SPACES OF FUNCTIONS WITH ASYMPTOTIC CONDITIONS AND EXISTENCE OF NON COMPACT MAXIMAL SUBMANIFOLDS OF A LORENTZIAN MANIFOLD

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ABSTRACT. We study some properties of weighted Sobolev spaces with asymptotic conditions appropriate to physical conditions on non compact space times. An existence and uniqueness theorem of non compact maximal space-like hypersurfaces for asymptotically flat metrics in a neighbourhood of the Minkowski's metric is proved.

1. INTRODUCTION

The existence of a maximal hypersurface is an important property for a space-time. For the initial value probelm, fundamental in General Relativity, on a submanifold with constant mean extrinsic curvature the system of constraints can be split into a linear system and a non linear equation ([3] [4]), for this non linear ellipic equation global results can be proved on closed manifolds ([2]). Of primary importance is the extension of these results to non compact manifolds. The existence of maximal submanifold is also essential in the positivity conjecture for the gravitational mass of an asymptotically flat space time.

In this paper we prove some theorems for the composition of spaces of functions satisfying asymptotic conditions appropriate for a large variety of physical problems on non compact space times. Using the method given by Y. Choquet-Bruhat [1] for compact manifolds we prove some theorems concerning the existence, uniqueness of non compact maximal space-like hypersurfaces for asymptotically flat metrics on $\mathbb{R}^{n+1}$ of a neighbourhood of Minkowski's metric in appropriate functional spaces.

2. FUNCTIONAL SPACES

We define, [5], [6], [7], for $1 \leqslant p \leqslant \infty$, $s \in \mathbb{N}$, $\delta \in \mathbb{R}$, the space $M^p_{s,\delta}(\mathbb{R}^n)$ to be the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm $\sum_{|\alpha| \leqslant s} \|\sigma^{\delta + |\alpha|} D^\alpha f\|_{L^p(\mathbb{R}^n)}$ with $\sigma(x) = (1 + |x|^2)^{1/2}$.

M. Cantor, [6], prove that the Laplacian is an isomorphism between $M^p_{s,\delta}$ and $M^p_{s-1,\delta+2}$ if $p > \dfrac{n}{n-2}$, $-\dfrac{n}{p} < \delta < n - 2 - \dfrac{n}{p}$.

We define for $1 \leqslant p < \infty$, $s \in \mathbb{N}$, $\delta \in \mathbb{R}$ the space $N^p_{s,\delta}(\mathbb{R}^n)$ to be the completion of $C_0^\infty(\mathbb{R}^n)$

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with respect to the norm $\|f\|_{L^\infty(\mathbb{R}^n)} + \sum_{0<|\alpha|\leq s} \|\sigma^{\delta+|\alpha|} D^\alpha f\|_{L^p(\mathbb{R}^n)}$. If Y is an open set of $\mathbb{R}$, the spaces $M^{p,\infty}_{s,\delta}(Y\times\mathbb{R}^n)$, $L^{p,\infty}_{s,\delta}(Y\times\mathbb{R}^n)$, $N^{p,\infty}_{s,\delta}(Y\times\mathbb{R}^n)$ are obtained by completion with respect to the norms

$$\sum_{|\alpha|+|\beta|=0}^{s} \|\sup_{t\in Y} |\sigma^{\delta+|\alpha|} D_x^\alpha D_t^\beta f(t,x)|\,\|_{L^p(\mathbb{R}^n)}$$

$$\|f\|_{L^\infty(\mathbb{R}^n\times Y)} + \sum_{|\alpha|+|\beta|>0}^{s} \|\sup_{t\in Y} |\sigma^{\delta+|\alpha|} D_x^\alpha D_t^\beta f(t,x)|\,\|_{L^p(\mathbb{R}^n)}$$

$$\sum_{|\beta|=0}^{s} \|\sup_{t\in Y} |D_t^\beta f(t,x)|\,\|_{L^\infty(\mathbb{R}^n)} + \sum_{\substack{|\alpha|+|\beta|=1\\|\alpha|>0}}^{s} \|\sup_{t\in Y} |\sigma^{\delta+|\alpha|} D_x^\alpha D_t^\beta f(t,x)|\,\|_{L^p(\mathbb{R}^n)}.$$

