operations are trivial.

Therefore it remains to show $N_\Theta(\mathfrak{B})^*$ is isomorphic to $\mathfrak{B}$. But it is easy to see that the map $b \mapsto (b, -b)$ is a required isomorphism from $\mathfrak{B}$ onto $N_\Theta(\mathfrak{B})^*$.

(ii) One can prove using Lemma 3.1 that the restriction of any congruence relation on $\mathfrak{A}$ to A^* is a congruence on $\mathfrak{A}^*$. So setting $\Theta = \beta_\mathfrak{A} \restriction A^*$ in virtue of Lemma 4.3 we obtain a Boolean congruence relation on $\mathfrak{A}^*$. Define a map $h\colon A \rightarrow N_\Theta(A^*)$ by $h(a) = (a^*, (\sim a)^*)$, for all $a \in A$. h is well-defined because $a^* \wedge^* (\sim a)^* = (a^* \wedge (\sim a)^*)^* = (a \wedge \sim a)^* = 0^* = 0$ and $a^* \vee^* (\sim a)^* = (a^* \vee (\sim a)^*)^* = (a \vee \sim a)^* \beta_\mathfrak{A} 1$. If $h(a) = h(b)$ then $a^* = b^*$ and $(\sim a)^* = (\sim b)^*$. This means $a \approx b$ and $\sim a \approx \sim b$; hence $a = b$. Furthermore, if (a, b) is in $N_\Theta(\mathbf{A}^*)$ then $a, b \in A^*$, $(a \wedge b)^* = a \wedge {}^* b = 0$ and $a \vee b = a^* \vee b^* = (a \vee b)^* \beta_\mathfrak{A} 1$. So $a \wedge b \approx 0$ and $a \vee b\beta_\mathfrak{A} 1$. Therefore by Lemma 3.5 there exists $c \in A$ such that $a \approx c \lessapprox \sim b$. Hence $a = a^* = c^*$ and $b = b^* = (\sim c)^*$, which proves $(a, b) = h(c)$. Thus h is a bijection. The verification that h preserves operations as an easy exercise is left to the reader.

3.7 COROLLARY. *If* Θ *and* Ψ *are Boolean congruences on the Heyting algebra* $\mathfrak{B}$, *then*

(i) *The Nelson algebra* $N_\Theta(\mathfrak{B})$ *is a subalgebra of the algebra* $N_\Psi(\mathfrak{B})$ *if and only if* $\Theta \subseteq \Psi$.

(ii) *If a Nelson algebra* $\mathfrak{A}$ *is a subalgebra of* $N_\Psi(\mathfrak{B})$ *and* $N_\Theta(\mathfrak{B})$ *is a subalgebra of* $\mathfrak{A}$, *then there exists a Boolean congruence* Φ *on* $\mathfrak{B}$ *such that* $\mathfrak{A} = N_\Phi(\mathfrak{B})$.

PROOF. (i) The sufficiency is obvious. The necessity follows by the fact that for any congruence Φ on $\mathfrak{B}$, $a\Phi b$ iff $(a \Rightarrow b) \wedge (b \Rightarrow a)\Phi 1$, for all $a, b \in B$. Indeed, if $a\Theta b$ then $(a \Rightarrow b) \wedge (b \Rightarrow a)\Theta 1$; and hence $((a \Rightarrow b) \wedge (b \Rightarrow a), 0) \in N_\Theta(B) \subseteq N_\Psi(B)$. So $(a \Rightarrow b) \wedge (b \Rightarrow a)\Psi 1$, which means $a\Psi b$.

(ii) By the assumption $A^* = N_\Theta(B)^* = N_\Psi(B)^* = \{(a, -a);\ a \in B\}$. And by 3.6 (ii) there is a Boolean congruence Φ_0 on $\mathfrak{A}^*$ such the map $A \ni (a, b) \mapsto ((a, -a), (b, -b)) \in N_{\Phi_0}(A^*)$ is a Nelson algebra isomorphism. Thus as Φ it sufficies to take the relation on B defined by $a\Phi b$ iff $(a, -a)\,\Phi_0(b, -b)$.

