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# Gelfand–Zetlin Basis for $U_q(g\ell(N+1))$ Modules

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**Abstract.** The Gelfand–Zetlin basis of $U_q(g\ell(N+1))$ modules is constructed via the lowering operator method.

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## 0. Introduction

The purpose of this paper is to give a constructive description of the Gelfand–Zetlin basis for irreducible modules of the quantum universal enveloping algebra $U_q(g\ell(N+1))$.

Gelfand–Zetlin [1] proposed an orthonormal basis for finite-dimensional irreducible representations of the Lie algebra $g_N = g\ell(N+1, \mathbb{C})$. The effectiveness of this method is due to the fact that the branching $g_N \downarrow g_{N-1}$ has simple spectrum.

Much has been studied on the construction of this basis, in particular, Zhelobenko [2] illustrated it in detail with the method of the lowering operators. Recently, Jimbo [3] has shown the existence of a $q$-analog of the Gelfand–Zetlin basis for $U_q(g\ell(N+1))$.

In this paper, with the lowering operator method we reprove Jimbo's results for $U_q(g\ell(N+1))$.

This paper is organized as follows: Section 1 deals with the definition of the quantum universal enveloping algebra $U_q(g\ell(N+1))$ and its irreducible representations. In Section 2, we will introduce the lowering operators for $U_q(g\ell(N+1))$ and investigate their properties. In Section 3, we will establish the branching law for $U_q(g\ell(N+1)) \downarrow U_q(g\ell(N))$, and show the existence of the Gelfand–Zetlin basis for irreducible $U_q(g\ell(N+1))$ modules of finite dimensions with the aid of the lowering operators. This basis is orthonormal under a natural Hermitian inner product and the modules turn out to be unitarizable. Finally, we get the explicit action of the generators on this basis.

The details and further investigations of the results in this paper will be discussed in a forthcoming paper.

## 1. Quantum Universal Enveloping Algebra $U_q(g\ell(N+1))$

Let $q$ be a positive number ($q \neq 1$). The quantum universal enveloping algebra $U_N = U_q(g\ell(N+1))$ is, by definition,

$$U_N = \mathbb{C}[q^{\pm 1/2\varepsilon_i}\ (0 \leqslant i \leqslant N), e_j, f_j\ (1 \leqslant j \leqslant N)]$$

with defining relations

$$q^{\pm 1/2\varepsilon_i} q^{\pm 1/2\varepsilon_j} = q^{\pm 1/2\varepsilon_j} q^{\pm 1/2\varepsilon_i}, \qquad q^{1/2\varepsilon_i} q^{-1/2\varepsilon_i} = q^{-1/2\varepsilon_i} q^{1/2\varepsilon_i} = 1,$$

$$q^{1/2\varepsilon_i} e_j q^{-1/2\varepsilon_i} = \begin{cases} q^{1/2} e_j \\ q^{-1/2} e_j, \\ e_j \end{cases} \qquad q^{1/2\varepsilon_i} f_j q^{-1/2\varepsilon_i} = \begin{cases} q^{-1/2} f_j, & i = j-1, \\ q^{1/2} f_j, & i = j, \\ f_j, & i \neq j-1, j, \end{cases}$$

$$[e_i, f_j] = \delta_{ij} \frac{k_i^2 - k_i^{-2}}{q - q^{-1}} \quad (k_i = q^{1/2(\varepsilon_{i-1} - \varepsilon_i)}),$$

$$[e_i, e_j] = [f_i, f_j] = 0, \quad \text{for } |i-j| \geqslant 2,$$

$$e_j^2 e_{j\pm 1} - [2] e_j e_{j\pm 1} e_j + e_{j\pm 1} e_j^2 = 0,$$

$$f_j^2 f_{j\pm 1} - [2] f_j f_{j\pm 1} f_j + f_{j\pm 1} f_j^2 = 0,$$

where

$$[m] = \frac{q^m - q^{-m}}{q - q^{-1}}$$

is a $q$-integer.