We denote:

$$M^p_{s,\delta}\times M^p_{t,\rho} = \{f\cdot g; f\in M^p_{s,\delta}, g\in M^p_{s,\rho}\}$$

$$M^{p,\infty}_{s,\delta}\circ M^p_{t,\rho} = \{f; f(x)=F(\varphi(x),x), F\in M^{p,\infty}_{s,\delta}(Y\times\mathbb{R}^n), \varphi\in M^p_{t,\rho}(\mathbb{R}^n), \varphi(\mathbb{R}^n)\subset Y\}.$$

We will prove the following:

LEMMA 1. *If* $r=\inf(s,t)$, $p>1$, $\frac{n}{p}-s-t+r<0$

$$M^p_{s,\delta}\times M^p_{t,\rho} \hookrightarrow M^p_{r,\delta+\rho}; \qquad N^p_{s,\delta}\times N^p_{t,\rho+\delta} \hookrightarrow N^p_{\delta+|\delta+\rho|_-}, \qquad \rho\geq 0;$$

$$N^p_{s,\delta}\times M^p_{t,\rho} \hookrightarrow M^p_{r,\rho+|\delta|_-}; \qquad |\delta|_- = \min(0,\delta).$$

Proof. If $p>1$, $\frac{n}{p}-s+\lambda<\frac{n}{q}$, $\nu\leq\delta+|\lambda|$ from Sobolev inequalities we have $\|\sigma^\nu D^\lambda f\|_{L^q(\mathbb{R}^n)} \leq \sum_{|\alpha|=\lambda}^{s} \|\sigma^{|\alpha|+\delta} D^\alpha f\|_{L^p(\mathbb{R}^n)}$. If $|\alpha|\leq r$, we have

$$\|\sigma^{|\alpha|+\nu} D^\alpha(fg)\|_p \leq C \sum_{\beta\leq\alpha} \|\sigma^\theta D^\beta f\|_a \cdot \|\sigma^\varphi D^{\alpha-\beta}\|_b$$

with $\frac{1}{a}+\frac{1}{b}=\frac{1}{p}$ and $\theta+\varphi\geq|\alpha|+\nu$. Each term can be bounded by $C\cdot\|f\|_{M^p_{s,\delta}}\|g\|_{M^p_{t,\rho}}$ if

$$\begin{cases} \frac{n}{p}-s+|\beta|<\frac{n}{a} & 1\leq a\leq\infty \\ \theta\leq\delta+\beta \end{cases} \qquad \begin{cases} \frac{n}{p}-t+|\alpha-\beta|<\frac{n}{b} & 1\leq b\leq\infty \\ \varphi\leq\rho+|\alpha-\beta|. \end{cases}$$

Thus $M^p_{s,\delta} \times M^p_{t,\rho} \hookrightarrow M^p_{r,\delta+\rho}$ if $\frac{n}{p} - s - t + r < 0$ and $\nu \leqslant \delta + \rho$.

LEMMA 2. *If* $p > 1, r \leqslant t \leqslant s, \delta \geqslant 0, \frac{n}{p} - t - s + r + 1 < 0$.

$$M^{p,\infty}_{t,\rho} \circ M^p_{s,\delta} \subset M^\nu_{r,\rho}; \qquad N^{p,\infty}_{t,\rho} \circ M^p_{s,\delta} \subset N^p_{r,\min(\rho,\delta)}$$

$$L^{p,\infty}_{t,\rho} \circ M^p_{s,\delta} \subset N^p_{r,\min(\rho,\delta)}; \qquad M^{p,\infty}_{t,\rho} \circ N^p_{s,\delta} \subset N^p_{r,\rho}, \qquad \frac{n}{p} - t < 0$$

$$N^{p,\infty}_{t,\rho} \circ N^p_{s,\delta} \subset N^p_{r,\min(\rho,\delta)}; \qquad L^{p,\infty}_{t,\rho} \circ N^p_{s,\delta} \subset N^p_{r,\min(\rho,\delta)}.$$