Obviously the set of all Boolean congruences of any Heyting algebra $\mathfrak{B}$, denote it by $\mathrm{Con}_B(\mathfrak{B})$, is a sublattice of the lattice $\mathrm{Con}(\mathfrak{B})$ of all congruence on $\mathfrak{B}$. $\mathrm{Con}_B(\mathfrak{B})$ is a complete lattice with the least element being an intersection of all Boolean congruence and with the greatest one equal B^2. The Nelson algebra $N_\Theta(\mathfrak{B})$ when Θ is the least element in $\mathrm{Con}_B(\mathfrak{B})$ will be denoted by $\overleftarrow{N}(\mathfrak{B})$, and when Θ is the greatest one by $\vec{N}(\mathfrak{B})$.

From Theorem 3.6 and Corollary 3.7 immediately follows

3.8 COROLLARY. *The set* $\{N_\Theta(\mathfrak{B});\ \Theta \in \mathrm{Con}_B(\mathfrak{B})\}$ *is a principal filter generated by* $\overleftarrow{N}(\mathfrak{B})$ *in the lattice of all subalgebras of* $\vec{N}(\mathfrak{B})$; *this filter treated as a sublattice of the lattice of subalgebras of* $\vec{N}(\mathfrak{B})$ *is isomorphic to the lattice* $\mathrm{Con}_B(\mathfrak{B})$.

By the above Corollary and Theorem 3.6 we also have

3.9 COROLLARY. *The number of all non-isomorphic Nelson algebras* $\mathfrak{A}$ *whose* $\mathfrak{A}^*$ *algebras are isomorphic to a given non-trivial Heyting algebra* $\mathfrak{B}$ *is not less than* 2 *and not greater than* $\mathrm{card}(\mathrm{Con}_B(\mathfrak{B}))$; *all such algebras can be found as subalgebras of* $\vec{N}(\mathfrak{B})$ *containing* $\overleftarrow{N}(\mathfrak{B})$.

REMARKS. 1. The bounds of the number of Nelson algebras from Corollary 3.9 are the best as possible, that is, they are reached for some Heyting algebras.

2. Congruence of any Heyting algebra can be represented by filters in this algebra, and filters corresponding to Boolean congruence are precisely those containing all dense elements i.e., elements a such that $-a = 0$ (see [21]). This allows to replace the notion of a Boolean congruence by the notion of a filter containing dense elements. Therefore the underlying set of the algebra $\overleftarrow{N}(\mathfrak{B})$ can be defined as the set of all pairs (a, b) satisfying a one of the equivalent conditions: (i) $a \wedge b = 0$ and $a \vee b$ is dense, (ii) $(a \Rightarrow b) \wedge (b \Rightarrow a) = 0$ and (iii) $-a = --b$.

4. Relationships in the categorical presentation

In this section we look into results of proceding sections from the categorical point of view. This approach leads us to state that the relationship between Nelson and Heyting algebras has a topological nature. There are two approaches to categorical topology, namely the constructive one via topological theories and the axiomatic one. We recall basic definitions of the first one which is due to O. Wyler [26], and for the second one we refer to [10], where the axioms of a topological functor and necessary results may be found.

A topological theory on a category $\boldsymbol{C}$ is a functor $T\colon \boldsymbol{C}^{\mathrm{op}} \to \boldsymbol{POS}$, where

POS is a category of partially ordered sets as objects and order preserving maps as morphisms, such that

(i) every $T(A)$ is a complete lattice,
(ii) every $T(f)$ preservs arbitrary meets.

Each topological theory T on $\boldsymbol{C}$ defines a top-category $\boldsymbol{C}^T$ over $\boldsymbol{C}$. Objects of $\boldsymbol{C}^T$ are pairs (A, a), where A is an object of $\boldsymbol{C}$ and $a \in T(A)$; while morphisms from (A, a) to (B, b) are morphisms $f\colon A \to B$ in $\boldsymbol{C}$ such that $a \leqslant T(f)(b)$.