Let $U_q(\underline{n}_+)$ (resp. $U_q(\underline{n}_-)$) be the subalgebra of $U_q(g\ell(N+1))$ generated by $e_j$'s (resp. $f_j$'s), and $U_q(\underline{h})$ be the subalgebra generated by $q^{\pm 1/2\varepsilon_i}$'s. In [5], Rosso has shown the following proposition.

PROPOSITION 1. *We have the triangular decomposition*

$$U_q(g\ell(N+1)) \simeq U_q(\underline{n}_-) \otimes U_q(\underline{h}) \otimes U_q(\underline{n}_+)$$

*as vector spaces.*

The sequence of subalgebras $U_1 \subset U_2 \subset \cdots \subset U_{N-1} \subset U_N$ is specified by

$$U_n = \mathbb{C}[q^{\pm 1/2\varepsilon_i} \quad (0 \leqslant i \leqslant n), \qquad e_j, \quad f_j\ (1 \leqslant j \leqslant n)].$$

We introduce an involution * by

$$(q^{\pm 1/2\varepsilon_i})^* = q^{\pm 1/2\varepsilon_i}, \qquad e_j^* = f_j, \quad f_j^* = e_j.$$

This * algebra is regarded as the 'compact real form' $U_q(u(N+1))$.

Let $V = V(\Lambda)$ be an irreducible left $U_N$ module of finite dimensions with highest weight

$$\Lambda = (\lambda_0, \lambda_1, \ldots, \lambda_N) \quad (\lambda_j \in \mathbb{Z}_+, \lambda_0 \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_N).$$

Denote by $|\mathrm{vac}\rangle$ the highest weight vector of $V$:

$$V = U_N|\mathrm{vac}\rangle, \qquad e_j|\mathrm{vac}\rangle = 0 \quad (1 \leqslant j \leqslant N),$$

$$q^{1/2\varepsilon_i}|\mathrm{vac}\rangle = q^{1/2\lambda_i}|\mathrm{vac}\rangle \quad (0 \leqslant i \leqslant N).$$

Let $V^*$ be the dual of $V$, that is, irreducible right $U_N$ module defined by

$$V^* = \langle \mathrm{vac}|U_N, \qquad \langle \mathrm{vac}|f_j = 0 \quad (1 \leqslant j \leqslant N),$$

$$\langle \mathrm{vac}|q^{1/2\varepsilon_i} = q^{1/2\lambda_i}\langle \mathrm{vac}| \quad (0 \leqslant i \leqslant N).$$

Due to Proposition 1, there exists a natural pairing $V^* \otimes V \to \mathbb{C}$ defined by $\langle \mathrm{vac}|b \otimes a|\mapsto \mathrm{vac}\rangle \mapsto \langle \mathrm{vac}|ba|\mathrm{vac}\rangle$.

## 2. Lowering Operators for $U_q(gl(N+1))$

For $0 \leqslant i \leqslant n \leqslant N$, the operators $d_{ni} \in U_q(\underline{h}) \otimes U_q(\underline{n}_-)$, which are referred to as lowering operators, are inductively defined as follows:

$$d_{nn} = 1, \qquad d_{n,n-1} = f_n,$$

$$d_{ni} = \langle \varepsilon_i - \varepsilon_{n-1} + n - i\rangle f_n d_{n-1,i} - \langle \varepsilon_i - \varepsilon_{n-1} + n - i - 1\rangle d_{n-1,i} f_n,$$

where

$$\langle \varepsilon_i - \varepsilon_j + l\rangle = \frac{q^{\varepsilon_i - \varepsilon_j + l} - q^{-(\varepsilon_i - \varepsilon_j + l)}}{q - q^{-1}}.$$

The next theorem and lemma are fundamental for our arguments.

