MATHEMATICAL PROPERTIES OF THE VACUUM POLARIZATION FUNCTION

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ABSTRACT. This paper is an investigation of certain mathematical properties of the vacuum polarization function $\Sigma(s)$. We show that $\Sigma(s)$ is a Herglotz function, has no complex zeroes, and belongs to the class of functions called 'typically real'. In addition, we obtain upper bounds on the higher derivatives of $\Sigma(s)$, at $s = 0$, given that we know the value of the first derivative at that point.

The vacuum polarization function $\Sigma(s)$ may be written [1]

$$\Sigma(s) = \frac{s}{\pi} \int_{4\mu^2}^{\infty} \frac{\mathrm{d}w}{w} \frac{\rho(w)}{(w-s)}, \tag{1}$$

where ρ is a positive weight function. (In the one photon approximation, ρ is related to the cross section for electron-positron going to hadrons [1, 2, 3].) In this paper, we obtain a number of mathematical properties of the vacuum polarization function which follow from the integral representation given in Equation (1).

In more detail, we make the following assumptions: (i) The real function $\rho(s)$ exists for $\{s{:}s \geqslant 4\,\mu^2\}$ and is non-negative on that interval. (ii) The integral, as defined by Equation (1), exists.

Let $s = x + iy$ and define $h(w)$ as,

$$h(w) = \frac{\rho(w)}{\pi w}. \tag{2}$$

x and y are, respectively, the real and imaginary parts of the variable s and $h(w)$ is a non-negative function of w for $w \geqslant 4\mu^2$. With this change in notation, Equation (1) may be written,

$$\begin{aligned}\Sigma(x+iy) = \int_{4\mu^2}^{\infty} \left[\frac{x(w-x)-y^2}{(w-x)^2+y^2}\right] h(w)\,\mathrm{d}w \\ + iy \int_{4\mu^2}^{\infty} \frac{wh(w)\,\mathrm{d}w}{(w-x^2)+y^2}.\end{aligned} \tag{3}$$

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An immediate consequence of Equation (3) is the following result:

$$\begin{cases} \operatorname{Im} \Sigma(s) > 0 & \text{if } \operatorname{Im} s > 0, \\ \operatorname{Im} \Sigma(s) < 0 & \text{if } \operatorname{Im} s < 0. \end{cases} \tag{4}$$

Thus, it is easy to see that $\Sigma(s)$ is a Herglotz function [4, 5]. This class of functions admits a general well-known integral representation, along with well-defined asymptotic upper and lower bounds [4, 5].

Note that since $\operatorname{Im} \Sigma(s) \neq 0$ for $\operatorname{Im} s \neq 0$, then $\Sigma(s)$ can have no complex zeroes. If $\Sigma(s)$ has any zeroes, other than the one at $s = 0$, then they must be real.

It is easy to see that $\Sigma(s)$ is real only for real s. This result follows from Equations (3) and (4).

The function $f(z)$, analytic in $|z| < 1$, is called 'typically real' if it is real in the open interval $-1 < z < 1$ and if one of the following two sets of inequalities holds [6]:

$$\begin{cases} \operatorname{Im} f(z) > 0 & \text{if } \operatorname{Im} z > 0, \\ \operatorname{Im} f(z) < 0 & \text{if } \operatorname{Im} z < 0, \end{cases} \tag{5a}$$

$$\begin{cases} \operatorname{Im} f(z) > 0 & \text{if } \operatorname{Im} z < 0, \\ \operatorname{Im} f(z) < 0 & \text{if } \operatorname{Im} z > 0. \end{cases} \tag{5b}$$

Typically real functions $f(z)$ obey the following theorem: Let $f(z)$ be analytic in $|z| < 1$ and have the representation,

$$f(z) = z + a_2 z^2 + \ldots + a_n z^n + \ldots . \tag{6}$$

The coefficients a_n satisfy the following inequalities [7, 8],

$$|a_n| \leqslant n. \tag{7}$$

The inequality in Equation (8) may be replaced by the equality only for the functions,

$$f(z) = \frac{z}{(1 \pm z)^2} . \tag{8}$$

Since the above theorem is probably well-known to only a small group of specialists and since the proof is rather straightforward, we give it in full below. We follow closely the proof in Reference 8.