4.1 THEOREM. *The category $\boldsymbol{N}$ of Nelson algebras is equivalent to the top-category $\boldsymbol{H}^{\mathrm{Con}_{\boldsymbol{B}}}$ over the category $\boldsymbol{H}$ of Heyting algebras, where $\mathrm{Con}_{\boldsymbol{B}}\colon \boldsymbol{H}^{\mathrm{op}} \to \boldsymbol{POS}$ is a functor with a morphism assignment defined by $\mathrm{Con}_{\boldsymbol{B}}(k) = k^{-1}\colon \mathrm{Con}_{\boldsymbol{B}}(\mathfrak{B}_2) \to \mathrm{Con}_{\boldsymbol{B}}(\mathfrak{B}_1)$, for all Heyting algebra homomorphisms $k\colon \mathfrak{B}_1 \to \mathfrak{B}_2$.*

PROOF. Clearly, $\mathrm{Con}_{\boldsymbol{B}}$ is a topological theory on $\boldsymbol{H}$. To prove $\boldsymbol{N}$ and $\boldsymbol{H}^{\mathrm{Con}_{\boldsymbol{B}}}$ are equivalent define functors $F\colon \boldsymbol{N} \to \boldsymbol{H}^{\mathrm{Con}_{\boldsymbol{B}}}$ and $E\colon \boldsymbol{H}^{\mathrm{Con}_{\boldsymbol{B}}} \to \boldsymbol{N}$ as follows: $F(\mathfrak{A}) = (\mathfrak{A}^*, \beta_{\mathfrak{A}} \restriction A^*)$, for any Nelson algebra $\mathfrak{A}$ and $F(h) = h \restriction A_1^*$, for any Nelson algebra homomorphism $h\colon \mathfrak{A}_1 \to \mathfrak{A}_2$; while $E(\mathfrak{B}, \Theta) = N_\Theta(\mathfrak{B})$, for any Heyting algebra $\mathfrak{B}$ and $E(k) = \langle k, k \rangle$, for any morphism $k\colon (\mathfrak{B}_1, \Theta_1) \to (\mathfrak{B}_2, \Theta_2)$ of $\boldsymbol{H}^{\mathrm{Con}_{\boldsymbol{B}}}$, where $\langle k, k \rangle (a, b) = \langle k(a), k(b) \rangle$ for all $(a, b) \in N_{\Theta_1}(B_1)$. Then by Theorem 3.6 the functor E is equivalence of categories with equivalence--inverse F.

Let us extend the definition of * onto homomorphism as follows: for any Nelson algebra homomorphism $h\colon \mathfrak{A}_1 \to \mathfrak{A}_2$ h^* is a restriction of h to the set A_1^*. Then we get a well-defined faithful functor $*\colon \boldsymbol{N} \to \boldsymbol{H}$. By Theorem 4.1 we have

4.2 COROLLARY. *The faithful functor $*\colon \boldsymbol{N} \to \boldsymbol{H}$ is topological.*

By general results each topological functor has both a left adjoint functor and a right adjoint one. In our situation adjoint functors to * can be described as extensions of the operators $\overleftarrow{N}$ and $\overrightarrow{N}$. If $k\colon \mathfrak{B}_1 \to \mathfrak{B}_2$ is a Heyting algebra homomorphism then $\overrightarrow{N}(k)\colon \overrightarrow{N}(\mathfrak{B}_1) \to \overrightarrow{N}(\mathfrak{B}_2)$ defined by $\overrightarrow{N}(k)(a, b) = (k(a), k(b))$, for all $(a, b) \in \overrightarrow{N}(\mathfrak{B}_1)$ and $\overleftarrow{N}(k) = \overrightarrow{N}(k) \restriction \overleftarrow{N}(B_1)\colon \overleftarrow{N}(\mathfrak{B}_1) \to \overleftarrow{N}(\mathfrak{B}_2)$ are Nelson algebra homomorphisms. Then we have well-defined functors $\overleftarrow{N}, \overrightarrow{N}\colon \boldsymbol{H} \to \boldsymbol{N}$.

4.3 PROPOSITION. *The functor $\overleftarrow{N}$ is left adjoint, and the functor $\overrightarrow{N}$ is right adjoint to the functor *.*

PROOF. For any Nelson algebra $\mathfrak{A}$ define $p_{\mathfrak{A}}\colon \mathfrak{A} \to \overrightarrow{N}(\mathfrak{A}^*)$ by $p_{\mathfrak{A}}(a) = (a^*, (\sim a)^*)$, for all $a \in A$. Then one can verify that for any Heyting algebra $\mathfrak{B}$ and any Nelson algebra homomorphism $h\colon \mathfrak{A} \to \overrightarrow{N}(\mathfrak{B})$ the homomorphism $\hat{h}\colon \mathfrak{A}^* \to \mathfrak{B}$ defined by $\hat{h}(a) = pr_1(h(a))$, for all $a \in A^*$ (where pr_1 is a projection on the first component) is unique such that $\overrightarrow{N}(\hat{h}) p_{\mathfrak{A}} = h$. Thus $\overrightarrow{N}$ is a right adjoint to *, and the natural transformation $p = (p_{\mathfrak{A}})$ is a unit of this adjunction.

