Non-negative Solutions of the Evolution p-Laplacian Equation. Initial Traces and Cauchy Problem when $1 < p < 2$

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0. Introduction 225

Chapter I. Weak Solutions and Initial Traces 229
I.1. Non-negative local weak solutions 229
I.2. Estimating $|Du|$ 232
I.3. An equivalent formulation of local solutions 236
I.4. Harnack inequality and initial traces 239

Chapter II. Uniqueness 245
II.1. The uniqueness theorem 245
II.2. Preliminaries 245
II.3. An auxiliary proposition 248
II.4. Proof of Theorem II.1.1 257

Chapter III. The Cauchy Problem 257
III.1. Introduction 257
III.2. Existence of solutions $(u_0 \in C_0^\infty(\mathbf{R}^N))$ 258
III.3. *A priori* estimates 260
III.4. Approximating problems 268
III.5. The limit as $n \to \infty$ 270
III.6. Locally bounded solutions 278
III.7. Condition (III.6.19) is sharp 283
III.8. The case u_0 a measure and $p > 2N/(N+1)$ 286

0. Introduction

We continue here the investigation initiated in [11] concerning the solvability of the Cauchy problem and the existence of initial traces for non-negative weak solutions of the non-linear evolution equation

$$u_t - \operatorname{div}(|Du|^{p-2} Du) = 0 \quad \text{in } S_T \equiv \mathbf{R}^N \times (0, T), \quad 0 < T < \infty, \quad N \geqq 1. \tag{0.1}$$

We study here the case $1 < p < 2$, consider only non-negative solutions and investigate the solvability of the Cauchy problem when (0.1) is associated with an initial datum

$$u_0 \in L^1_{\mathrm{loc}}(\mathbf{R}^N), \quad u_0 \geqq 0. \tag{0.2}$$

It turns out that the Cauchy problem is solvable whenever (0.2) holds, regardless of the growth of $x \to u_0(x)$ as $|x| \to \infty$. The weak solutions are shown to be unique whenever the initial datum is taken in the sense of $L^1_{\mathrm{loc}}(\mathbf{R}^N)$. We also prove that every non-negative weak solution of (0.1) possesses, as initial trace, a σ-finite Borel measure $\mu \geqq 0$.

The case $1 < p < 2$ is noticeably different from the case $p > 2$, both in terms of results and techniques. The main difference stems from the fact that, unlike solutions in the case $p > 2$, solutions of (0.1) are not, in general, locally bounded.

Specifically, if

$$u_0 \in L^r_{\mathrm{loc}}(\mathbf{R}^N), \quad r \geqq 1 \quad \text{and} \quad p > \frac{2N}{N+r}, \tag{0.3}$$

then the solution u of (0.1)–(0.2) belongs to $L^\infty_{\mathrm{loc}}(S_T)$, $\forall\, t > 0$, whereas if either one of (0.3) is violated, then $u \notin L^\infty_{\mathrm{loc}}(S_T)$ (see Theorem III.6.1 and § III.7). Moreover if $u_0 \notin L^r_{\mathrm{loc}}(\mathbf{R}^N)$, $\forall\, r > 1$, then the solution $u(t) \notin L^r_{\mathrm{loc}}(\mathbf{R}^N)$, $\forall\, r > 1$.

This in turn implies the lack of an estimate of the type

$$|Du| \in L^p_{\mathrm{loc}}(S_T). \tag{0.4}$$

We have spoken of solutions of (0.1). However, if (0.4) fails, one of the main problems of the theory is to make precise what it is meant by solution.

Questions of solvability of the Cauchy problem and existence of initial traces for the porous medium equation

$$u_t = \Delta u^m, \quad m > 1, \quad u \geqq 0$$

have been studied by ARONSON & CAFFARELLI [2], BENILAN, CRANDALL & PIERRE [6] and DAHLBERG & KENIG [9]. Results for the case $0 < m < 1$ are due to HERRERO & PIERRE [17] and PIERRE [26].

The porous medium equation can be interpreted in a rather obvious way in the sense of distributions. In fact since the principal part is linear in u^m, one can use a variety of "linear" techniques [6, 9, 26, 27].

In our case, as a starting point, a precise meaning has to be given to Du to make sense out of (0.1).

When $p > 2$ this is resolved by $C^{1,\alpha}_{\mathrm{loc}}(S_T)$-type estimates [12, 14]. If p is "close to 1" (see (0.3)), such estimates fail and a different approach has to be used.

We have given a new formulation of non-negative weak solutions. Such solutions are "regular" in the sense that the truncations

$$\forall\, k > 0, \quad u_k = \min\{u, k\}, \tag{0.5}$$

satisfy

$$|Du_k| \in L^p_{\mathrm{loc}}(S_T), \quad \frac{\partial}{\partial t} u_k \in L^1_{\mathrm{loc}}(S_T). \tag{0.6}$$

Then (0.1) can be interpreted weakly against testing functions that vanish whenever "u is large". A suitable choice of such testing functions is

$$(\varphi - u)_+ \equiv \max\{(\varphi - u); 0\}, \qquad \varphi \in C_0^\infty(S_T).$$

The notion is introduced and discussed in Chapter I. We prove that our solutions coincide with the distributional ones if (0.4) holds (Lemma I.1.2) and that the truncations u_k, $\forall\, k > 0$, are distributional super-solutions of (0.1) (Lemma I.1.3).

We derive a spectrum of properties of such "local" weak solutions, regardless of their initial datum. For example, they satisfy (Lemma I.2.2)

$$\left| Du^{\frac{p-1-\alpha}{p}} \right| \in L^p_{\mathrm{loc}}(S_T), \qquad \forall\, \alpha \in (0, p-1). \tag{0.7}$$

Since estimates of the type of (0.7) "deteriorate" as $\alpha \searrow 0$, we investigate the behaviour of $\left| Du^{\frac{p-1}{p}} \right|$ (*i.e.*, $\alpha = 0$) on the sets $[u > k]$, $\forall\, k > 0$ (Lemma I.2.3 and Corollaries I.2.4, I.2.5).

A relevant fact is the estimate

$$\forall\, 0 < s < t < \infty, \quad \forall\, 0 < r < R < \infty, \quad \forall\, \varepsilon > 0,$$

$$\int_s^t \int_{\{|x|<r\}} |Du|^{p-1}\, dx\, d\tau \leq \gamma \left(1 + \frac{t-s}{\varepsilon^{2-p}(R-r)^p}\right)^{\frac{p-1}{p}} \int_s^t \int_{|x|<R} (t-\tau)^{\frac{1}{p}-1} (u+\varepsilon)^{\frac{2}{p}(p-1)}\, dx\, d\tau \tag{0.8}$$

where γ is a constant depending only upon N and p (Lemma I.4.1).

We remark that in [11], when $p > 2$, an estimate of the local integral-norm of $|Du|^{p-1}$ was the crucial fact to derive a local L^∞ estimate for the solutions.

In the present case, it is precisely (0.8) that permits us to prove a local L^1 estimate for u and a weak global Harnack-type inequality. Specifically,

Estimate in $L_1^{\mathrm{loc}}(\mathbf{R}^N)$ (Lemma III.3.1)

$$\forall\, 0 < s < t < \infty, \qquad \forall\, \varrho > 0, \qquad \varkappa = N(p-2) + p,$$

$$\sup_{\tau \in (s,t)} \int_{\{|x|<\varrho\}} u(x,\tau)\, dx \leq \gamma \left\{ \int_{\{|x|<2\varrho\}} u(x,s)\, dx + \left(\frac{t-s}{\varrho^\varkappa}\right)^{\frac{1}{2-p}} \right\}. \tag{0.9}$$

Harnack-type estimate (Theorem I.4.1)

$$\forall\, 0 < s < t < \infty, \qquad \forall\, \varrho > 0, \qquad \varkappa = N(p-2) + p,$$

$$\sup_{\tau \in (s,t)} \int_{\{|x|<\varrho\}} u(x,\tau)\, dx \leq \gamma \left\{ \int_{\{|x|<2\varrho\}} u(x,t)\, dx + \left(\frac{t-s}{\varrho^\varkappa}\right)^{\frac{1}{2-p}} \right\}. \tag{0.10}$$

In (0.9)–(0.10), γ is a constant depending only upon N and p.

The gradient estimate (0.8) and the Harnack inequality (0.10) permit us to develop a theory of (unique) initial traces (see Theorem I.4.1), whereas the

$L_1^{loc}(\mathbf{R}^N)$ estimate (0.9) permits us to establish existence of solutions to the Cauchy problem (0.1)–(0.2) (see Chapter III).

The solutions are constructed by using an increasing sequence $\{u_{0,n}\}$ of approximations to the initial datum u_0. This, the comparison principle, and (0.9) yield the $L^1_{loc}(S_T)$ convergence of the approximating solutions $\{u_n\}$. A one-sided bound on u_0 (and hence on u) is crucial to this process.

Further, the non-negativity of u_0 is essential in proving some sort of compactness in the t-variable for $\{u_n\}$, via the BÈNILAN & CRANDALL [5] regularizing effect. This result requires in an essential way the positivity of u_0 and the homogeneous structure of the equation in (0.1).

We prove uniqueness of weak solutions if they take their initial datum in the sense of $L^1_{loc}(\mathbf{R}^N)$ (see Chapter II). Namely if u and v solve (0.1) weakly and if $t \to (u - v)(t) \to 0$ in $L^1_{loc}(\mathbf{R}^N)$ as $t \searrow 0$, then the difference $w = u - v$ satisfies $\forall\, q > 1$, $\forall\, t > 0$, $\forall\, \varrho > 0$,

$$\int_{\{|x|<\varrho\}} |w|^q (t)\, dx \leqq \gamma(q) \left(t/\varrho^{\frac{N(p-2)+pq}{q}} \right)^{\frac{q}{2-p}}, \tag{0.11}$$

where γ depends only upon N, p, q (see § II.4). The theorem follows by letting $\varrho \to \infty$ after we choose q so large that $N(p-2) + pq > 0$.

The proof of (0.11) is rather delicate (*i.e.*, to prove that $w \in L^q_{loc}(S_T)$), since the class of available testing functions is rather small (indeed $(\varphi - u)_+$ are, uniformly in φ, only in $L^1_{loc}(S_T)$).

If in (0.3), $r = 1$ and $p > \dfrac{2N}{N+1}$, all the arguments remain valid if $u_0 \in L^1_{loc}(\mathbf{R}^N)$ with no restriction of sign. The strong $L^1_{loc}(S_T)$ convergence of the approximating solutions $\{u_n\}$ is realized by the fact that (see [13])

$$u_n \in C^\alpha_{loc}(S_T), \qquad \text{uniformly in } n, \text{ for some } \alpha \in (0, 1).$$

In fact, u_0 could be a σ-finite Borel measure μ, since the estimates discussed remain valid. In particular, if $t > 0$ the approximating solutions are equi-bounded (Theorem III.6.1) and hence equi-Hölder continuous ([13]).

It would be of interest to study the question of existence of solutions to the Cauchy problem if the initial datum is a measure μ and if

$$p \leqq 2N/(N+1).$$

In view of the results of PIERRE [26] and BARAS & PIERRE [3, 4], it seems that some sort of capacitary restriction has to be placed on μ (see also BREZIS & FRIEDMAN [8]). The methods of [3, 4, 26] do not apply to this situation since the operator is not linear in the principal part. We intend to discuss the matter by a different approach in a forthcoming paper.

If $u_0 \in L^r(\mathbf{R}^N)$, $r \geqq 1$, existence and uniqueness of the Cauchy problem is known through the semigroup theory [5, 15]. However, if $p \leqq 2N/(N+1)$, it is not clear to us what is the a.e. S_T meaning of Du.

For further comments on the connection between our solution and the semigroup solution we refer to § III.1.

Finally, we notice that all the results of this note hold true for equations of the type (see LIONS [21])

$$u_t - \sum_{i=1}^{N} (|u_{x_i}|^{p-2} u_{x_i})_{x_i} = 0 \quad \text{in } S_T.$$

Besides their intrinsic mathematical interest, equations of the type of (0.1) have a connection to physical problems (see MARTINSON & PAVLOV [22, 23], ANTONSEV [1], LADYZHENSKAYA [20] and, for the stationary case, PAYNE & PHILIPPIN [25]).

Chapter I. Weak Solutions and Initial Traces

I.1. Non-negative local weak solutions. Let $0 < T < \infty$, $S_T \equiv \mathbf{R}^N \times (0, T)$, $k > 0$ and $\forall f \in L^1_{\text{loc}}(S_T)$ set

$$f_k = \begin{cases} f & \text{if } f < k \\ k & \text{if } f \geqq k. \end{cases}$$

Define

$$X_{\text{loc}}(S_T) \equiv L^p_{\text{loc}}(0, T; W^{1,p}_{\text{loc}}(\mathbf{R}^N)) \cap L^\infty_{\text{loc}}(S_T), \tag{I.1.1}$$

$$\overset{\circ}{X}_{\text{loc}}(S_T) \equiv \{\varphi \in X_{\text{loc}}(S_T) \mid \exists\, r > 0 : \varphi \in L^p_{\text{loc}}(0, T; \overset{\circ}{W}{}^{1,p})(|x| < r)\}. \tag{I.1.2}$$

Consider the parabolic equation

$$\begin{aligned} &u_t - \operatorname{div}(|Du|^{p-2} Du) = 0 \quad \text{in } S_T \\ &u \geqq 0, \quad 1 < p < 2. \end{aligned} \tag{I.1.3}$$

A measurable function $u : S_T \to \mathbf{R}^+$ is a *local* weak solution of (I.1.3) in S_T if

$$u \in C(0, T : L^1_{\text{loc}}(\mathbf{R}^N)), \tag{I.1.4}$$

$$|Du_k| \in L^p_{\text{loc}}(S_T), \quad \forall k > 0, \tag{I.1.5}$$

$$\frac{\partial}{\partial t} u_k \in L^1_{\text{loc}}(S_T), \quad \forall k > 0, \tag{I.1.6}$$

$$\forall \varphi \in C_0^\infty(S_T) \tag{I.1.7}$$

$$\iint_{S_T} \{u_t(\varphi - u)_+ + |Du|^{p-2} Du\, D(\varphi - u)_+\}\, dx\, d\tau = 0.$$

By density arguments this implies $\forall \varphi \in \overset{\circ}{X}_{\text{loc}}(S_T)$, $\forall\, 0 < s < t \leqq T$,

$$\int_s^t \int_{\mathbf{R}^N} \{u_t(\varphi - u)_+ + |Du|^{p-2} Du\, D(\varphi - u)_+\}\, dx\, d\tau = 0. \tag{I.1.8}$$

We denote by Σ the set of all non-negative local weak solutions.

Here and in what follows, if $u \in \Sigma$, the symbol Du denotes a measurable function from S_T into $\mathbf{R}^N$ that coincides with Du_k on the set $[u < k]$, $\forall k \in \mathbf{R}^+$.

Lemma I.1.1. *Let* $u \in \Sigma$. *Then* $\forall \psi \in X_{\text{loc}}(S_T)$, $\forall \eta \in C_0^\infty(S_T)$,

$$\iint_{S_T} \{u_t(\psi - u)_+ \eta + |Du|^{p-2} Du\, D[(\psi - u)_+ \eta]\}\, dx\, d\tau = 0. \tag{I.1.9}$$

Proof. Let $\mathscr{K} \subset \mathscr{K}'$ be compact subsets of S_T such that dist $(\partial\mathscr{K}, \partial\mathscr{K}') = d > 0$ and let $(x, t) \to \zeta(x, t) \in C_0^\infty(\mathscr{K}')$ be such that, $0 \leqq \zeta \leqq 1$, $\zeta \equiv 1$ on $\mathscr{K}$.

Choose $\psi \in X_{\text{loc}}(S_T)$ and in (I.1.8) take

$$\varphi = (\psi - u)_+ \eta + u_k \zeta$$

where

$$\eta \in C_0^\infty(\mathscr{K}), \qquad k = \|\psi\|_{\infty,\mathscr{K}'}. \tag{I.1.10}$$

We have a.e. in $\mathscr{K}' \setminus \mathscr{K}$

$$(\varphi - u)_+ = ((\psi - u)_+ \eta + u_k \zeta - u)_+ = (u_k \zeta - u)_+ = 0.$$

Moreover a.e. in $\mathscr{K}$, $(\varphi - u)_+ = ((\psi - u)_+ \eta + u_k - u)_+$, and in view of (I.1.10) this vanishes unless $u < \psi$. In such a case $u_k = u$ and

$$(\varphi - u)_+ = (\psi - u)_+ \eta \quad \text{ a.e. } \mathscr{K}. \tag{I.1.11}$$

We conclude that (I.1.11) holds a.e. S_T and (I.1.9) follows. □

Let B_ϱ denote the ball $\{|x| < \varrho\}$ and denote by

$x \to \zeta(x)$ a piecewise smooth cutoff function in $B_{(1+\sigma)\varrho}$, $\sigma > 0$ such that $\zeta(x) = 1$, $x \in B_\varrho$, $\zeta(x) = 0$, $|x| \geqq (1 + \sigma)\varrho$, $0 \leqq \zeta \leqq 1$, $|D\zeta| \leqq (\sigma\varrho)^{-1}$.

(I.1.12)

By density arguments (I.1.9) implies $\forall \psi \in X_{\text{loc}}(S_T)$, $\forall \varrho, \sigma > 0$, $\forall 0 < s < t \leqq T$,

$$\int_s^t \int_{\mathbf{R}^N} \{u_t(\psi - u)_+ \zeta^p + |Du|^{p-2} Du\, D[(\psi - u)_+ \zeta^p]\}\, dx\, d\tau = 0. \tag{I.1.13}$$

Conversely, if $\psi \in C_0^\infty(S_T)$, we may write (I.1.13) for s, t such that supp $\psi \subset \mathbf{R}^N \times (s, t)$. By taking ζ as in (I.1.12) with $\varrho > 2 \cdot \text{diam}(\text{supp}\, \psi)$ we obtain (I.1.7) We conclude that the formulations (I.1.7)–(I.1.9), (I.1.13) are equivalent.

Lemma I.1.2. *Let $u \in \Sigma$ satisfy $|Du| \in L^p_{\text{loc}}(S_T)$, $u_t \in L^1_{\text{loc}}(S_T)$. Then*

$$u_t - \text{div}\,(|Du|^{p-2} Du) = 0 \quad \textit{in } \mathscr{D}'(S_T).$$

Proof. In (I.1.9) take $\psi = u_n + 1 \in X_{\text{loc}}(S_T)$, $n \in \mathbf{N}$. We obtain $\forall \eta \in C_0^\infty(S_T)$

$$\iint_{S_T} \{u_t \eta + |Du|^{p-2} Du\, D\eta\}\, (u_n - u + 1)_+\, dx\, d\tau = \iint_{S_T \cap [n < u < n+1]} |Du|^p \eta\, dx\, d\tau.$$

Since $|Du| \in L^p_{\text{loc}}(S_T)$, the right-hand side tends to zero as $n \to \infty$. The left-hand side converges to

$$\iint_{S_T} \{u_t \eta + |Du|^{p-2} Du\, D\eta\}\, dx\, d\tau = 0. \quad \square$$

Lemma I.1.3. *Let* $u \in \Sigma$. *Then* $\forall\, k > 0$, u_k *is a distributional super-solution of* (I.1.3) *in* S_T.

Proof. Let $k > 0$, $\alpha, \varepsilon, \in (0, 1)$ and in (I.1.9) take

$$\psi = u_k + [(k-u)_+ + \varepsilon]^\alpha \in X_{\mathrm{loc}}(S_T).$$

From (I.1.9) we obtain $\forall\, \eta \in C_0^\infty(S_T)$, $\eta \geqq 0$

$$\iint_{S_T} \{u_t\eta + |Du|^{p-2} Du\, D\eta\} (\psi - u)_+ \, dx\, d\tau$$
$$= \iint_{S_T \cap [k<u<\psi]} |Du|^p \eta \, dx\, d\tau + \alpha \iint_{S_T} |Du_k|^p [(k-u)_+ + \varepsilon]^{\alpha-1} \eta\, dx\, d\tau \geqq 0.$$

We let $\varepsilon \to 0$ first as $\alpha \in (0, 1)$ remains fixed. Since

$$(\psi - u)_+ \to (k-u)_+^\alpha \quad \text{a.e. } S_T,$$

we deduce

$$\iint_{S_T} \{u_t\eta + |Du|^{p-2} Du\, D\eta\} (k-u)_+^\alpha \, dx\, d\tau \geqq 0 \qquad \forall\, \alpha \in (0, 1).$$

Now letting $\alpha \to 0$ gives

$$\iint_{S_T} \left\{\frac{\partial}{\partial t} u_k \eta + |Du_k|^{p-2} Du_k\, D\eta\right\} dx\, d\tau \geqq 0 \qquad \forall \eta \in C_0^\infty(S_T), \quad \eta \geqq 0. \qquad \text{(I.1.14)}$$

□

The next proposition permits a rather large class of testing functions in (I.1.9).

If $k_0 \in \mathbf{R}^+$ let $\mathscr{F}(k_0)$ denote the set of all the Lipschitz-continuous functions $f\colon \mathbf{R}^+ \to \mathbf{R}$ such that $f(k) = 0$, $\forall\, k > k_0$, and set

$$\mathscr{F} = \bigcup_{k_0 \in \mathbf{R}^+} \mathscr{F}(k_0).$$

Proposition I.1.4. *Let* $u \in \Sigma$. *Then* $\forall f \in \mathscr{F}$, $\forall\, \eta \in C_0^\infty(S_T)$,

$$\iint_{S_T} \{u_t f(u)\eta + |Du|^{p-2} Du\, D(f(u)\eta)\}\, dx\, d\tau = 0. \qquad \text{(I.1.15)}$$

Proof. Assume first that $f \in C^2(0, \infty)$. Write (I.1.9) for $\psi = k$, $k \in \mathbf{R}^+$, multiply by $f''(k)$ and integrate over $(0, \infty)$ in dk. By interchanging the order of integration with the aid of the Fubini theorem we obtain

$$\iint_{S_T} \left\{u_t\eta \int_u^\infty f''(k)(k-u)\, dk + |Du|^{p-2} Du\, D\left[\eta \int_u^\infty f''(k)(k-u)\, dk\right]\right\} dx\, d\tau = 0.$$

Since

$$\int_u^\infty f''(k)\,(k-u)\, dk = f(u),$$

the assertion follows for $f \in C^2(0, \infty)$. The general case is proved by approximation. □

I.2. Estimating $|Du|$. In the estimates to follow we denote by $\gamma = \gamma(N, p)$ a generic positive constant that can be determined *a priori* only in terms of N and p. For a measurable set Ω we let $\chi(\Omega)$ denote the characteristic function of Ω.

Lemma I.2.1 $\exists\, \gamma = \gamma(N, p)$ *such that* $\forall\, k > 0$, $\forall\, \varrho > 0$, $\forall\, 0 < s < t \leqq T$, $\forall\, u \in \Sigma$

$$\int_s^t \int_{B_\varrho} |Du_k|^p \, dx\, d\tau \leqq \gamma k^p |B_\varrho| \left(\frac{t-s}{\varrho^p} + k^{2-p}\right).$$

Proof. From (I.1.13) with $\psi = k$ and ζ as in (I.1.12)

$$\int_s^t \int_{\mathbf{R}^N} |Du_k|^p \zeta^p \, dx\, d\tau \leqq p \int_s^t \int_{\mathbf{R}^N} |Du_k|^{p-1} \zeta^{p-1} (k-u)_+ |D\zeta| \, dx\, d\tau$$
$$- \tfrac{1}{2} \int_s^t \int_{\mathbf{R}^N} \frac{\partial}{\partial \tau} (k-u)_+^2 \zeta^p \, dx\, d\tau$$
$$\leqq \tfrac{1}{2} \int_s^t \int_{\mathbf{R}^N} |Du_k|^p \zeta^p \, dx\, d\tau + 2^{p-1} p^p \int_s^t \int_{\mathbf{R}^N} (k-u)_+^p |D\zeta|^p dx\, d\tau$$
$$+ \tfrac{1}{2} \int_{\mathbf{R}^N \times \{s\}} (k-u)_+^2 \zeta^p \, dx.$$

From this the lemma follows readily. □

If A is a measurable subset of S_T, let

$$\fint_A u = \frac{1}{|A|} \int_A u$$

denote the integral average of u over A. Define $\forall\, 0 < s < t \leqq T$, $\forall\, \varrho > 0$

$$M_{s,t}(\varrho) = \sup_{\tau \in (s,t)} \fint_{B_\varrho} u(x, \tau)\, dx. \tag{I.2.1}$$

Since u_k are, $\forall\, k > 0$, distributional supersolutions of (I.1.3), $\forall\, 0 < s < t \leqq T$, $\forall\, \psi \in X_{\mathrm{loc}}(S_T)$, $\psi \geqq 0$, ζ as in (I.1.12)

$$\int_s^t \int_{\mathbf{R}^N} \left\{\frac{\partial u_k}{\partial t} \psi \zeta^p + |Du_k|^{p-2} Du_k\, D(\psi \zeta^p)\right\} dx\, d\tau \geqq 0. \tag{I.2.2}$$

Lemma I.2.2. *Let* $u \in \Sigma$. *Then* $\forall\, \alpha \in (0, p-1)$

$$\left|Du^{\frac{p-1-\alpha}{p}}\right| \in L^p_{\mathrm{loc}}(S_T)$$

and $\exists\, \gamma = \gamma(N, p)$ *such that* $\forall\, 0 < s < t \leqq T$, $\forall\, \varrho > 0$,

$$\int_s^t \fint_{B_\varrho} \left|Du^{\frac{p-1-\alpha}{p}}\right|^p dx\, d\tau \leqq \frac{\gamma}{\alpha^p} \left[M_{s,t}(2\varrho) + \left(\frac{t-s}{\varrho^p}\right)^{\frac{1}{2-p}}\right]^{1-\alpha}. \tag{I.2.3}$$

Proof. Let $k > 0$, $\varepsilon \in (0, 1)$ be fixed and in (I.2.2) take

$$\psi = \begin{cases} u_k^{-\alpha}, & u > \varepsilon \\ \varepsilon^{-\alpha}, & u \leqq \varepsilon, \end{cases}$$

to obtain

$$\begin{aligned} &\alpha \int_s^t \int_{\mathbf{R}^N} |Du|^p u^{-\alpha-1} \zeta^p \chi[\varepsilon < u < k] \, dx \, d\tau \\ &\leqq p \int_s^t \int_{\mathbf{R}^N} |Du|^{p-1} u^{-\alpha} \zeta^{p-1} |D\zeta| \chi[\varepsilon < u < k] \, dx \, d\tau \\ &+ p \int_s^t \int_{\mathbf{R}^N} |Du_\varepsilon|^{p-1} \varepsilon^{-\alpha} \zeta^{p-1} |D\zeta| \, dx \, d\tau \\ &+ \frac{1}{1-\alpha} \int_s^t \int_{\mathbf{R}^N} \frac{\partial}{\partial t} \psi^{1-\alpha} \zeta^p \, dx \, d\tau + \int_s^t \int_{\mathbf{R}^N} \frac{\partial}{\partial t} u_\varepsilon \varepsilon^{-\alpha} \zeta^p \, dx \, d\tau. \end{aligned} \tag{I.2.4}$$

By Young's inequality, the first integral on the right-hand side of (I.2.4) is majorized by

$$\frac{\alpha}{2} \int_s^t \int_{\mathbf{R}^N} |Du|^p u^{-\alpha-1} \zeta^p \chi[\varepsilon < u < k] \, dx \, d\tau + \left(\frac{2}{\alpha}\right)^{p-1} p^p \int_s^t \int_{\mathbf{R}^N} u^{p-1-\alpha} |D\zeta|^p \, dx \, d\tau.$$

By virtue of Lemma I.2.1 the second integral tends to zero as $\varepsilon \to 0$ at the rate of $\varepsilon^{p-1-\alpha}$. Combining these calculations and taking $x \to \zeta(x)$ as the cutoff function in $B_{2\varrho}$ which equals 1 on B_ϱ, we deduce

$$\begin{aligned} &\alpha \int_s^t \int_{B_\varrho} |Du|^p u^{-\alpha-1} \chi[\varepsilon < u < k] \, dx \, d\tau \\ &\leqq O(\varepsilon^{p-1-\alpha}) + \frac{\gamma}{\alpha^{p-1}} \left\{ \left(\sup_{\tau \in (s,t)} \int_{B_{2\varrho}} u(x,\tau) \, dx \right)^{1-\alpha} (2\varrho)^{\alpha N} \right. \\ &\left. + \left(\frac{t-s}{\varrho^p}\right) \left(\sup_{\tau \in (s,t)} \int_{B_{2\varrho}} u(x,\tau) \, dx \right)^{p-1-\alpha} (2\varrho)^{N(2-p+\alpha)} \right\} \\ &\leqq \frac{\gamma}{\alpha^{p-1}} \left\{ [M_{s,t}(2\varrho)]^{1-\alpha} + \left(\frac{t-s}{\varrho^p}\right) [M_{s,t}(2\varrho)]^{p-1-\alpha} \right\} \varrho^N + O(\varepsilon^{p-1-\alpha}). \end{aligned} \tag{I.2.5}$$

If $\left(\frac{t-s}{\varrho^p}\right) \leqq [M_{s,t}(2\varrho)]^{2-p}$, the quantity in braces in the member further right in (2.5) is majorized by $[M_{s,t}(2\varrho)]^{1-\alpha}$. If $\left(\frac{t-s}{\varrho^p}\right) > [M_{s,t}(2\varrho)]^{p-2}$, it is majorized by

$$[M_{s,t}(2\varrho)]^{1-\alpha} + \left(\frac{t-s}{\varrho^p}\right)^{\frac{1-\alpha}{2-p}}.$$

In either case

$$\int_s^t \int_{B_\varrho} |Du|^p u^{-(\alpha+1)} \chi[\varepsilon < u < k]\, dx\, d\tau$$

$$\leqq O\left(\varepsilon^{p-(\alpha+1)}\right) + \frac{\gamma}{\alpha^p} \varrho^N \left\{ M_{s,t}(2\varrho) + \left(\frac{t-s}{\varrho^p}\right)^{\frac{1}{2-p}} \right\}^{1-\alpha}, \qquad \text{(I.2.6)}$$

and the lemma follows by letting $\varepsilon \to 0$ first and then $k \to \infty$. □

Estimate (I.2.3) deteriorates as $\alpha \to 0$. The next lemma gives some information for the case $\alpha = 0$.

