SOME COMMENTS ON THE MANY-DIMENSIONAL SOLITONS

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ABSTRACT. The possibilities for the existence of truly localized soliton solutions in the realistic three-dimensional case are discussed. The gauge invariant theory of a non-linear chiral field is shown to be a good candidate for a model with solitons.

The exact solution of the two-dimensional Sine-Gordon model [1, 2] gives us a new mechanism whereby one (or a few) field can describe a rich spectrum of particles. Localized solutions of the corresponding classical equations of motion (solitons) correspond to the particles with the following properties:

1. They are heavy compared with the weakly interacting field masses.
2. They interact strongly.
3. They have an exactly conserved quantum number – the topological charge.

These properties suggest the attractive possibility of building unified models of weak, electromagnetic and strong interactions based on a Lagrangian with fundamental fields for leptons, photons and intermediate bosons only [3]. The existence of soliton solutions is a necessary property for such a Lagrangian.

I want to present here some arguments for the conclusion that such a Lagrangian must include a non-linear chiral field. These arguments will be based mostly on the properties of the topological charge.

The conserved topological current of the Sine-Gordon model can be written in two forms

$$J_\mu = \frac{1}{2\pi} \epsilon_{\mu\nu}\, \delta_\nu \underline{u} = \frac{1}{2\pi} \epsilon_{\mu\nu} \delta_\nu \chi \bar{\chi}^{-1}, \quad \chi = e^{iu}. \tag{1}$$

This allows two generalizations to three-dimensional case. The first consists in using a linear 'isovector' field u_a, $a = 1, 2, 3$, with the current

$$J_\mu = \epsilon_{\mu\nu\rho\sigma}\, \epsilon^{abc}\, \delta_\nu u^a\, \delta_\rho u^b\, \delta_\sigma u^c. \tag{2}$$

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This current is conserved irrespective of the equations of motion (thus it is topological) and the corresponding charge density J_0 is a pure divergence

$$J_0 = \mathrm{div} P, \quad P_i = \epsilon_{ikl} \epsilon^{abc} u^a \delta_k u^b \delta_l u^c .$$

The charge

$$Q = \int J_0 \, \mathrm{d}^3 x$$

is non-zero only for u fields with non-vanishing asymptotics when $|x| \to \infty$, which is realized for instance by the Higgs mechanism. It is exactly this charge which is used in the monopole solutions of 't Hooft and Polyakov. There are two obstacles for such solutions to be used as the building blocks of hadron physics.

1. They have magnetic charge and so are accompanied by the const/r^2 tail of the magnetic field. Magnetic charge and topological charge are proportional so that a magnetically neutral bound state has a zero topological charge. Thus the latter cannot serve as a baryon number.

2. They are very heavy in the case of the standard $W+EM$ models. Indeed for the fields u with vanishing $\nabla u = \delta u + [A, u]$ the current (2) is equivalent

$$J_\mu = \epsilon_{\mu\nu\rho\sigma} F^a_{\nu\rho} \nabla_\sigma u^a ,$$

so that we have an estimate for the static part of the Hamiltonian

$$M_{\mathrm{sol}} = \int \left(\frac{1}{4e^2} (F_{ik})^2 + \frac{1}{2} (\nabla_i u)^2 + V(u) \right) \mathrm{d}^3 x \geqslant$$

$$\geqslant \frac{1}{e} \left| \int \epsilon_{ikl} F^a_{ik} \nabla_l u^a \, \mathrm{d}^3 x \right| = \frac{M_u}{e} |Q| ,$$

where

$$M_u^2 = (u^a)^2 \Big|_{r=\infty}$$

and M_u is of the order of the mass of the intermediate boson.

The second possibility of generalizing (1) consists in using a non-linear chiral field $\chi(x)$ with values in a compact group G. The expression for the corresponding topological current looks as follows

$$J_\mu = \epsilon_{\mu\nu\rho\sigma} \, \mathrm{tr} \, [T_\nu, T_\rho] \, T_\sigma , \tag{4}$$

where

$$T_\mu = \delta_\mu \chi \chi^{-1}$$

lies in the corresponding Lie algebra. The charge density J_0 is not a divergence and the charge $Q = \int J_0 \, d^2x$ has only integer values when properly normalized.

The stability of a non-trivial soliton solution will be guaranteed if the static Hamiltonian of the model allows for an estimate like (3). There is no reason to believe that such an estimate is true for the ordinary chiral Lagrangian of Sugawara and S. Weinberg

$$\mathcal{L} = \frac{1}{2\lambda^2} \operatorname{tr} T_\mu^2 . \tag{5}$$

Indeed, it contains only quadratic terms in T_μ whereas the current (4) is cubic in T_μ.

The form of the current (4) suggests the natural modification of the Lagrangian (5). Namely, the Lagrangian

$$\mathcal{L} = \operatorname{tr} \left(\frac{1}{2\lambda^2} T_\mu^2 + \frac{\epsilon^2}{2} ([T_\mu, T_\nu])^2 \right) \tag{6}$$

leads to the static Hamiltonian

$$H_{\text{static}} = \int \operatorname{tr} \left(\frac{1}{2\lambda^2} T_i^2 + \frac{\epsilon^2}{2} ([T_i, T_k])^2 \right) d^3x$$

which allows an estimate

$$\frac{\epsilon}{\lambda} |Q| = \frac{\epsilon}{\lambda} \left| \int \epsilon_{ikl} \operatorname{tr} [T_i, T_k] T_l \, d^3x \right| \leqslant H_{\text{static}} .$$

Note that ϵ is in the numerator of the left-hand side in this estimate, in contrast with e in the right-hand side of (3). This shows that for small ϵ one can expect a small mass for the soliton.