Proof. $D^\alpha f$ is a linear combination with constant coefficients of terms: $D^\beta_x D^\gamma_y F(x, \gamma(x)) \prod_k D^{\lambda_k} y$ with $(\Sigma \lambda_k) + \beta = \alpha, \sum_k 1 = |\gamma|$. By Holder inequality, if $|\gamma| > 0$, we have $\|\sigma^\nu \prod_k D^{\lambda_k} y\|_q \leqslant \prod_k \|\sigma^{\nu_k} D^{\lambda_k} y\|_{q_k}, \sum_k \frac{1}{q_k} = \frac{1}{q}, \nu \leqslant \sum_k \nu_k$. To prove the estimate $\|\sigma^\nu \prod_k D^{\lambda_k} y\|_q \leqslant C\, \|y\|^{|\gamma|}_{M^p_{s,\delta}}$ it is sufficient that $\sum_k \left[\frac{n}{p} - s + \lambda_k\right]_+ < \frac{n}{q}, \nu \leqslant \sum_k (\delta + \lambda_k)$, or using a lemma of Dionne [8] p. 22:

$$\frac{n}{p} - s + (|\alpha| - |\beta|) - |\gamma| + 1 < \frac{n}{q} \qquad \nu \leqslant |\gamma|\, \delta + (|\alpha| - |\beta|).$$

Thus

$$\|\sigma^{\nu+|\alpha|} D^\beta_x D^\gamma_{y}, F \prod_k D^{\lambda_k} y\|_p \leqslant C\, \|\sup_y |\sigma^{\theta_{\beta,\gamma}} D^\beta_x D^\gamma_y F|\, \|_{\Gamma_{\beta,\gamma}} \|\sigma^\theta \prod_k D^{\lambda_k} y\|_q$$

with

$$1/r_{\beta,\gamma} + 1/q = 1/p \qquad p \leqslant r_{\beta,\gamma}$$

$$\theta_{\beta,\gamma} + \theta \geqslant \nu + |\alpha|$$

Thus

$$\|\sigma^{\nu+|\alpha|} D^\beta_x D^\gamma_y F \prod_k D^{\lambda_k} y\|_p \leqslant C\, \|\sup_{y \in \gamma} |\sigma^{\theta_{\beta,\gamma}} D^\beta_x D^\gamma_y F|\, \|_{r_{\beta,\gamma}}\, \|y\|^{|\gamma|}_{M^p_{s,\delta}}$$

with

$$n/\Gamma_{\beta,\gamma} < s - r + |\beta| + |\gamma| - 1; \qquad \theta_{\beta,\gamma} \geqslant \nu - |\gamma|\delta + |\beta|. \qquad (*)$$

Using a lemma of Dionne [8] p. 17 we prove the following estimate

LEMMA 2'. *If* $1 < p$, $\frac{n}{p} - \frac{n}{q} < t - |\beta| - |\gamma|$, $\mu \leqslant \delta + |\beta|$,

$$\| \sup_{t \in \gamma} |\sigma^{\mu} D_x^{\beta} D_y^{\gamma} f| \|_{L^q(\mathbb{R}^n)} \leqslant C \sum_{|\alpha| = |\beta|}^{t - |\gamma|} \| \sup_{t \in \gamma} \|\sigma^{\delta + |\alpha|} D_x^{\alpha} D_y^{\gamma} f| \|_{L^p(\mathbb{R}^n)}.$$

Thus we have an estimate of $\| \sup_{y \in \gamma} \|\sigma^{\theta_{\beta,\gamma}} D_x^{\beta} D_y^{\gamma} F| \|_{r_{\beta,\gamma}}$ by a linear combination with constant coefficients of terms

$$\| \sup_{y \in \gamma} |^{\sigma \rho + |\chi|} D_x^{\chi} D_y^{\gamma} F| \|_p \text{ with } |\chi| = |\beta|, \dots, t - |\gamma|, \text{ if}$$

$$\theta_{\beta,\gamma} \leqslant \rho + |\beta|, \frac{n}{p} - \frac{n}{r_{\beta,\gamma}} < t - (|\beta| + |\gamma|) \tag{**}$$

with $r_{\beta,\gamma} = p$ if $|\beta| + |\gamma| = r = t$.

The conditions (*), (**) are satisfied if we have

$$\text{(I) } |\beta| + \rho \geqslant \nu - |\gamma| \delta + |\beta|; \qquad \text{(II) } \frac{n}{p} - t - s + r + 1 < 0.$$

Thus for $|\gamma| > 1$, the condition (I) is satisfied if $\delta \geqslant 0$ and $\nu \leqslant \rho$ (for $|\gamma| = 0$ we have to estimate $\| \sup_{y \in \gamma} |\sigma^{\nu + |\alpha|} D_x^{\alpha} F(x, y)| \|_p$ and it is sufficient to supposse $\nu \leqslant \rho$).