Lemma I.2.3. *Let* $u \in \Sigma$. *There exists* $\gamma = \gamma(N, p)$ *such that* $\forall\, 0 < s < t \leqq T$, $\forall\, \varrho > 0$, $\forall\, n \geqq 1$,

$$\int_s^t \fint_{B_\varrho} \left| Du^{\frac{p-1}{p}} \right|^p \chi[n < u < n+1]\, dx\, d\tau$$

$$\leqq \gamma \ln\left(1 + \frac{1}{n}\right) \left[M_{s,t}(2\varrho) + \left(\frac{t-s}{\varrho^p}\right)^{\frac{1}{2-p}} \right].$$

Proof. In (I.2.2) we let $x \to \zeta(x)$ be as in (I.1.12) with $\sigma = 1$, and take $\psi = \ln^+\left(\frac{n+1}{u^{(n)}}\right)$, where

$$u^{(n)} = \begin{cases} n, & \text{if } 0 < u \leqq n \\ u, & \text{if } u > n \end{cases}$$

and

$$\forall\, a > 0, \qquad \ln^+ a = \max\{\ln a; 0\}.$$

We get

$$\int_s^t \int_{B_\varrho} |Du|^p u^{-1} \chi\, [n < u < n+1]\, dx\, d\tau$$

$$\leqq \int_s^t \int_{B_{2\varrho}} \frac{\partial}{\partial t} u_{n+1} \ln^+\left(\frac{n+1}{u^{(n)}}\right) \zeta^p\, dx\, d\tau + \frac{p}{\varrho} \int_s^t \int_{B_{2\varrho}} |Du|^{p-1} \ln^+\left(\frac{n+1}{u^{(n)}}\right) dx\, d\tau$$

$$= I_n^{(1)} + \frac{p}{\varrho} I_n^{(2)}. \qquad \text{(I.2.7)}$$

Setting for simplicity of notation

$$A = B_{2\varrho} \times (s, t),$$

we have

$$I_n^{(2)} \leqq \ln\left(1 + \frac{1}{n}\right) \iint_{A \cap [u < n+1]} |Du|^{p-1} u^{-(\alpha+1)\frac{(p-1)}{p}} u^{(\alpha+1)\frac{(p-1)}{p}}\, dx\, d\tau$$

$$\leqq \gamma \ln\left(1 + \frac{1}{n}\right) \left(\iint_A \left| Du^{\frac{p-(\alpha+1)}{p}} \right|^p dx\, d\tau \right)^{\frac{p-1}{p}} \left(\iint_A u^{(\alpha+1)(p-1)}\, dx\, d\tau \right)^{\frac{1}{p}}$$

If $\alpha \in (0, p-1)$ is so small that $(\alpha+1)(p-1) \leqq 1$, both integrals in braces are finite. Taking into account Lemma I.2.2 in estimating the first integral, we have

$$\frac{p}{\varrho} I_n^{(2)} \leqq \gamma \varrho^N \ln\left(1+\frac{1}{n}\right)$$
$$\cdot \left[M_{s,t}(2\varrho) + \left(\frac{t-s}{\varrho^p}\right)^{\frac{1}{2-p}}\right]^{(1-\alpha)\frac{(p-1)}{p}} \left(\frac{1}{\varrho^p} \int_s^t \fint_{B_{2\varrho}} u^{(\alpha+1)(p-1)}\, dx\, d\tau\right)^{\frac{1}{p}}.$$

The last integral above is estimated by

$$\left(\frac{1}{\varrho^p} \int_s^t \fint_{B_{2\varrho}} u^{(\alpha+1)(p-1)}(x,\tau)\, dx\, d\tau\right)^{\frac{1}{p}} \leqq \left(\frac{t-s}{\varrho^p}\right)^{\frac{1}{p}} (M_{s,t}(2\varrho))^{(\alpha+1)(\frac{p-1}{p})}$$
$$\leqq \left[M_{s,t}(2\varrho) + \left(\frac{t-s}{\varrho^p}\right)^{\frac{1}{2-p}}\right]^{(\alpha+1)\frac{(p-1)}{p} + \frac{2-p}{p}}.$$

Therefore

$$\frac{p}{\varrho} I_n^{(2)} \leqq \gamma \varrho^N \ln\left(1+\frac{1}{n}\right) \left[M_{s,t}(2\varrho) + \left(\frac{t-s}{\varrho^p}\right)^{\frac{1}{2-p}}\right]^{\frac{1}{p}}.$$

As for $I_n^{(1)}$, we write

$$I_n^{(1)} = \iint_{A \cap [u<n]} u_t \ln\left(1+\frac{1}{n}\right) \zeta^p\, dx d\tau + \iint_{A \cap [u>n]} u_t \ln^+\left(\frac{n+1}{u}\right) \zeta^p\, dx\, d\tau$$
$$= \ln\left(1+\frac{1}{n}\right) \iint_A \frac{\partial}{\partial t} u_n \zeta^p\, dx\, d\tau + \iint_A \frac{\partial}{\partial t} u^{(n)} \ln^+\left(\frac{n+1}{u}\right) \zeta^p\, dx\, d\tau$$
$$\leqq \gamma \varrho^N \ln\left(1+\frac{1}{n}\right) M_{s,t}(2\varrho) + \iint_A \frac{\partial}{\partial t}\left(\int_n^u \ln^+\left(\frac{n+1}{\xi}\right) d\xi\right)_+ \zeta^p\, dx\, d\tau.$$

The last integral is majorized by

$$\ln\left(1+\frac{1}{n}\right) \int_{B_{2\varrho}\times\{t\}} (u-n)_+(t)\, dx \leqq \gamma \varrho^N \ln\left(1+\frac{1}{n}\right) M_{s,t}(2\varrho).$$

Therefore

$$I_n^{(1)} \leqq \gamma \varrho^N \ln\left(1+\frac{1}{n}\right) \left[M_{s,t}(2\varrho) + \left(\frac{t-s}{\varrho^p}\right)^{\frac{1}{2-p}}\right].$$

Substituting the estimates of $I_n^{(i)}$, $i = 1, 2$, in (I.2.7) delivers the lemma. □

Corollary I.2.4. *Let* $u \in \Sigma$ *and define*

$$(x,t) \to z(x,t) = \int_e^{u(x,t)} (\xi \ln^{1+\varepsilon} \xi)^{\frac{1}{p}}\, dx, \quad \varepsilon \in (0, p-1).$$

Then $|Dz| \in L^p_{loc}(S_T)$, *and there exists* $\gamma = \gamma(N,p)$ *such that* $\forall\, 0 < s < t \leqq T$, $\forall\, \varrho > 0$,

$$\int_s^t \fint_{B_\varrho} |Dz|^p\, dx\, d\tau \leqq \gamma \varepsilon^{-1} \left[M_{s,t}(2\varrho) + \left(\frac{t-s}{\varrho^p}\right)^{\frac{1}{2-p}}\right]. \tag{I.2.8}$$

Proof. Divide both sides of the inequality of Lemma I.2.3 by $\ln^{1+\varepsilon} n$, and add over all $n = 2, 3, \ldots$. □

Estimate (I.2.8) deteriorates as $\varepsilon \to 0$. The following corollary gives some information in the case $\varepsilon = 0$.

Corollary I.2.5. *Let* $u \in \Sigma$. *Then* $\forall\, 0 < s < t \leqq T$, $\forall\, C > 1$, $\forall\, \varrho > 0$

$$\overline{\lim_{k\to 0}} \int_s^t \int_{B_\varrho} |Du|^p \frac{1}{u \ln u} \chi[k < u < Ck]\, dx\, d\tau = 0.$$

Proof. Without loss of generality we may assume that $k, Ck \in \mathbf{N}$. Divide both sides of the inequality of Lemma I.2.3 by $\ln n$ and add for $n = k, k+1, \ldots, Ck$. This gives

$$\int_s^t \int_{B_\varrho} |Du|^p (u \ln u)^{-1} \chi[k < u < Ck]\, dx\, d\tau$$

$$\leqq \gamma (\ln \ln Ck - \ln \ln k) \left[M_{s,t}(2\varrho) + \left(\frac{t-s}{\varrho^p}\right)^{\frac{1}{2-p}} \right]$$

$$= \gamma \ln \left(1 + \frac{\ln C}{\ln k}\right) \left[M_{s,t}(2\varrho) + \left(\frac{t-s}{\varrho^p}\right)^{\frac{1}{2-p}} \right]. \qquad \square$$

I.3. An equivalent formulation of local solutions. Since in general $|Du| \notin L^p_{\text{loc}}(S_T)$, (I.1.3) need not hold in $\mathscr{D}'(S_T)$. We give another formulation of local weak solutions, equivalent to (I.1.7), that will be needed in the process of constructing weak solutions of the Cauchy problem. It says that a small power of u satisfies an equation similar to (I.1.3) in $\mathscr{D}'(S_T)$.

Let $\alpha \in (0, p-1)$ be fixed and define the numbers

$$\delta = \frac{(\alpha+1)(p-1)}{p}, \quad \delta \in \left(\frac{p-1}{p},\, p-1\right), \tag{I.3.1}$$

$$c_0 = \frac{1}{1-\delta} \left(\frac{p-1-\alpha}{p}\right)^{\frac{p}{p-1-\alpha}(1-\delta)},$$

$$\beta = \frac{p}{p-1-\alpha}(1-\delta), \tag{I.3.2}$$

$$d_0 = \frac{\delta p}{p-1-\alpha}.$$

Also let

$$v = \frac{p}{p-1-\alpha} u^{\frac{p-1-\alpha}{p}}. \tag{I.3.3}$$

Proposition I.3.1. *A measurable function* $u: S_T \to \mathbf{R}^+$ *is a local weak solution of* (I.1.3) *in* S_T *if and only if*

$$v \in L_{\mathrm{loc}}^{\overline{\frac{p}{p-1-\alpha}}}(S_T),$$

$$|Dv| \in L^p_{\mathrm{loc}}(S_T), \qquad |Dv|^p v^{-1} \in L^1_{\mathrm{loc}}(S_T), \tag{I.3.4}$$

$$\frac{\partial}{\partial t} v_k^{p/(p-1-\alpha)} \in L^1_{\mathrm{loc}}(S_T), \qquad \forall\, k > 0,$$

$$c_0 \frac{\partial}{\partial t} v^\beta - \operatorname{div}\left(|Dv|^{p-2} Dv\right) = d_0 \, |Dv|^p v^{-1} \quad \textit{in } \mathscr{D}'(S_T). \tag{I.3.5}$$

Remark I.3.1. One could obtain (I.3.5) *formally* from (I.1.3) by multiplication by $u^{-\delta}$ and formal calculations.

Proof of Proposition I.3.1. In (I.1.9) take $\psi = \frac{1}{\lambda}(u^{(\varepsilon)})^{-\delta}$, where $\lambda > 0$ and for $\varepsilon > 0$

$$u^{(\varepsilon)} = \begin{cases} u & \text{if } u > \varepsilon \\ \varepsilon & \text{if } 0 \leqq u \leqq \varepsilon. \end{cases}$$

Set

$$\mathscr{E}_\lambda = \{(x, t) \in S_T \mid (\psi - u)^+ > 0\},$$

and observe that

$$u(x, t) \leqq \lambda^{-1/(1+\delta)}, \qquad \text{a.e. } (x, t) \in \mathscr{E}_\lambda. \tag{I.3.6}$$

We obtain from (I.1.9) $\forall\, \eta \in C_0^\infty(S_T)$ upon multiplication by λ,

$$\iint_{\mathscr{E}_\lambda} \{u_t (u^{(\varepsilon)})^{-\delta} \eta + |Du|^{p-2} Du (u^{(\varepsilon)})^{-\delta} D\eta\}\, dx\, d\tau \tag{I.3.7}$$

$$= \delta \iint_{\mathscr{E}_\lambda} |Du|^p u^{-\delta-1} \eta \chi[u > \varepsilon]\, dx\, d\tau + \lambda \iint_{\mathscr{E}_\lambda} u_t u \eta\, dx\, d\tau + \lambda \iint_{\mathscr{E}_\lambda} |Du|^p \eta\, dx\, d\tau.$$

We first let $\varepsilon \to 0$ and then $\lambda \to 0$. By virtue of (I.3.6)

$$\left|\lambda \iint_{\mathscr{E}_\lambda} u u_t \eta\, dx\, d\tau\right| \leqq \left|\lambda \iint_{[u \leqq \varepsilon]} u u_t \eta\, dx\, d\tau\right| + \left|\lambda \iint_{\left[\varepsilon < u < \lambda^{-\frac{1}{1+\delta}}\right]} u u_t \eta\, dx\, d\tau\right|$$

$$\leqq \tfrac{1}{2}\lambda \left|\iint_{S_T} u_\varepsilon^2 |\eta_t|\, dx\, d\tau\right| + \tfrac{1}{2}\lambda \iint_{S_T} \left(\min\left[u^{(\varepsilon)}; \lambda^{1-\frac{1}{1+\delta}}\right]\right)^2 |\eta_t|\, dx\, d\tau$$

$$\leqq \tfrac{1}{2}\varepsilon^2 \lambda \iint_{S_T} |\eta_t|\, dx\, d\tau + \tfrac{1}{2}\lambda^{1-\frac{1}{1+\delta}} \iint_{S_T} u\, |\eta_t|\, dx\, d\tau$$

$$\to 0 \quad \text{as} \quad \varepsilon, \lambda \to 0.$$

By (I.3.6) and Lemma I.2.2, if $\delta_0 \in (o, \delta)$, then

$$\lambda \iint_{\mathscr{E}_\lambda} |Du|^p \eta\, dx\, d\tau = \lambda \iint_{\mathscr{E}_\lambda} |Du|^p u^{-(1+\delta_0)} u^{(1+\delta_0)} \eta\, dx\, d\tau$$

$$\leqq \gamma \lambda^{1-\frac{1+\delta_0}{1+\delta}} \iint_{S_T} \left|D u^{\frac{p-1-\delta_0}{p}}\right|^p \eta\, dx\, d\tau$$

$$\to 0 \quad \text{as} \quad \lambda \to 0.$$

We treat the first integral on the left hand side of (I.3.7) as follows

$$\iint_{\mathscr{E}_\lambda} u_t (u^{(\varepsilon)})^{-\delta} \eta \, dx \, d\tau = \iint_{\left[u^{(\varepsilon)} < \lambda^{-\frac{1}{1+\delta}}\right]} u_t^{(\varepsilon)} (u^{(\varepsilon)})^{-\delta} \eta \, dx \, d\tau + \varepsilon^{-\delta} \iint_{S_T} \frac{\partial u_\varepsilon}{\partial t} \eta \, dx \, d\tau$$

$$= -\frac{1}{1-\delta} \iint_{S_T} \left(u_{\lambda^{-\frac{1}{1+\delta}}}^{(\varepsilon)}\right)^{1-\delta} \eta_t \, d\tau - \varepsilon^{-\delta} \iint_{S_T} u_\varepsilon \eta_t \, dx \, d\tau.$$

Letting $\varepsilon \to 0$ first and then $\lambda \to 0$ yields

$$\int_{\mathscr{E}_\lambda} u_t (u^{(\varepsilon)})^{-\delta} \eta \, dx \, d\tau \;\to\; -\frac{1}{1-\delta} \iint_{S_T} u^{1-\delta} \eta_t \, dx \, d\tau.$$

Next

$$\iint_{\mathscr{E}_\lambda} |Du|^{p-2} Du (u^{(\varepsilon)})^{-\delta} D\eta \, dx \, d\tau$$

$$= \iint_{\left[u^{(\varepsilon)} < \lambda^{-\frac{1}{1+\delta}}\right]} |Du^{(\varepsilon)}|^{p-2} Du^{(\varepsilon)} (u^{(\varepsilon)})^{-\delta} D\eta \, dx \, d\tau + \varepsilon^{-\delta} \iint_{S_T} |Du_\varepsilon|^{p-2} Du_\varepsilon D\eta \, dx \, d\tau.$$

As $\varepsilon \to 0$

$$\iint_{\mathscr{E}_\lambda} |Du|^{p-2} Du (u^{(\varepsilon)})^{-\delta} D\eta \, dx \, d\tau \;\to\; \iint_{\left[u < \lambda^{-\frac{1}{1+\delta}}\right]} |Dv|^{p-2} Dv \, D\eta \, dx \, d\tau.$$

Since $\chi\left[u < \lambda^{-\frac{1}{1+\delta}}\right]$ increases to 1 a.e. S_T, letting $\lambda \to 0$ we find

$$\iint_{\left[u < \lambda^{-\frac{1}{1+\delta}}\right]} |Dv|^{p-2} Dv \, D\eta \, dx \, d\tau \;\to\; \iint_{S_T} |Dv|^{p-2} Dv \, D\eta \, dx \, d\tau.$$

Finally by similar reasoning, as $\varepsilon \to 0$ and $\lambda \to 0$

$$\delta \iint_{\mathscr{E}_\lambda} |Du|^p u^{-\delta-1} \eta \chi[u > \varepsilon] \, dx \, d\tau \;\to\; d_0 \iint_{S_T} |Dv|^p v^{-1} \eta \, dx \, d\tau. \tag{I.3.8}$$

Letting $\varepsilon \to 0$ and $\lambda \to 0$ in (I.3.7) yields $\forall \eta \in C_0^\infty(S_T)$

$$\iint_{S_T} \{-c_0 v^\beta \eta_t + |Dv|^{p-2} Dv \, D\eta\} \, dx \, d\tau = d_0 \iint_{S_T} |Dv|^p v^{-1} \eta \, dx \, d\tau. \tag{I.3.9}$$

Suppose now that $v : S_T \to \mathbf{R}^+$ satisfies (I.3.4), (I.3.5) and set

$$u = \left(\frac{p-1-\alpha}{p} v\right)^{\frac{p}{p-1-\alpha}}, \quad \alpha \in (0, p-1). \tag{I.3.10}$$

It follows from (I.3.4) that $\forall \, k > 0$

$$u \in L^1_{\text{loc}}(S_T), \quad |Du|_k \in L^p_{\text{loc}}(S_T), \quad \frac{\partial}{\partial t} u_k \in L^1_{\text{loc}}(S_T). \tag{I.3.11}$$

Let $\varepsilon \in (0, 1)$ and in (I.3.9) take the test function

$$\eta = (\varphi - u)^+ (u^{(\varepsilon)})^\delta,$$

where $\varphi \in C_0^\infty(S_T)$, and δ is defined in (I.3.1). Such a choice is admissible by a standard approximation process.

Treating separately the various parts of (I.3.9) with the indicated choice of test function we obtain, by direct calculation

(a) $$\iint_{S_T} \{-c_0 v^\beta (u^{(\varepsilon)})^\delta \frac{\partial}{\partial t}(\varphi - u)_+ - \delta c_0 v^\beta (u^{(\varepsilon)})^{\delta-1} u_t^{(\varepsilon)}(\varphi - u)_+\} dx\, d\tau$$

$$= - \iint_{S_T} u \frac{\partial}{\partial t}(\varphi - u)_+ \chi[u > \varepsilon]\, dx\, d\tau - \frac{1}{1-\delta} \iint_{S_T} u_\varepsilon^{1-\delta} \varepsilon^\delta \frac{\partial}{\partial t}(\varphi - u)_+ \chi[u \leqq \varepsilon]\, dx\, d\tau$$

$$\to \iint_{S_T} u_t(\varphi - u)_+\, dx\, d\tau \quad \text{as } \varepsilon \to 0.$$

(b) $$\iint_{S_T} |Dv|^{p-2} Dv (u^{(\varepsilon)})^\delta D(\varphi - u)_+\, dx\, d\tau$$

$$+ \delta \iint_{S_T} |Dv|^{p-2} Dv (u^{(\varepsilon)})^{\delta-1} Du^{(\varepsilon)}(\varphi - u)_+\, dx\, d\tau$$

$$= \iint_{S_T} |Du|^{p-2} Du\, D(\varphi - u)_+ \chi[u \geqq \varepsilon]\, dx\, d\tau$$

$$+ \varepsilon^\delta \iint_{S_T} |Dv|^{p-2}\, Dv\, D(\varphi - u)_+ \chi[0 \leqq u \leqq \varepsilon]\, dx\, d\tau$$

$$+ d_0 \iint_{S_T} |Dv|^p v^{-1} u^\delta (\varphi - u)^+ \chi[u > \varepsilon]\, dx\, d\tau$$

$$\to \iint_{S_T} |Du|^{p-2} Du\, D(\varphi - u)_+\, dx\, d\tau$$

$$+ d_0 \iint_{S_T} |Dv|^p v^{-1} u^\delta (\varphi - u)_+\, dx\, d\tau.$$

Combining these calculations in (I.3.9) and letting $\varepsilon \to 0$ delivers the proposition. □

I.4. Harnack inequality and initial traces. In the definition of local weak solutions of (I.1.3) in S_T, no reference has been made to initial data. We will show that each $u \in \Sigma$ has a *unique* non-negative σ-finite Borel measure μ as initial trace. Existence of such a trace will be a consequence of the following Harnack type estimate.

Theorem I.4.1. *Let* $u \in \Sigma$. *There exists* $\gamma = \gamma(N, p)$, *such that* $\forall \varrho > 0$, $\forall\, 0 < s < t \leqq T$,

$$\sup_{\tau \in (s,t)} \int_{B_\varrho} u(x, \tau)\, dx \leqq \gamma \left(\int_{B_{2\varrho}} u(x, t)\, dx + \left(\frac{t-s}{\varrho^\varkappa}\right)^{\frac{1}{2-p}} \right), \quad \varkappa = N(p-2) + p. \tag{I.4.1}$$

The uniqueness of the initial trace μ as well as the proof of (I.4.1) relies on the next gradient estimates.

Lemma I.4.1. *Let* $u \in \Sigma$. $\exists \gamma = \gamma(N, p)$ *such that* $\forall\, 0 < s < t \leqq T$, $\forall 0 < r < R$, $\forall\, \varepsilon > 0$

$$\int_s^t \int_{B_r} (t-\tau)^{\frac{1}{p}} |Du|^p (u+\varepsilon)^{-\frac{2}{p}} \, dx\, d\tau$$

$$\leqq \gamma \left(1 + \frac{t-s}{\varepsilon^{2-p}(R-r)^p}\right) \int_s^t \int_{B_R} (t-\tau)^{\frac{1}{p}-1} (u+\varepsilon)^{\frac{2}{p}(p-1)} \, dx\, d\tau, \quad \text{(I.4.2)}$$

$$\int_s^t \int_{B_r} |Du|^{p-1} \, dx\, d\tau$$

$$\leqq \gamma \left(1 + \frac{t-s}{\varepsilon^{2-p}(R-r)^p}\right)^{\frac{p-1}{p}} \int_s^t \int_{B_R} (t-\tau)^{\frac{1}{p}-1} (u+\varepsilon)^{\frac{2}{p}(p-1)} \, dx\, d\tau. \quad \text{(I.4.3)}$$

Proof of Lemma I.4.1. Write (I.2.2) with $x \to \zeta(x)$ a non-negative piecewise smooth cutoff function in B_R which equals one on B_r and is such that $|D\zeta| \leqq (R-r)^{-1}$. Take also, $\forall\, k > 0$,

$$\psi = (t-\tau)^{\frac{1}{p}} (u_k + \varepsilon)^{1-\frac{2}{p}} \in X_{\text{loc}}(S_T),$$

where $\varepsilon > 0$ is arbitrary.