To prove $\bar{N}$ is left adjoint to $*$, for any Nelson algebra $\mathfrak{A}$ define $s_{\mathfrak{A}}$: $\bar{N}(\mathfrak{A}^*) \to \mathfrak{A}$ to be a composition of the inclusion $\bar{N}(\mathfrak{A}^*) \subsetneq N_\Theta(\mathfrak{A}^*)$, where $\Theta = \beta_{\mathfrak{A}} \upharpoonright A^*$ and the isomorphism $N_\Theta(\mathfrak{A}^*) \to \mathfrak{A}$ being inverse to the isomorphism defined in the proof of 3.6 (ii). So, for any $(a, b) \in \bar{N}(A^*)$, $s_{\mathfrak{A}}(a, b)$ is the unique element in the set $[a]_{\approx} \cap [\sim b]_{\approx}$ (see Lemma 3.5). By standart calculation one can verify that for any Heyting algebra $\mathfrak{B}$ and any Nelson algebra homomorphism k: $\bar{N}(\mathfrak{B}) \to \mathfrak{A}$ the homomorphism $\hat{k}$: $\mathfrak{B} \to \mathfrak{A}^*$ defined by $\hat{k}(b) = k(b, -b)^*$, for all $b \in B$ is unique such that $s_{\mathfrak{A}} \bar{N}(\hat{k}) = k$. Therefore $\bar{N}$ is a left adjoint functor to $*$, and the natural transformation $s = (s_{\mathfrak{A}})$ is a counit of this adjunction.

Components of the natural transformations p and s are embeddings of Nelson algebras. Moreover, since the functor $*$ is faithful and the functor $\vec{N}$ is full they are bimorphisms in the category N. Now we describe Nelson algebras $\mathfrak{A}$ such that the components $p_{\mathfrak{A}}$ and $s_{\mathfrak{A}}$ are isomorphisms.

4.4 PROPOSITION. *For any Nelson algebra $\mathfrak{A}$ the following are equivalent:*
(i) *$p_{\mathfrak{A}}$ is an isomorphism*
(ii) *There exists element $a \in A$ such that $a = \sim a$*
(iii) *For any dual* N-*space* $(\mathfrak{X}, g)$ *of the algebra $\mathfrak{A}$, $X^+ \cap X^- = \emptyset$*
(iv) *$\beta_{\mathfrak{A}}$ is a full congruence relation on $\mathfrak{A}$.*
Moreover, if $\mathfrak{A}$ is non-trivial then each of the above conditions is equivalent to
(v) *$\vec{N}(\mathbf{2})$ is embeddable into $\mathfrak{A}$, where $\mathbf{2}$ denotes two-element Boolean algebra.*

PROOF. (i)$\Rightarrow$(ii). For the required element take $p_{\mathfrak{A}}^{-1}(0, 0)$.

(ii)$\Rightarrow$(iii). If $a = \sim a$ then for any prime filter F in $\mathfrak{A}$, a is in F iff a is not in $g(F) = A \backslash \sim F$. Thus there is no prime filter F such that $F = g(F)$. This proves (iii) since any dual N-space of the algebra $\mathfrak{A}$ is isomorphic in ***P-N*** to the N-space of prime filters of $\mathfrak{A}$.

(iii)$\Rightarrow$(iv). One can prove that if $\mathfrak{A} = \mathcal{O}_{\mathrm{N}}(\mathfrak{X}, g)$ for some N-space $(\mathfrak{X}, g)$, then the relation Θ defined by $U \Theta V$ iff $U \cap X^+ \cap X^- = V \cap X^+ \cap X^-$, for all clopen increasing U and V in $\mathfrak{X}$, is a congruence relation on $\mathfrak{A}$. Moreover, it is the least congruence such that $\mathfrak{A}/\Theta \Vdash \neg x = \sim x$. Therefore by Lemma 3.4 $\beta_{\mathfrak{A}} = \Theta$. Thus, if $X^+ \cap X^- = \emptyset$, then $\beta_{\mathfrak{A}}$ must be full.

