DIFFERENTIAL-DIFFERENCE EVOLUTION EQUATIONS. II (DARBOUX TRANSFORMATION FOR THE TODA LATTICE)

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ABSTRACT. Toda lattice equation is represented in the form of the condition of compatibility of the system of linear equations corresponding to a non-Hermitian Lax representation. The Darboux invariance of this linear system is defined and proved in the text, and enables us to construct some new formulas for the solutions of the Toda lattice equation. These formulas involving determinants are applicable to an arbitrary initial solution of the Toda equation for example to a solution growing at infinity.

1. INTRODUCTION

In the preceding article [1] we have shown that the concept of the Darboux-invariance previously formulated in the framework of matrix differential operators [2] can be applied also to the explicit integration of linear differential-difference or difference-difference evolution equations with scalar or matrix-valued coefficients. Here we show that this approach leads very naturally to some new formulas for the solutions of the Toda lattice equation

$$\ddot{X}_n = \exp(X_{n-1} - X_n) - \exp(X_n - X_{n+1}), \qquad n = 0, \pm 1, \pm 2, \ldots \tag{1}$$

describing one-dimensional anharmonic lattice with the exponential interaction of the atoms. Equation (1) provides one of the most remarkable examples of completely integrable Hamiltonian systems admitting the so-called Lax representation. It was already integrated for initial conditions rapidly decreasing at infinity by Manakov [3] and Flaschka [4], and for periodic and almost periodic initial conditions by Tanaka and Date [5], Novikov [6] and Krichever [7]. The formulas we present below are different from those of Refs. 3–7 and are applicable to arbitrary initial conditions.

As we already noticed in [1] the concept of a Darboux transformation is not uniquely

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determined and below we use a version that is slightly different from [1] *, but is more convenient in the applications to the Toda lattice case.

2. DARBOUX INVARIANCE OF TODA LATTICE EQUATION

Using the new variables $v_n = \dot{x}_n$, $u_n = \exp(X_n - X_{n+1})$ we can rewrite eqn. (1) in the following way

$$\dot{v}_n = u_{n-1} - u_n, \qquad \dot{u}_n = u_n(v_n - v_{n+1}). \tag{2}$$

Equations (2) are the conditions of compatibility of the following system of linear equations

$$\dot{f}_n = V_n f_n + f_{n-1}, \tag{3}$$

$$v_n f_n + u_n f_{n+1} + f_{n-1} = \lambda f_n. \tag{4}$$

The system (3)–(4) is obviously related with the non-Hermitian Lax representation of the system (2)

$$\dot{L} = [A, L], \qquad L_{nm} = V_n \delta_{nm} + u_n \delta_{n+1,m} + \delta_{n-1,m}, \qquad A_{nm} = V_n \delta_{nm} + \delta_{n,m+1}. \tag{5}$$

Let $\varphi_n(1)$ be some fixed solution of eqns. (3)–(4) corresponding to $\lambda = \lambda_1$.

The Darboux transformation $f_n \to \psi_n$ is defined by the formula

$$\psi_n = f_n - \sigma(n) f_{n+1}, \quad \sigma(n) = \varphi_n(1)\varphi_{n+1}^{-1}(1). \tag{6}$$

THEOREM 1. *Equations (3), (4) are invariant with respect to Darboux transformation i.e.* ψ_n *satisfies the following system of the linear equations:*

$$\dot{\psi}_n = \tilde{V}_n \psi_n + \psi_{n-1}, \tag{7}$$

$$\tilde{V}_n \psi_n + \tilde{u}_n \psi_{n+1} + \psi_{n-1} = \lambda \psi_n, \tag{8}$$

* For the solutions with the property $X_n = X_{n+2}$ these two versions coincide if we choose the periodic solutions with the same period for φ_n and f_n (see below) but this situation, which corresponds to the genus 1, i.e. to the case of elliptical algebraic curves, is not very interesting.

where the coefficients $\tilde{V}_n$, $\tilde{U}_n$ are defined by the formulas

$$\tilde{V}_n = V_n - [\sigma(n) - \sigma(n-1)], \tag{9}$$

$$\tilde{U}_n = \frac{\sigma(n)}{\sigma(n+1)} u_{n+1}. \tag{10}$$

Remark. Introducing the operator $\mathrm{d}/\mathrm{d}n$, $(\mathrm{d}/\mathrm{d}n)\varphi_n = \varphi_n - \varphi_{n-1}$, and the 'logarithmic derivative' by $(\mathrm{d}/\mathrm{d}n)\ln\varphi_n = (\varphi_n - \varphi_{n-1})/\varphi_n$, we can rewrite (9) in the form

$$\tilde{V}_n = V_n - (\mathrm{d}^2/\mathrm{d}n^2)\ln\varphi_n(1),$$

which is in close analogy with the corresponding formula for the Darboux-transformed Schrödinger operator (see for example [8]).