Treating the various terms separately, we have

(i) $$\int_s^t \int_{B_R} (t-\tau)^{\frac{1}{p}} \frac{\partial}{\partial t} (u_k + \varepsilon)\, (u_k+\varepsilon)^{1-\frac{2}{p}} \zeta^p(x) dx\, d\tau$$

$$\leqq \frac{1}{2(p-1)} \int_s^t \int_{B_R} (t-\tau)^{\frac{1}{p}-1} (u_k+\varepsilon)^{\frac{2}{p}(p-1)} \zeta^p(x)\, dx\, d\tau, \qquad \forall\, k > 0;$$

(ii) $$\int_s^t \int_{B_R} (t-\tau)^{\frac{1}{p}} |Du_k|^{p-2} Du_k D((u_k+\varepsilon)^{1-\frac{2}{p}} \zeta^p(x))\, dx\, d\tau$$

$$\leqq \frac{(p-2)}{p} \int_s^t \int_{B_R} (t-\tau)^{\frac{1}{p}} |Du_k|^p (u_k+\varepsilon)^{-\frac{2}{p}} \zeta^p dx\, d\tau$$

$$+ p \int_s^t \int_{B_R} (t-\tau)^{\frac{1}{p}} |Du_k|^{p-1} \zeta^{p-1} (u_k+\varepsilon)^{1-\frac{2}{p}} |D\zeta|\, dx\, d\tau$$

$$\leqq \frac{(p-2)}{2p} \int_s^t \int_{B_R} (t-\tau)^{\frac{1}{p}} |Du_k|^p (u_k+\varepsilon)^{-\frac{2}{p}} \zeta^p\, dx\, d\tau$$

$$+ \left(\frac{2p}{2-p}\right)^{p-1} p^p \int_s^t \int_{B_R} (t-\tau)^{\frac{1}{p}} (u_k+\varepsilon)^{p-\frac{2}{p}} |D\zeta|^p\, dx\, d\tau.$$

This last integral is estimated above by

$$\gamma \frac{(t-s)}{\varepsilon^{2-p}(R-r)^p} \int_s^t \int_{B_R} (t-\tau)^{\frac{1}{p}-1} (u_k+\varepsilon)^{\frac{2}{p}(p-1)}\, dx\, d\tau.$$

Combining these calculations gives

$$\int_s^t \int_{B_R} (t-\tau)^{\frac{1}{p}} |Du_k|^p (u_k+\varepsilon)^{-\frac{2}{p}} \zeta^p(x)\, dx\, d\tau$$

$$\leqq \frac{\gamma}{(2-p)^p}\left(1+\frac{t-s}{\varepsilon^{2-p}(R-r)^p}\right) \int_s^t \int_{B_R} (t-\tau)^{\frac{1}{p}-1} (u+\varepsilon)^{\frac{2}{p}(p-1)}\, dx\, d\tau. \tag{I.4.4}$$

To prove (I.4.2) it suffices to let $k\to\infty$. Indeed, since

$$|Du_k|^p (u_k+\varepsilon)^{-\frac{2}{p}} \in L^1_{\text{loc}}(S_T)$$

uniformly in k, and since $\chi[u<k]\,\zeta^p \nearrow \zeta^p$ as $k\to\infty$, the term on the left of (I.4.4) can be written as

$$\int_s^t \int_{B_r} (t-\tau)^{\frac{1}{p}} |Du|^p (u+\varepsilon)^{-\frac{2}{p}} \zeta^p \chi[u<k]\, dx\, d\tau$$

$$\to \int_s^t \int_{B_R} (t-\tau)^{\frac{1}{p}} |Du|^p (u+\varepsilon)^{-\frac{2}{p}} \zeta^p\, dx\, d\tau$$

$$\geqq \int_s^t \int_{B_r} (t-\tau)^{\frac{1}{p}} |Du|^p (u+\varepsilon)^{-\frac{2}{p}}\, dx\, d\tau.$$

To prove (I.4.3) note that $\forall\, k>0$

$$\iint_{s\,B_R}^{t} |Du_k|^{p-1} \zeta^{p-1}\, dx\, d\tau$$

$$= \iint_{s\,B_R}^{t} (t-\tau)^{\frac{1}{p}\frac{p-1}{p}} \frac{|Du_k|^{p-1} (t-\tau)^{-\frac{1}{p}\frac{p-1}{p}}}{(u+\varepsilon)^{\frac{2}{p}\frac{p-1}{p}}} (u+\varepsilon)^{\frac{2}{p}\frac{p-1}{p}} \zeta^{p-1}\, dx\, d\tau$$

$$\leqq \left(\iint_{s\,B_R}^{t} (t-\tau)^{\frac{1}{p}} \frac{|Du_k|^p}{(u+\varepsilon)^{\frac{2}{p}}} \zeta^p\, dx\, d\tau\right)^{\frac{p-1}{p}} \left(\iint_{s\,B_R}^{t} (t-\tau)^{\frac{1}{p}-1} (u+\varepsilon)^{\frac{2}{p}(p-1)}\, dx\, d\tau\right)^{\frac{1}{p}}.$$

We estimate the first integral on the right-hand side by (I.4.4), and let $k\to\infty$ on the left-hand side. □

Proof of Theorem I.4.1. Let $\varrho>0$ be fixed. Define the sequences

$$\varrho_n = \left(\sum_{i=0}^{n} 2^{-i}\right)\varrho, \qquad \bar{\varrho}_n = \frac{\varrho_n+\varrho_{n+1}}{2},$$

$$B_n = B_{\varrho_n}, \qquad \bar{B}_n = B_{\bar{\varrho}_n},$$

and let $x\to\zeta_n(x)$ be the standard cutoff function in $\bar{B}_n$ which equals one on B_n and is such that $|D\zeta_n| \leqq 2^{n+2}\varrho$.

Fix $0 < s < t \leqq T$ and write (I.2.2) over the time interval (τ, t), $s \leqq \tau < t$, with ζ replaced by ζ_n and with $\psi \equiv 1$. We obtain by standard calculations

$$\int_{B_n} u_k(\tau)\,dx \leqq \int_{\bar{B}_n} u_k(t)\,dx + \frac{2^{n+2}}{\varrho}\int_\tau^t \int_{\bar{B}_n} |Du_k|^{p-1}\,dx\,d\tau. \tag{I.4.5}$$

We set

$$H_n = \sup_{\tau\in(s,t)} \int_{B_n} u(x,\tau)\,dx. \tag{I.4.6}$$

Then we let $k \to \infty$ in (I.4.5) and estimate the last term by (I.4.3) of Lemma I.4.1 with $r = \bar{\varrho}_n$, $R = \varrho_{n+1}$, $(R - r) = \varrho/2^{n+2}$. We obtain from (I.4.5), $\forall\, \varepsilon > 0$

$$\begin{aligned} H_n &\leqq \int_{B_n} u(t)\,dx + \frac{\gamma 2^n}{\varrho}\left(1 + \frac{(t-s)\,2^{np}}{\varepsilon^{2-p}\varrho^p}\right)^{\frac{p-1}{p}} \int_s^t \int_{B_{n+1}} (t-\tau)^{\frac{1}{p}-1} (u+\varepsilon)^{\frac{2}{p}(p-1)}\,dx\,d\tau \\ &\leqq \int_{B_{2\varrho}} u(x,t)\,dx + \gamma\frac{2^{np}}{\varrho}\left(1 + \frac{t-s}{\varepsilon^{2-p}\varrho^p}\right)^{\frac{p-1}{p}} \int_s^t \int_{B_{n+1}} (t-\tau)^{\frac{1}{p}-1}(u+\varepsilon)^{\frac{2}{p}(p-1)}\,dx\,d\tau. \end{aligned} \tag{I.4.7}$$

By the Hölder inequality and (I.4.6)

$$\int_s^t \int_{B_{n+1}} (t-\tau)^{\frac{1}{p}-1}(u+\varepsilon)^{\frac{2}{p}(p-1)}\,dx\,d\tau \leqq \gamma (t-s)^{\frac{1}{p}} \varrho^{\frac{N(2-p)}{p}} (H_{n+1} + \varepsilon\varrho^N)^{\frac{2}{p}(p-1)}.$$

By Young's inequality, $\forall\, \delta \in (0, 1)$

$$\begin{aligned} &\frac{\gamma 2^{np}}{\varrho}\left(1 + \frac{t-s}{\varepsilon^{2-p}\varrho^p}\right)^{\frac{p-1}{p}} \int_s^t \int_{B_{n+1}} (t-\tau)^{\frac{1}{p}-1}(u+\varepsilon)^{\frac{2}{p}(p-1)}\,dx\,d\tau \\ &\quad\leqq \gamma 2^{np}\left(1 + \frac{t-s}{\varepsilon^{2-p}\varrho^p}\right)^{\frac{p-1}{p}} \left(\frac{t-s}{\varrho^\varkappa}\right)^{\frac{1}{p}} (H_{n+1} + \varepsilon\varrho^N)^{\frac{2}{p}(p-1)} \\ &\quad\leqq \delta(H_{n+1} + \varepsilon\varrho^N) + \gamma 2^{n\frac{p^2}{2-p}} \delta^{-\frac{2(p-1)}{2-p}} \left(1 + \frac{t-s}{\varepsilon^{2-p}\varrho^p}\right)^{\frac{p-1}{2-p}} \left(\frac{t-s}{\varrho^\varkappa}\right)^{\frac{1}{2-p}} \\ &\quad\leqq \delta H_{n+1} + \gamma(\delta)\,2^{n\frac{p^2}{2-p}} \left\{\left(\frac{t-s}{\varrho^\varkappa}\right)^{\frac{1}{2-p}} + \varepsilon\varrho^N + \left(\frac{t-s}{\varrho^\varkappa}\right)^{\frac{1}{2-p}} \left(\frac{t-s}{\varepsilon^{2-p}\varrho^p}\right)^{\frac{p-1}{2-p}}\right\}. \end{aligned}$$

This in (I.4.7) gives

$$H_n \leqq \delta H_{n+1} + b^n C_0(\varepsilon), \qquad n = 0, 1, 2, \ldots, \qquad \forall\, \varepsilon > 0, \tag{I.4.8$_n$}$$

where

$$b = 2^{p^2/(2-p)}, \qquad C_0(\varepsilon, \delta) = \int_{B_{2\varrho}} u(x,t)\,dx + B_0(\varepsilon, \delta) \tag{I.4.9}$$

$$B_0(\varepsilon, \delta) = \gamma(\delta)\left\{\left(\frac{t-s}{\varrho^\varkappa}\right)^{\frac{1}{2-p}} + \varepsilon\varrho^N + \left(\frac{t-s}{\varrho^\varkappa}\right)^{\frac{1}{2-p}} \left(\frac{t-s}{\varepsilon^{2-p}\varrho^p}\right)^{\frac{p-1}{2-p}}\right\}. \tag{I.4.10}$$

Iterating the inequalities (I.4.8$_n$)

$$H_0 \leqq \delta^n H_n + \delta^{-1} C_0(\varepsilon, \delta) \sum_{i=0}^{n+1} (b\,\delta)^i. \tag{I.4.11}$$

Choose $\delta = \frac{1}{2b}$ and let $n \to \infty$ to conclude

$$H_0 \leqq \frac{2}{\delta} \int_{B_{2\varrho}} u(x, t)\,dx + \frac{2}{\delta} B_0(\varepsilon), \qquad \forall\, \varepsilon > 0.$$

We finally minimize the function $\varepsilon \to B_0(\varepsilon)$ given by (I.4.10). The minimum is achieved for

$$\varepsilon = \bar{\varepsilon} \equiv (p-1)^{\frac{1}{p}} \left(\frac{t-s}{\varrho^p}\right)^{\frac{1}{2-p}}$$

and

$$B_0(\bar{\varepsilon}, \delta) = \gamma \left(\frac{t-s}{\varrho^{\varkappa}}\right)^{\frac{1}{2-p}}, \qquad \gamma = \gamma(N, p). \quad \square$$

Theorem I.4.2. *Every* $u \in \Sigma$ *has a unique σ-finite non-negative Borel measure* μ *as initial trace at* $t = 0$.

Proof. From Theorem I.4.1 it follows that $\forall\, \eta \in C_0(\mathbf{R}^N)$, the net

$$\left\{ \int_{\mathbf{R}_N} u(\tau)\, \eta\, dx \right\}_{\tau \in (0,t)}$$

is equibounded, with bound depending only upon $\|\eta\|_{\infty, \mathbf{R}^N}$. A subnet indexed with $\{\tau'\}$ converges to a non-negative σ-finite Borel measure μ, in the sense of measures, *i.e.*, as $\tau' \searrow 0$

$$\int_{\mathbf{R}^N} u(\tau')\, \eta\, dx \to \int_{\mathbf{R}^N} \eta\, d\mu, \qquad \forall\, \eta \in C_0(\mathbf{R}^N).$$

Suppose now that there exist another subnet, indexed with $\{\tau''\}$ and a non-negative σ-finite Borel measure v, such that

$$\int_{\mathbf{R}_N} u(\tau'')\, \eta\, dx \to \int_{\mathbf{R}^N} \eta\, dv, \qquad \forall\, \eta \in C_0(\mathbf{R}^N).$$

We will prove that $\mu \equiv v$.

Let $\sigma \in (0, 1)$ and write (I.2.2) with $\psi \equiv 1$ and ζ as in (I.1.12). Letting $k \to \infty$, standard calculations give $\forall\, 0 < s < t \leqq T$

$$\int_{B_\varrho} u(s)\, dx \leqq \int_{B_{(1+\sigma)\varrho}} u(t)\, dx + \frac{1}{\sigma\varrho} \int_s^t \int_{B_{(1+\sigma)\varrho}} |Du|^{p-1}\, dx\, d\tau. \tag{I.4.12}$$

We estimate the last term by using (I.4.3) of Lemma I.4.1 with $r = (1+\sigma)\varrho$, $R = 2(1+\sigma)\varrho$ and $\varepsilon = \left(\frac{t-s}{\varrho^p}\right)^{\frac{1}{2-p}}$. We obtain $\forall\, \delta \in (0, 1)$

$$\frac{1}{\sigma\varrho}\int_s^t \int_{B_{(1+\sigma)\varrho}} |Du|^{p-1}\, dx\, d\tau$$

$$\leqq \frac{\gamma}{\sigma\varrho}\int_s^t (t-\tau)^{\frac{1}{p}-1}\, d\tau \left\{\left(\sup_{\tau\in(s,t)} \int_{B_{(1+\sigma)\varrho}} u(x,\tau)\, dx\right)^{\frac{2}{p}(p-1)} \varrho^{\frac{N(2-p)}{p}} + \left(\frac{t-s}{\varrho^p}\right)^{\frac{1}{2-p}\frac{2}{p}(p-1)} \varrho^N\right\}$$

$$\leqq \frac{\gamma}{\sigma}\left(\frac{t}{\varrho^\varkappa}\right)^{\frac{1}{p-2}} + \frac{\gamma}{\sigma}\left(\frac{t}{\varrho^\varkappa}\right)^{\frac{1}{p}} \left(\sup_{\tau\in(s,t)} \int_{B_{(1+\sigma)\varrho}} u(x,\tau)\, dx\right)^{\frac{2}{p}(p-1)}$$

$$\leqq \frac{\gamma}{\sigma}\left(\frac{t}{\varrho^\varkappa}\right)^{\frac{1}{p-2}} + \left(\frac{\gamma}{\sigma}\right)^{\frac{p}{2-p}} \delta^{-\frac{p(p-1)}{2-p}} \left(\frac{t}{\varrho^\varkappa}\right)^{\frac{1}{2-p}} + \delta\left(\sup_{\tau\in(s,t)} \int_{B_{(1+\sigma)\varrho}} u(x,\tau)\, dx\right).$$

Finally, taking into account Theorem I.4.1, we deduce that $\forall\, \delta \in (0, 1)$ $\exists\, \gamma = \gamma(N, p, \delta)$ such that

$$\frac{1}{\sigma\varrho}\int_s^t \int_{B_{(1+\sigma)\varrho}} |Du|^{p-1}\, dx\, d\tau \leqq \delta \int_{B_{2(1+\sigma)\varrho}} u(x,t)\, dx + \gamma(\delta)\, \sigma^{-\frac{p}{2-p}} \left(\frac{t}{\varrho^\varkappa}\right)^{\frac{1}{2-p}}.$$

Substitute this estimate in (I.4.12) to obtain $\forall\, 0 < s < t \leqq T$, $\forall\, \sigma \in (0, 1)$, $\forall\, \delta \in (0, 1)$, we obtain

$$\int_{B_\varrho} u(s)\, dx \leqq \int_{B_{(1+\sigma)\varrho}} u(t)\, dx + \delta \int_{B_{2(1+\sigma)\varrho}} u(t)\, dx + \frac{\gamma(\delta)}{\sigma^{\frac{p}{2-p}}}\left(\frac{t}{\varrho^\varkappa}\right)^{\frac{1}{2-p}}. \tag{I.4.13}$$

Let $s \searrow 0$ along $\{\tau'\}$ by keeping t fixed. Then let $t \searrow 0$ along $\{\tau''\}$, to get

$$\int_{B_\varrho} d\mu \leqq \int_{B_{(1+\sigma)\varrho}} d\nu + \delta \int_{B_{2(1+\sigma)\varrho}} d\nu.$$

Since $\sigma, \delta \in (0, 1)$ are arbitrary we conclude that

$$\int_{B_\varrho} d\mu \leqq \int_{B_\varrho} d\nu, \quad \forall\, \varrho > 0.$$

Interchanging the role of $\{\tau'\}$ and $\{\tau''\}$ proves that

$$\int_{B_\varrho} d\mu = \int_{B_\varrho} d\nu, \quad \forall\, \varrho > 0$$

and $\mu \equiv \nu$. □

Chapter II. Uniqueness

II.1. The uniqueness theorem. Let Σ^* denote the subclass of Σ of those non-negative local weak solutions of (I.1.3) in S_T, satisfying

$$\begin{aligned}&\frac{\partial}{\partial t}u_k(x,t)\leqq \gamma u_k(x,t), \qquad a.e.\ (x,t)\in S_T\\ &\text{for some } \gamma=\gamma(N,p,t), \qquad \forall\, k\in \mathbf{R}^+,\end{aligned} \tag{II.1.1}$$

$$\overline{\lim_{k\to\infty}} \iint_{\mathscr{K}\cap[k<u<Ck]} |Du|^p \frac{1}{u}\,dx\,d\tau = 0, \tag{II.1.2}$$

for every compact subset $\mathscr{K}$ of S_T and for all $C \geqq 1$. In the next section we will construct solutions of the Cauchy problem associated with (I.1.3) for initial data $u_0 \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$, $u_0 \geqq 0$, that satisfy both (II.1.1, 1.2); therefore Σ^* is not empty. Corollary I.2.5 suggests that (II.1.2) is *almost* satisfied by all solutions in Σ. It would be of interest to know whether the inclusion

$$\Sigma^* \subset \Sigma$$

is strict.

Theorem II.1.1. *Let* $u_1, u_2 \in \Sigma^*$ *satisfy*

$$(u_1-u_2)(t)\to 0 \quad \textit{in } L^1_{\mathrm{loc}}(\mathbf{R}^N) \quad \textit{as} \quad t\to 0.$$

Then $u_1 = u_2$ *a.e.* S_T.

II.2. Preliminaries. Let $\varrho > 0$, $B_\varrho \equiv \{|x| < \varrho\}$, $0 < s < t \leqq T$ and let $x \to \zeta(x)$ be a non-negative piecewise smooth cutoff function in $B_{(1+\sigma)\varrho}\times(s,t)$, $\sigma \in (0,1)$, that equals one on B_ϱ and is such that

$$|D\zeta| \leqq \frac{1}{\sigma\varrho}. \tag{II.2.1}$$

We assume for the moment that $\sigma \in (0,1)$ has been fixed.

Lemma II.2.1. *Let* $u \in \Sigma^*$. *For all* $0 < s < t \leqq T$, $\forall\, \varrho > 0$, $\forall\, C > 1$,

$$\overline{\lim_{k\to\infty}} \int_s^t \int_{B_\varrho} |u_t|\, \chi[k<u<Ck]\,dx\,d\tau = 0.$$

Proof. Consider (I.1.14) written for u_k replaced by u_{Ck}, $C > 1$, against test functions

$$\eta = \zeta \ln\left(\frac{k}{2w_{k,C}}\right),$$

where $x \to \zeta(x)$ is as in (II.2.1) with $\sigma = 1$ and $\forall\, C > 1$:

$$w_{k,C} \equiv \begin{cases} \frac12 k, & 0 \leqq u \leqq \frac12 k,\\ u, & \frac12 k < u < Ck,\\ Ck, & u \geqq Ck.\end{cases}$$

It follows from these definitions that $\eta \leqq 0$, a.e. S_T and $\eta = 0$, a.e. on the set $[0 < u < \frac{1}{2}k]$.

Standard calculations give $\forall\, 0 < s < t \leqq T$, $\forall\, \varrho > 0$

$$\int_s^t \int_{B_{2\varrho}} \frac{\partial}{\partial t} u_{Ck} \eta \, dx \, d\tau \leqq \int_s^t \int_{B_{2\varrho}} |Du|^p \frac{1}{u} \chi\, [\tfrac{1}{2}k < u < Ck] \, dx \, d\tau$$
$$+ \ln 2C \int_s^t \int_{B_{2\varrho}} |Du_{Ck}|^{p-1} \chi\, [u > \tfrac{1}{2}k] \, |D\zeta| \, dx \, d\tau. \tag{II.2.2}$$

The first integral on the right-hand side of (II.2.2) tends to zero as $k \to \infty$ by virtue of (II.1.2) since $u \in \Sigma^*$.

As for the second integral, we estimate it above (formally) by

$$\frac{\ln 2C}{\varrho} \int_s^t \int_{B_{2\varrho}} |Du_{Ck}|^{p-1} \chi[u > \tfrac{1}{2}k] \, dx \, d\tau$$
$$= \frac{\ln 2C}{\varrho} \int_s^t \int_{B_{2\varrho}} |Du_{Ck}|^{p-1} u^{-\frac{(\alpha+1)(p-1)}{p}} u^{\frac{(\alpha+1)(p-1)}{p}} \chi[u > \tfrac{1}{2}k] \, dx \, d\tau$$
$$\leqq \frac{\ln 2C}{\varrho} \left(\frac{p}{p-1-\alpha}\right)^{p-1} \left(\int_s^t \int_{B_{2\varrho}} \left| Du^{\frac{p-1-\alpha}{p}} \right|^p dx \, d\tau\right)^{\frac{p-1}{p}}$$
$$\times \left(\int_s^t \int_{B_{2\varrho}} u^{(\alpha+1)(p-1)} \chi[u > \tfrac{1}{2}k] \; dx \; d\tau\right)^{\frac{1}{p}}.$$

If we choose $\alpha \in (0, p-1)$ to be so small that

$$(\alpha + 1)(p - 1) \leqq 1,$$

the estimate is rigorous and the rightmost side of (II.2.3) tends to zero as $k \to \infty$, since $u \in L^1_{\text{loc}}(S_T)$ and Lemma I.2.2 holds.

Combining these remarks, we see that (II.2.2) implies

$$\int_s^t \int_{B_{2\varrho}} u_t \ln\left(\frac{k}{2w_{k,C}}\right) \zeta\chi[u_t < 0] \, \chi[\tfrac{1}{2}k < u < Ck] \, dx \, d\tau$$
$$\leqq \int_s^t \int_{B_{2\varrho}} u_t \left|\ln \frac{k}{2w_{k,C}}\right| \zeta\chi[u_t \geqq 0] \, \chi[u > \tfrac{1}{2}k] \, dx \, d\tau + O\left(\frac{1}{k}\right).$$

In view of the definition of $w_{k,C}$ this gives in turn

$$\int_s^t \int_{B_\varrho} |u_t| \, \chi[k < u < Ck] \, dx \, d\tau \leqq \gamma \int_s^t \int_{B_{2\varrho}} u_t \chi[u_t > 0] \, \chi[u > \tfrac{1}{2}k] \, dx \, d\tau + O\left(\frac{1}{k}\right).$$

The last integral is estimated by means of (II.1.1) and the lemma follows. □

Remark II.2.1. The assertion of the lemma is trivial if $u_t \in L^1_{\text{loc}}(S_T)$.

We give next a weak formulation for the difference of two solutions u_1, u_2. First we recall that by Lemma I.1.3, the truncated function

$$u_{2,k} \equiv \begin{cases} u_2 & \text{if } 0 < u_2 < k \\ k & \text{if } u_2 \geqq k \end{cases}$$

is a distributional supersolution of (I.1.1) $\forall\, k > 0$. We write (I.1.14) for $u_{2,k}$ against the testing functions

$$\eta = (\psi - u_1)^+ \zeta^p, \qquad \forall\, \psi \in X_{\text{loc}}(S_T)$$

where ζ is as in (II.2.1). Such a choice is admissible in view of the definition of $X_{\text{loc}}(S_T)$ and the regularity properties (I.1.4)–(I.1.6) of u_i, $i = 1, 2$, modulo a standard density argument. On the other hand the weak formulation of u_1 (I.1.13) holds against the same testing functions. Therefore, setting

$$w \equiv u_1 - u_2, \qquad w_{(k)} \equiv u_1 - u_{2(k)}, \qquad k \in \mathbf{R}^+,$$

we obtain by difference the weak formulation

$$\int_s^t \int_{B_{(1+\sigma)\varrho}} \left\{ \frac{\partial}{\partial t} w_{(k)} (\psi - u_1)_+ \zeta^p + \mathbf{J}_k D(\psi - u_1)_+ \zeta^p \right\} dx\, d\tau$$

$$= -p \int_s^t \int_{B_{(1+\sigma)\varrho}} \mathbf{J}_k (\psi - u_1)_+ \zeta^{p-1} D\zeta\, dx\, d\tau, \qquad \forall\, \psi \in X_{\text{loc}}(S_T) \tag{II.2.4}$$

where

$$\begin{aligned} \mathbf{J}_k &\equiv |Du_1|^{p-2} Du_1 - |Du_{2,k}|^{p-2} Du_{2,k} \\ &= \int_0^1 \frac{d}{d\xi} \{ |D(\xi u_1 + (1-\xi) u_{2,k})|^{p-2} D(\xi u_1 + (1-\xi) u_{2,k}) \} \, d\xi \\ &= \left(\int_0^1 |D(\xi u_1 + (1-\xi) u_{2,k})|^{p-2} d\xi \right) Dw_{(k)} \\ &\quad + (p-2) \left(\int_0^1 |D(\xi u_1 + (1-\xi) u_{2,k})|^{p-4} D(\xi u_1 \right. \\ &\quad \left. + (1-\xi) u_{2,k}) (\xi u_1 + (1-\xi) u_{2,k})_{x_j} d\xi \right) w_{(k),x_j}. \end{aligned} \tag{II.2.5}$$

Here the summation convention is adopted. Set also

$$A_0 \equiv \int_0^1 |D(\xi u_1 + (1-\xi) u_{2,k})|^{p-2} d\xi .$$

Lemma II.2.2. $A_0 \leqq \dfrac{2}{p-1} |Dw_{(k)}|^{p-2}$.

Proof. If $|Du_{2,k}| \geqq |Dw_{(k)}|$, we have

$$|D(\xi u_1 + (1-\xi) u_{2,k})| = |Du_{2,k} + \xi\, Dw_{(k)}| \geqq \big| |Du_{2,k}| - \xi\, |Dw_{(k)}| \big| \geqq (1-\xi)\, |Dw_{(k)}| .$$

Therefore

$$A_0 \leqq \left(\int_0^1 (1-\xi)^{p-2}\,d\xi\right) |Dw_{(k)}|^{p-2} = \frac{1}{p-1}|Dw_{(k)}|^{p-2}.$$

If $|Du_{2,k}| < |Dw_{(k)}|$,

$$\begin{aligned}\int_0^1 &|Du_{2,k} + \xi\, Dw_{(k)}|^{p-2}\,d\xi \\ &\leqq \int_0^1 \big||Du_{2,k}| + \xi\,|Dw_{(k)}|\big|^{p-2}\,d\xi \\ &\leqq \frac{1}{(p-1)\,|Dw_{(k)}|}\Big\{-\int_0^{\xi_0}\frac{d}{d\xi}(|Du_{2,k}| + \xi\,|Dw_{(k)}|)^{p-1}\,d\xi \\ &\quad + \int_{\xi_0}^1 \frac{d}{d\xi}(\xi\,|Dw_{(k)}| + |Du_{2,k}|)^{p-1}\,d\xi\Big\},\end{aligned}$$

where $\xi_0 \in (0, 1)$ is defined by

$$\xi_0 \equiv \frac{|Du_{2,k}|}{|Dw_{(k)}|} \in (0, 1).$$

By direct calculation $A_0 \leqq \dfrac{2}{p-1}|Dw_{(k)}|^{p-2}$. □

From the definitions set forth and Lemma II.2.1 we have

$$\begin{aligned}&\mathbf{J}_k\, Dw_{(k)} \geqq (p-1)\,A_0\,|Dw_{(k)}|^2, \\ &|\mathbf{J}_k| \leqq A_0\,|Dw_{(k)}| \leqq \frac{4}{p-1}|Dw_{(k)}|^{p-1}.\end{aligned} \tag{II.2.6}$$

We will use these inequalities without specific mention.

II.3. An auxiliary proposition.

Proposition II.3.1. *Let* $u_i \in \Sigma^*$, $i = 1, 2$, *satisfy*

$$w(t) \equiv (u_1 - u_2)(t) \to 0 \quad \text{in} \quad L^1_{\mathrm{loc}}(\mathbf{R}^N) \quad \text{as } t \to 0.$$

Then $w \in L^\infty(0, T; L^q_{\mathrm{loc}}(\mathbf{R}^N))$, $\forall\, q \in [1, \infty)$. *Moreover* $\forall\, q \geqq 1$, $\exists\, \gamma = \gamma(N, p, q)$, *such that*

$$\int_{B_\varrho(t)} |w(t)|^q\,dx \leqq \frac{\gamma}{(\sigma\varrho)^p}\int_0^t \int_{B_{(1+\sigma)\varrho}} |w|^{q+(p-2)}\,dx\,d\tau, \tag{II.3.1}$$

for all $\varrho > 0$ *and for all* $\sigma \in (0, 1)$.

The proof is based on an iteration procedure and uses recursive inequalities obtained from (II.2.4) with suitable choices of test functions ψ.