(iv)$\Rightarrow$(i). Because $\beta_{\mathfrak{A}} \upharpoonright A^*$ is a full congruence on $\mathfrak{A}^*$.

Finally, (v) implies (ii) in general, and the inverse holds if $\mathfrak{A}$ is non-trivial.

4.5 PROPOSITION. *For any Nelson algebra $\mathfrak{A}$ the following are equivalent:*
(i) *$s_{\mathfrak{A}}$ is an isomorphism*
(ii) $\mathfrak{A} \Vdash (x \to \sim x) \wedge (\sim x \to x) = x \wedge \sim x$
(iii) $\mathfrak{A} \Vdash (x \to \sim x) \wedge (\sim x \to x) \prec x$
(iv) *For any dual* N-*space* $(\mathfrak{X}, g)$ *of an algebra $\mathfrak{A}$, $X^+ \cap X^- = \mathrm{Max}(X^+)$ $(\mathrm{Min}(X^-))$*
(v) *$\vec{N}(\mathbf{2})$ is not a homomorphic image of any subalgebra of $\mathfrak{A}$.*

PROOF. If $\mathfrak{A}$ is trivial then the assertion is true. So, assume $\mathfrak{A}$ is non-trivial.

(i) $\Rightarrow$ (ii). Since the Nelson algebra $\vec{N}(\mathfrak{B})$ for any Heyting algebra $\mathfrak{B}$ satisfies an equation from (ii).

(ii) $\Rightarrow$ (i). If $\mathfrak{A}$ satisfies an equation from (ii) then $N_\Theta(\mathfrak{A}^*)$, for $\Theta = \beta_{\mathfrak{A}} \restriction A^*$, satisfies it, as well. This implies $N_\Theta(A^*) \subseteq \vec{N}(A^*)$. Hence $s_{\mathfrak{A}}$ is an isomorphism.

(ii) $\Rightarrow$ (v). Suppose there exists subalgebra $\mathfrak{C}$ of $\mathfrak{A}$ and homomorphism h from $\mathfrak{C}$ onto $\vec{N}(\mathbf{2})$. Let a be an element such that $h(a) = (0, 0)$. Then, since $h(a) = \sim h(a)$ we have

$$\begin{aligned}(1, 0) &= ((0, 0) \to (0, 0)) \wedge ((0, 0) \to (0, 0)) \\ &= (h(a) \to \sim h(a)) \wedge (\sim h(a) \to h(a)) \\ &= h((a \to \sim a) \wedge (\sim a \to a)) \\ &= h(a \wedge \sim a) \\ &= h(a) \wedge \sim h(a) \\ &= (0, 0); \text{ which is impossible.}\end{aligned}$$

(v) $\Rightarrow$ (iv). By Proposition 4.4 $X^+ \cap X^- \neq \emptyset$. In general $X^+ \cap X^- \subseteq \mathrm{Max}(X^+)$. Suppose there exists $x \in \mathrm{Max}(X^+)$ such that $x \neq g(x)$. Then the relation Θ in $\mathcal{O}(X)$ defined by $U \Theta V$ iff $U \cap \{x, g(x)\} = V \cap \{x, g(x)\}$ is a congruence on $\mathcal{O}_{\mathrm{N}}(\mathfrak{X}, g)$ such that the quotient algebra is isomorphic to $\vec{N}(\mathbf{2})$. But it is in contradition with assumption, because $\mathcal{O}_{\mathrm{N}}(\mathfrak{X}, g)$ is isomorphic to $\mathfrak{A}$.