THEOREM 2. *For fixed n, the operators $d_{ni}$ $(0 \leqslant i \leqslant n)$ mutually commute.*

LEMMA 3. *The lowering operators have the following properties*:

(i) $e_j d_{ni} \equiv 0 \bmod I_{n-1}$ *for* $1 \leqslant j \leqslant n-1$,

*where* $I_{n-1} = \Sigma_{j=1}^{n-1} U_N e_j$ *is a left ideal of* $U_N$.

(ii) $e_n d_{ni} = d_{ni}(1) e_n + d_{n-1,i}\langle \varepsilon_i - \varepsilon_n + n - i - 1\rangle$,

*where* $d_{ni}(1) = \langle \varepsilon_i - \varepsilon_{n-1} + n - i + 1\rangle f_n d_{n-1,i} - \langle \varepsilon_i - \varepsilon_{n-1} + n - i\rangle d_{n-1,i} f_n$.

(iii) $d_{ni}(1) d_{n-1,i} = d_{n-1,i} d_{ni}$.

Putting $c_{in} = d_{ni}^*$, we readily see that

$$c_{in}^m d_{ni}^m \equiv \prod_{l=1}^{m} \prod_{k=1}^{n-i} \langle \varepsilon_i - \varepsilon_{i+k} + k - l + 1\rangle (e_{i+1} \cdots e_n)^m d_{ni}^m \bmod I_{n-1}.$$

One can show the following relations by virtue of Lemma 3 and induction on $m$.

PROPOSITION 4.

(i) $$(e_{i+1}\cdots e_n)^m d_{ni}^m \equiv (e_{i+1}\cdots e_n)^{m-1} e_{i+1}\cdots e_{n-1} d_{ni}(1)^m e_n + \\ + \prod_{k=1}^{n-i-1} \langle \varepsilon_i - \varepsilon_{i+k} + k - m\rangle \sum_{l=0}^{m-1} \langle \varepsilon_i - \varepsilon_n + n - i - 1 - 2l\rangle \times \\ \times (e_{i+1}\cdots e_n)^{m-1} d_{ni}^{m-1} \bmod I_{n-1}.$$

(ii) $$(e_{i+1}\cdots e_n)^m d_{ni}^m \equiv [m]! \prod_{l=1}^{m} \prod_{k=1}^{n-i} \langle \varepsilon_i - \varepsilon_{i+k} + k - l\rangle \bmod I_n,$$

*where* $[m]! = [m][m-1]\cdots[1]$.

Thus we obtain the key to the arguments in the next section.

PROPOSITION 5. For multi-indices $\alpha = (\alpha_0, \ldots, \alpha_{n-1})$, $\beta = (\beta_0, \ldots, \beta_{n-1})$, $\alpha \geqslant \beta$ stands for the lexicographic order. Setting $d_n^\alpha = d_{n,0}^{\alpha_0}\cdots d_{n,n-1}^{\alpha_{n-1}}$ ($c_n^\alpha$ is similarly defined), we have for $\alpha \geqslant \beta$,

$$c_n^\alpha d_n^\beta \equiv \delta_{\alpha\beta}[\alpha]! \prod_{i=0}^{n-1} \left\{ \prod_{l=1}^{\alpha_i} \prod_{k=1}^{n-i} \langle \varepsilon_i - \varepsilon_{i+k} + k - l\rangle \times \right. \\ \left. \times \prod_{l=1}^{\alpha_i} \prod_{k=1}^{n-i-1} \langle \varepsilon_i - \varepsilon_{i+k} + k - l + 1 + \alpha_{i+k}\rangle \right\} \bmod I_n,$$

*where* $[\alpha]! = [\alpha_0]!\cdots[\alpha_{n-1}]!$.

