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A $*$-Product on SL(2) and the Corresponding Nonstandard Quantum-U($\mathfrak{sl}(2)$)

CH. OHN*
Université Libre de Bruxelles, Campus Plaine, CP 218, B-1050 Brussels, Belgium

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Abstract. We obtain Zakrzewski's deformation of Fun SL(2) through the construction of a $*$-product on SL(2). We then give the deformation of U($\mathfrak{sl}(2)$) dual to this, as well as a Poincaré basis for both algebras.

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1. A $*$-Product on SL(2) and the Corresponding Hopf Algebra

Let $G = \mathrm{SL}(2)$ and $A_0 = \mathrm{Fun}(G)$. Formally, A_0 is the associative algebra with 1 generated by a, b, c, d, with relations

$$[a, b] = [a, c] = [a, d] = [b, c] = [b, d] = [c, d] = 0, \quad ad - bc = 1. \tag{1}$$

Let $\mathfrak{g} = \mathfrak{sl}(2)$ and $U_0 = \mathrm{U}(\mathfrak{g})$. Let H, X, Y be a basis of $\mathfrak{g}$ such that

$$[H, X] = 2X, \qquad [H, Y] = -2Y, \qquad [X, Y] = H. \tag{2}$$

Then $r = H \otimes X - X \otimes H$ is a (skew-symmetric) solution of the classical Yang–Baxter equation, i.e.

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0,$$

hence it defines a left-invariant Poisson structure r^λ and a Poisson-Lie structure $r^\lambda - r^\rho$ on G.

In [3], Drinfeld gives a general procedure to construct a left-invariant $*$-product on G deforming r^λ. Such a $*$-product is given by a left-invariant bidifferential operator F_h^λ on G, where $F_h \in U_0 \otimes U_0$ verifies

$$F_h = 1 + \frac{hr}{2} + \mathrm{O}(h^2),$$

$$(\Delta_0 \otimes 1)F_h(F_h \otimes 1) = (1 \otimes \Delta_0)F_h(1 \otimes F_h),$$

$$(\varepsilon_0 \otimes 1)F_h = (1 \otimes \varepsilon_0)F_h = 1$$

(where Δ_0 and ε_0 denote the standard comultiplication and counit on U_0).

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Following the construction of [3], we get

$$F_h = \exp\left(\frac{1}{2}\Delta_0 H - \frac{1}{2}\left(H\frac{\sinh hX}{hX}\otimes e^{-hX} + e^{hX}\otimes H\frac{\sinh hX}{hX}\right)\frac{h\Delta_0 X}{\sinh h\Delta_0 X}\right).$$

The bidifferential operator $F_h^\lambda(F_h^{-1})^\rho$ then defines a $*$-product on G deforming $r^\lambda - r^\rho$ and actually turns A_0 (with its usual coalgebra structure) into a noncommutative Hopf algebra, denoted A_h.

Still following [3], the element $R_h = {}^\tau F_h^{-1}\cdot F_h = 1 + hr + O(h^2)$ is a solution of the quantum Yang–Baxter equation, i.e.

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$

In the two-dimensional representation of $\mathfrak{g}$ defined by

$$H = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\qquad X = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\qquad Y = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},\tag{3}$$

we have

$$R_h = \begin{bmatrix}1 & h & -h & h^2\\ 0 & 1 & 0 & h\\ 0 & 0 & 1 & -h\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

This is the R-matrix described by Zakrzewski in [5] as the representation of the element $R_h' = e^{hr}$. Unfortunately, R_h' is not a solution of the quantum Yang–Baxter equation. (One can check this using a three-dimensional representation of $\mathfrak{g}$.)

In [5], the matrix R_h is used to determine relations for A_h (following the method of [4]). We recall that A_h is the associative algbra with 1 generated by a, b, c, d, with relations

$$\begin{aligned}
&[c, a] = hc^2, \qquad [c, d] = hc^2,\\
&[b, a] = h - ha^2, \qquad [b, d] = h - hd^2,\\
&[a, d] = hac - hdc, \qquad [c, b] = hac + hcd,\\
&ad - bc = 1 + hac.
\end{aligned}\tag{4}$$

Clearly, (4) deforms (1). The sufficiency of these relations will follow from Proposition 2 below.