(iv) $\Rightarrow$ (iii) Assume (iv). Since $\mathfrak{A}$ is isomorphic to $\mathcal{O}_{\mathrm{N}}(\mathfrak{X}, g)$ it sufficies to show $\mathcal{O}_{\mathrm{N}}(\mathfrak{X}, g)$ satisfies the inequality from (iii). Let U be clopen increasing in $\mathfrak{X}$. Then

$$\begin{aligned}(U \to \sim U) \cap (\sim U \to U) &= (X \backslash {\downarrow}(U \cap g(U) \backslash \sim U)) \cap (X \backslash {\downarrow}(\sim U \cap g(\sim U) \backslash U)) \\ &= X \backslash ({\downarrow}(U \cap g(U) \backslash \sim U) \cup {\downarrow}(\sim U \cap g(\sim U) \backslash U)) \\ &= X \backslash ({\downarrow}(U \cap g(U)) \cup {\downarrow}(X \backslash (U \cup g(U)))).\end{aligned}$$

At first, note X^+ is contained in ${\downarrow}(U \cap g(U)) \cup {\downarrow}(X \backslash (U \cup g(U)))$. Indeed, if $x \in X^+$ then $x \leqslant x_0$, for some x_0 maximal in $\mathfrak{X}^+$. So by (iv) $x \leqslant x_0 = g(x_0) \leqslant g(x)$. Hence, if $x_0 \in U$ then $x \in {\downarrow}(U \cap g(U))$, and if $x_0 \notin U$ then $x \in {\downarrow}(X \backslash (U \cup g(U)))$; which proves the required inclusion.

Therefore $(U \to \sim U) \cap (\sim U \to U) \cap X^+ = \emptyset \subseteq U \cap X^+$. However this means $(U \to \sim U) \cap (\sim U \to U) \prec U$ (see the proof of 1.3).

(iii) $\Rightarrow$ (ii). It is proved in [22] (comp. Lemma 4.3 p. 270).

4.6 LEMMA. *Let f: $\mathfrak{X} \to \mathfrak{Y}$ be a morphism in **P-H**, and let S and R be closed subsets of* $\mathrm{Max}(X)$ *and* $\mathrm{Max}(Y)$, *respectively. Then the function $f_{S,R}$: $X \nearrow S \to Y \nearrow R$ defined by*

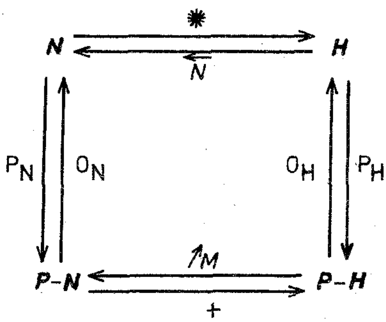
$$f_{S,R}(x) = \begin{cases} f(x); & \text{if } x \in X \\ g_R(f(g_S(x))); & \textit{otherwise} \end{cases}$$

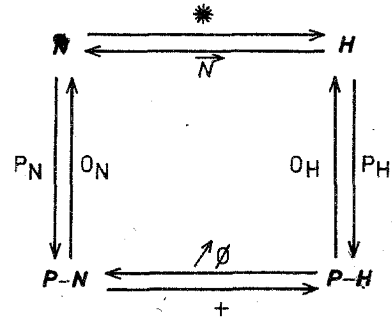
*is a morphism in **P-N** if and only if $f(S) \subseteq R$.*

PROOF. By straightforward verification.

Let us define $\nearrow\emptyset$ and $\nearrow M$ setting $\nearrow\emptyset(\mathfrak{X}) = \mathfrak{X}\nearrow\emptyset$ and $\nearrow M(\mathfrak{X}) = \mathfrak{X}\nearrow \mathrm{Max}(X)$, for any H-space $\mathfrak{X}$; and $\nearrow\emptyset(f) = f_{\emptyset,\emptyset}$ and $\nearrow M(f) = f_{\mathrm{Max}(X),\mathrm{Max}(Y)}$, for any morphism $f\colon \mathfrak{X}\to\mathfrak{Y}$ in ***P-H***. By the above lemma $\nearrow\emptyset$ and $\nearrow M$ are well-defined functors from the category ***P-H*** to the category ***P-N***. We also have the functor $^+$: ***P-N***$\to$***P-H*** which has appeared in natural way in Section 1. By Corollary 4.2, Propositions 4.3, 4.4 and 4.5 the following holds