## 3. Gelfand–Zetlin Basis

The natural pairing between $V$ and $V^*$ induces an Hermitian inner product on $V$ by

$$(a|\text{vac}\rangle, b|\text{vac}\rangle) = \langle \text{vac}|a^* b|\text{vac}\rangle.$$

Let $\mu_{iN} = \lambda_i$. The sequence of integer vectors

$$\mu = \begin{pmatrix} \mu_N \\ \mu_{N-1} \\ \vdots \\ \mu_1 \\ \mu_0 \end{pmatrix} = \begin{pmatrix} \mu_{0N} & \mu_{1N} & \cdots\cdots & \mu_{NN} \\ \mu_{0,N-1} & \mu_{1,N-1} & \cdots & \mu_{N-1,N-1} \\ & \ddots & \iddots & \\ & \mu_{01} & \mu_{11} & \\ & & \mu_{00} & \end{pmatrix}$$

is called the Gelfand–Zetlin scheme attached to the module $V(\Lambda)$, if each pair of vectors $\mu_{k-1}, \mu_k$ satisfies the condition $\mu_{i,k} \geqslant \mu_{i,k-1} \geqslant \mu_{i+1,k}$ for all $i, k$. For each scheme, put

$$d^\mu = d_1^{\mu_1-\mu_0} d_2^{\mu_2-\mu_1}\cdots d_N^{\mu_N-\mu_{N-1}},$$

where

$$\mu_n - \mu_{n-1} = (\mu_{0n} - \mu_{0,n-1}, \ldots, \mu_{n-1,n} - \mu_{n-1,n-1}).$$

From Lemma 3, it follows that

$$e_j d_N^{\mu_N - \mu_{N-1}}|\text{vac}\rangle = 0 \quad (1 \leqslant j \leqslant N-1).$$

Hence, we see a weak form of the branching law

$$V(\Lambda)|_{U_q(g\ell(N))} \supset \bigoplus_{\substack{0 \leqslant \alpha_i \leqslant \lambda_i - \lambda_{i+1} \\ 0 \leqslant i \leqslant N-1}} V(\lambda_1 + \alpha_0, \ldots, \lambda_N + \alpha_{N-1}).$$

As $q$ tends to 1 (the classical limit), $U_q(g\ell(N+1))$ goes to the universal enveloping algebra $U(g\ell(N+1))$, and the module $V(\Lambda)$ to the irreducible $U(g\ell(N+1))$ module $\tilde{V}(\Lambda)$. For $\tilde{V}(\Lambda)$, the branching law has been already established:

$$\tilde{V}(\Lambda)|_{U(g\ell(N))} = \bigoplus_{\substack{0 \leqslant \alpha_i \leqslant \lambda_i - \lambda_{i+1} \\ 0 \leqslant i \leqslant N-1}} \tilde{V}(\lambda_1 + \alpha_0, \ldots, \lambda_N + \alpha_{N-1}).$$

The weight spaces of $V(\Lambda)$ have the same dimensions as the corresponding weight spaces of $\tilde{V}(\Lambda)$ and we have $\dim V(\Lambda) = \dim \tilde{V}(\Lambda)$ ([4]). Thus, we obtain the following theorem.

THEOREM 6. *(Branching law for $U_N \downarrow U_{N-1}$) We have*

$$V(\Lambda)|_{U_{N-1}} = \bigoplus_{\substack{0 \leqslant \alpha_i \leqslant \lambda_i - \lambda_{i+1} \\ 0 \leqslant i \leqslant N-1}} V(\lambda_1 + \alpha_0, \ldots, \lambda_N + \alpha_{N-1}).$$

Using successively Proposition 5 and Theorem 6, we show the following proposition.

PROPOSITION 7. *The vectors $|\mu\rangle = (1/N_\mu)d^\mu|\text{vac}\rangle$ form a basis of the module $V(\Lambda)$ and are orthonormal under the Hermitian inner product. Here*

$$N_\mu^2 = \prod_{n=1}^{N} \tau_n(\mu_{n-1}, \mu_n),$$

$$\tau_n(\mu_{n-1}, \mu_n)$$

$$= \prod_{0 \leqslant i \leqslant j \leqslant n-1} \frac{[\mu_{in} - \mu_{j,n-1} + j - i]!}{[\mu_{i,n-1} - \mu_{j,n-1} + j - i]!} \prod_{0 \leqslant i < j \leqslant n} \frac{[\mu_{in} - \mu_{jn} + j - i - 1]!}{[\mu_{i,n-1} - \mu_{jn} + j - i - 1]!}.$$