The comultiplication, counit and antipode given in [5], respectively, by

$$\Delta\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}a\otimes a + b\otimes c & a\otimes b + b\otimes d\\ c\otimes a + d\otimes c & c\otimes b + d\otimes d\end{pmatrix},$$

$$\varepsilon\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$

$$S\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}d - hc & -b + ha - hd + h^2c\\ -c & a + hc\end{pmatrix},$$

turn A_h into a Hopf algebra.

Remarks. (1) The algebra A_1 has also been studied in [2].

(2) The set of relations of A_h is invariant under the permutation $(a, d)(b)(c)$.

2. A Deformation of U(sl(2))

Let U_h be the associative algebra with 1 generated by H, Y, T, T^{-1} and satisfying the relations

$$[H, T] = T^2 - 1, \qquad [H, T^{-1}] = T^{-2} - 1,$$

$$[Y, T] = -\frac{h}{2}(HT + TH), \qquad [Y, T^{-1}] = \frac{h}{2}(HT^{-1} + T^{-1}H),$$

$$[H, Y] = -\frac{1}{2}(YT + TY + YT^{-1} + T^{-1}Y), \qquad TT^{-1} = T^{-1}T = 1.$$

PROPOSITION 1. *The comultiplication, counit and antipode given, respectively, by*

$$\Delta(H) = H \otimes T + T^{-1} \otimes H, \qquad \Delta(Y) = Y \otimes T + T^{-1} \otimes Y,$$

$$\Delta(T) = T \otimes T, \qquad \Delta(T^{-1}) = T^{-1} \otimes T^{-1},$$

$$\varepsilon(H) = \varepsilon(Y) = 0, \qquad \varepsilon(T) = \varepsilon(T^{-1}) = 1,$$

$$S(H) = -THT^{-1}, \qquad S(Y) = -TYT^{-1},$$

$$S(T) = T^{-1}, \qquad S(T^{-1}) = T,$$

turn U_h into a Hopf algebra.

Remarks. (1) Applying the method described in [4] to derive a quantum universal enveloping algebra from A_h only gives rise to the subalgebra of U_h generated by H, T, T^{-1}.

(2) An extension $\hat{U}_h$ of U_h can be defined by introducing a new generator $X = (\log T)/h$. U_h then is the associative algebra with 1 generated by H, X, Y, with relations

$$[H, X] = \frac{2 \sinh hX}{h},$$

$$[H, Y] = -Y(\cosh hX) - (\cosh hX)Y, \tag{5}$$

$$[X, Y] = H,$$

which are indeed a deformation of (2). The additional Hopf structure is given by

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \qquad \varepsilon(X) = 0, \qquad S(X) = -X.$$

3. Poincaré Basis for A_h and U_h

The relations defining A_h and U_h allows an immediate application of Bergman's diamond lemma (see [1]) to prove the following

PROPOSITION 2. *The elements* $a^k b^l c^m$ *and* $d^k b^l c^m$ $(k, l, m \in \mathbf{N})$ *form a basis of* A_h. *Similarly, the elements* $Y^k H^l T^\mu$ $(k, l \in \mathbf{N}, \mu \in \mathbf{Z})$ *form a basis of* U_h.

4. The Duality between A_h and U_h

Let $(H_i, m_i, u_i, \Delta_i, \varepsilon_i, S_i)$ $(i = 1, 2)$ be two Hopf algebras. Recall that a *duality* between them is a nondegenerate bilinear form $\langle\ ,\ \rangle$ on $H_1 \times H_2$ satisfying

$$\langle m_1(a \otimes b), x\rangle = \langle a \otimes b, \Delta_2(x)\rangle, \qquad \langle u_1(1), x\rangle = \varepsilon_2(x),$$
$$\langle \Delta_1(a), x \otimes y\rangle = \langle a, m_2(x \otimes y)\rangle, \quad \varepsilon_1(a) = \langle a, u_2(1)\rangle,$$
$$\langle S_1(a), x\rangle = \langle a, S_2(x)\rangle.$$

PROPOSITION 3. *The relations*

$$\begin{aligned}
&\left\langle H, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
&\left\langle Y, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\rangle = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\
&\left\langle T, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\rangle = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}, \\
&\left\langle T^{-1}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\rangle = \begin{pmatrix} 1 & -h \\ 0 & 1 \end{pmatrix}
\end{aligned} \tag{6}$$

can be extended according to the rules above to a bilinear form on $U_h \times A_h$, *which is a duality between* U_h *and* A_h.

The relations (6) deform (3).

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