If the condition (II) is satisfied we can choose $r_{\beta,\gamma}$ if $s - r + |\beta| + |\gamma| - 1 > 0$, this relation is satisfied excepting the case $s = r$, $|\beta| = 0$, $|\gamma| = 1$. In this case we have

$$\|\sigma^{|\alpha| + \nu} D_y F D_y^{\alpha} \|_p \leqslant C \, \|\sigma^{\nu - \delta} D_y F\|_{\infty} \, \|\sigma^{|\alpha| + \delta} D_y^{\alpha} \|_p \leqslant C \, \|F\|_{M_{t,\rho}^{p,\infty}} \, \|y\|_{M_{s,\delta}^p}$$

since the hypothesis implies $\nu - \delta \leqslant \rho$, $t - 1 > \frac{n}{p}$.

The other relations are proved by the same way. From the demonstration of Lemma 2 we have the estimates

$$\|F(x, \varphi(x))\|_{N_{r,\min(\rho,\delta)}^p} \leqslant C \|F\|_{N_{t,\rho}^{p,\infty}} (1 + \|\varphi\|_{M_{s,\delta}^p})^r$$

$$\|F(x, \varphi(x))\|_{N_{r,\min(\rho,\delta)}^p} \leqslant C \|F\|_{L_{t,\rho}^{p,\infty}} (1 + \|\varphi\|_{M_{s,\delta}^p})^r$$

$$\|F(x, \varphi(x))\|_{M_{t,\rho}^p} \leqslant C \|F\|_{M_{t,\rho}^{p,\infty}} (1 + \|\varphi\|_{M_{s,\delta}^p})^r.$$

LEMMA 3. *If* $p > 1, \frac{n}{p} - s + 1 < 0, \delta \geqslant 0$

(a) $F \in M^{p,\infty}_{s,\rho}$ *the map* $\varphi \mapsto F(x, \varphi(x))$ *belongs to*

$\mathscr{C}^1(M^p_{s,\delta}, M^p_{s-2,\rho})$ *and* $\mathscr{C}^1(N^p_{s,\delta}, N^p_{s-2,\rho})$

(b) $F \in N^{p,\infty}_{s,\delta}$ *the map* $\varphi \mapsto F(x, \varphi(x))$ *belongs to*

$\mathscr{C}^1(M^p_{s,\delta}, N^p_{s-2,\delta})$ *and* $\mathscr{C}^1(N^p_{s,\delta}, N^p_{s-2,\delta})$

(c) $F \in L^{p,\infty}_{s,\delta}$ *the map* $\varphi \mapsto F(x, \varphi(x))$ *belongs to*

$\mathscr{C}^1(M^p_{s,\delta}, N^p_{s-2,\delta})$ *and* $\mathscr{C}^1(N^p_{s,\delta}, N^p_{s-2,\delta})$.

Proof. We have

$$F(x, \varphi(x) + \psi(x)) - F(x, \varphi(x)) - \psi(x) F'_y(x, \varphi(x))$$

$$= \int_0^1 (1-t) F''_{y^2}(x, \varphi(x) + t\,\psi(x)) \cdot \psi^2(x)\,\mathrm{d}t.$$

On the other hand we have

$$\|F''_{y^2}(x, \varphi(x) + t\psi(x))\,\psi^2(x)\|_{M^p_{s-2,\rho}} \leqslant C\|\psi\|^2_{M^p_{s,\delta}} \|F''_{y^2}(x, \varphi(x) + t\psi(x)\|_{M^p_{s-2,\rho-\delta}}$$

$$\leqslant C\|\psi\|^2_{M^p_{s,\delta}} \|F''_{y^2}\|_{M^{p,\infty}_{s-2,\rho}} (1 + \|\varphi\|_{M^p_{s,\delta}} + \|\psi\|_{M^p_{s,\delta}})^{s-2}$$

and

$$\|F''_{y^2}(x, \varphi(x) + t\psi(x))\,\psi^2(x)\|_{N^p_{s-2,\delta}} \leqslant C\|\psi\|^2_{M^p_{s,\delta}} \|F''_{y^2}(x, \varphi(x) + t\psi(x)\|_{N^p_{s-2,\delta}}$$

$$< C\|\psi\|^2_{M^p_{s,\delta}} \|F''_{y^2}\|_{N^{p,\infty}_{s-2,\delta}} (1 + \|\psi\|_{M^p_{s,\delta}} + \|\varphi\|_{M^p_{s,\delta}})^{s-2}.$$