Testing functions in (II.2.4). If h is any positive number let

$$w^+_{(k),h} \equiv (u_1 - u_{2,k})^+_h = \begin{cases} 0 & \text{if } w_{(k)} \leqq 0 \\ w^+_{(k)} & \text{if } w_{(k)} < h \\ h & \text{if } w_{(k)} \geqq h \end{cases} \tag{II.3.2}$$

and in (II.2.4) consider the test function

$$\psi \equiv u_{1,\frac{1}{\varepsilon}} + \frac{1}{\varepsilon}(w^+_{(k),n} + \varepsilon)^a (w^+_{(k),m} + \varepsilon)^b \in X_{\rm loc}(S_T), \tag{II.3.3}$$

where

$$\varepsilon \in (0, 1), \quad a, b > 0, \quad n, m \in \mathbf{N}, \quad n > m + 1.$$

We obtain

$$\int_{\mathbf{R}^N \times \{t\}} w_{(k)}(\psi - u_1)_+ \zeta^p \, dx - \int_{\mathbf{R}^N \times \{s\}} w_{(k)}(\psi - u_1)_+ \zeta^p \, dx$$
$$- \int_s^t \int_{\mathbf{R}^N} w_{(k)} \frac{\partial}{\partial \tau}(\psi - u_1)_+ \zeta^p \, dx \, d\tau + \int_s^t \int_{\mathbf{R}^N} \mathbf{J}_k \, D(\psi - u_1)_+ \zeta^p \, dx \, d\tau$$
$$\leqq -p \int_s^t \int_{\mathbf{R}^N} \mathbf{J}_k \, (\psi - u_1)_+ \zeta^{p-1} \, D\zeta \, dx \, d\tau. \tag{II.3.4}$$

In using ψ as a test function in (II.3.4) we keep in mind that the truncated functions $u_{i,h}$, $i = 1, 2$, $\forall\, h > 0$ are regular in the sense of (I.1.4)–(I.1.6). In particular the first two integrals on the left-hand side of (II.3.4) are well defined $\forall\, 0 < s < t \leqq T$. We will eliminate the parameters ε, k, s, n, m by letting $\varepsilon \to 0$, $k \to \infty$, $s \to 0$, $n, m \to \infty$ in the indicated order.

The limit as $\varepsilon \to 0$. We multiply both sides of (II.3.4) by ε and let $\varepsilon \to 0$, while k, s, n, m remain fixed. From the definition (II.3.3) of ψ it follows that $\forall\, \tau \in (0, T]$ the net $[w_{(k)}(\varepsilon\psi - \varepsilon u_1)_+](\cdot, \tau)$, is equibounded in $L^1_{\rm loc}(\mathbf{R}^N)$, converges to

$$[w_{(k)}(w^+_{(k),n})^a (w^+_{(k),m})^b](\cdot, \tau) \qquad \text{a.e. } B_{2\varrho}$$

and is majorized a.e. $\mathbf{R}^N$ by

$$w_{(k)}(w^+_{(k),n} + 1)^a (w^+_{(k),m} + 1)^b (\cdot, \tau) \in L^1_{\rm loc}(\mathbf{R}^N).$$

Therefore for all $0 < \tau \leqq T$, as $\varepsilon \to 0$

$$\int_{\mathbf{R}^N \times \{\tau\}} w_{(k)}(\psi - u_1)_+ \zeta^p \, dx \quad \to \quad \int_{\mathbf{R}^N \times \{\tau\}} w_{(k)}(w^+_{(k),n})^a (w^+_{(k),m})^b \zeta^p \, dx. \tag{II.3.5}$$

This determines the limit for the first two terms on the left-hand side of (II.3.4). To examine the remaining terms we let $\bar{u}_i$, $i = 1, 2$ be arbitrarily selected but fixed representatives out of the equivalence classes u_i, define $\bar{w}$, $\bar{w}_{(k)}$ accordingly, and let

$$\mathscr{E}_\varepsilon = \left\{(x, \tau) \,\middle|\, \bar{u}_1(x, \tau) < \frac{1}{\varepsilon}\right\}$$

$$\mathscr{F}_\varepsilon = \left\{(x, \tau) \in S_T \,\middle|\, \frac{1}{\varepsilon} \leqq \bar{u}_1(x, \tau) \leqq \frac{1}{\varepsilon} + \frac{1}{\varepsilon}(\bar{w}_{(k),n}(x, \tau) + \varepsilon)^a (\bar{w}_{(k),m}(x, \tau) + \varepsilon)^b\right\}$$

$$\mathscr{G}_\varepsilon = \mathscr{E}_\varepsilon \cup \mathscr{F}_\varepsilon. \tag{II.3.6}$$

Next

$$L_\varepsilon \equiv -\varepsilon \int_s^t \int_{\mathbf{R}^N} w_{(k)} \frac{\partial}{\partial \tau}(\psi - u_1)_+ \zeta^p \, dx \, d\tau$$

$$= -a \int_s^t \int_{\mathbf{R}^N} w^+_{(k),n}(w^+_{(k),n} + \varepsilon)^{a-1} (w^+_{(k),m} + \varepsilon)^b \frac{\partial}{\partial \tau} w^+_{(k),n} \zeta^p \chi(\mathcal{G}_\varepsilon) \, dx \, d\tau$$

$$-b \int_s^t \int_{\mathbf{R}^N} w^+_{(k),m}(w^+_{(k),n} + \varepsilon)^a (w^+_{(k),m} + \varepsilon)^{b-1} \frac{\partial}{\partial \tau} w^+_{(k),m} \zeta^p \chi(\mathcal{G}_\varepsilon) \, dx \, d\tau$$

$$- \int_s^t \int_{\mathbf{R}^N} w_{(k)}(1 - \varepsilon u_1)_t \, \chi(\mathcal{F}_\varepsilon) \, \varsigma^p \, dx \, d\tau$$

$$\equiv L_\varepsilon^{(1)} + L_\varepsilon^{(2)} + L_\varepsilon^{(3)}.$$

We claim that $L_\varepsilon^{(3)} \to 0$ as $\varepsilon \to 0$. Indeed

$$|L_\varepsilon^{(3)}| \leqq \int_s^t \int_{B_{2\varrho}} \varepsilon \, |w_{(k)}| \left| \frac{\partial}{\partial \tau} u_1 \right| \chi(\mathcal{F}_\varepsilon) \, dx \, d\tau.$$

On the set $\mathcal{F}_\varepsilon$ we have

$$\frac{1}{\varepsilon} \leqq u_1 \leqq \frac{1}{\varepsilon} + \frac{1}{\varepsilon}(n+1)^{a+b} \equiv \frac{\gamma}{\varepsilon}$$

$$\varepsilon \, |w_{(k)}| \leqq \gamma \qquad \text{a.e. } \mathcal{F}_\varepsilon.$$

Therefore

$$|L_\varepsilon^{(3)}| \leqq \gamma \int_s^t \int_{B_{2\varrho}} \left| \frac{\partial}{\partial \tau} u_1 \right| \chi \left[\frac{1}{\varepsilon} \leqq u_1 \leqq \frac{\gamma}{\varepsilon} \right] dx \, d\tau,$$

and the assertion follows from Lemma II.2.1.

Since k, n, m are fixed, the integrands in $L_\varepsilon^{(i)}$, $i = 1, 2$ are in $L^1_{\mathrm{loc}}(S_T)$ uniformly in ε. Moreover they have a.e. limits that are in $L^1_{\mathrm{loc}}(S_T)$ and their absolute value is majorized a.e. S_T, uniformly in ε, by functions in $L^1_{\mathrm{loc}}(S_T)$. Therefore as $\varepsilon \to 0$

$$L_\varepsilon^{(1)} + L_\varepsilon^{(2)} \to \mathcal{L} \equiv -\frac{a}{a+1} \int_s^t \int_{\mathbf{R}^N} \frac{\partial}{\partial \tau} (w^+_{(k),n})^{a+1} (w^+_{(k),m})^b \zeta^p \, dx \, d\tau$$
$$- \frac{b}{b+1} \int_s^t \int_{\mathbf{R}^N} (w^+_{(k),n})^a \frac{\partial}{\partial \tau} (w^+_{(k),m})^{b+1} \zeta^p \, dx \, d\tau. \tag{II.3.7}$$

$$(w^+_{(k),n})^a \frac{\partial}{\partial \tau} (w^+_{(k),m})^{b+1} = (w^+_{(k),m})^a \frac{\partial}{\partial \tau} (w^+_{(k),m})^{b+1}$$

$$= \frac{b+1}{a+b+1} \frac{\partial}{\partial \tau} (w^+_{(k),m})^{a+b+1}, \qquad \text{a.e. } S_T,$$

we obtain from (II.3.7)

$$\begin{aligned}\mathscr{L} \equiv & -\frac{a}{a+1} \int_{\mathbf{R}^N \times\{t\}} (w_{(k),n}^{+})^{a+1} (w_{(k),m}^{+})^{b} \zeta^p \, dx \\ & - \frac{b}{(a+1)(a+b+1)} \int_{\mathbf{R}^N \times\{t\}} (w_{(k),m}^{+})^{a+b+1} \zeta^p \, dx \\ & + \frac{a}{a+1} \int_{\mathbf{R}^N \times\{s\}} (w_{(k),n}^{+})^{a+1} (w_{(k),m}^{+})^{b} \zeta^p \, dx . \\ & + \frac{b}{(a+1)(a+b+1)} \int_{\mathbf{R}^N \times\{s\}} (w_{(k),m}^{+})^{a+b+1} \zeta^p \, dx \end{aligned} \tag{II.3.7'}$$

We combine this with (II.3.5) and conclude that the sum of the first three terms on the left-hand side of (II.3.4) has a limit as $\varepsilon \to 0$ that is minorized by

$$\frac{1}{a+b+1} \int_{\mathbf{R}^N \times\{t\}} (w_{(k),m}^{+})^{a+b+1} \zeta^p \, dx - \frac{1}{a+b+1} \int_{\mathbf{R}^N \times\{s\}} w_{(k)}^{+} (w_{(k),n}^{+})^{a+b} \zeta^p \, dx . \tag{II.3.8}$$

We turn to estimate below the lim-inf as $\varepsilon \to 0$ of the last integral on the left-hand side of (II.3.4)

$$\begin{aligned} & \varepsilon \int_s^t \int_{\mathbf{R}^N} \mathbf{J}_k \, D(\psi - u_1)_+ \zeta^p \, dx \, d\tau \\ & = a \int_s^t \int_{\mathbf{R}^N} \mathbf{J}_k \, Dw_{(k),n}^{+} (w_{(k),n}^{+} + \varepsilon)^{a-1} (w_{(k),m}^{+} + \varepsilon)^{b} \zeta^p \chi(\mathscr{G}_\varepsilon) \, dx \, d\tau \\ & \quad + b \int_s^t \int_{\mathbf{R}^N} \mathbf{J}_k \, Dw_{(k),m}^{+} (w_{(k),n}^{+} + \varepsilon)^{a} (w_{(k),m}^{+} + \varepsilon)^{b-1} \zeta^p \chi(\mathscr{G}_\varepsilon) \, dx \, d\tau \\ & \quad + \int_s^t \int_{\mathbf{R}_N} \mathbf{J}_k \, D(1 - \varepsilon u_1) \zeta^p \chi(\mathscr{F}_\varepsilon) \, dx \, d\tau \\ & \geqq a(p-1) \int_s^t \int_{\mathbf{R}_N} A_0 \, |Dw_{(k),n}^{+}|^2 (w_{(k),n}^{+} + \varepsilon)^{a-1} (w_{(k),m}^{+} + \varepsilon)^{b} \zeta^p \chi(\mathscr{G}_\varepsilon) \, dx \, d\tau \\ & \quad - \frac{4\varepsilon}{p-1} \int_s^t \int_{\mathbf{R}^N} A_0 \, |Dw_{(k)}| \, |Du_1| \, \zeta^p \chi(\mathscr{F}_\varepsilon) \, dx \, d\tau \equiv H_\varepsilon^{(1)} + H_\varepsilon^{(2)} . \end{aligned} \tag{II.3.9}$$

By weak lower semicontinuity

$$\varliminf_{\varepsilon \to 0} H_\varepsilon^{(1)} \geqq a(p-1) \int_s^t \int_{\mathbf{R}^N} A_0 \, |Dw_{(k),n}^{+}|^2 (w_{(k),n}^{+})^{a-1} (w_{(k),m}^{+})^{b} \zeta^p \, dx \, d\tau . \tag{II.3.10}$$

We claim that $H_\varepsilon^{(2)} \to 0$ as $\varepsilon \to 0$. Using Lemma II.2.2, we have

$$\begin{aligned} |H_\varepsilon^{(2)}| & \leqq \varepsilon C(N, p) \int_s^t \int_{B_{2\varrho}} |Du_1 - Du_{2,k}|^{p-1} \, |Du_1| \, \chi(\mathscr{F}_\varepsilon) \, dx \, d\tau \\ & \leqq C \int_s^t \int_{B_{2\varrho}} |Du_1|^p \, \varepsilon \chi(\mathscr{F}_\varepsilon) dx \, d\tau + C\varepsilon \int_s^t \int_{B_{2\varrho}} |Du_{2,k}|^p \, \chi(\mathscr{F}_\varepsilon) \, dx \, d\tau \equiv H_{\varepsilon,1}^{(2)} + H_{\varepsilon,2}^{(2)} . \end{aligned}$$

Since $|Du_{2,k}| \in L^p_{\text{loc}}(S_T)$ the second term tends to zero as $\varepsilon \to 0$. As for $H^{(2)}_{\varepsilon,1}$ write

$$H^{(2)}_{\varepsilon,1} \leqq C \int_s^t \int_{B_{2\varrho}} |Du_1|^p \frac{1}{u_1} \varepsilon u_1 \chi \left(\frac{1}{\varepsilon} \leqq u_1 < \frac{\gamma}{\varepsilon}\right) dx\, d\tau$$

where $\gamma = 1 + (n+1)^a (m+1)^b$. This implies

$$H^{(2)}_{\varepsilon,1} \leqq C(p) \int_s^t \int_{B_{2\varrho}} \left|Du_1^{\frac{p-1}{p}}\right|^p \chi\left(\frac{1}{\varepsilon} \leqq u_1 \leqq \frac{\gamma}{\varepsilon}\right) dx\, d\tau \to 0 \quad \text{as } \varepsilon \to 0,$$

since $u_1 \in \Sigma^*$.

We finally estimate above the lim-sup as $\varepsilon \to 0$ of the integral on the right-hand side of (II.3.4). Using the definition (II.3.3) of ψ and (II.2.6), we see that

$$\left| p \int_s^t \int_{\mathbf{R}^N} \mathbf{J}_k(\psi - u_1)_+ \zeta^{p-1} D\zeta\, dx\, d\tau \right|$$

$$\leqq \gamma \int_s^t \int_{\mathbf{R}^N} A_0 |Dw_{(k)}| (w^+_{(k),n} + \varepsilon)^a (w^+_{(k),m} + \varepsilon)^b \zeta^{p-1} |D\zeta| \chi(\mathscr{G}_\varepsilon)\, dx\, d\tau$$

$$+ \gamma \int_s^t \int_{\mathbf{R}^N} A_0 |Dw_{(k)}| \varepsilon u_1 \chi(\mathscr{F}_\varepsilon) \zeta^{p-1} |D\zeta|\, dx\, d\tau. \tag{II.3.11}$$

The last integral tends to zero as $\varepsilon \to 0$. Indeed it can be majorized by

$$C(p)\,[1 + (n+1)^a (m+1)^b] \int_s^t \int_{B_{2\varrho}} |Dw_{(k)}|^{p-1} \chi(\mathscr{F}_\varepsilon)\, dx\, d\tau$$

$$\leqq \bar{\gamma} \int_s^t \int_{B_{2\varrho}} |Du_1|^{p-1} \chi\left[u_1 \geqq \frac{1}{\varepsilon}\right] dx\, d\tau + \bar{\gamma} \int_s^t \int_{B_{2\varrho}} |Du_{2,k}|^{p-1} \chi\left[u_1 \leqq \frac{1}{\varepsilon}\right] dx\, d\tau. \tag{II.3.12}$$

The second integral on the right-hand side of (II.3.12) tends to zero as $\varepsilon \to 0$, since $u_1 \in L^1_{\text{loc}}(S_T)$. As for the first integral, let $\alpha_0 \in (0, p-1)$ be so small that $(\alpha_0 + 1)(p-1) < 1$. Then

$$\bar{\gamma} \int_s^t \int_{B_{2\varrho}} |Du_1|^{p-1} \chi\left[u_1 > \frac{1}{\varepsilon}\right] dx\, d\tau$$

$$= \bar{\gamma} \int_s^t \int_{B_{2\varrho}} |Du_1|^{p-1} u_1^{-\frac{(\alpha_0+1)(p-1)}{p}} u_1^{\frac{(\alpha_0+1)(p-1)}{p}} \chi\left[u_1 > \frac{1}{\varepsilon}\right] dx\, d\tau$$

$$= \bar{\bar{\gamma}} \int_s^t \int_{B_{2\varrho}} \left|Du_1^{\frac{p-1-\alpha_0}{p}}\right|^{p-1} u_1^{\frac{(\alpha_0+1)(p-1)}{p}} \chi\left[u_1 > \frac{1}{\varepsilon}\right] dx\, d\tau$$

$$\leqq \bar{\bar{\gamma}} \left\| Du_1^{\frac{p-1-\alpha_0}{p}} \right\|^{p-1}_{p, B_{2\varrho} \times (s,t)} \left(\int_s^t \int_{B_{2\varrho}} u_1^{(\alpha_0+1)(p-1)} \chi\left[u_1 > \frac{1}{\varepsilon}\right] dx\, d\tau \right)^{\frac{1}{p}} \to 0 \quad \text{as } \varepsilon \to 0.$$

We examine the lim-sup as $\varepsilon \to 0$ of the first integral on the right-hand side of (II.3.11). The numbers $k \in \mathbf{R}^+$, $n \in \mathbf{N}$ being fixed, if ε is small enough

$$\{(x, \tau) \in S_T \mid 0 \leqq \bar{w}_{(k)}(x, \tau) \leqq n\} \subset \mathscr{G}_\varepsilon .$$

Moreover since, $w^+_{(k),n} \in L^p_{\mathrm{loc}}(0, T; W^{1,p}_{\mathrm{loc}}(\mathbf{R}^N))$,

$$|Dw^+_{(k),n}| = 0 \quad \text{a.e. on } \{(x, \tau) \in S_T \mid \bar{w}_{(k)}(x, \tau) > n\}.$$

We write

$$\begin{aligned}
&\int_s^t \int_{\mathbf{R}^N} A_0 \, |Dw^+_{(k)}| \, (w^+_{(k),n} + \varepsilon)^a \, (w^-_{(k),m} + \varepsilon)^b \, \zeta^{p-1} \, |D\zeta| \, \chi(\mathscr{G}_\varepsilon) \, dx \, d\tau \\
&\quad = \int_s^t \int_{\mathbf{R}^N} A_0 \, |Dw^+_{(k),n}| \, (w^+_{(k),n} + \varepsilon)^a \, (w^-_{(k),m} + \varepsilon)^b \, \zeta^{p-1} \, |D\zeta| \, dx \, d\tau \\
&\qquad + \int_s^t \int_{\mathbf{R}^N} A_0 \, |Dw^+_{(k)}| \, (n + \varepsilon)^a \, (m + \varepsilon)^b \, \zeta^{p-1} \, |D\zeta| \, \chi[w^+_{(k)} > n] \, \chi(\mathscr{G}_\varepsilon) \, dx \, d\tau \\
&\qquad + \int_s^t \int_{\mathbf{R}^N} A_0 \, |Dw^+_{(k)}| \, \varepsilon^{a+b} \, \zeta^{p-1} \, |D\zeta| \, \chi(\mathscr{G}_\varepsilon) \, dx \, d\tau \\
&\quad = K^{(1)}_\varepsilon + K^{(2)}_\varepsilon + K^{(3)}_\varepsilon . \qquad \text{(II.3.13)}
\end{aligned}$$

As for $K^{(1)}_\varepsilon$ the integrand tends to $A_0 \, |Dw^+_{(k),n}| \, (w^+_{(k),n})^a \, (w^-_{(k),m})^b \, \zeta^{p-1} \, |D\zeta|$ a.e. $B_{2\varrho} \times (s, t)$ in a decreasing way. Therefore

$$K^{(1)}_\varepsilon \to \int_s^t \int_{\mathbf{R}^N} A_0 \, |Dw^+_{(k),n}| \, (w^+_{(k),n})^a \, (w^-_{(k),m})^b \, \zeta^{p-1} \, |D\zeta| \, dx \, d\tau .$$

The last integral tends to zero as $\varepsilon \to 0$. Indeed

$$K^{(3)}_\varepsilon \leqq C(p) \frac{\varepsilon^{a+b}}{\sigma\varrho} \int_s^t \int_{B_{2\varrho}} (|Du_{1,k}|^{p-1} + |Du_{2,k}|^{p-1}) \, dx \, d\tau \to 0 \qquad \text{as } \varepsilon \to 0 .$$

The operation $Dw^+_{(k)}$ coincides with the weak derivative of $w^+_{(k)}$ only on the sets $\mathscr{A}^{(l)}$ where $w^+_{(k)}$ is bounded by a constant l, i.e.

$$Dw^+_{(k)} \, \chi(\mathscr{A}^{(l)}) = Dw^+_{(k),(l)} .$$

Since $Dw^+_{(k)}$ is not well defined a.e. in the whole strip, S_T, we estimate $K^{(2)}_\varepsilon$ above as follows:

$$\begin{aligned}
K^{(2)}_\varepsilon &\leqq \gamma \frac{(m+1)^b}{\sigma\varrho} \int_s^t \int_{B_{2\varrho}} |Du_1 - Du_{2,k}|^{p-1} \, u_1^a \chi[u_1 > n + u_{2,k}] \, \chi(\mathscr{G}_\varepsilon) \, dx \, d\tau \\
&\leqq \gamma \frac{(m+1)^b}{\sigma\varrho} \int_s^t \int_{B_{2\varrho}} |Du_1|^{p-1} \, u_1^a \chi[u_1 > n] \, \chi(\mathscr{G}_\varepsilon) \, dx \, d\tau \\
&\quad + \gamma \frac{(m+1)^b}{\sigma\varrho} \int_s^t \int_{B_{2\varrho}} |Du_{2,k}|^{p-1} \, u_1^a \chi[u_1 > n + u_{2,k}] \, \chi(\mathscr{G}_\varepsilon) \, dx \, d\tau .
\end{aligned}$$

If $\alpha_0 \in (0, p-1)$, write

$$\int_s^t \int_{B_{2\varrho}} |Du_1|^{p-1} u_1^a \chi[u_1 > n]\, \chi(\mathscr{G}_\varepsilon)\, dx\, d\tau$$

$$= \int_s^t \int_{B_{2\varrho}} |Du_1|^{p-1} u_1^{-\frac{(\alpha_0+1)(p-1)}{p}} u_1^{\frac{(\alpha_0+1)(p-1)+ap}{p}} \chi[u_1 > n]\, \chi(\mathscr{G}_\varepsilon)\, dx\, d\tau$$

$$\leqq \gamma \left(\int_s^t \int_{B_{2\varrho}} \left| Du_1^{\frac{p-1-\alpha_0}{p}} \right|^p dx\, d\tau \right)^{\frac{p-1}{p}} \left(\int_s^t \int_{B_{2\varrho}} u_1^{(\alpha_0+1)(p-1)+ap} \chi[u_1 > n]\, dx\, d\tau \right)^{\frac{1}{p}}.$$

Choose α_0 and $a > 0$ so small that

$$(\alpha_0 + 1)(p-1) + ap \leqq 1. \tag{II.3.14}$$

Then $u^{(\alpha_0+1)(p-1)+ap} \in L^1_{\mathrm{loc}}(S_T)$ and $\forall\, \varepsilon \in (0, 1)$

$$\int_s^t \int_{B_{2\varrho}} |Du_1|^{p-1} u_1^a \chi[u_1 > n]\, \chi(\mathscr{G}_\varepsilon)\, dx\, d\tau \leqq O\left(\frac{1}{n}\right).$$

Analogously

$$\int_s^t \int_{B_{2\varrho}} |Du_{2,k}|^{p-1} u_1^a\, \chi[u_1 > n + u_{2,k}]\, \chi(\mathscr{G}_\varepsilon)\, dx\, d\tau$$

$$\leqq \gamma \left(\int_s^t \int_{B_{2\varrho}} \left| Du^{\frac{p-1-\alpha_0}{p}} \right|^p dx\, d\tau \right)^{\frac{p-1}{p}} \left(\int_s^t \int_{B_{2\varrho}} u_1^{ap} u_2^{(\alpha_0+1)(p-1)} \chi[u_1 > n + u_{2,k}]\, dx\, d\tau \right)^{\frac{p-1}{p}}$$

$$\leqq \gamma \left(\int_s^t \int_{B_{2\varrho}} \left| Du^{\frac{p-1-\alpha_0}{p}} \right|^p dx\, d\tau \right)^{\frac{p-1}{p}} \left(\int_s^t \int_{B_{2\varrho}} u_1^{(\alpha_0+1)(p-1)+ap} \chi[u_1 > n]\, dx\, d\tau \right)^{\frac{1}{p}}$$

$$= O\left(\frac{1}{n}\right).$$

We conclude that

$$\overline{\lim_{\varepsilon \to 0}}\, K_\varepsilon^{(2)} \leqq \gamma(m+1)\, O\left(\frac{1}{n}\right), \quad k \in \mathbf{R}, \tag{II.3.15}$$

provided α_0 and $a > 0$ are so chosen that (II.3.14) holds. Combining these estimates and limiting processes as parts of (II.3.4), we obtain

$$\begin{aligned} &\frac{1}{a+b+1} \int_{\mathbf{R}^N \times \{t\}} (w^+_{(k),m})^{a+b+1} \zeta^p\, dx \\ &\qquad + a(p-1) \int_s^t \int_{\mathbf{R}^N} A_0\, |Dw^+_{(k),n}|^2 (w^+_{(k),n})^{a-1} (w^+_{(k),m})^b \zeta^p\, dx\, d\tau \\ &\leqq \frac{1}{a+b+1} \int_{\mathbf{R}^N \times \{s\}} w_k^+ (w^+_{(k),n})^{a+b} \zeta^p\, dx \\ &\qquad + \frac{C}{\sigma\varrho} \int_s^t \int_{\mathbf{R}^N} A_0\, |Dw^+_{(k),n}|\, (w^+_{(k),n})^a (w^+_{(k),m})^b \zeta^{p-1}\, dx\, d\tau + \gamma(m+1)^b\, O\left(\frac{1}{n}\right). \end{aligned} \tag{II.3.16}$$

Remark. We remark that the term involving $O\left(\frac{1}{n}\right)$ holds $\forall\, k>0$, $\forall\, 0<s<t\leqq T$.