Let $z = re^{i\theta}$, where r is fixed $(0 < r < 1)$ and $0 < \theta < \pi$. Using the results given in Equations (5a) and (6), we obtain,

$$f(z) = \sum_{n=1}^{\infty} a_n z^n = \sum_{n=1}^{\infty} a_n r^n e^{in\theta}, \tag{9}$$

where $a_1 = 1$. Now define $p(\theta)$ as,

$$p(\theta) \equiv \sin\theta \ \operatorname{Im} f(z) = \sum_{n=1}^{\infty} a_n r^n \sin\theta \sin(n\theta), \qquad 0 < \theta < \pi. \tag{10}$$

The coefficients a_n are real, consequently, $p(\theta)$ is real in the interval $0 < \theta < \pi$. In addition, since $\sin\theta \neq 0$ for $0 < \theta < \pi$ and $p(\theta)$ is continuous, then $p(\theta)$ does not vanish in this interval and is of constant sign throughout the interval. In addition, it easily follows from Equation (10) that $p(\theta) = p(-\theta)$. This shows that either $p(\theta) \geqslant 0$ or $p(\theta) \leqslant 0$ for all values of θ. Using the addition theorem of the cosine function, we obtain,

$$p(\theta) = \frac{1}{2} \sum_{n=1}^{\infty} a_n r^n [\cos(n-1)\theta - \cos(n+1)\theta] = \frac{r}{2}[1 + a_2 r \cos\theta] + \frac{r}{2} \sum_{n=2}^{\infty} \left(a_{n+1} - \frac{a_{n-1}}{r^2} \right) r^n \cos n\theta. \tag{11}$$

It follows easily that,

$$\int_0^{2\pi} p(\theta)\, d\theta = \pi r, \tag{12}$$

and, therefore, $p(\theta) \geqslant 0$ for $0 \leqslant \theta < 2\pi$. Using the fact that $(1 \pm \cos n\theta) \geqslant 0$, we obtain

$$0 \leqslant \frac{2}{\pi r} \int_0^{2\pi} p(\theta)[1 \pm \cos n\theta]\, d\theta = 2 \pm \left(a_{n+1} - \frac{a_{n-1}}{r^2} \right) r^n, \tag{13}$$

or

$$\left| a_{n+1} - \frac{a_{n-1}}{r^2} \right| r^n \leqslant 2. \tag{14}$$

Since this relation must hold for all r between 0 and 1, we may take the limit as $r \to 1$ and obtain,

$$|a_{n+1} - a_{n-1}| \leqslant 2. \tag{15}$$

For $n = 2$, we obtain $|a_2| \leqslant 2$. Since $a_1 = 1$ and the difference of two terms whose subscripts differ by two is not larger than two, it follows that,

$$|a_n| \leqslant n, \qquad n = 2, 3, \ldots . \tag{16}$$

Q.E.D.

We may now use the results given by Equations (7) and (16) to obtain upper bounds on the higher derivatives of $\Sigma(s)$ at $s = 0$. From Equation (1), it follows that $\Sigma(s)$ is analytic in $|s| < 4\mu^2$; consequently, $\Sigma(s)$ may be expanded in a Taylor series about $s = 0$,

$$\Sigma(s) = \Sigma^{(1)}s + \frac{1}{2!}\Sigma^{(2)}s^2 + \ldots + \frac{1}{n!}\Sigma^{(n)}s^n + \ldots, \tag{17}$$

where

$$\Sigma^{(n)} \equiv \frac{\mathrm{d}^n\,\Sigma(0)}{\mathrm{d}\,s^n}, \qquad n = 1, 2, 3, \ldots . \tag{18}$$

Let us transform to the variable $z = s/(4\mu^2)$ and construct the new function $f(z)$ defined in $|z| < 1$, as,

$$\begin{aligned} f(z) \equiv \frac{\Sigma(4\mu^2 z)}{4\mu^2\,\Sigma^{(1)}} &= z + \left(\frac{4\mu^2}{2!}\right)\left[\frac{\Sigma^{(2)}}{\Sigma^{(1)}}\right] z^2 + \ldots \\ &+ \frac{(4\mu^2)^{n-1}}{n!}\left[\frac{\Sigma^{(n)}}{\Sigma^{(1)}}\right] z^n + \ldots . \end{aligned} \tag{19}$$

Since Σ is 'typically real', so is the function $f(z)$. Comparing Equations (6), (7) and (19), we obtain the following upper bounds on the higher derivatives of $\Sigma(s)$ at $s = 0$, in terms of $\Sigma^{(1)}$, the first derivative,

$$\left|\frac{\Sigma^{(n)}}{\Sigma^{(1)}}\right| \leqslant \frac{n(n!)}{(4\mu^2)^{n-1}} . \tag{20}$$

The result given in Equation (20) is new and quite interesting, since, within the context of analytic S-matrix theory, Cauchy integrals of the type given by Equation (1) often occur. The bounds of Equation (20) may be useful in placing constraints on models of pion-pion interactions and electromagnetic form factors [9].

In summary, we have obtained a number of mathematical properties of the vacuum polarization function. These properties were obtained from the integral representation given in Equation (1). It is important to keep in mind that the obtained results are an essential consequence of the fact that $h(w)$, as given in Equation (2), is a non-negative function of w for $w \geqslant 4\mu^2$.

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