Thus the conclusion from $\|F''_{y^2}\|_{M^{p,\infty}_{s-2,\rho}} \leqslant \|F\|_{M^{p,\infty}_{s,\rho}}$ and $\|F''_{y^2}\|_{N^{p,\infty}_{s-2,\rho}} \leqslant \|F\|_{N^{p,\infty}_{s,\rho}}$.

Remarks. (1) $M^p_{s,\delta}(R^{n+1}) \subset M^{p,\infty}_{s-1,\delta}(R \times R^n)$ if $s \geqslant 1, p > 1$
(2) $N^p_{s,\delta}(R^{n+1}) \subset L^{p,\infty}_{s-1,\delta}(R \times R^n)$ if $s \geqslant 1, p > 1$
(3) If $\delta' < \delta$, $s' < s$, $p > 1$ the injection of $M^p_{s,\delta}(R^n)$ into $M^p_{s',\delta'}(R^n)$ is compact. Let us take $\{f_n\}$ a sequence of $M^p_{1,\delta}$ with $\|f_n\|_{M^p_{1,\delta}} \leqslant 1$, there exists $f \in M^p_{1,\delta}$ and for every $R > 0$, using the

theorem of Rellich, a subsequence $\{f_{n_k}\}$ which converges weakly to f in $M^p_{1,\delta}$ and verifies $\lim \|f_{n_k} - f\|_{L^p(B_R)} = 0, B_R = \{|x| \leqslant R\}$. On the other hand we have $\|f_n - f\|_{M^p_{0,\delta}(R^n \setminus B_R)} \leqslant \frac{2}{R} \delta - \delta'$ and we can construct a subsequence $\{f_{n_k}\}$such that $\lim |f_{n_k} - f|_{M^p_{0,\delta}(R^n)} = 0$.

3. MAXIMAL SAPCE-LIKE HYPERSURFACES

If g is a Lorentzian metric on $\mathbb{R}^{n+1}$, S a space-like hypersurface, we denote by P_s the mean extrinsic curvature of S in (R^{n+1}, g), $P_S = \operatorname{Tr} K_S$, K_S is the second fundamental form of S as a submanifold of (R^{n+1}, g).

We denote by (t, x) a point of $R \times R^n$, for the mean extrinsic curvature of the hypersurface $s(t, x) = t - \varphi(x) = 0$ we have [1]

$$P(g, \varphi) = \{\mu^{-1/2}(g, \varphi) \cdot \{(\gamma^{ij}(g, \varphi)\partial^2_{ij}\varphi - \gamma^{\alpha\beta}(g, \varphi)\Gamma^i_{\alpha\beta}\, \partial_i\varphi + \gamma^{\alpha\beta}(g, \varphi)\Gamma^0_{\alpha\beta}\}\}_{\, t=\varphi(x)}$$

with

$$\mu(g, \varphi) = g^{\alpha\beta}\partial_\alpha s \partial_\beta s; \qquad \gamma(g, \varphi) = g^{\alpha\beta} - \mu^{-1} g^{\beta\rho}\partial_\lambda s \partial_\rho s.$$

The Christoffel symbols of the metric g are denoted $\Gamma^\gamma_{\alpha\beta}$. We denote by $\eta = (\eta_{\alpha\beta})$ the Minkowskian metric on R^{n+1}.

PROPOSITION 1. *For* $^n/p - s + 1 < 0, \delta \geqslant 0, p > 1$, *there exists a neighbourhood U of* 0 *in* $L^{p,\infty}_{s+1,\delta+2}(R \times R^n) \times M^p_{s,\delta}(R^n)$ *such that the map*: $(g - \eta, \varphi) \longrightarrow P(g, \varphi)$ *belongs to* $\mathscr{C}^1(U, M^p_{s-2,\delta+2}(R^n))$.