The limits as $k\to\infty$ and $s\to 0$. If $n\in\mathbf{N}$ and $k>0$ are fixed, we let $\overline{w}^+_{(k),n}$ and $\overline{D}w^+_{(k),n}$ be arbitrarily selected but fixed representatives of the equivalence classes $w^+_{(k),n}$ and $Dw^+_{(k),n}$ and introduce the sets

$$\mathscr{E}_1\equiv\left\{(x,\tau)\in S_T \mid \overline{w}^+_{(k),n}(x,\tau)\leqq\frac{a(p-1)\,\sigma\varrho}{4C}\,\zeta(x)\,|\overline{D}w^+_{(k),n}|\,(x,\tau)\right\}$$

$$\mathscr{E}_2\equiv\left\{(x,\tau)\in S_T \mid |\overline{D}w^+_{(k),n}|\,(x,\tau)\leqq\frac{4C}{a(p-1)\sigma\varrho\zeta(x)}\,\overline{w}^+_{(k),n}(x,\tau)\right\},$$

where C is the constant appearing in the last integral on the right-hand side of (II.3.16). This integral is estimated as follows:

$$\begin{aligned}
&\frac{C}{\sigma\varrho}\int_s^t\int_{\mathbf{R}^N} A_0\,|Dw^+_{(k),n}|\,(w^+_{(k),n})^a\,(w^+_{(k),m})^b\,\zeta^{p-1}\,dx\,d\tau\\
&\leqq\frac{C}{\sigma\varrho}\int_s^t\int_{\mathbf{R}^N} A_0\,|Dw^+_{(k),n}|\,(w^+_{(k),n})^a\,(w^+_{(k),m})^b\,\zeta^{p-1}\chi(\mathscr{E}_1)\,dx\,d\tau\\
&\quad+\frac{2C}{(p-1)\,\sigma\varrho}\int_s^t\int_{\mathbf{R}^N} |Dw^+_{(k),n}|^{p-1}\,(w^+_{(k),n})^a\,(w^+_{(k),m})^b\,\zeta^{p-1}\chi(\mathscr{E}_2)\,dx\,d\tau\\
&\leqq\frac{a(p-1)}{2}\int_s^t\int_{\mathbf{R}^N} A_0\,|Dw^+_{(k),n}|^2\,(w^+_{(k),n})^{a-1}\,(w^+_{(k),m})^b\,\zeta^p\,dx\,d\tau\\
&\quad+\frac{4^pC^p}{a^{p-1}(p-1)^p\,(\sigma\varrho)^p}\int_s^t\int_{\mathbf{R}^N}(w^+_{(k),n})^{p-1+a}\,(w^+_{(k),m})^b\,dx\,d\tau.
\end{aligned}$$

We carry this estimate in (II.3.16), move the integral involving $|Dw^+_{(k),n}|^2$ onto the left-hand side and drop the resulting nonnegative term to obtain

$$\begin{aligned}
&\frac{1}{a+b+1}\int_{\mathbf{R}^N\times\{t\}}(w^+_{(k),m})^{a+b+1}\,\zeta^p\,dx\\
&\leqq\frac{1}{a+b+1}\int_{\mathbf{R}^N\times\{s\}} w^+_k\,(w^+_{(k),n})^{a+b}\,\zeta^p\,dx\\
&\quad+\frac{\gamma(p)}{(\sigma\varrho)^p}\int_s^t\int_{B_{(1+\sigma)\varrho}}(w^+_{(k),n})^{p-1+a}\,(w^+_{(k),m})^b\,dx\,d\tau+\gamma(m+1)^b\,O\left(\frac{1}{n}\right).
\end{aligned}\tag{II.3.17}$$

We now let $k\to\infty$ while $s>0$, n, m, $\in\mathbf{N}$ remain fixed. Since $w^+_k\searrow w^+$, we may pass to the limit under the integrals in (II.3.17) and obtain the same integral inequality written for w^+. In particular the first integral on the right-hand side takes the form

$$\frac{1}{a+b+1}\int_{\mathbf{R}^N\times\{s\}} w^+(w^+_n)^{a+b}\,\zeta^p\,dx.\tag{II.3.18}$$

As $s \to 0$, the integral in (II.3.18) tends to zero since it can be majorized by

$$\frac{n^{a+b}}{a+b+1} \int_{\mathbf{R}^N \times \{s\}} w^+ \zeta^p \, dx \to 0 \quad \text{as } s \to 0.$$

These limiting processes yield

$$\int_{\mathbf{R}^N \times \{t\}} (w_m^+)^{a+b+1} \zeta^p \, dx$$
$$\leq \frac{\gamma(p)(a+b+1)}{(\sigma\varrho)^p} \int_0^t \int_{B_{(1+\sigma)\varrho}} (w_n^+)^{p-1+a} (w_m^+)^b \, dx \, d\tau$$
$$+ \gamma(a+b+1)(m+1)^b \, O\left(\frac{1}{n}\right). \tag{II.3.19}$$

Proof of Proposition II.3.1. We let $n \to \infty$ in (II.3.19), while $m \in \mathbf{N}$ remains fixed. The integrand in the last integral tends to $(w^+)^{p-1+a}(w_m^+)^b$ a.e. in $B_{(1+\sigma)\varrho} \times (0, t)$ in an increasing fashion. Moreover if a is so small that

$$p - 1 + a \in (0, 1), \tag{II.3.20}$$

it is dominated, uniformly in n, by the function

$$(w^+)^{p-1+a} (w_m^+)^b \in L^1_{\text{loc}}(S_T).$$

The limit process gives

$$\int_{\mathbf{R}^N \times \{t\}} (w_m^+)^{a+b+1} \zeta^p \, dx \leqq \frac{\gamma(p)(a+b+1)}{(\sigma\varrho)^p} \int_0^t \int_{B_{(1+\sigma)\varrho}} (w^+)^{p-1+a} (w_m^+)^b \, dx \, d\tau. \tag{II.3.21}$$

This inequality holds true $\forall\, m \in \mathbf{N}$, $\forall\, b \geqq 0$, $\forall\, \sigma \in (0, 1)$, $\forall\, \varrho > 0$. The positive number a is fixed and satisfies the restrictions (II.3.14) and (II.3.20).

The sequence $\{w_m^+\}$ increases to w^+ a.e. S_T. Therefore we may pass to the limit as $m \to \infty$ under the integrals in (II.3.21) for those $b \geqq 0$ for which

$$(w^+)^{p-1+a+b} \in L^1_{\text{loc}}(S_T).$$

If $b_i \geqq 0$ is one such b, letting $m \to \infty$ we find that

$$(w^+)^{a+b_i+1} \in L^1_{\text{loc}}(S_T),$$

which implies that

$$(w^+)^{p-1+a+b_{i+1}} \in L^1_{\text{loc}}(S_T), \qquad b_{i+1} = b_i + 2 - p > b_i.$$

Let $b_0 \geqq 0$ be defined by $p - 1 + a + b_0 = 1$. Then the previous remarks show that

$$(w^+)^{p-1+a+b_0+i(2-p)} \equiv (w^+)^{1+i(2-p)} \in L^1_{\text{loc}}(S_T), \qquad i = 0, 1, 2, \ldots.$$

Interchanging the role of u_1 and u_2 proves the Proposition. $\square$

II.4. Proof of Theorem II.1.1. From (II.3.1) by Hölder's inequality, since $p \in (1, 2)$

$$\int_{B_\varrho} |w(t)|^q\, dx \leqq \frac{\gamma}{(\sigma\varrho)^p} \int_0^t \left(\int_{B_{(1+\sigma)\varrho}} |w(\tau)|^q\, dx \right)^{\frac{q+(p-2)}{q}} \varrho^{\frac{N(2-p)}{q}}\, d\tau$$

$$\leqq \frac{\gamma}{\sigma^p} \left(\sup_{0<\tau<t} \int_{B_{(1+\sigma)\varrho}} |w(\tau)|^q\, dx \right)^{1-\frac{2-p}{q}} t\varrho^{\frac{N(2-p)}{q}-p}. \tag{II.4.1}$$

Let $\varrho > 0$ be fixed and for $n = 1, 2, \ldots$ define

$$\varrho_n = \left(\sum_{i=0}^n 2^{-i} \right) \varrho, \quad B_n = B_{\varrho_n}, \quad \sigma_n = 2^{-(n+1)p},$$

$$\Lambda_n = \sup_{0<\tau\leqq t} \int_{B_n} |w(\tau)|^q\, dx,$$

and rewrite (II.4.1) over B_n and B_{n+1} to obtain

$$\Lambda_n \leqq \gamma\, 2^{np} \left(t/\varrho^{\frac{N(p-2)+pq}{p}} \right) (\Lambda_{n+1})^{1-\frac{2-p}{q}}. \tag{II.4.2}$$

Let $\delta_0 \in (0, 1)$. Then since $1 < p < 2$

$$\Lambda_n \leqq \delta_0 \Lambda_{n+1} + \gamma(N, p, \delta_0)\, (2^p)^{\frac{nq}{2-p}} \left(t/\varrho^{\frac{N(p-2)+pq}{q}} \right)^{\frac{q}{2-p}}, \tag{II.4.3}$$

and iteration of (II.4.3) gives

$$\Lambda_0 \leqq \delta_0^n \Lambda_{n+1} + \gamma(N, p, \delta_0) \left(t/\varrho^{\frac{N(p-2)+pq}{q}} \right)^{\frac{q}{2-p}} \sum_{i=0}^{n+1} \left(\delta_0\, 2^{\frac{pq}{2-p}} \right)^i.$$

If δ_0 is chosen so that $\delta_0\, 2^{\frac{pq}{2-p}} = \frac{1}{2}$, the last sum is majorized by a convergent series. Letting $n \to \infty$ proves that for every $q \in [1, \infty)$ there exists a constant $\gamma = \gamma(N, p, q)$, independent of ϱ, such that for all $t \in (0, T)$

$$\int_{B_\varrho} |w(t)|^q\, dx \leqq \gamma \left(t/\varrho^{\frac{N(p-2)+pq}{q}} \right)^{\frac{q}{2-p}}. \tag{II.4.4}$$

To prove the theorem we choose q so large that

$$\frac{N(p-2)+pq}{q} > 0$$

and then, such a $q \in [1, \infty)$ being fixed, we let $\varrho \to \infty$ in (II.4.4). □

Chapter III. The Cauchy Problem

III.1. Introduction. We will construct a solution to the Cauchy problem

$$\begin{aligned} u_t - \operatorname{div}(|Du|^{p-2} Du) &= 0 \quad \text{in } S_T, \quad 1 < p < 2 \\ u(\cdot, 0) &= u_0(\cdot) \end{aligned} \tag{III.1.1}$$

under the assumptions

$$u_0 \geqq 0, \quad u_0 \in L^1_{\text{loc}}(\mathbf{R}^N). \tag{III.1.2}$$

In particular no growth condition is imposed on $x \to u_0(x)$ as $|x| \to \infty$.

The solution will be in the class Σ and it is meant in the weak sense made precise in Section I.1. In fact, it will be in Σ^* (see Section II.1) and therefore it will be unique.

If $u_0 \in L^1(\mathbf{R}^N)$, the existence of a unique semigroup solution is known (see [5, 15]), even without sign restriction on u_0. However it is not clear to us what is the a.e. S_T meaning of Du, if any, for such semigroup solutions, nor how the equation has to be interpreted. For this reason we have chosen a simple approach based on *a priori* estimates in spaces of integrable functions.

Let $\{u_n\}$ be a sequence of sufficiently smooth approximating solutions. There are two main difficulties in showing existence. The first is an *a priori* bound of the type

$$u_n \in L^1_{\text{loc}}(S_T), \quad \text{uniformly in } n;$$

and the second is to identify the weak limit in $L^1_{\text{loc}}(S_T)$ of the sequence of nonlinear terms $\{|Du_n|^{p-2} Du_n\}$.

The first is overcome by a Harnack type estimate whose proof requires that $u_0 \geqq 0$. To deal with the second we need some time-regularity of the approximating solutions, which is supplied by the Bènilan-Crandall regularizing effect [5] if $u_0 \geqq 0$.

It would be of interest to know whether solutions to (III.1.1) exist if $u_0 \in L^1_{\text{loc}}(\mathbf{R}^N)$ with no restriction of sign, or even in the case of $u_0 \in L^1(\mathbf{R}^N)$ if the semigroup solution can be given a concrete a.e. meaning. We establish existence of a (unique) solution to (III.1.1) in Sections III.2–4.

In the remaining sections we prove some properties of these solutions, such as how $u_0 \in L^q_{\text{loc}}(\mathbf{R}^N)$, $q > 1$ reflects on the regularity of u (§ III.6–III.7).

III.2. Existence of solutions ($u_0 \in C_0^\infty(\mathbf{R}^N)$). As a starting point we consider solutions of (III.1.1) for smooth initial data.

Let $n_0 \in \mathbf{N}$ be so large that $\operatorname{supp} u_0 \subset B_{n_0} \equiv \{|x| < n_0\}$, and $\forall\, n \geqq n_0$ consider the boundary value problems

$$\begin{gathered} \frac{\partial}{\partial t} u_n - \operatorname{div}(|Du_n|^{p-2} Du_n) = 0, \quad 1 < p < 2, \\ \text{in } Q_n \equiv B_n \times (0, T), \\ u_n(x, t) = 0, \quad |x| = n, \quad t \in (0, T), \\ u_n(\cdot, 0) = u_0(\cdot). \end{gathered} \tag{III.2.1}$$

Existence and uniqueness of a weak solution to (III.2.1) is established for example in (Lions [21]) by means of Galerkin approximations (see also [19]).

The solutions u_n, $n = n_0, n_0 + 1, \ldots$ satisfy

$$u_n \in W^{1,2}(0, T; L^2(B_n)) \cap L^\infty(0, T; W^{1,p}(B_n))$$

with the *a priori* estimates

$$\left\|\frac{\partial}{\partial t} u_n\right\|^2_{2,Q_n} + \operatorname*{ess\,sup}_{0<t<T} \|Du_n\|^p_{p,B_n} \leqq \|Du_0\|^p_{p,\mathbf{R}^N}, \tag{III.2.2}$$

$$\|u_n\|_{q,Q_n} \leqq \|u_0\|_{q,\mathbf{R}^N}, \qquad \forall\, q \in [1, \infty),$$

and the equation is interpreted in the following sense:

$$\forall\, n \geqq n_0, \qquad \forall\, 0 < R \leqq n, \qquad \forall\, 0 < t \leqq T,$$

$$\forall\, \varphi \in W^{1,2}(0, T; L^2(B_R)) \cap L^p(0, T; \overset{\circ}{W}{}^{1,p}(B_R)),$$

$$\int_{B_R} u_n(x, t)\, \varphi(x, t)\, dx + \int_0^t \int_{B_R} \{-u_n\varphi_t + |Du_n|^{p-2}\, Du_n\, D\varphi\}\, dx\, d\tau$$

$$= \int_{B_R} u_0(x)\, \varphi(x, 0)\, dx. \tag{III.2.3}$$

By the maximum principle $\forall\, n \geqq n_0$

$$u_n \geqq 0, \qquad \|u_n\|_{\infty,Q_n} \leqq \|u_0\|_{\infty,\mathbf{R}^N}. \tag{III.2.4}$$

By the comparison principle applied to the pair u_{n+1}, u_n over Q_n we have

$$u_{n+1} \geqq u_n \qquad \text{a.e. } Q_n. \tag{III.2.5}$$

We view the u_n as defined in the whole strip S_T by extending them to be zero on $|x| \geqq n$, $n = n_0$, $n_0 + 1, \ldots$. It follows that

$$\{u_n\} \nearrow u \in L^2(S_T) \cap L^\infty(S_T), \qquad \text{a.e. } S_T \tag{III.2.6}$$

and if $Q_R \equiv B_R \times (0, T)$, $\forall\, R > 0$

$$u_{n,t} \rightharpoonup u_t \text{ in } L^2(Q_R), \qquad Du_n \rightharpoonup Du, \quad \text{in } L^p(Q_R), \tag{III.2.7}$$

$$|Du_n|^{p-2}\, Du_n \rightharpoonup \vec{\chi} \qquad \text{in } L^{\frac{p}{p-1}}(Q_R). \tag{III.2.8}$$

Letting $n \to \infty$ in (III.2.3)

$$\int_{B_R} u(x,t)\, \varphi(x, t)\, dx + \int_0^t \int_{B_R} \{-u\varphi_\tau + \vec{\chi}\, D\varphi\}\, dx\, d\tau = \int_{B_R} u_0\varphi(x, 0)\, dx, \tag{III.2.9}$$

for all $R > 0$ and $\forall\, \varphi$ specified in (III.2.3).

Fix $R > 0$ and $\forall\, n > R$ in (III.2.3) choose

$$\varphi = u_n\zeta$$

where $x \to \zeta(x)$ is a non-negative piecewise smooth function vanishing for $|x| \geqq R$. Standard calculations give

$$\lim_{n\to\infty} \int_0^t \int_{B_R} |Du_n|^p\, \zeta\, dx\, d\tau = \tfrac{1}{2} \int_{B_R} (u_0^2 - u^2(t))\, \zeta\, dx - \iint_{Q_R} (\vec{\chi}\, D\zeta)\, u\, dx\, d\tau.$$

On the other hand, by taking $\varphi = u\zeta$ in (III.2.9) we obtain

$$\frac{1}{2}\int_{B_R} (u_0^2 - u^2(t))\,\zeta\,dx - \iint_{Q_R} (\vec{\chi}\,D\zeta)\,u\,dx\,d\tau = \int_0^T\int_{B_R} (\vec{\chi}\,Du)\,\zeta\,dx\,d\tau. \tag{III.2.10}$$

Therefore

$$\lim_{n\to\infty}\iint_{Q_R} |Du_n|^p\,\zeta\,dx\,d\tau = \iint_{Q_R} \vec{\chi}\cdot Du\,\zeta\,dx\,d\tau. \tag{III.2.11}$$

If $\zeta \geqq 0$, $\forall\, n \geqq n_0$, $\forall\, \varphi \in C_0^\infty(S_T)$

$$\iint_{Q_R} (|Du_n|^{p-2}\,Du_n - |D\varphi|^{p-2}\,D\varphi)\,(Du_n - D\varphi)\,\zeta\,dx\,d\tau \geqq 0.$$

Expanding this expression, letting $n\to\infty$ and using (III.2.11), gives $\forall\,\varphi \in C_0^\infty(S_T)$

$$\iint_{Q_R} (\vec{\chi} - |D\varphi|^{p-2}\,D\varphi)\,(Du - D\varphi)\,\zeta\,dx\,d\tau \geqq 0.$$

Hence $\vec{\chi} = |Du|^{p-2}\,Du$ by MINTY's lemma [24]. We conclude that if $u_0 \in C_0^\infty(\mathbf{R}^N)$ the problem (III.1.1) has a solution u in the sense of the integral identity (III.2.10) satisfying

$$u_t \in L^2(0, T; L^2(\mathbf{R}^N)),$$
$$|Du| \in L^p(0, T; L^p(\mathbf{R}^N)),$$
$$u \in L^\infty(S_T).$$

Moreover by the results of [13], $u \in C^\alpha(\overline{S}_T)$ where $\alpha \in (0, 1)$ depends only upon N, p and the Hölder norm $\|u_0\|_{C^\beta(\overline{B}_{n_0})}$, for some $\beta \in (0, 1)$.

This construction procedure was also used in SABININA [28] in the framework of the porous-medium equation.

III.3. *A priori* estimates. We let $(x, t) \to u(x, t)$ be the unique solution of (III.1.1) with $u_0 \geqq 0$ and $u_0 \in C_0^\infty(\mathbf{R}^N)$. In view of (III.2.1) such a u satisfies the equation in the sense of (III.2.10) with $\vec{\chi} \equiv |Du|^{p-2}\,Du$, or equivalently in the sense of § I.1, so that the estimates of Chapter I are valid for it.

Lemma III. 3.1. *$\exists\,\gamma$ depending only upon N and p such that* $\forall\, 0 < t \leqq T$, $\forall\, R > 0$,

$$\sup_{0<\tau<t}\int_{B_R} u(x, \tau)\,dx \leqq \gamma\left\{\int_{B_{2R}} u_0\,dx + \left(\frac{t}{R^\varkappa}\right)^{\frac{1}{2-p}}\right\}, \tag{III.3.1}$$

where

$$\varkappa = N(p-2) + p. \tag{III.3.2}$$

Proof. For $n = 0, 1, 2, \ldots,$ let

$$R_n = R\sum_{i=0}^{n} 2^{-i}, \quad \overline{R}_n = \frac{R_n + R_{n+1}}{2}, \quad Q_n \equiv B_{R_n}\times(0, t)$$

and let $x \to \zeta_n(x)$ be the standard non-negative piecewise smooth cutoff function in $B_{\bar{R}_n}$ that equals one on B_{R_n} and such that

$$\zeta_n(x) = 0, \ |x| > \bar{R}_n, \qquad |D\zeta_n| \leqq 2^{n+2}/R.$$

Choose $\zeta_n = \varphi$ in (III.2.9) to obtain by standard calculations

$$\sup_{0<\tau<t} \int_{B_{R_n}} u(x,\tau)\,dx \leqq \int_{B_{2R}} u_0\,dx + \frac{2^{n+4}}{R} \int_0^t \int_{B_{\bar{R}_n}} |Du|^{p-1}\,dx\,d\tau. \quad \text{(III.3.3)}$$

Set

$$M_n = \sup_{0<\tau<t} \int_{B_{R_n}} u(x,\tau)\,dx,$$

and estimate the last integral by (I.4.3) of Lemma I.4.1, with $s = 0$, $r = \bar{R}_n$, $R = R_{n+1}$, and

$$\varepsilon = \left(\frac{t}{R^p}\right)^{\frac{1}{2-p}}.$$

We obtain

$$M_n \leqq \int_{B_{2R}} u_0\,dx + \frac{\gamma\,2^{np}}{R}\left\{\int_0^t \int_{B_{R_{n+1}}} (t-\tau)^{\frac{1}{p}-1} \left(\frac{t}{R^p}\right)^{\frac{2(p-1)}{p(2-p)}} + t^{\frac{1}{p}} \sup_{0<\tau<t} \int_{B_{R_{n+1}}} u^{\frac{2}{p}(p-1)}(x,\tau)\,dx\right\}$$

$$\leqq \int_{B_{2R}} u_0\,dx + \gamma\,2^{np}\left\{\left(\frac{t}{R^\varkappa}\right)^{\frac{1}{2-p}} + \left(\frac{t}{R^\varkappa}\right)^{\frac{1}{p}} M_{n+1}^{\frac{2}{p}(p-1)}\right\}. \quad \text{(III.3.4)}$$

If $\delta_0 \in (0,1)$, by Young's inequality

$$\gamma\,2^{np}\left(\frac{t}{R^\varkappa}\right)^{\frac{1}{p}} M_{n+1}^{\frac{2}{p}(p-1)} \leqq \delta_0\,M_{n+1} + \gamma(N,p,\delta_0)\,2^{\frac{p^2}{2-p}n}\left(\frac{t}{R^\varkappa}\right)^{\frac{1}{2-p}},$$

where $\gamma(N,p,\delta_0) = [\gamma^p\,\delta_0^{2(p-1)}]^{\frac{1}{2-p}}$.

Combining these calculations, we find the recursive inequalities

$$M_n \leqq \delta_0\,M_{n+1} + \gamma(N,p,\delta_0)\,2^{\frac{p^2}{2-p}n}\left[\int_{B_{2R}} u_0\,dx + \left(\frac{t}{R^\varkappa}\right)^{\frac{1}{2-p}}\right], \quad \text{(III.3.5)}$$

$$n = 0, 1, 2, \ldots .$$

The proof is concluded by means of an interpolation argument similar to that in Theorem I.4.1. □

Corollary III.3.1. $\exists\,\gamma = \gamma(N,p)$ *such that*

$$\forall\, 0 < s < t \leqq T, \qquad \forall\, R > 0, \quad \text{(III.3.6)}$$

$$\frac{1}{R}\int_s^t \int_{B_R} |Du|^{p-1}\,dx\,d\tau \leqq \gamma\left(\frac{t-s}{R^\varkappa}\right)^{\frac{1}{p}}\left\{\int_{B_{2R}} u_0\,dx + \left(\frac{t}{R^\varkappa}\right)^{\frac{1}{2-p}}\right\}^{\frac{2}{p}(p-1)}.$$

Proof. Combine (I.4.3) of Lemma I.4.1 and Lemma III.3.1. □

From the results of § I.2 and inequality (III.3.1) we obtain gradient estimates depending only upon the initial datum u_0. In particular Lemma I.2.2 yields

Lemma III.3.2. *$\exists \gamma = \gamma(N, p)$ such that $\forall\, 0 < t \leqq T$, $\forall\, R > 0$, $\forall\, \alpha \in (0, p-1)$,*

$$\int_0^t \int_{B_R} \left| D u^{\frac{p-1-\alpha}{p}} \right|^p dx\, d\tau \leqq \frac{\gamma}{[((p-1)-\alpha)\,\alpha]^p} \left[\int_{B_{2R}} u_0\, dx + \left(\frac{t}{R^{\varkappa}}\right)^{\frac{1}{2-p}} \right]^{1-\alpha}. \tag{III.3.7}$$

As remarked before, the estimate deteriorates as $\alpha \searrow 0$. The next lemma supplies some information on the case $\alpha = 0$, and it will be needed later to show that the solution of (III.1.1) is in Σ^*.

Lemma III.3.3. *Let $\alpha \in (0, p-1)$ be so small that $(\alpha + 1)(p-1) < 1$. $\exists \gamma = \gamma(N, p, \alpha)$ such that $\forall\, 0 < t \leqq T$, $\forall\, k,\ R > 0$, $\forall\, C > 1$,*

$$\begin{aligned} &\int_0^t \int_{B_R} |Du|^p u^{-1} \chi[k < u < Ck]\, dx\, d\tau \\ &\quad \leqq \gamma k^{-\left(\frac{1-(\alpha+1)(p-1)}{p}\right)} \left(\frac{t}{R^p}\right)^{\frac{1}{p}} \left\{ \int_{B_{2R}} u_0\, dx + \left(\frac{t}{R^{\varkappa}}\right)^{\frac{1}{2-p}} \right\}^{\frac{p-\alpha(p-1)}{p}} \\ &\qquad + \ln C \int_{B_{2R}} u_0 \chi[u_0 > k]\, dx. \end{aligned} \tag{III.3.8}$$

The constant $\gamma(\alpha) \nearrow \infty$ as $\alpha \searrow 0$ or as $\alpha \nearrow p-1$.

Proof. If $C > 1$ is fixed, let

$$u_{Ck}^{(k)} \equiv \begin{cases} k & \text{if } 0 < u < \leqq k \\ u & \text{if } k < u < Ck \\ Ck & \text{if } u \geqq Ck \end{cases}$$

and in (III.2.9) take the test function

$$\left(\ln \frac{u_{Ck}^{(k)}}{k}\right) \zeta(x),$$

where $x \to \zeta(x)$ is the standard cutoff function in B_{2R} that equals one on B_R. Standard calculations give

$$\begin{aligned} &\int_0^t \int_{B_R} |Du|^p \frac{1}{u} \chi[k < u < Ck]\, dx\, d\tau \\ &\leqq \frac{2\gamma}{R} \int_0^t \int_{B_{2R}} |Du|^{p-1} \chi[u > k]\, dx\, d\tau - \int_0^t \int_{B_{2R}} \frac{\partial}{\partial t} \left(\int_k^u \ln \frac{\min\{\xi; Ck\}}{k}\, d\xi \right) \zeta(x)\, dx\, d\tau \\ &\equiv G_k^{(1)} + G_k^{(2)}. \end{aligned}$$

Let α be any positive number satisfying $\alpha \in (0, p-1)$, $(\alpha+1)(p-1) < 1$. Then by virtue of Lemma III.3.2

$$G_k^{(1)} \leqq \frac{2\gamma}{R} \int_0^t \int_{B_{2R}} |Du|^{p-1} u^{-\frac{(\alpha+1)(p-1)}{p}} u^{\frac{(\alpha+1)(p-1)}{p}} \chi[u>k]\, dx\, d\tau$$

$$= \left(\frac{p}{p-1-\alpha}\right)^{p-1} \frac{2\gamma}{R} \int_0^t \int_{B_{2R}} \left| Du^{\frac{p-1-\alpha}{p}} \right|^{p-1} u^{\frac{(\alpha+1)(p-1)}{p}} \chi[u>k]\, dx\, d\tau$$

$$\leqq \frac{\gamma(\alpha, p)}{R} \left(\int_0^t \int_{B_{2R}} \left| Du^{\frac{p-1-\alpha}{p}} \right|^p dx\, d\tau \right)^{\frac{p-1}{p}} \left(\int_0^t \int_{B_{2R}} u^{(\alpha+1)(p-1)} \chi[u>k]\, dx\, d\tau \right)^{\frac{1}{p}}$$

$$\leqq \frac{\gamma(\alpha, p)}{R} \left[\int_{B_{4R}} u_0\, dx + \left(\frac{t}{R^k}\right)^{\frac{1}{2-p}} \right]^{\frac{p-1}{p}(1-\alpha)} \left[\int_0^t \int_{B_{2R}} u^{(\alpha+1)(p-1)} \chi[u>k]\, dx\, d\tau \right]^{\frac{1}{p}}$$

$$\leqq \frac{\gamma(\alpha, p)}{R} \left[\int_{B_{4R}} u_0\, dx + \left(\frac{t}{R^\varkappa}\right)^{\frac{1}{2-p}} \right]^{\frac{p-1}{p}(1-\alpha)} \left[\sup_{0<\tau<t} \int_{B_{2R}} u(x,\tau)\, dx \right]^{\frac{(\alpha+1)(p-1)}{p}} t^{\frac{1}{p}}$$

$$\cdot \left(\sup_{0<\tau<t} \int_{B_{2R}} \chi[u>k]\, dx \right)^{\frac{1-(\alpha+1)(p-1)}{p}}$$

$$\leqq C(\alpha, p) \left(\frac{t}{R^p}\right)^{\frac{1}{p}} \left[\int_{B_{4R}} u_0\, dx + \left(\frac{t}{R^\varkappa}\right)^{\frac{1}{2-p}} \right]^{\frac{2(p-1)}{p}}$$

$$\cdot \left[\sup_{0<\tau<t} \int_{B_{2R}} \chi[u>k]\, dx \right]^{\frac{1-(\alpha+1)(p-1)}{p}}.$$

The last step follows by use of Lemma III.3.1. Using it again, we obtain

$$\sup_{0<\tau<t} \int_{B_{2R}} \chi[u>k]\, dx \leqq \frac{\gamma}{k} \left\{ \int_{B_{4R}} u_0\, dx + \left(\frac{t}{R^\varkappa}\right)^{\frac{1}{2-p}} \right\}.$$

Therefore

$$G_k^{(1)} \leqq \gamma k^{-\frac{1-(\alpha+1)(p-1)}{p}} \left(\frac{t}{R^p}\right)^{\frac{1}{p}} \left\{ \int_{B_{4R}} u_0\, dx + \left(\frac{t}{R^\varkappa}\right)^{\frac{1}{2-p}} \right\}^{\frac{p-\alpha(p-1)}{p}}.$$

As for $G_k^{(2)}$, it is estimated above by

$$\int_{B_{2R}} \left(\int_k^{u_0} \ln \frac{\min\{\xi; Ck\}}{k}\, d\xi \right)_+ dx \leqq \ln C \int_{B_{2R}} u_0\, \chi[u_0 > k]\, dx.$$

This proves the lemma. □

The next estimates concern regularity in the time variable. The proof of the following lemma is known and is included here for completeness.