The Lagrangian (6) was introduced by Skyrme [6] in the particular case of $G = SU(2)$. Skyrme used a parametrization of χ of the form

$$\chi = (\phi_0, \vec{\phi}), \quad \phi_0^2 + \vec{\phi}\vec{\phi} = 1$$

and a Lagrangian which looked as follows:

$$\mathcal{L} = \frac{1}{2\lambda^2} \delta_\mu \phi \delta_\mu \phi + \frac{\epsilon^2}{2} [(\delta_\mu \phi \delta_\mu \phi)^2 - (\delta_\mu \phi \delta_\nu \phi)^2] .$$

It can be shown to be a particular case of (6). Skyrme has found a spherically symmetrical solution of the corresponding equations of motion

$$\vec{\phi} = \sin g(z) \frac{\vec{\chi}}{r} ; \quad \phi_0 = \cos g(r),$$

such that $g(r) = O(r^{-2})$, when $r \to \infty$. The last result reflects the fact that the first term in (6) dominates for small T_μ and the asymptotic behaviour is governed by equations corresponding to (5), which does not contain any dimensional parameters.

The situation with a r^{-2} tail can be cured if the equations of motion for the χ field contain a source rapidly vanishing at infinity. Such a sourse can be provided by a gauge field.

The gauge invariance of the usual kind, i.e., with respect to the transformation

$$\chi \to \Omega \chi$$

evidently destroys the topological charge. The same unfortunately is true for the Hopf invariant of the $O(3)/O(2)$ n field which was a base for the $EM + W$ model in Reference 7). Another possibility is to use

$$\chi \to \Omega \chi \Omega^{-1} \tag{7}$$

as a gauge transformation. This makes the field χ a natural non-linear generalization of the Higgs field u. We shall show that there exists a Lagrangian of the χ field interacting with the Yang-Mills field A_μ which is gauge invariant with respect to the ordinary transformation law for the Yang-Mills field and law (7) for the χ field, and which admits an estimate from below for the corresponding static Hamiltonian.

To do this, let us introduce the covariant generalization of the current T_μ

$$\begin{aligned} L_\mu &= \nabla_\mu \chi \chi^{-1} = (\partial_\mu \chi + A_\mu \chi - \chi A_\mu)\chi^{-1} = \partial_\mu \chi \chi^{-1} + A_\mu - \chi A_\mu \chi^{-1} \\ R_\mu &= \chi^{-1} \nabla_\mu \chi = \chi^{-1} \partial_\mu \chi + \chi^{-1} A_\mu \chi - A_\mu = \chi^{-1} L_\mu \chi. \end{aligned} \tag{8}$$

It can be shown that the topological current (4) is equivalent to (i.e., leads to the same charge as)

$$J_\mu = \epsilon_{\mu\nu\rho\sigma} \operatorname{tr}([L_\nu, L_\rho] L_\sigma - 3F_{\nu\rho}(L_\sigma + R_\sigma)). \tag{9}$$

Note that the last expression is manifestly gauge invariant.

The Lagrangian

$$\mathcal{L} = \operatorname{tr}\left(\frac{1}{2\lambda^2} L_\mu^2 + \frac{1}{4e^2} F_{\mu\nu}^2 + \frac{\epsilon^2}{2} ([L_\mu, L_\nu])^2\right) \tag{10}$$

allows for the desired estimate. To prove it, it is sufficient to realize that

$$\operatorname{tr} L_\mu^2 = \operatorname{tr} R_\mu^2$$

and use the inequality

$$ab \leqslant \frac{\gamma}{2} a^2 + \frac{1}{2\gamma} b^2$$

with an appropriate γ for both terms in (9). One finds that

$$\frac{\epsilon}{\lambda} \frac{1}{(36 + e^2 \epsilon^2)^{1/2}} |Q| \leqslant H_{\text{static}}.$$

In the model for $EM + W$ interaction λ plays the role of the weak coupling constant. We see that the soliton mass (if it exists) is of order $\epsilon(M_w/e)$ and can be made reasonably small for small ϵ.

The interesting soliton solution has to have an asymptotical behaviour of the form

$$\chi \to \chi_0, \qquad r \to \infty, \tag{11}$$

where χ_0 is a constant, but is different from unity. It is clear from (8) and (10) that the stationary subgroup H of χ_0 corresponds to the massless fields. So to make the Yang-Mills field acquire a mass, we must have H smaller than G. The last requirement constitutes an obstacle for the actual finding of the soliton for the most interesting case $O(3)$. Indeed, the condition (11) spoils the spherical symmetry and we cannot use the standard spherically symmetric ansatz.

I am still optimistic about the possibility of finding the appropriate separation of variables in the equations of motion. Actual work in this direction is in progress now.

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