4.7 THEOREM. (i) *In each of the diagrams*



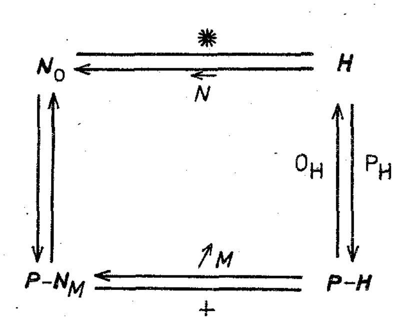


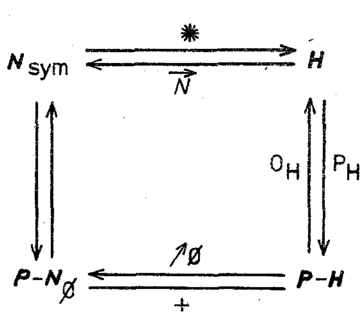
both the inner and outer squares are (up to natural isomorphism) commutative. They remain commutative after rotating one or both columns.

(ii) *The functor* $^+$ *is topological with* $\nearrow\emptyset$ *and* $\nearrow M$ *as left and right adjoint, respectively.*

A Nelson algebra is called symmetric if it has an element a such that $a = \sim a$ (such element, if it exists, is unique). Let us denote by $\boldsymbol{N}_{\mathrm{sym}}$ a full subcategory of $\boldsymbol{N}$ whose objects are symmetric Nelson algebras, and by $\boldsymbol{N}_0$ the equational subcategory of $\boldsymbol{N}$ defined by the equation $(x\to \sim x)\wedge(\sim x\to x) = x\wedge \sim x$. Moreover, let ***P-N***$_\emptyset$ and ***P-N***$_M$ denote full subcategories of ***P-N*** whose objects satisfy $X^+\cap X^- = \emptyset$ and $X^+\cap X^- = \mathrm{Max}(X^+)$, respectively. Then by Propositions 4.3, 4.4 and 4.5 the following holds

4.8 THEOREM. *Each of diagrams*





with the left column being the suitable restrictions of the functors $\mathscr{P}_{\mathrm{N}}$ *and* $\mathscr{O}_{\mathrm{N}}$ *is (up to natural isomorphism) commutative. In both diagrams any row settles an equivalence and any column a dual equivalence of categories.*

REMARKS. 1. The functors $\vec{N}$, $\bar{N}$ (denoted by N, N^+, respectively) and the functor P which is naturally isomorphic to our functor * have been also considered by Goranko in [7]. In that paper some special properties of these functors have been stated (for details see [7]). Using these properties the author, among others proved that up to isomorphism of algebras the class $\{\vec{N}(\mathfrak{B});\ \mathfrak{B}\in \boldsymbol{H}\}$ (algebras of the form $\vec{N}(\mathfrak{B})$ are called special N-lattices) coincides with the class $\boldsymbol{N}_{\text{sym}}$ (Theorem 15 p. 297), and the class $\{\bar{N}(\mathfrak{B});\ \mathfrak{B}\in \boldsymbol{H}\}$ (algebras of the form $\bar{N}(\mathfrak{B})$ are called normal N-lattices) coincides with the class of all Nelson algebras which satisfy the equation $\neg\big((x\to\sim x)\wedge(\sim x\to x)\big)=1$ (Theorem 37 p. 303). From the first of these results it follows the equivalence of the conditions (i), (ii) and (v) in our Proposition 4.4, and from the second one that each condition in our Proposition 4.5 is equivalent to each condition of Lemma 30 in [7].

2. The functor $\vec{N}$ (denoted by N) and the functor H, in fact defined as the composition $\boldsymbol{N}\xrightarrow{\mathscr{P}_N}\boldsymbol{P}\text{-}\boldsymbol{N}\xrightarrow{+}\boldsymbol{P}\text{-}\boldsymbol{H}\xrightarrow{\mathscr{O}_H}\boldsymbol{H}$ (hence naturally isomorphic to our functor *) appear also in the paper by Cignoli [4]. The author, by the restriction of the more general adjoint situation exists between the category $\boldsymbol{Kl}$ and the category of bounded distributive lattices (Theorem 1.7 p. 269), has obtained that the functor H is the left adjoint to the functor $\vec{N}$; and hence also that the categories $\boldsymbol{N}_{\text{sym}}$ (in [4], objects of this category are termed as centered Nelson algebras) and $\boldsymbol{H}$ are naturally equivalent (Theorem 3.14 p. 278).

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