Lemma III.3.4 (BÈNILAN & CRANDALL [5]).

$$u_t \leqq \frac{1}{2-p} \frac{u}{t} \quad \text{a.e } S_T. \qquad \text{(III.3.9)}$$

Proof. The unique solution v of (III.1.1) with initial datum

$$x \to v(x, 0) = \lambda^{\frac{1}{p-2}} u_0(x), \qquad \lambda > 0,$$

is given by

$$(x, t) \to v(x, t) = \lambda^{\frac{1}{p-2}} u(x, \lambda t)$$

because of the homogeneity of (III.1.1). If $\lambda \geqq 1$, $v(x, 0) \leqq u_0(x)$ and $v(\cdot, t) \leqq u(\cdot, t)$ in $\mathbf{R}^N$, $\forall t \in (0, T)$. Choose $\lambda = \left(1 + \frac{h}{t}\right)$ for a small positive number h. Then

$$\begin{aligned} u(x, t+h) - u(x, t) &= u(x, \lambda t) - u(x, t) \\ &= \lambda^{\frac{1}{2-p}} \lambda^{\frac{1}{p-2}} u(x, \lambda t) - u(x, t) \\ &= \lambda^{\frac{1}{2-p}} v(x, t) - u(x, t) \\ &\leqq \left(\lambda^{\frac{1}{2-p}} - 1\right) u(x, t). \end{aligned}$$

By the mean value theorem,

$$u(x, t+h) - u(x, t) \leqq \frac{h}{2-p} (1+\xi)^{\frac{p-1}{2-p}} u(x, t) \quad \text{for some } \xi \in \left(0, \frac{h}{t}\right). \tag{III.3.10}$$

If $h < 0$ (and $|h| \ll 1$), $\lambda < 1$, $v(x, 0) \geqq u_0(x)$ and (III.3.10) holds with the inequality sign reversed. Divide by h and let $h \to 0$ in $\mathscr{D}'(S_T)$ to prove (III.3.9). □

Remark III.3.1. In Lemma III.3.4 are essential
(i) the homogeneity of the operator in (III.1.1),
(ii) the positivity of the initial datum u_0.

Lemma III.3.5. *Let $\alpha \in (0, p-1)$. There is a constant $\gamma = \gamma(N, p)$ such that*

$$\forall R > 0, \quad \forall 0 < s < t \leqq T, \quad \forall \theta \geqq \alpha + 1,$$

$$\int_s^t \int_{B_R} (u+1)^{-\theta} u_t^2 \, dx \, d\tau \leqq \gamma \left(\frac{t}{R^2}\right)^{\frac{p}{2-p}} R^N + \frac{\gamma}{s[\alpha(p-1-\alpha)]^p} \left\{ \int_{B_{2R}} u_0 \, dx + \left(\frac{t}{R^\varkappa}\right)^{\frac{1}{2-p}} \right\}^{1-\alpha}. \tag{III.3.11}$$

Proof. Let $0 < s < t \leqq T$ and $R > 0$ be fixed. Consider the cylinders

$$Q_0 \equiv B_R \times (s, t), \qquad Q_1 \equiv B_{\frac{3}{2}R} \times (\tfrac{1}{2} s, t),$$

and let $(x, \tau) \to \zeta(x, \tau)$ be a non-negative, piecewise smooth cutoff function in

Q_1 which equals one on Q_0 and is such that

$$|D\zeta| \leqq \frac{2}{R}, \quad 0 \leqq \zeta_t \leqq \frac{2}{s}.$$

At first we will proceed formally. The calculations below will be made rigorous later. Multiply (III.1.1) by the testing function

$$u_t(u+1)^{-\theta}\zeta^2,$$

and integrate by parts over Q_1. We obtain

$$\begin{aligned}
\iint_{Q_1} (u+1)^{-\theta} u_t^2 \zeta^2 \,dx\,d\tau &= -\iint_{Q_1} |Du|^{p-2} Du\, D(u_t(u+1)^{-\theta}\zeta^2)\,dx\,d\tau \\
&= -\frac{1}{p}\iint_{Q_1} \frac{\partial}{\partial t}|Du|^p (u+1)^{-\theta}\zeta^2\,dx\,d\tau \\
&\quad + \theta \iint_{Q_1} |Du|^p (u+1)^{-\theta-1} u_t \zeta^2\,dx\,d\tau \\
&\quad - 2\iint_{Q_1} |Du|^{p-2} Du u_t (u+1)^{-\theta}\zeta D\zeta\,dx\,d\tau \\
&\leqq \frac{\theta}{p}(p-1)\iint_{Q_1} |Du|^p (u+1)^{-\theta-1} u_t \zeta^2\,dx\,d\tau \\
&\quad + \frac{2}{p}\iint_{Q_1} |Du|^p (u+1)^{-\theta}\zeta\zeta_\tau\,dx\,d\tau \\
&\quad + \frac{4}{R}\iint_{Q_1} |Du|^{p-1}(u+1)^{-\frac{\theta}{2}}((u+1)^{-\theta}u_t^2\zeta^2)^{\frac{1}{2}}\,dx\,d\tau \\
&= \mathscr{R}^{(1)} + \mathscr{R}^{(2)} + \mathscr{R}^{(3)}.
\end{aligned}$$

By Lemma III.3.4

$$\mathscr{R}^{(1)} \leqq \frac{\theta(p-1)}{p(2-p)}\frac{1}{s}\iint_{Q_1} |Du|^p (u+1)^{-\theta}\zeta^2\,dx\,d\tau.$$

By Young's inequality

$$\mathscr{R}^{(3)} \leqq \tfrac{1}{2}\iint_{Q_1} (u+1)^{-\theta}u_t^2\zeta^2\,dx\,d\tau + \frac{8}{R^2}\iint_{Q_1} |Du|^{2(p-1)}(u+1)^{-\theta}\,dx\,d\tau.$$

Since $1 < p < 2$, this last integral is majorized by

$$\frac{1}{s}\iint_{Q_1} |Du|^p (u+1)^{-\theta}\,dx\,d\tau + \gamma(p)\left(\frac{t}{R^2}\right)^{\frac{p}{2-p}} R^N.$$

Combining these estimates, we find that

$$\iint_{Q_0} (u+1)^{-\theta} u_t^2\,dx\,d\tau \leqq \gamma\left(\frac{t}{R^2}\right)^{\frac{p}{2-p}} R^N + \frac{\gamma}{s}\iint_{Q_1} |Du|^p (u+1)^{-\theta}\,dx\,d\tau. \tag{III.3.12}$$

By Lemma III.3.2 if $\alpha \in (0, p-1)$, this is estimated by

$$\frac{\gamma}{s} \iint_{Q_1} |Du|^p u^{-(\alpha+1)} (u+1)^{-[\theta-(\alpha+1)]} \, dx \, d\tau$$

$$\leq \frac{\gamma}{s} \left(\frac{p}{p-1-\alpha}\right)^p \iint_{Q_1} \left| Du^{\frac{p-1-\alpha}{p}} \right|^p dx \, d\tau$$

$$\leq \frac{\gamma}{s[\alpha(p-1-\alpha)]^p} \left\{ \int_{B_{2R}} u_0 \, dx + \left(\frac{t}{R^\varkappa}\right)^{\frac{1}{2-p}} \right\}^{1-\alpha},$$

and the lemma follows by formal calculations.

To make the calculations rigorous it will suffice to show that

$$Du_t \in L^2_{\text{loc}}(S_T). \tag{III.3.13}$$

Indeed, if so, we may take in the weak formulation (III.2.9) the testing function

$$\frac{u(t+h) - u(t)}{h} \zeta^2$$

where $0 < h < \frac{1}{2} s$ and ζ is the cutoff function in Q_1. The limit as $h \to 0$ is justified and we may proceed as before.

To prove (III.3.13) we refer back to the construction procedure of § III.2. Let u_n be the unique solution of (III.2.1). By the results of [5], $\frac{\partial}{\partial t} u_n$ is bounded over $B_n \times (s, t)$ with bounds depending only upon $\|u_0\|_{\infty, \mathbf{R}^N}$. Therefore

$$\frac{\partial}{\partial t} u_n \in L^\infty_{\text{loc}}(S_T) \quad \text{uniformly in } n.$$

Here, as before, we view u_n, $\frac{\partial}{\partial t} u_n$, Du_n as defined in the whole strip S_T by extending them so as to be zero on $|x| > n$.

By the interior elliptic estimates of [10] and the boundary estimates of [14] (see remarks on page 104)

$$|Du_n| \in L^\infty_{\text{loc}}(S_T) \quad \text{uniformly in } n.$$

Next write the first of (III.2.1) for the time levels $t+h$ and $t, h \in (0, 1)$ and set

$$w = \frac{u_n(t+h) - u_n(t)}{h}.$$

By difference

$$w_t - h^{-1} \operatorname{div} \vec{J}_h = 0 \quad \text{in } B_n \times (0, T-h), \tag{III.3.14}$$

where

$$\vec{J}_h \equiv |Du_n(t+h)|^{p-2} Du_n(t+h) - |Du_n(t)|^{p-2} Du_n(t).$$

Fix $0 < \frac{1}{2}s < t \leqq T - h$ and in the weak formulation of (III.3.14) take the test function $w(t - \frac{1}{2}s)_+$ which vanishes on $|x| = n$ and on $t \leqq \frac{1}{2}s$. Standard calculations give

$$\int_{\frac{1}{2}s}^{T-h} \int_{B_n} (t - \tfrac{1}{2}s)_+ A_{0,n} |Dw|^2 \, dx \, d\tau \leqq \gamma \int_0^{T-h} \int_{B_n} \left| \frac{\partial}{\partial t} \fint_t^{t+h} u_n(x, \tau) \, d\tau \right|^2 dx \, d\tau \tag{III.3.15}$$

where

$$A_{0,n} = \int_0^1 |D(\xi u_n(t + h) + (1 - \xi) u_n(t))|^{p-2} \, d\xi$$

(compare with (II.2.4) and (II.2.6)). By virtue of (III.2.2)

$$\int_0^{T-h} \int_{B_n} \left| \frac{\partial}{\partial t} \fint_t^{t+h} u_n(x, \tau) \, dx \right| dx \, d\tau \leqq \gamma \left\| \frac{\partial}{\partial t} u_n \right\|_{2,Q_n}^2 \leqq \gamma \int_{\mathbf{R}^N} |Du_0|^p \, dx.$$

Since $1 < p < 2$,

$$A_{0,n} \geqq [2 \, \|Du_n\|_{\infty,\mathbf{R}^N \times (\frac{1}{2}s, T)}]^{p-2},$$

and it follows from (III.3.15) that

$$\int_s^T \int_{\mathbf{R}^N} \left| \frac{Du_n(t + h) - Du_n(t)}{h} \right|^2 dx \, d\tau$$

$$\leqq \frac{\gamma}{s} (\|Du_n\|_{\infty,\mathbf{R}^N \times (\frac{1}{2}s, T)})^{2-p} \, \|Du_0\|_{\infty,\mathbf{R}^N}^p \, |\{\text{supp } u_0\}|. \quad \square$$

We record a simple consequence of Lemma III.3.5.

If $x \to \zeta(x)$ is the usual cutoff function in B_{2R} that equals one on B_R, we find from the weak formulation (III.2.9) with $\varphi \equiv \zeta$, $\forall \, 0 < s < t \leqq T$,

$$\int_s^t \int_{B_{2R}} (u_t)^- \zeta \, dx \, d\tau - \int_s^t \int_{B_{2R}} (u_t)^+ \zeta \, dx \, d\tau$$

$$= - \int_s^t \int_{B_{2R}} u_t \zeta \, dx \, d\tau$$

$$= \int_s^t \int_{B_{2R}} |Du|^{p-2} Du \, D\zeta \, dx \, d\tau.$$

Therefore

$$\int_s^t \int_{B_R} (u_t)^- \, dx \, d\tau \leqq \int_s^t \int_{B_{2R}} (u_t)^+ \, dx \, d\tau + \frac{1}{R} \int_s^t \int_{B_{2R}} |Du|^{p-1} \, dx \, d\tau. \tag{III.3.16}$$

By III.3.9 and III.3.1

$$\int_s^t \int_{B_{2R}} (u_t)^+ \,dx\,d\tau \leqq \frac{1}{s(2-p)} \int_s^t \int_{B_{2R}} u(x,\tau)\,dx\,d\tau$$
$$\leqq \gamma \frac{t-s}{s}\left(\int_{B_{4R}} u_0\,dx + \left(\frac{t}{R^\varkappa}\right)^{\frac{1}{2-p}}\right).$$

We estimate the last integral on the right-hand side of (III.3.16) by Corollary III.3.1 and obtain

$$\frac{1}{R}\int_s^t \int_{B_{2R}} |Du|^{p-1}\,dx\,d\tau \leqq \gamma \left(\frac{t-s}{R^\varkappa}\right)^{\frac{1}{p}} \left(\int_{B_{4R}} u_0\,dx + \left(\frac{t}{R^\varkappa}\right)^{\frac{1}{2-p}}\right)^{\frac{2(p-1)}{p}}$$
$$\leqq \gamma \left(\frac{t-s}{t}\right)^{\frac{1}{p}} \left(\int_{B_{4R}} u_0\,dx + \left(\frac{t}{R^\varkappa}\right)^{\frac{1}{2-p}}\right).$$

Combining these estimates we deduce

Lemma III.3.6. *There exists a constant* $\gamma = \gamma(N, p)$, *independent of* u_0, *such that*

$$\forall\, 0 < s < t \leqq T, \quad \forall\, R > 0,$$
$$\int_s^t \int_{B_R} |u_t|\,dx\,d\tau \leqq \gamma t^{\frac{p-1}{p}} \frac{(t-s)^{\frac{1}{p}}}{s} \left(\int_{B_{4R}} u_0\,dx + \left(\frac{t}{R^\varkappa}\right)^{\frac{1}{2-p}}\right). \tag{III.3.17}$$

III.4. Approximating problems. If $u_0 \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$, and $u_0 \geqq 0$ is given, we construct the increasing sequence of functions

$$u_{0,n} = \begin{cases} u_0 & \text{if } u_0 < n, \quad |x| < n \\ n & \text{if } u_0 \geqq n, \quad |x| < n \\ 0 & \text{if } \qquad\qquad |x| \geqq n \end{cases} \tag{III.4.1}$$

and consider the sequence of Cauchy problems

$$\frac{\partial}{\partial t} u_n - \operatorname{div}(|Du_n|^{p-2} Du_n) = 0 \quad \text{in } S_T, \quad 1 < p < 2$$
$$u_n(\cdot, 0) = u_{0,n}. \tag{III.4.2$_n$}$$

A solution of (III.4.2)$_n$ is a non-negative measurable function u_n satisfying

$$u_n \in C(0, T; L^1_{\mathrm{loc}}(\mathbf{R}^N)) \cap L^p(0, T; W^{1,p}_{\mathrm{loc}}(\mathbf{R}^N)), \tag{III.4.3}$$

$$\|u_n\|_{\infty,S_T} \leqq \|u_{0,n}\|_{\infty,\mathbf{R}^N}, \tag{III.4.4}$$

$$\forall\, 0 < t \leqq T, \quad \forall\, R > 0, \quad \forall\, \varphi \in W^{1,1}(0, T; L^1_{\mathrm{loc}}(\mathbf{R}^N)) \cap L^p(0, T; \mathring{W}^{1,p}(B_R)),$$

$$\int_{\mathbf{R}^N} u_n(t)\,\varphi(t)\,dx + \int_0^t \int_{\mathbf{R}^N} \{-u_n\varphi_t + |Du_n|^{p-2} Du_n\, D\varphi\}\,dx\,d\tau = \int_{\mathbf{R}^N} u_{0,n}\varphi(0)\,dx. \tag{III.4.5}$$

For each $n \in \mathbf{N}$ there is a unique solution u_n of (III.4.2)$_n$. Moreover, u_n satisfy the estimates of § III.3 uniformly in n.

Uniqueness follows from Theorem II.1. For existence, if $n \in \mathbf{N}$ is fixed, let $v_0 = u_{0,n}$ and let $\{v_{0,m}\}$, $m \in \mathbf{N}$ be a sequence of $C_0^\infty(\mathbf{R}^N)$ functions such that

$$\begin{aligned} v_{0,m} \to u_{0,n} \quad &\text{in } L^q(\mathbf{R}^N), \ \forall\, 1 \leqq q < \infty \\ \|v_{0,m}\|_{\infty,\mathbf{R}^N} &\leqq n+1. \end{aligned} \tag{III.4.6}$$

If v_m are the solutions of (III.1.1) with initial datum $v_{0,m}$, in the sense of (III.2.2)–(III.2.3), by standard energy estimate and the comparison principle

$$\forall\, R > 0, \qquad \sup_{0<\tau<T} \|v_m(\tau)\|_{2,B_R}^2 + \|Dv_m\|_{p,B_R\times(0,T)}^p \leqq \gamma(n+1)^2\, R^N \left(1 + \frac{T}{R^p}\right), \tag{III.4.7}$$

$$\|v_m\|_{\infty,S_T} \leqq n+1 \qquad \forall\, m \in \mathbf{N}, \tag{III.4.8}$$

In particular, by the results of [13],

$$v_m \in C^\alpha_{\text{loc}}(S_T) \tag{III.4.9}$$

for some $\alpha \in (0, 1)$, uniformly in m.

By a diagonalization process we may extract a subsequence, relabelled m, such that

$$v_m \to v \equiv u_n \quad \text{in } L^2(0, T; L^2_{\text{loc}}(\mathbf{R}^N)), \text{ and uniformly over compact subsets of } S_T;$$

$$Dv_m \rightharpoonup Dv \equiv Du_n \qquad \text{weakly in} \qquad L^p(0, T; L^p_{\text{loc}}(\mathbf{R}^N)).$$

Write (III.2.3) for v_m and let $m \to \infty$ to get

$$\int_{\mathbf{R}^N} (v\varphi)(t)\, dx + \int_0^t \int_{\mathbf{R}^N} \{-v\varphi_t + \vec{\chi}_n\, D\varphi\}\, dx\, d\tau = \int_{\mathbf{R}^N} v_0\varphi(0)\, dx, \tag{III.4.10}$$

for all φ as in (III.4.5).

The identification $\vec{\chi}_n \equiv |Dv|^{p-2}\, Dv$ is carried here as in § III.2 except for identity (III.2.10), obtained from (III.2.9) by taking $\varphi = u\zeta$, where $x \to \zeta(x)$ is a non-negative, piecewise smooth cutoff function vanishing for $|x| \geqq R$. In (III.4.10) we cannot take $\varphi = v\zeta$ since $v_t \in L^1_{\text{loc}}(S_T)$ (see Lemma III.3.6 and [5]), but

$$v_t \notin L^1(0, T; L^1_{\text{loc}}(\mathbf{R}^N)).$$

Write (III.4.10) for the time levels t and s and subtract. Using the fact that

$$\frac{\partial}{\partial t} u_n \in L^1(s, t; L^1(B_R)), \qquad \forall\, 0 < s < t \leqq T, \qquad \forall\, R > 0,$$

by standard calculations and density arguments we obtain

$$\int_s^t \int_{\mathbf{R}^N} \left\{\frac{\partial}{\partial t} u_n\varphi + \vec{\chi}_n\, D\varphi\right\} dx\, d\tau = 0 \qquad \forall\, \varphi \in \mathring{X}_{\text{loc}}(S_T) \quad (\text{see § I.1}), \tag{III.4.11}$$

$$\lim_{\tau\to 0} \int_{B_R} u_n(\tau)\, \varphi(\tau)\, dx = \int_{B_R} u_{0,n}\varphi(0)\, dx \qquad \forall\, R > 0, \qquad \forall\, n \in \mathbf{N}. \tag{III.4.12}$$

We use (II.4.11) to identify

$$\vec{\chi}_n \equiv |Du_n|^{p-2} Du_n \quad \text{in } \mathbb{R}^N \times (s, T) \quad \forall\, 0 < s < T \tag{III.4.13}$$

and derive (III.4.5) by letting $s \to 0$.

III.5. The limit as $n \to \infty$. By (III.4.1) $u_n \nearrow u$ a.e. S_T. In view of (III.3.1) and Lemma III.3.6, $u \in C(0, T; L^1_{\text{loc}}(\mathbf{R}^N))$, and there exists a constant $\gamma = \gamma(N, p)$ such that $\forall\, 0 < t \leqq T$, $\forall\, R > 0$,

$$\sup_{0<\tau<t} \int_{B_R} u(x, \tau)\, dx \leqq \gamma \left(\int_{B_{2R}} u_0\, dx + \left(\frac{t}{R^\varkappa}\right)^{\frac{1}{2-p}} \right). \tag{III.5.1}$$

By Lemma III.3.2 the sequence $\left\{u_n^{\frac{p-1-\alpha}{p}}\right\}$, $\alpha \in (0, p-1)$ is equibounded in $L^p(0, T; W^{1,p}(B_R))$, $\forall\, R > 0$. Since the whole sequence $u_n \to u$ in $L^1_{\text{loc}}(\mathbf{R}^N)$

$$u_n^{\frac{p-1-\alpha}{p}} \rightharpoonup u^{\frac{p-1-\alpha}{p}} \quad \text{weakly in } L^p(0, T; W^{1,p}(B_R)), \quad \forall\, R > 0.$$

This implies that the sequences

$$w_{k,n} = u_n \wedge k = \min\{u_n, k\}$$

are equibounded in $L^p(0, T; W^{1,p}(B_r))$, $\forall\, R > 0$ and

$$w_{k,n} \rightharpoonup u \wedge k \quad \text{weakly in } L^p(0, T; W^{1,p}(B_R)) \quad \forall\, R > 0, \quad \forall\, k > 0. \tag{III.5.2}$$

Analogously, by Lemma III.3.5 the sequence

$$\left(\frac{\partial}{\partial t}(u_n + 1)^{\frac{2-\theta}{2}}\right)_{n \in \mathbb{N}}$$

is equibounded in $L^2_{\text{loc}}(S_T)$, $\forall\, \theta \geqq \alpha + 1$, $\forall\, \alpha \in (0, p-1)$, Therefore

$$\forall\, 0 < s < t \leqq T, \quad \forall\, R > 0,$$

$$\frac{\partial}{\partial t}(u_n + 1)^{\frac{2-\theta}{2}} \rightharpoonup \frac{\partial}{\partial t}(u + 1)^{\frac{2-\theta}{2}} \quad \text{weakly in } L^2(s, t; L^2(B_R)). \tag{III.5.3}$$

This implies that

$$\left(\frac{\partial}{\partial t} w_{k,n}\right) \in L^2_{\text{loc}}(S_T)$$

uniformly in n and

$$\frac{\partial}{\partial t} w_{k,n} \rightharpoonup \frac{\partial}{\partial t}(u \wedge k) \quad \text{weakly in } L^2_{\text{loc}}(S_T) \quad \forall\, k > 0. \tag{III.5.3'}$$

Choose $\psi \in \mathring{X}_{\text{loc}}(S_T)$ and in (III.4.14) consider the test function

$$\varphi = (\psi - u)_+. \tag{III.5.4}$$

Fix $0 < s < t \leqq T$ and let

$$k = \|\psi\|_{\infty, \mathbf{R}^N \times (s,t)}.$$

Then $(\psi - u)_+ = (\psi - u \wedge k)_+ \in \mathring{X}_{\text{loc}}(S_T)$ so that φ in (III.5.4) is an admissible test function in (III.4.11) and we obtain

$$\int_s^t \int_{\mathbf{R}^N} \left\{ \frac{\partial}{\partial t} u_n (\psi - u)_+ + |Du_n|^{p-2} Du_n D(\psi - u)_+ \right\} dx\, d\tau = 0. \tag{III.5.5}$$

Since $u_n \leqq u$, $\forall n \in \mathbf{N}$, a.e. S_T

$$\frac{\partial}{\partial t} u_n (\psi - u)_+ = \frac{\partial}{\partial t} u_n (\psi - u \wedge k)_+$$

$$= \frac{\partial}{\partial t} (u_n \wedge k)(\psi - u)_+ \quad \text{a.e. } S_T.$$

Therefore in view of (III.5.3)′

$$\lim_{n\to\infty} \int_s^t \int_{\mathbf{R}^N} \frac{\partial}{\partial t} u_n (\psi - u)_+ \, dx\, d\tau = \int_s^t \int_{\mathbf{R}^N} (u \wedge k)_t (\psi - u \wedge k)_+ \, dx\, d\tau$$

$$\equiv \int_s^t \int_{\mathbf{R}^N} u_t (\psi - u)_+ \, dx\, d\tau.$$

Analogously, since $u_n \leqq u$ a.e. S_T, $\forall n \in \mathbf{N}$

$$|Du_n|^{p-2} Du_n D(\psi - u)_+ = |D(u_n \wedge k)|^{p-2} D(u_n \wedge k) D(\psi - u \wedge k)_+.$$

By virtue of (III.5.2) the sequence

$$\{|D(u_n \wedge k)|^{p-2} D(u_n \wedge k)\}_{n \in \mathbf{N}}$$

is equibounded in $L^{\frac{p}{p-1}}_{\text{loc}}(S_T)$ and

$$D(u \wedge k) \in L^p(0, T; L^p_{\text{loc}}(\mathbf{R}^N)).$$

Let $R > 0$ be so large that

$$\operatorname{supp}\{x \to \psi(x, t)\} \subset B_R, \qquad \forall\, 0 < t \leqq T.$$

Then by possibly passing to a subsequence (depending upon s, R, k)

$$|D(u_n \wedge k)|^{p-2} D(u_n \wedge k) \rightharpoonup \vec{\chi}(k, s, R) \quad \text{weakly in } L^{\frac{p}{p-1}}\left(s, T; L^{\frac{p}{p-1}}(B_R)\right).$$

Letting $n \to \infty$ in (III.4.11) with $\vec{\chi}_n$ given by (III.4.13), we find that

$$\int_s^t \int_{B_R} \{u_t (\psi - u)_+ + \vec{\chi}(k, s, R) D(\psi - u)_+ \, dx\, d\tau = 0 \qquad \forall\, \psi \in \mathring{X}_{\text{loc}}(S_T).$$

(III.5.6)

To show that the function u so obtained is a weak solution of (III.1.1) we have to prove that

$$\forall\, 0 < s \leqq T, \quad \forall\, R > 0, \quad \forall\, \psi \in \overset{\circ}{X}_{\text{loc}}(S_T) \quad \text{such that} \tag{A}$$

$$\{\operatorname{supp} x \to \psi(x,t)\} \subset B_R, \quad \forall\, 0 < t < T, \quad k = \|\psi\|_{\infty, \mathbf{R}^N \times (s,T)},$$

$$\vec{\chi}(k,s,R)\, D(\psi - u)_+ \equiv |D(u \wedge k)|^{p-2} D(u \wedge k)\, D(\psi - u)_+$$

$$\equiv |Du|^{p-2} Du\, D(\psi - u)_+.$$

This will identify the nonlinear term in (III.5.6).

$$\forall\, R > 0, \quad \lim_{\tau \to 0} \int_{B_R} |u(\tau) - u_0|\, dx = 0. \tag{B}$$

This will specify the sense in which the initial datum is taken.