Proof. Let us denote $g_{\alpha\beta} = n_{\alpha\beta} + h_{\alpha\beta}$, $g^{\alpha\beta} = n^{\alpha\beta} + k^{\alpha\beta}$, the maps $(h_{\alpha\beta}, \varphi) \longrightarrow h_{\alpha\beta} \circ \varphi$ belongs to $\mathscr{C}^1(L^{p,\infty}_{s+1,\delta} \times M^p_{s,\delta}; N^p_{s-1,\delta})$ (Lemma 2). The map $t \leftrightarrow (1+t)^{-1/2}$ belongs to

$$N^{p,\infty}_{r,\nu}(Y \times R^n) \text{ if } Y =]-\epsilon, +\epsilon[, \quad 0 < \epsilon < 1, \quad (r, \nu) \in \mathbb{N} \times \mathbb{R}.$$

There exists a neighbourhood U of 0 in $L^{p,\infty}_{s+1,\delta} \times M^p_{s,\delta}$ such that the map $\psi(g - \eta, \varphi) = \mu(g, \varphi) - 1$ belongs to $\mathscr{C}^1(U, N^p_{s-1,\delta})$ with $\psi(U) \subset Y$. Thus using Lemma 2 and Lemma 3 we have: $(g - \eta, \varphi) \leftrightarrow \mu(g, \varphi)^{-1/2}$ belongs to $\mathscr{C}^1(U, N^p_{s-1,\delta})$.

Using the same technique we prove that $(g - \eta, \varphi) \leftrightarrow \gamma^{\alpha\beta}(g, \varphi)$ is a map of $\mathscr{C}^1(U, N^p_{s-1,\delta})$.

Since the functions $h_{\alpha\beta}$ are elements of $L^{p,\infty}_{s+1,\delta+2}$, the functions $\partial_\gamma g_{\alpha\beta}$ belong to $M^{p,\infty}_{s,\delta+2}$ and there exists a neighbourhood V of 0 in $L^{p,\infty}_{s+1,\delta+2} \times M^p_{s,\delta}$ such that $(g - \eta, \varphi) \leftrightarrow \Gamma^\alpha_{\beta,\gamma}(g, \varphi)$ is a map of $\mathscr{C}^1(V, M^p_{s-2,\delta+2})$.

From Lemma 1, we have: $(g - \eta, \varphi) \leftrightarrow P(g, \varphi)$ is a map of $\mathscr{C}^1(U, M^p_{s-2,\delta+2})$, U being a neighbourhood of 0 in $L^{p,\infty}_{s+1,\delta+2} \times M^p_{s,\delta}$.

Remark. We have the same proposition for an asymptotically flat metric g such that

$g - \eta \in M^{p;\infty}_{s+1,\delta+2}$.

The formal linearization with respect to Ψ of $P(g, \varphi)$ for $g = \eta, \varphi = 0$ is, [1], $P'_\varphi(\eta, 0)$.
$\psi = -\Delta\psi$, Δ is the Laplacian. Using the isomorphism of the Laplacian between $M^p_{s,\delta}$ and $M^p_{s-2,\delta+2}$ and the Proposition 1 we will prove, by a simple application of the implicit functional theorem, the following

PROPOSITION 2. *If* $p > \frac{n}{n-2}$, $0 \leqslant \delta < n - 2 - \frac{n}{p}$, $\frac{n}{p} - s + 1 < 0$, *there exists a neighbourhood* U *of* 0 *in* $L^{p;\infty}_{s+1,\delta+2}$ $(R \times R^n)$ *(resp.* $M^{p;\infty}_{s+1,\delta+2}(R \times R^n)$*) (resp.* $M^{p;\infty}_{s+1,\delta+2}(R \times R^n)$*) such that for* $g - \eta \in U$*, the space-time* (R^{n+1}, g) *admits a maximal space like hypersurface* S *with equation* $t - \varphi(x) = 0$, $\varphi \in M^p_{s,\delta}(R^n)$*. The map* φ *is unique in a neighbourhood of* 0 *in* $M^p_{s,\delta}(R^n)$.

Remark. For physical case ($n = 3$) we can choose $p > 3$, $s > 2$ and $0 < \delta < 1 - \frac{3}{p}$

For the existence and uniqueness of maximal non compact submanifolds results, in weighted Sobolev spaces, have also been obtained, under different hypothesis and by different methods, by M. Cantor, A. Fisher, J. Marsden, N. O'Murchada and J. York. [9].

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