From the definition of our inner product, it turns out that the module $V(\Lambda)$ is unitarizable:

COROLLARY 8. *For $\xi, \eta \in V(\Lambda)$ and $a \in U_N$, $(\xi, a\eta) = (a^*\xi, \eta)$.*

For a Gelfand–Zetlin scheme $\mu$, we set

$$\xi = d_{n+1}^{\mu_{n+1} - \mu_n} \cdots d_N^{\mu_N - \mu_{N-1}}|\text{vac}\rangle.$$

Then we have the following proposition.

PROPOSITION 9.

$$e_n d_n^{\mu_n - \mu_{n-1}} \xi$$
$$= \sum_{j=0}^{n-1} \frac{N_{\mu'}^2}{[\mu_{j,n-1} - \mu_{n-1,n-1} + n - j] N^2_{\mu' + \delta_j(n-1)}} d_{n-1,j} d_n^{\mu_n - \mu_{n-1} - \delta_j} \xi,$$

*where*

$$\mu' = \begin{pmatrix} \ddots & \ddots & & & \cdot^{\cdot^{\cdot}} \\ \mu_{0n} & \mu_{1n} & \cdots\cdots\cdots\cdots & & \mu_{nn} \\ & \mu_{0,n-1} & \mu_{1,n-1} & \cdots\cdots & \mu_{n-1,n-1} \\ & & \mu_{0,n-1} & \mu_{1,n-1} & \\ & & \ddots & \ddots & \\ & & & \mu_{0,n-1} & \end{pmatrix},$$

$$\mu_n - \mu_{n-1} - \delta_j = (\mu_{kn} - \mu_{k,n-1} - \delta_{jk})_{0 \leqslant k \leqslant n-1},$$

*and* $\mu' + \delta_j(n-1)$ *indicates that we should replace the* $\mu_{j,n-1}$ *in the* $(n-1)$*th row with* $\mu_{j,n-1} + 1$ *in* $\mu'$.

Now we can write down the action of generators of $U_q(g\ell(N+1))$ on the basis vectors $|\mu\rangle$. Substituting the formula for $N_\mu$ to Proposition 9, we obtain the following proposition.

THEOREM 10. (cf. Jimbo [3])

(i) $q^{1/2\varepsilon_n}|\mu\rangle = q^{1/2\{\Sigma_{k=0}^{n} \mu_{kn} - \Sigma_{k=0}^{n-1} \mu_{k,n-1}\}}|\mu\rangle \quad (0 \leqslant n \leqslant N),$

(ii) $e_n|\mu\rangle = \sum_{j=0}^{n-1} a_{\mu+\delta_j(n-1),\mu} |\mu + \delta_j(n-1)\rangle \quad (1 \leqslant n \leqslant N),$

(iii) $f_n|\mu\rangle = \sum_{j=0}^{n-1} a_{\mu,\mu-\delta_j(n-1)} |\mu - \delta_j(n-1)\rangle \quad (1 \leqslant n \leqslant N),$

*where*

$$a_{\mu+\delta_j(n-1),\mu}$$
$$= \left[ -\frac{\Pi_{k=0}^{n} [\mu_{kn} - \mu_{j,n-1} + j - k] \, \Pi_{k=0}^{n-2} [\mu_{k,n-2} - \mu_{j,n-1} + j - k - 1]}{\Pi_{k=0,(k \neq j)}^{n-1} [\mu_{k,n-1} - \mu_{j,n-1} + j - k - 1][\mu_{k,n-1} - \mu_{j,n-1} + j - k]} \right]^{1/2}.$$

*and* $\mu \pm \delta_j(k)$ *indicates to replace only* $\mu_{jk}$ *with* $\mu_{jk} \pm 1$ *in* $\mu$.

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