III.5 (i). Identification of $\vec{\chi}(k,s,R)$. We refer back to the equivalent formulation of weak solution in $\mathbf{R}^N \times (s,t)$ $\forall\, 0 < s < t \leqq T$, introduced in § I.3.

Let $\alpha \in (0, p-1)$ be fixed and let δ, c_0, β, d_0 be as in (I.3.1)–(I.3.2). For $n = 1, 2, \ldots$ let

$$v_n = \frac{p}{p-1-\alpha} (u_n + 1)^{\frac{p-1-\alpha}{p}}.$$

Then

$$c_0 \frac{\partial}{\partial t} v_n^\beta - \operatorname{div}(|Dv_n|^{p-2} Dv_n) = d_0\, |Dv_n|^p \frac{1}{v_n} \quad \text{in} \quad \mathscr{D}'(S_T), \quad \forall\, n \in \mathbf{N}. \tag{III.5.7}$$

By the estimates of § III.3 we have

$$Dv_n \text{ equibounded in } L^p_{\text{loc}}(S_T), \tag{III.5.8}$$

$$v_n \in L_{\text{loc}}^{\frac{p}{p-1-\alpha}}(S_T), \quad \text{uniformly in } n. \tag{III.5.9}$$

By the construction procedure the sequence $\{v_n\}$ is non-decreasing and

$$v_n \nearrow v = \frac{p}{p-1-\alpha}(u+1)^{\frac{p-1-\alpha}{p}} \quad \text{a.e. } S_T.$$

Therefore as $n \to \infty$ for the whole sequence $\{v_n\}$

$$Dv_n \rightharpoonup Dv \quad \text{weakly in } L^p_{\text{loc}}(S_T),$$

$$v_n \to v \quad \text{in } L_{\text{loc}}^{\frac{p}{p-1-\alpha}}(S_T).$$

Let $0 < s < t \leqq T$ and $R > 0$ be fixed and set

$$Q_R \equiv B_R \times (s,t).$$

Since

$$|Dv_n|^{p-2} Dv_n \quad \text{is equibounded in} \quad L_{\text{loc}}^{\frac{p}{p-1}}(S_T)$$

by possibly passing to a subsequence (relabelled with n)

$$|Dv_n|^{p-2} Dv_n \rightharpoonup \Xi \quad \text{weakly in} \quad L^{\frac{p}{p-1-\alpha}}(Q_R).$$

We will identify Ξ as $|Dv|^{p-2} Dv$.

If $n \in \mathbf{N}$ is fixed we have

$$v_n \in L^\infty(S_T) \quad \text{(depending on } n)$$

$$\frac{\partial}{\partial t} v_n^\beta \in L^\infty_{\text{loc}}(S_T) \quad \text{(see [5], depending on } n).$$

Therefore in the weak formulation of (III.5.7) we may take the test function

$$v_n\zeta \quad \text{where } x \to \zeta(x) \in C_0^\infty(B_R). \tag{III.5.10}$$

By standard density arguments we obtain

$$\frac{c_0\beta}{\beta+1}\int_{B_R} v_n^{\beta+1}(t)\,\zeta\,dx - \frac{c_0\beta}{\beta+1}\int_{B_R} v_n^{\beta+1}(s)\,\zeta\,dx + \int_s^t \int_{B_R} |Dv_n|^{p-2} Dv_n\, v_n\, D\zeta\,dx\,d\tau$$
$$= (d_0 - 1)\int_s^t \int_{B_R} |Dv_n|^p \zeta\,dx\,d\tau. \tag{III.5.11}$$

As $n \to \infty$

$$\frac{c_0\beta}{\beta+1}\int_{B_R} v^{\beta+1}(t)\,\zeta\,dx - \frac{c_0\beta}{\beta+1}\int_{B_R} v^{\beta+1}(s)\,\zeta\,dx + \int_{Q_R} \Xi\, D\zeta\,dx\,d\tau$$
$$= \left(\frac{\alpha p}{p-1-\alpha}\right) \lim_{n\to\infty} \int_s^t \int_{B_R} |Dv_n|^p \zeta\,dx\,d\tau. \tag{III.5.12}$$

Consider now (III.4.11) with $\vec{\chi}_n$ given by (III.4.13), and in its weak formulation over Q_R take the test function

$$(u+1)^{-\alpha}\zeta, \quad \alpha \in (0, p-1), \quad \zeta \in C_0^\infty. \tag{III.5.13}$$

Observe that by virtue of Lemma III.3.5 applied to u_n, by weak lower semicontinuity we have

$$\frac{\partial}{\partial t}(u+1)^{\frac{2-\theta}{2}} \in L^2_{\text{loc}}(S_T), \quad \forall\, \theta \geqq \alpha + 1.$$

By the Hölder inequality, if $\mathscr{K}$ is a compact subset of S_T

$$\iint_{\mathscr{K}} (u+1)^{-\alpha} |(u+1)_t|\,dx\,d\tau = \iint_{\mathscr{K}} (u+1)^{\frac{-1-\alpha}{2}} |(u+1)_t|\,(u+1)^{\frac{1-\alpha}{2}}\,dx\,d\tau$$
$$\leqq \left(\iint_{\mathscr{K}} (u+1)^{-(1+\alpha)}(u+1)_t^2\,dx\,d\tau\right)^{\frac12} \left(\iint_{\mathscr{K}} (u+1)^{1-\alpha}\,dx\,d\tau\right) < \infty.$$

Therefore

$$\frac{\partial}{\partial t}(u+1)^{1-\alpha} \in L^1_{\text{loc}}(S_T). \tag{III.5.14}$$

Using the test function (III.5.13) in (III.4.11), we obtain

$$\int_s^t \int_{B_R} (u_n+1)_t (u+1)^{-\alpha} \zeta \, dx \, d\tau + \int_s^t \int_{B_R} |Du_n|^{p-1} Du_n (u+1)^{-\alpha} D\zeta \, dx \, d\tau$$

$$= \alpha \iint_{Q_R} |D(u_n+1)|^{p-2} D(u_n+1) \, D(u+1)^{-(\alpha+1)} D(u+1) \, \zeta \, dx \, d\tau. \quad \text{(III.5.15)}$$

We transform the various terms in (III.5.15) as follows:

$$(\mathrm{a}_n) \qquad \int_s^t \int_{B_R} \frac{\partial}{\partial t}(u_n+1)(u+1)^{-\alpha} \zeta \, dx \, d\tau = \int_{B_R(t)} (u_n+1)(u+1)^{-\alpha} \zeta \, dx$$

$$- \int_{B_R(s)} (u_n+1)(u+1)^{-\alpha} \zeta \, dx + \alpha \int_s^t \int_{B_R} (u_n+1) \frac{\partial}{\partial t}(u+1)(u+1)^{-\alpha-1} \zeta \, dx \, d\tau$$

$$= \int_{B_R(\tau)} (u_n+1)(u+1)^{-\alpha} \zeta \, dx \Big|_s^t + \frac{\alpha}{1-\alpha} \int_s^t \int_{B_R} \left(\frac{u_n+1}{u+1}\right) \frac{\partial}{\partial t}(u+1)^{1-\alpha} \zeta \, dx \, d\tau.$$

We let $n \to \infty$. Since $\dfrac{u_n+1}{u+1} \nearrow 1$ a.e. S_T and $\dfrac{\partial}{\partial t}(u+1)^{1-\alpha} \in L^1_{\mathrm{loc}}(S_T)$ by (III.5.14), we obtain

$$(\mathrm{a}_\infty) \qquad \int_s^t \int_{B_R} \frac{\partial}{\partial t}(u_n+1)(u+1)^{-\alpha} \zeta \, dx \, d\tau$$

$$\to \frac{1}{1-\alpha} \int_{B_R(t)} (u+1)^{1-\alpha} \zeta \, dx - \frac{1}{1-\alpha} \int_{B_R(s)} (u+1)^{1-\alpha} \zeta \, dx$$

$$\equiv \frac{p-1-\alpha}{p} \left[\frac{c_0\beta}{\beta+1} \int_{B_R} v^{\beta+1}(t) \, \zeta \, dx - \frac{c_0\beta}{\beta+1} \int_{B_R} v^{\beta+1}(s) \, \zeta \, dx \right].$$

Next

$$(\mathrm{b}) \qquad \iint_{Q_R} |Du_n|^{p-2} Du_n (u+1)^{-\alpha} D\zeta \, dx \, d\tau$$

$$= \iint_{Q_R} |D(u_n+1)|^{p-2} D(u_n+1) \, (u_n+1)^{-\frac{(\alpha+1)(p-1)}{p}},$$

$$(u_n+1)^{\frac{(\alpha+1)(p-1)}{p}} (u+1)^{-\alpha} D\zeta \, dx \, d\tau$$

$$= \iint_{Q_R} |Dv_n|^{p-2} Dv_n (u_n+1)^{\frac{(\alpha+1)(p-1)}{p}} (u+1)^{-\alpha} D\zeta \, dx \, d\tau$$

$$\to \iint_{Q_R} \Xi (u+1)^{\frac{p-1-\alpha}{p}} D\zeta \, dx \, d\tau \equiv \frac{p-1-\alpha}{p} \iint_{Q_R} \vec{\Xi} \, D\zeta \, dx \, d\tau.$$

Finally

(c) $$\alpha \iint_{Q_R} |D(u_n+1)|^{p-2} D(u_n+1)\,(u+1)^{-(\alpha+1)} D(u+1)\,\zeta\,dx\,d\tau$$
$$= \alpha \iint_{Q_R} |Dv_n|^{p-2} Dv_n \left(\frac{u_n+1}{u+1}\right)^{\frac{(\alpha+1)(p-1)}{p}} Dv\,\zeta\,dx\,d\tau \to \alpha \iint_{Q_R} \Xi\, D\zeta\,dx\,d\tau.$$

Letting $n \to \infty$ in (III.5.15). Multiplying by $\frac{p}{p-1-\alpha}$, we obtain

$$\frac{c_0\beta}{\beta+1} \int_{B_R(t)} v^{\beta+1}\zeta\,dx - \frac{c_0\beta}{\beta+1} \int_{B_R(s)} v^{\beta+1}\zeta\,dx + \iint_{Q_R} \Xi v\, D\zeta\,dx\,d\tau$$
$$= \frac{\alpha p}{p-1-\alpha} \iint_{Q_R} \Xi\, Dv\,\zeta\,dx\,d\tau. \qquad \text{(III.5.16)}$$

We compare this with (III.5.12) to conclude that

$$\lim_{n\to\infty} \int_s^t \int_{B_R} |Dv_n|^p \zeta\,dx\,d\tau = \int_s^t \int_{B_R} \Xi\, Dv\,\zeta\,dx\,d\tau. \qquad \text{(III.5.17)}$$

Next $\forall R>0$, $\forall \eta \in X_{\text{loc}}(S_T)$, $\forall \zeta \in C_0^\infty$, $\zeta \geqq 0$, $\forall n \in \mathbf{N}$

$$\iint_{Q_R} (|Dv_n|^{p-2} Dv_n - |D\eta|^{p-2} D\eta)(Dv_n - D\eta)\,\zeta\,dx\,d\tau \geqq 0.$$

Expanding the integrand and letting $n \to \infty$ with the aid of (III.5.17), we find that

$$\iint_{Q_R} (\Xi - |D\eta|^{p-2} D\eta)(Dv - D\eta)\,\zeta\,dx\,d\tau \geqq 0.$$

This and MINTY's device [24], imply that

$$\Xi = |Dv|^{p-2} Dv.$$

Now that the limit Ξ has been identified, we conclude that for the whole sequence

$$|Dv_n|^{p-2} Dv_n \rightharpoonup |Dv|^{p-2} Dv \quad \text{weakly in } L_{\text{loc}}^{\frac{p}{p-1}}(S_T). \qquad \text{(III.5.18)}$$

We return to (III.5.6) and identify $\vec{\chi}$. To this end we let $n \to \infty$ in the nonlinear term of (III.5.5) as follows:

$$\int_s^t \int_{\mathbf{R}^N} |Du_n|^{p-2} Du_n\, D(\psi-u)_+\,dx\,d\tau$$
$$= \int_s^t \int_{\mathbf{R}^N} |D(u_n+1)|^{p-2} D(u_n+1)\,(u_n+1)^{-\frac{(\alpha+1)(p-2)}{p}} (u_n+1)^{\frac{(\alpha+1)(p-1)}{p}}$$
$$\times D(\psi-u)_+\,dx\,d\tau$$
$$= \int_s^t \int_{\mathbf{R}^N} (|Dv_n|^{p-2} Dv_n)\,(u_n+1)^{\frac{(\alpha+1)(p-1)}{p}} D(\psi-u)_+ dx\,d\tau$$
$$\forall \alpha \in (0, p-1), \qquad \forall \psi \in \mathring{X}_{\text{loc}}(S_T).$$

Since $u_n \nearrow u$ and since $u \wedge k \in L^p_{\text{loc}}(S_T)$, $k = \|\psi\|_{\infty,\mathbf{R}^N\times(s,t)}$, we have

$$(u_n+1)^{\frac{(\alpha+1)(p-1)}{p}} D(\psi-u)_+ = (u_n \wedge k+1)^{\frac{(\alpha+1)(p-1)}{p}} D(\psi-u\wedge k)_+$$

$$\to (u\wedge k+1)^{\frac{(\alpha+1)(p-1)}{p}} D(\psi-u\wedge k)_+ \quad \text{in} \quad L^p_{\text{loc}}((S_T)).$$

We let $n\to\infty$ and use (III.5.18) to obtain

$$\int_s^t \int_{\mathbf{R}^N} |Du_n|^{p-2} Du_n D(\psi-u)_+ \, dx\, d\tau$$

$$\to \int_s^t \int_{\mathbf{R}^N} |Dv|^{p-2} Dv(u+1)^{\frac{(\alpha+1)(p-1)}{p}} D(\psi-u\wedge k)_+ \, dx\, d\tau$$

$$= \int_s^t \int_{\mathbf{R}^N} |Du|^{p-2} Du\, D(\psi-u)_+ \, dx d\tau.$$

III.5 (ii). Continuity in $L^1_{\text{loc}}(\mathbf{R}^N)$ at $t=0$. Let $x\to K(|x|)$ be a mollifying kernel supported in the ball $\{|x|<1\}$ and for $\varepsilon\in(0,1)$ and $f\in L^1_{\text{loc}}(\mathbf{R}^N)$ let

$$K_\varepsilon(x) = \varepsilon^{-N} K\left(\frac{|x|}{\varepsilon}\right), \qquad f_\varepsilon = K_\varepsilon * f.$$

The kernel $K(\cdot)$ can be chosen so that

$$\forall f\in L^1_{\text{loc}}(\mathbf{R}^N), \qquad \forall R>0, \qquad \|f\|_{\infty,B_R} \leqq \varepsilon^{-N}\|f\|_{1,B_{2R}},$$

$$\|Df\|_{\infty,B_R} \leqq C\varepsilon^{-(N+1)}\|f\|_{1,B_{2R}} \tag{III.5.7}$$

for a constant C depending only upon N. Let

$$u_0^{(\varepsilon)} = \begin{cases} u_0 & \text{if } |x| < \dfrac{1}{\varepsilon} \\[2ex] 0 & \text{if } |x| \geqq \dfrac{1}{\varepsilon}, \end{cases}$$

define

$$u_{0,\varepsilon} = K_\varepsilon * u_0^{(\varepsilon)} \in C_0^\infty(\mathbf{R}^N),$$

and let u_ε be the solutions of (III.1.1) corresponding to the initial data $u_{0,\varepsilon}\in C_0^\infty(\mathbf{R}^N)$.

We write (III.4.14) for u_n (constructed in § III.4) and u_ε (constructed in III.2), subtract, and take the testing function

$$\varphi = [(u_n-u_\varepsilon)_+ + \delta]^\sigma \zeta$$

where $\sigma, \delta\in(0,1)$ and $x\to\zeta(x)$ is the usual cutoff function in B_{2R} that equals one on B_R.

Such a choice is admissible by possibly approximating u_n by the smooth solutions v_m introduced in § III.4.

We perform an integration by parts and let $\delta \to 0$, $s \to 0$, $\sigma \to 0$, to obtain

$$\int_{B_R} (u_n(t) - u_\varepsilon(t))_+ \, dx \leqq \int_{B_R} (u_{0,n} - u_{0,\varepsilon})_+ \, dx + \frac{2\gamma}{R} \int_0^t \int_{B_{2R}} (|Du_n|^{p-1} + |Du_\varepsilon|^{p-1}) \, dx \, d\tau .$$

We apply Corollary III.3.1 with $s = 0$, interchange the role of u_n and u_ε and for $t > 0$ fixed let $n \to \infty$ to get

$$\int_{B_R} |u(t) - u_\varepsilon(t)| \, dx \leqq \int_{B_{2R}} |u_0 - u_{0,\varepsilon}| \, dx + \gamma \left(\frac{t}{R^\varkappa}\right)^{\frac{1}{p}} \left\{ \int_{B_{2R}} u_0 dx + \left(\frac{t}{R^\varkappa}\right)^{\frac{1}{2-p}} \right\}^{\frac{2}{p}(p-1)} \tag{III.5.8}$$

Now recall that since $\{u_\varepsilon\}$ are solutions of (III.1.1) with initial datum $u_{0,\varepsilon} \in C_0^\infty(\mathbf{R}^N)$, it follows from (III.2.2) that

$$\int_0^t \int_{B_R} \left| \frac{\partial}{\partial t} u_\varepsilon \right|^2 dx \, d\tau \leqq \int_{\mathbf{R}^N} |Du_{0,\varepsilon}|^p \, dx . \tag{III.5.9}$$

By (III.5.7)

$$\int_0^t \int_{B_R} \left| \frac{\partial}{\partial t} u_\varepsilon \right|^2 dx \, d\tau \leqq \gamma \varepsilon^{-(N+1)p-N} \left(\int_{|x|<\frac{1}{\varepsilon}} u_0 \, dx \right)^p ,$$

and

$$\int_{B_R} |u_\varepsilon(t) - u_{\varepsilon,0}| \, dx \leqq \gamma(R) \, \varepsilon^{-\left(\frac{N(p+1)+p}{2}\right)} t^{\frac{1}{2}} \left(\int_{|x|<\frac{1}{\varepsilon}} u_0 \, dx \right)^{\frac{p}{2}} .$$

Combining this in (III.5.8), we deduce that there exists a constant γ depending only upon R, N, $\int_{B_{2R}} u_0 \, dx$ and independent of ε such that

$$\int_{B_R} |u(t) - u_0| \, dx \leqq 2 \int_{B_{2R}} |u_0 - u_{0,\varepsilon}| \, dx + \gamma(R) \, \varepsilon^{-\left(\frac{N(p+1)+p}{2}\right)} \left(\int_{|x|<\frac{1}{\varepsilon}} u_0 \, dx \right)^{\frac{p}{2}} t^{\frac{1}{2}} + \gamma \left(\frac{t}{R^\varkappa}\right)^{\frac{1}{p}} \left\{ \int_{B_{2R}} u_0 \, dx + \left(\frac{t}{R^\varkappa}\right)^{\frac{1}{2-p}} \right\}^{\frac{2}{p}(p-1)} .$$

This proves the continuity of $t \to u(t)$ in $L^1_{\mathrm{loc}}(\mathbf{R}^N)$ near $t = 0$.

For $t > 0$ the continuity of this map follows from Lemma III.3.6.

III.5 (iii). $u \in \Sigma^*$. By Lemma III.3.4 $\forall n \in \mathbf{N}$, $\forall k > 0$

$$\frac{\partial}{\partial t} (u_n \wedge k) \leqq \frac{1}{2-p} \frac{u_n}{t} .$$

As $n \to \infty$

$$(u \wedge k)_t \leqq \frac{1}{2-p} \frac{u}{t} \quad \text{a.e. } S_T .$$

Next from Lemma III.3.3 it follows that $\forall \, C > 1$

$$\int_s^t \int_{B_R} |Du_n|^p \frac{1}{u_n} \chi[k < u_n] \, \chi[u < Ck] \, dx \, d\tau = O\left(\frac{1}{k}\right) .$$

Here we have used the fact that $u_n \nearrow u$ implies $[u_n < Ck] \supset [u < Ck]$. Letting $n \to \infty$ for $k > C$, $C > 1$ fixed yields by lower semicontinuity

$$\int_s^t \int_{B_R} |Du|^p \frac{1}{u} \chi[k < u < Ck] \, dx \, d\tau = O\left(\frac{1}{k}\right).$$

Remark III.5.1. We conclude the construction of solutions of the Cauchy problem (III.1.1) by observing that the requirement

$$u \in \Sigma^*$$

in Theorem II.1 is necessary and sufficient for uniqueness.

Indeed, if solutions in Σ are unique, they can be constructed starting from their traces on $t = \tau \in (0, T)$ to yield $u \in \Sigma^*$. *Vice versa* by Theorem II.1 solutions in Σ^* are unique.

III.6. Locally bounded solutions. We investigate here the regularity of solutions of (III.1.1) whenever

$$u_0 \in L^r_{\text{loc}}(\mathbf{R}^N), \quad r \geqq 1.$$

In particular, we show that if $p > \dfrac{2N}{N + r}$, then $u \in L^\infty_{\text{loc}}(S_T)$. Such a condition on p and r is the same as the positivity of the number

$$\varkappa_r \equiv N(p - 2) + rp, \quad \varkappa_r > 0, \tag{III.6.1}$$

which will play a central role in what follows.

Lemma III.6.1. *If* $u_0 \in L^r_{\text{loc}}(\mathbf{R}^N)$, $r \geqq 1$, $\exists\, \gamma = \gamma(N, p, r)$ *such that* $\forall\, 0 < t \leqq T$, $\forall\, R > 0$,

$$\sup_{0<\tau<t} \int_{B_R} u^r(x, \tau) \, dx \leqq \gamma \left\{ \int_{B_{2R}} u_0^r \, dx + \left(\frac{t^r}{R^{\varkappa_r}}\right)^{\frac{1}{2-p}} \right\}. \tag{III.6.2}$$

Theorem III.6.2. *Let* $r \geqq 1$ *and* $1 < p < 2$ *satisfy* $\varkappa_r > 0$ *and assume* $u_0 \in L^r_{\text{loc}}(\mathbf{R}^N)$. *Then* $\forall\, t > 0$, $x \to u(x, t) \in L^\infty_{\text{loc}}(\mathbf{R}^N)$ *and* $\exists\, \gamma = \gamma(N, p, r)$ *such that*

$$\forall\, 0 < t, \quad \forall\, R > 0,$$

$$\sup_{x \in B_R} u(x, t) \leqq \gamma t^{-\frac{N}{\varkappa_r}} \left(\int_{B_{2R}} u_0^r \, dx\right)^{\frac{p}{\varkappa_r}} + \gamma \left(\frac{t}{R^p}\right)^{\frac{1}{2-p}}. \tag{III.6.3}$$

Remark III.6.1. The first term on the right-hand side of (III.6.3) is formally the same as an estimate for the case $p > 2$ (see [11], Theorem 1). In that context ($p > 2$) one could find only solutions locally in time, in general. In the present situation, the solutions are all global in time and the second term on the right-hand side of (III.6.3) controls the possible growth of such solutions for large times.

In our calculations below, we will assume that u solves (III.1.1) with $u_0 \in C_0^\infty(\mathbf{R}^N)$ so that

$$\begin{gathered} u_t - \operatorname{div}(|Du|^{p-2} Du) = 0 \quad \text{a.e. } S_T, \\ u_t \in L^\infty_{\text{loc}}(S_T), \quad u \in W^{1,p}_{\text{loc}}(S_T), \quad Du_t \in L^2_{\text{loc}}(S_T). \end{gathered} \tag{III.6.4}$$

By the approximation procedure developed earlier, whence (III.6.2) is established for such smooth solutions, it continues to hold for solutions of (III.1.1) with $u_0 \in L^r_{\text{loc}}(\mathbf{R}^N)$.

Also, if (III.6.3) is proved for smooth u, then all the approximations to the solutions of (III.1.1) with $u_0 \in L^r_{\text{loc}}(\mathbf{R}^N)$ are locally uniformly Hölder continuous in S_T (see [13]).

Proof of Lemma III.6.1. If $r = 1$, this is precisely Lemma III.3.1, and so let us assume that $r > 1$. Let $\sigma \in (0, 1)$, $R > 0$ be fixed, multiply (III.6.4) by

$$(x, t) \to u^{r-1}(x, t)\, \zeta(x), \qquad r > 1,$$

where $x \to \zeta(x)$ is the cutoff function in $B_{(1+\sigma)R}$ that equals one on B_R. Also, without loss of generality we may assume that $u > 0$ (if not, replace u by $u + \varepsilon$ and then let $\varepsilon \to 0$). Standard calculations give

$$\int_{B_R} u^r(t)\, dx \leqq \frac{\gamma}{(\sigma R)^p} \int_0^t \int_{B_{(1+\sigma)R}} u^{p-2+r}\, dx\, d\tau + \int_{B_{2R}} u_0^r\, dx.$$

From this, $\forall\, 0 < t \leqq T$, $\forall\, R > 0$, $\forall\, \sigma \in (0, 1)$,

$$\left(\sup_{0<\tau<t} \int_{B_R} u^r(\tau)\, dx\right) \leqq \int_{B_{2R}} u_0^r\, dx + \frac{\gamma}{\sigma^p}\left(\frac{t^r}{R^{\varkappa_r}}\right)^{\frac{1}{r}} \left(\sup_{0<\tau<t} \int_{B_{(1+\sigma)R}} u^r(\tau)\, dx\right)^{\frac{p-2+r}{r}}.$$

The lemma now follows by an interpolation argument similar to that in Theorem I.4.1. □

Proof of Theorem III.6.2. Assume first that

$$1 \leqq r \leqq p.$$

Let $t_1 > 0$ and $\sigma \in (0, 1)$ be fixed and consider the sequence of radii and time levels

$$R_n = R(1 + \sigma\, 2^{-n}), \qquad t_n = \tfrac{1}{2}\, t\, (1 - \sigma\, 2^{-n}).$$

Set

$$B_n \equiv B_{R_n}, \qquad Q_n \equiv B_n \times (t_n, t), \qquad n = 0, 1, 2, \ldots,$$

and let $(x, \tau) \to \zeta_n(x, \tau)$ be a non-negative piecewise smooth cutoff function in Q_n that equals one on Q_{n+1} and is such that

$$0 \leqq \zeta_{n,t} \leqq \frac{2^{n+2}}{\sigma t}, \qquad |D\zeta_n| \leqq \frac{2^{n+1}}{\sigma R}.$$

Consider also the sequence of increasing levels

$$k_n = k - \frac{k}{2^n}, \quad n = 0, 1, 2, \ldots,$$

where $k > 0$ will be chosen later.

Multiply (III.6.4) by $(u - k_n)_+ \zeta_n^p$ and integrate by parts over Q_n. Standard calculations give

$$\sup_{t_n<\tau<t} \int_{B_n(\tau)} (u - k_n)_+^2 \zeta_n^p \, dx + \iint_{Q_n} |D(u - k_n)_+ \zeta_n|^p \, dx \, d\tau$$
$$\leqq \frac{\gamma 2^{np}}{(\sigma R)^p} \iint_{Q_n} (u - k_n)_+^p \, dx \, d\tau + \frac{\gamma 2^n}{\sigma t} \iint_{Q_n} (u - k_n)_+^2 \, dx \, d\tau. \quad \text{(III.6.5)}$$

Majorize the last integral on the right-hand side by

$$\frac{\gamma 2^{np}}{\sigma^p t} \|u\|_{\infty,Q_0}^{2-p} \iint_{Q_n} (u - k_n)_+^p \, dx \, d\tau,$$

and the first integral by

$$\frac{\gamma 2^{np}}{\sigma^p t} \left(\frac{t}{R^p}\right) \iint_{Q_n} (u - k_n)_+^p \, dx \, d\tau \leqq \frac{\gamma 2^{np}}{\sigma^p t} \|u\|_{\infty,Q_0}^{2-p} \iint_{Q_n} (u - k_n)_+^p \, dx \, d\tau,$$

where we have assumed that

$$\|u\|_{\infty,Q_0} > \left(\frac{t}{R^p}\right)^{\frac{1}{2-p}}. \quad \text{(III.6.6)}$$

If (III.6.6) is violated, there is nothing to prove. We estimate now the left-hand side of (III.6.5).

For all $\tau \in (t_n, t)$

$$\int_{B_n(\tau)} (u - k_n)_+^2 \zeta_n^p \, dx \geqq \int_{B_n(\tau) \cap [u > k_{n+1}]} (u - k_{n+1})_+^p (u - k_n)_+^{2-p} \zeta_n^p \, dx$$
$$\geqq \left(\frac{k}{2^{n+1}}\right)^{2-p} \int_{B_n(\tau)} (u - k_{n+1})_+^p \zeta_n^p \, dx.$$

These remarks in (III.6.5) yield

$$\left(\frac{k}{2^n}\right)^{2-p} \sup_{t_n<\tau<t} \int_{B_n(\tau)} [(u - k_{n+1})_+ \zeta_n]^p \, dx + \iint_{Q_n} |D[(u - k_{n+1})_+ \zeta_n]|^p \, dx \, d\tau$$
$$\leqq \gamma 2^{np} M \iint_{Q_n} (u - k_n)_+^p \, dx \, d\tau, \quad \text{(III.6.7)}$$

where we have set

$$M = \frac{\|u\|_{\infty,Q_0}^{2-p}}{\sigma^p t}. \quad \text{(III.6.8)}$$

Since $(u - k_{n+1})_+ \zeta_n$ vanishes on the lateral boundary of Q_n, by the embedding of GAGLIARDO & NIRENBERG (see [19] page 62),

$$\left(\iint_{Q_n} [(u - k_{n+1})_+ \zeta_n]^q \, dx \, d\tau\right)^{\frac{p}{q}} \leqq \gamma \left(\iint_{Q_n} |D[(u - k_{n+1})_+ \zeta_n]|^p \, dx \, d\tau\right)^{\frac{p}{q}}$$
$$\cdot \left(\sup_{0<\tau<t} \int_{B_n(\tau)} [(u - k_{n+1})_+ \zeta_n]^p \, dx\right)^{1-\frac{p}{q}}. \quad \text{(III.6.9)}$$

where

$$q = p\left(\frac{N+p}{N}\right). \tag{III.6.10}$$

Estimating the last two terms on the right-hand side of (III.6.9) by (III.6.7), we deduce that

$$\begin{aligned}\iint_{Q_n} (u-k_{n+1})_+^p \zeta_n^p \, dx\, d\tau &\leqq \left(\iint_{Q_n} [(u-k_{n+1})_+ \zeta_n]^q \, dx\, d\tau\right)^{\frac{p}{q}} \\ &\quad \cdot \left(\text{meas}\left\{(x,\tau)\in Q_n \mid u(x,\tau) > k_{n+1}\right)\right\}^{1-\frac{p}{q}} \\ &\leqq \gamma\, 2^{n\left((2-p)\frac{q-p}{q}+p\right)} \left(M \iint_{Q_n} (u-k_n)_+^p \, dx\, d\tau\right) k^{(p-2)\frac{q-p}{q}} \\ &\quad \cdot (\text{meas}\,[u > k_{n+1}] \cap Q_n)^{1-\frac{p}{q}}. \end{aligned} \tag{III.6.11}$$

Since

$$\iint_{Q_n} (u-k_n)_+^p \, dx\, d\tau \geqq (k_{n+1}-k_n)^p \,\text{meas}\,([u > k_{n+1}] \cap Q_n),$$

inequality (III.6.11) yields

$$\iint_{Q_{n+1}} (u-k_{n+1})_+^p \, dx\, d\tau \leqq \gamma b^n M k^{-2\frac{q-p}{q}} \left(\iint_{Q_n} (u-k_n)_+^p \, dx\, d\tau\right)^{1+\frac{q-p}{q}}$$

where $b = 2^{p+2\frac{q-p}{q}} > 1$.

It follows from Lemma 5.6 of [19] page 95 that if k is chosen so that

$$\iint_{Q_0} u^p \, dx = CM^{-\frac{q}{q-p}} k^2, \quad C = b^{-\left(\frac{q}{q-p}\right)^2} \gamma^{-\frac{q}{q-p}}, \tag{III.6.12}$$

then

$$\iint_{Q_n} (u-k_n)_+^p \, dx\, d\tau \to 0 \quad \text{as } n \to \infty,$$

i.e.,

$$\|u\|_{\infty, Q_\infty} \leqq k.$$

Recalling the definition of M in (III.6.8), we conclude that there is a constant $\gamma = \gamma(N, p)$ such that $\forall\, 0 < t \leqq T$, $\forall\, R > 0$,

$$\begin{aligned}\|u\|_{\infty, B_R \times \left(\frac{t}{2}, t\right)} &\leqq \frac{\gamma}{(\sigma^p t)^{\frac{q}{2(q-p)}}} \left(\|u\|_{\infty, B_{(1+\sigma)R} \times \left(\frac{t}{2}(1-\sigma), t\right)}\right)^{\frac{(2-p)q}{2(q-p)}} \\ &\quad \cdot \left(\int_{\frac{t}{2}(1-\sigma)}^{t} \int_{B_{(1+\sigma)R}} u^p \, dx\, d\tau\right)^{\frac{1}{2}}. \end{aligned} \tag{III.6.13}$$

Consider the sequence of radii and time levels

$$R_n = R\left(\sum_{i=0}^{n} 2^{-i}\right), \qquad t_n = \frac{t}{2}\left(1 - \sum_{i=1}^{n} 2^{-i}\right).$$

We write (III.6.13) over the pair of cylinders

$$Q_n \equiv B_{R_n} \times (t_n, t) \subset Q_{n+1} \equiv B_{R_{n+1}} \times (t_{n+1}, t).$$

For these $\sigma = 2^{-(n+1)}$. If for notational simplicity we set

$$Y_n = \|u\|_{\infty, B_{R_n} \times (t_n, t)},$$

we obtain

$$Y_n \leqq \frac{\gamma l^n}{t^{\frac{q}{2(q-p)}}} Y_{n+1}^{\frac{q(2-p)}{2(q-p)}} \left(\iint_{Q_{n+1}} u^p \, dx \, d\tau\right)^{\frac{1}{2}}, \qquad l = 2^{\frac{pq}{2(q-p)}}. \tag{III.6.14}$$

If $1 < r \leqq p$ we majorize the last integral in (III.6.14) by

$$\left(\iint_{Q_{n+1}} u^r \, dx \, d\tau\right)^{\frac{1}{2}} Y_{n+1}^{\frac{p-r}{2}},$$

and rewrite the inequality that results as

$$Y_n \leqq \frac{\gamma l^n}{t^{\frac{q}{2(q-p)}}} Y_{n+1}^{\frac{q(2-p)}{2(q-p)} + \frac{p-r}{2}} \left(\frac{1}{t} \int_0^t \int_{B_{2R}} u^r \, dx \, d\tau\right)^{\frac{1}{2}} t^{\frac{1}{2}}. \tag{III.6.15}$$

Using the value of q given by (III.6.10), we find that

$$Y_n \leqq \gamma l^n Y^{1 - \frac{\varkappa_r}{2p}} \left(t^{-\frac{N}{p}} \fint_0^t \int_{B_{2R}} u^r \, dx \, d\tau\right)^{\frac{1}{2}}.$$

By Young's inequality, $\forall\, n = 0, 1, 2, \ldots$, $\forall\, \delta \in (0, 1)$ if $\varkappa_r > 0$,

$$Y_n \leqq \delta Y_{n+1} + \gamma(\delta)\, l^{\frac{2p}{\varkappa_r} n} \left(t^{-\frac{N}{p}} \fint_0^t \int_{B_{2R}} u^r \, dx \, d\tau\right)^{\frac{p}{\varkappa_r}}.$$

By an interpolation argument entirely analogous to the one in Theorem I.4.1, we conclude that $\forall\, 0 < t \leqq T$, $\forall\, R > 0$

$$\sup_{x \in B_R} u(x, t) \leqq \gamma t^{-\frac{N}{\varkappa_r}} \left(\sup_{0 < \tau < t} \int_{B_{2R}} u^r(\tau) \, dx\right)^{\frac{p}{\varkappa_r}}, \tag{III.6.16}$$

provided (III.6.6) holds.

From this and Lemma III.6.1

$$\|u(t)\|_{\infty,B_R} \leqq \gamma t^{-\frac{N}{\varkappa_r}} \left\{ \int_{B_{4R}} u_0^r \, dx + \left(\frac{t^r}{R^{\varkappa_r}} \right)^{\frac{1}{2-p}} \right\}^{\frac{p}{\varkappa_r}}$$

$$\leqq \gamma \left\{ t^{-\frac{N}{\varkappa_r}} \left(\int_{B_{4R}} u_0^r \, dx \right)^{\frac{p}{\varkappa_r}} + \left(\frac{t^{-\frac{N}{p}} t^{\frac{r}{2-p}}}{R^{\frac{\varkappa_r}{2-p}}} \right)^{\frac{p}{\varkappa_r}} \right\}$$

$$\leqq \gamma \left\{ t^{-\frac{N}{\varkappa_r}} \left(\int_{B_{4R}} u_0^r \, dx \right)^{\frac{p}{\varkappa_r}} + \left(\frac{t}{R^p} \right)^{\frac{1}{2-p}} \right\}.$$

This last inequality holds true even if (III.6.6) is violated and the theorem follows by suitably redefining R and the constant γ.

If $r > p$ we take in (III.6.4) the test functions $(u - k_n)_+^{r-1} \zeta_n^p$ and proceed as before with obvious modifications.

III.7. Condition (III.6.1) is sharp. We will show that if u is a weak solution of (III.1.1) with

$$u_0 \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$$

and if $N(p-2) + p = 0$, then in general

$$x \to u(x, t) \notin L^\infty_{\mathrm{loc}}(\mathbf{R}^N), \quad t > 0.$$

Let $\alpha, \beta, \varepsilon > 0$ and consider the function

$$z = \frac{(\varepsilon^2 - \varrho^2)_+^\alpha}{\varrho^N |\ln \varrho^2|^\beta}, \quad \varrho \equiv |x|. \tag{III.7.1}$$

One verifies that if $\varepsilon < 1$ and $\beta > 1$

$$z \in L^1(\mathbf{R}^N), \quad \text{but} \quad z \notin L^r(\mathbf{R}^N), \quad \forall\, r > 1. \tag{III.7.2}$$

Let u be the unique solution of (III.1.1) with

$$u_0 = z, \quad N \geqq 2, \qquad p = \frac{2N}{N+1} \quad (\text{i.e. } N(p-2) + p = 0). \tag{III.7.3}$$

Let $h > 0$ and set

$$v = (1 - ht)_+ z. \tag{III.7.4}$$

Proposition III.7.1. *The numbers* $\varepsilon \in (0, 1)$, $\beta > 1$, $\alpha > 1$, $h > 0$ *can be chosen so that*

$$v_t - \operatorname{div}(|Dv|^{p-2} Dv) \leqq 0 \quad \text{in } \mathscr{D}'(S_T).$$

This fact being assumed for the moment, it follows from the comparison principle that

$$u \geqq v \quad \text{a.e. } S_T.$$

Hence $u \in L^1(0, T; L^1_{\text{loc}}(\mathbf{R}^N))$ but

$$x \to u(x, t) \notin L^r_{\text{loc}}(\mathbf{R}^N) \quad \forall r > 1.$$

The comparison principle here is applied as follows. By the definition of weak solution, the truncated functions $u_k = u \wedge k$ are, $\forall k > 0$, distributional supersolutions of (III.1.1) (see § I.1). Setting

$$w = v - u, \quad w_{(k)} = v - u_k,$$

we find

$$\forall\, 0 < s < t \leqq T, \quad \forall \psi \in X_{\text{loc}}(S_T), \quad \forall \zeta \in C_0^\infty(\mathbf{R}^N), \quad \zeta \geqq 0, \tag{III.7.5}$$

$$\int_s^t \int_{\mathbf{R}^N} \left\{ \frac{\partial}{\partial t} w_{(k)} (\psi - v)_+ \zeta + [|Dv|^{p-2} Dv - |Du_k|^{p-2} Du_k]\, D((\psi - v)_+ \zeta) \right\} dx\, d\tau \leqq 0.$$

Observe that $w(\tau) \to 0$ as $\tau \to 0$ in $L^1_{\text{loc}}(\mathbf{R}^N)$. Therefore we may proceed as in the proof of the uniqueness theorem in § II.3 and prove the analog of Proposition II.3.1, *i.e.*, $\forall\, 0 < t \leqq T$, $\forall q \geqq 1$, $\exists \gamma = \gamma(N, p, q)$ such that $\forall R > 0$, $\forall \sigma \in (0, 1)$,

$$\int_{B_R} (w^+(t))^q\, dx \leqq \frac{\gamma}{(\sigma R)^p} \int_0^t \int_{B_{(1+\sigma)R}} (w^+)^{p-2+q}\, dx\, d\tau. \tag{III.7.6}$$

Proceeding as in the proof of Theorem II.1.1, we find $w^+ = 0$, i.e., $v \leqq u$ a.e. S_T.

Proof of Proposition III.7.1. By direct calculation on the set $\varrho > 0$,

$$Dz = -\frac{z}{\varrho^2} Fx,$$

where

$$F = \left\{ N + \frac{2\beta}{\ln \varrho^2} + \frac{2\alpha\varrho^2}{\varepsilon^2 - \varrho^2} \right\}. \tag{III.7.7}$$

We will choose

$$\varepsilon^2 = e^{-2k}$$

and $k \in \mathbf{N}$ so large that $F > 0$. Also

$$|Dz|^{p-2} Dz = -\frac{z^{p-1} F^{p-1}}{\varrho^p} x,$$

$$\operatorname{div}(|Dz|^{p-2} Dz) = -(p-1) \frac{z^{p-2} F^{p-1}}{\varrho^p} Dz \cdot x + p \frac{z^{p-1} F^{p-1}}{\varrho^{p+1}} D\varrho \cdot x$$

$$- N \frac{z^{p-1} F^{p-1}}{\varrho^p} - (p-1) \frac{z^{p-1} F^{p-2}}{\varrho^p} DF \cdot x.$$

Since

(i) $$Dz \cdot x = -zF,$$

(ii) $$D\varrho \cdot x = \varrho,$$

(iii) $$DF \cdot x = \frac{-4\beta}{\ln^2 \varrho^2} + \frac{4\alpha\varrho^2}{(\varepsilon^2 - \varrho^2)} + \frac{4\alpha\varrho^4}{(\varepsilon^2 - \varrho^2)^2}$$

we obtain

$$\operatorname{div}(|Dz|^{p-2} Dz) = \frac{z^{p-1}F^{p-2}}{\varrho^p}\{(p-1)F^2 - (N-p)F - (p-1)DF \cdot x\}. \tag{III.7.8}$$

We calculate the expression in braces on the right-hand side of (III.7.8) using the definition of F and the fact that $N(p-2)+p=0$, to obtain

$$\operatorname{div}(|Dz|^{p-2} Dz) = 2(p-1)\frac{z^{p-1}F^{p-2}}{\varrho^p}\left\{\frac{2\beta(\beta+1)}{\ln^2 \varrho^2}\right. \tag{III.7.8'}$$

$$\left. + \frac{2\alpha(\alpha-1)\varrho^4}{(\varepsilon^2-\varrho^2)_+^2} + \frac{4\alpha\beta\varrho^2}{\ln \varrho^2(\varepsilon^2-\varrho^2)_+} + \frac{N\beta}{\ln \varrho^2} + \frac{(N-2)\alpha\varrho^2}{(\varepsilon^2-\varrho^2)_+}\right\}$$

$$\geqq 2(p-1)\frac{z^{p-1}F^{p-2}}{\varrho^p}$$

$$\cdot\left\{\frac{2\alpha\varrho^2}{(\varepsilon^2-\varrho^2)_+}\left[(\alpha-1)\frac{\varrho^2}{(\varepsilon^2-\varrho^2)_+} + \frac{2\beta}{\ln \varrho^2}\right] + \frac{N\beta}{\ln \varrho^2}\right\}.$$

Consider the sets

$$\mathscr{E}_k^{(1)} \equiv \{e^{-2(k+1)} \leqq \varrho^2 < e^{-2k}\}, \qquad k \in \mathbf{N},$$

$$\mathscr{E}_k^{(2)} \equiv \{\varrho^2 < e^{-2(k+1)}\}.$$

On $\mathscr{E}_k^{(1)}$

$$\operatorname{div}(|Dz|^{p-2} Dz) \geqq 2(p-1)\frac{z^{p-1}F^{p-2}}{\varrho^p}\left\{\frac{2\alpha(\alpha-1)}{e^4} - \frac{2\alpha\beta}{k} - \frac{N\beta}{2k}\right\}.$$

Thus $\alpha > 2$, $\beta > 1$ and $k > 1$ can be chosen so that

$$\operatorname{div}(|Dz|^{p-2} Dz) \geqq 0 \quad \text{on } \mathscr{E}_k^{(1)}.$$

On $\mathscr{E}_k^{(2)}$, if k is sufficiently large,

$$F \geqq N - \frac{\beta}{k+1} \geqq (N-1),$$

so that

$$\operatorname{div}(|Dz|^{p-2} Dz) \geqq -2(p-1)\frac{z^{p-1}F^{p-2}}{\varrho^p}\left\{-\frac{4\alpha\beta}{k+1} - \frac{N\beta}{2(k+1)}\right\}$$

$$\geqq -\frac{\gamma}{(k+1)}\frac{z^{p-1}}{\varrho^p}$$

where

$$\gamma = 2(p-1)(N-1)^{p-2}(4\alpha+N)\beta.$$

Finally with v given by (III.7.4) we compute in $\{0 < \varrho < \varepsilon\}$

$$\begin{aligned}\mathscr{L}(v) &\equiv v_t - \operatorname{div}(|Dv|^{p-2} Dv) \\ &= -hz - (1-ht)_+^{p-1} \operatorname{div}(|Dz|^{p-2} Dz).\end{aligned}$$

On $\mathscr{E}_k^{(1)}$, $\mathscr{L}(v) \leqq 0$ and on $\mathscr{E}_k^{(2)}$

$$\mathscr{L}(v) \leqq z\left[-h + \frac{\gamma}{(k+1)} \frac{z^{p-2}}{\varrho^p}\right].$$

By calculation on $\mathscr{E}_k^{(2)}$

$$\frac{\gamma}{(k+1)} \frac{z^{p-2}}{\varrho^p} \leqq \frac{\gamma}{(k+1)} (k+1)^{2-p} e^{2k(2-p)} \equiv \gamma^*(k)$$

where we have used the fact that $N(p-2)+p=0$. Therefore

$$\mathscr{L}(v) \leqq v(-h + \gamma^*(k)).$$

Choosing $h = \gamma^*(k)$ proves that

$$\mathscr{L}(v) \leqq 0 \qquad \text{on } 0 < \varrho < \varepsilon.$$

If the inequality is viewed in the sense of § I.1, then the test functions all vanish near $\varrho = 0$ and the result follows.

III.8. The case u_0 a measure and $p > 2N/(N+1)$. We prove the solvability of the Cauchy problem

$$\begin{gathered} u_t - \operatorname{div}(|Du|^{p-2} Du) = 0 \quad \text{in } \mathbf{R}^N \times (0, \infty) \\ u(\cdot, 0) = \mu \geqq 0, \qquad \frac{2N}{N+1} < p < 2, \end{gathered} \tag{III.8.1}$$

where μ is a σ-finite Borel measure in $\mathbf{R}^N$ with no growth condition as $|x| \to \infty$. We let $S_\infty \equiv \mathbf{R}^N \times (0, \infty)$.

Theorem III.8.1. *For every non-negative σ-finite Borel measure μ in $\mathbf{R}^N$, there is a measurable function $u: S_\infty \to \mathbf{R}^+$ satisfying*

$$u \in C^\alpha_{\text{loc}}(S_\infty) \quad \textit{for some } \alpha = \alpha(N,p) \in (0,1), \tag{III.8.2}$$

$$|Du| \in L^p_{\text{loc}}(S_\infty), \tag{III.8.3i}$$

$$(x,t) \to Du(x,t) \in C^\alpha_{\text{loc}}(S_\infty) \quad \textit{for some } \alpha = \alpha(N,p) \in (0,1), \tag{III.8.3ii}$$

$$u_t \in L^2_{\text{loc}}(S_\infty). \tag{III.8.4}$$

$\forall\, 0 < s < t < \infty$, $\forall\, \varphi \in C^\infty(\mathbf{R}^N \times [0, T])$ *such that* $\operatorname{supp}\{x \to \varphi(x,t)\} \subset B_\varrho$, $\forall\, t \geqq 0$ *for some* $\varrho > 0$,

$$\text{(i)} \quad \int_{\mathbf{R}^N} (u\varphi)(t)\, dx + \int_s^t \int_{\mathbf{R}^N} \{-u\varphi_t + |Du|^{p-2} Du\, D\varphi\}\, dx\, d\tau = \int_{\mathbf{R}^N} (u\varphi)(s)\, dx, \tag{III.8.5}$$

$$\text{(ii)} \qquad \lim_{s \to 0} \int_{B_\varrho} (u\varphi)(s)\, dx = \int_{B_\varrho} \varphi(0,x)\, d\mu.$$

Proof. Let $\{\mu_n\}$, $n \in \mathbf{N}$ be a sequence of $C^\infty(\mathbf{R}^N)$ functions and let there be a constant γ independent of n such that for all $\varrho > 0$

$$\text{(i)} \qquad \int_{B_\varrho} \mu_n \, dx \leqq \gamma \int_{B_\varrho} d\mu,$$

$$\text{(ii)} \qquad \int_{\mathbf{R}^N} \varphi\mu_n \, dx \to \int_{\mathbf{R}^N} \varphi \, d\mu, \quad \forall \varphi \in C_0^\infty(\mathbf{R}^N). \tag{III.8.6}$$

Let u_n, $n = 1, 2, \ldots,$ be the unique weak solution of

$$\begin{gathered} \frac{\partial}{\partial t} u_n - \operatorname{div}(|Du_n|^{p-2} Du_n) = 0 \quad \text{in } \mathbf{R}^N \times (0, \infty) \\ u_n(\cdot, 0) = \mu_n. \end{gathered} \tag{III.8.7$_n$}$$

Since $p > 2N/(N+1)$, from Theorem III.6.1 with $r = 1$ (see Remark III.6.2), and (III.8.6)-(i) it follows that

$$\{u_n\}_{n\in\mathrm{N}} \text{ is locally } \textit{equibounded} \text{ in } S_\infty.$$

Since also $p > \max\left\{1; \dfrac{2N}{N+2}\right\}$ by the results of [12, 13], there exists $\alpha = \alpha(N, p) \in (0, 1)$ independent of n such that

$$(x, t) \to u_n(x, t) \in C^\alpha_{\mathrm{loc}}(S_\infty) \qquad \text{uniformly in } n$$

$$(x, t) \to Du_n(x, t) \in C^\alpha_{\mathrm{loc}}(S_\infty) \qquad \text{uniformly in } n.$$

By a standard diagonalization process a subsequence can be selected and relabelled by n such that

$$u_n, Du_n \to u, Du \quad \text{uniformly on every compact subset } \mathscr{K} \text{ of } S_\infty,$$

$$\frac{\partial}{\partial t} u_n \rightharpoonup \frac{\partial}{\partial t} u, \quad \text{weakly in } L^2(\mathscr{K}).$$

Therefore (III.8.5)-(i) holds for u, $\forall\, 0 < s < t < \infty$.

Let φ be a test function as in (III.8.5). Multiply the first of (III.8.7)$_n$ by φ and integrate over $B_\varrho \times (0, s)$ to obtain

$$\left| \int_{B_\varrho} (u_n\varphi)(s) \, dx - \int_{B_\varrho} \mu_n \varphi(0) \, dx \right| \leqq \int_0^s \int_{B_\varrho} |Du_n|^{p-1} \, |D\varphi| \, dx \, d\tau.$$

By Corollary III.3.1 and (III.8.6)-(i), the right-hand side is majorized by

$$\gamma(\varrho, N, p) \, s^{\frac{1}{p}} \left(1 + \int_{B_{2\varrho}} d\mu\right)^{\frac{2}{p}(p-1)}.$$

Letting $n \to \infty$

$$\left| \int_{B_\varrho} (u\varphi)(s) \, dx - \int_{B_\varrho} \varphi(0, x) \, d\mu \right| \leqq \gamma s^{\frac{1}{p}} \left(1 + \int_{B_{2\varrho}} d\mu\right)^{\frac{2}{p}(p-1)},$$

whence (III.8.5-ii).

Added in proof

PHILIPPE BÉNILAN has informed us that, in a note in preparation with T. GALLOUËT, he has also introduced a notion of solution for elliptic equations of the type

$$\operatorname{div}(|Du|^{p-2}Du) = f \in L^1(\Omega), \qquad \forall p > 1,$$

where Ω is a domain in $\mathbf{R}^N$. Such a notion is irrespective of the sign of u and could give an a.e. meaning to Du for the semigroup solution of (0.1) if one had the *global* information $u_0 \in L^1(\mathbf{R}^N)$.

A remark of STEFAN LUCKHAUS. The arguments of §III.5(i) show that indeed $Dv_n \to Dv$ strongly in $L^p_{loc}(S_T)$. This implies that that $\forall k > 0$, $Du_{n,k} \to Du_k$ strongly in $L^p_{loc}(S_T)$.

In discussion with DI BENEDETTO, S. LUCKHAUS found a simple way to show the latter. First, in (III.4.2$_n$) take the test functions $(u_{n,k} - u_k)\varphi$, where u is the limit of the u_n and $\varphi \in C_0^\infty(S_T)$ is not negative. The *a priori* estimates of §III.3 and the construction procedure of §III.4 can be used to establish these as admissible test functions, and some calculations show that

$$\iint_{S_T} |Du_{n,k}|^p \varphi \, dx \, d\tau = \iint_{S_T} |Du_{n,k}|^{p-2} Du_{n,k} Du_k \, \varphi \, dx \, d\tau + O\left(\frac{1}{n}\right). \qquad (1)$$

Now since $[u < k] \subset [u_n > k]$, we have

$$\iint_{S_T} |Du_{n,k}|^{p-2} Du_{n,k} Du_k \, \varphi \, dx \, d\tau \leqq \frac{p-1}{p} \iint_{S_T} |Du_{n,k}|^p \varphi \, dx \, d\tau + \frac{1}{p} \iint_{S_T} |Du_k|^p \varphi \, dx \, d\tau.$$

Put this in (1) and let $n \to \infty$ along a subsequence reindexed by n to get

$$\begin{aligned} \iint_{S_T} |Du_k|^p \varphi \, dx \, d\tau &\leqq \liminf_{n\to\infty} \iint_{S_T} |Du_{n,k}|^p \varphi \, dx \, d\tau \\ &\leqq \limsup_{n\to\infty} \iint_{S_T} |Du_{n,k}|^p \varphi \, dx \, d\tau \leqq \iint_{S_T} |Du_k|^p \varphi \, dx \, d\tau. \end{aligned}$$

This and Lemma II.2.2 can be used to identify more efficiently the limit of the nonlinear term in §III.5(i). The advantage of working with the functions v_n and v, however is that they are not truncated and therefore they supply information on the set where the solution is unbounded. Such a singularity set is non-void in general, as shown by the arguments in §III.7.

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