Proof of Theorem I may be obtained by inserting the expression (6) for ψ_n in the system (7), (8). Using eqn. (4) we can represent both sides of eqns. (7) and (8) as a linear combination of f_{n+1} and f_n. Equating the coefficients of f_{n+1} gives the formulas (9) and (10). So as to achieve the proof it is necessary to use eqn. (4) for φ_n and φ_{n+1} with $\lambda = \lambda_1$, and to check that the system (7)–(8) is satisfied. We do not present more details in the present paper since all calculations are similar to those presented in Ref. 1.

3. ITERATIONS OF THE DARBOUX TRANSFORMATION

Let us consider the sequence $\varphi_n(1), \varphi_n(2), \ldots, \varphi_n(m)$ of the different solutions of the system (7)–(8) corresponding to the different values of λ: $\lambda_1, \lambda_2, \ldots, \lambda_m$. The functions $\psi_n(1), \psi_n(1,2), \psi_n(1,2,\ldots,m)$ are defined by the recurrent relation:

$$\psi_n(1) = \varphi_n(1),\ \psi_n(1,2) = \begin{vmatrix} \varphi_{n+1}(1), \varphi_n(1) \\ \varphi_{n+1}(2), \varphi_n(2) \end{vmatrix} \varphi_{n+1}^{-1}(1),$$

$$\psi_n(1,3) = \begin{vmatrix} \varphi_{n+1}(1), \varphi_n(1) \\ \varphi_{n+1}(3), \varphi_n(3) \end{vmatrix} \varphi_{n+1}^{-1},$$

$$\psi_n(1,\ldots,m-1,m) = \begin{vmatrix} \psi_{n+1}(1,2,\ldots,m-2,m-1), \psi_n(1,2,\ldots,m-2,m-1) \\ \psi_{n+1}(1,2,\ldots,m-2,m), \psi_n(1,2,\ldots,m-2,m) \end{vmatrix} \psi_{n+1}^{-1}(1,2,\ldots,m-2,m-1)$$

For example

$$\psi_n(1,2,3) = \begin{vmatrix} \psi_{n+1}(1,2), \psi_n(1,2) \\ \psi_{n+1}(1,3), \psi_n(1,3) \end{vmatrix} \psi_{n+1}^{-1}(1,2)$$

THEOREM II. *The function* $\psi_n(1, 2, ..., m-1, m) \equiv \psi_{n,[m]}$ *may be represented in the form*

$$\psi_{n,[m]} = \frac{\Delta_m(n)}{\Delta_{m-1}(n+1)}, \qquad \Delta_m = \det A, \qquad A_{ik} = \varphi_{n+m-k}(i), \qquad i, k = 1, 2, ..., m. \tag{11}$$

This theorem can be either derived from the well known Sylvester's theorem (for the proof and formulation of Sylvester's theorem see [9, 10]) or proved directly by induction but we do not present the corresponding simple calculations here.

Taking now $\lambda_m = \lambda$ we obtain from (11) all the solutions of the system which correspond to the result of the m-times repeated Darboux-transformation of the initial system (3)–(4).

Now from the fact that the 'starting' functions u_n, V_n form the solution of the system (2) it turns out that the functions

$$V_{n,[m]} = V_n - \sum_{i=1}^{m} [\sigma_i(n) - \sigma_i(n-1)], \qquad \sigma_i(n) = \frac{\psi_{n,[i]}}{\psi_{n+1,[i]}}, \tag{12}$$

$$U_{n,[m]} = U_{n+m} \prod_{i=1}^{m} \frac{\sigma_i(n+m-i)}{\sigma_i(n+m+1-i)} \tag{13}$$

form the new solution of the same system.

The formulas (12)–(13) represent one of the most general versions of the Bäcklund transformation for the Toda lattice. So as to write down the formula explicitly it is only necessary to calculate all solutions in the starting system (3)–(4).

For the solutions we can take the well known functions corresponding to $V_n = 0, u_n = 1$, or the solutions f corresponding to the periodic problem. We can also consider a different class of solutions for which $X_n = \text{cn}\, t$. Thus we obtain a large class of growing solutions for the Toda-lattice equation, in complete analogy with the method given in [8] to construct growing solutions of the KdV equation.

In conclusion, let us notice that the Theorem I can be trivially generalized to the case of 'higher' Toda equations because the evolution equations of the more general form

$$\dot{f}_n = \sum_{-K}^{m} b_K(n) f_{n+K} + f_{n-K-1}$$

are also invariant with respect to the same Darboux transformation [for a proof compare with Ref. 1].

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Erratum

'Darboux transformation and the explicit solutions of differential-difference and difference-difference evolution equations. I', *Lett. Math. Phys.* **3**, 217–222 (1979).

On p. 219 (just above eqns. (11)) Δ_2/Δ_1 should read $\Delta_2(n)/\Delta_1(n-1)$, and in eqns. (11) $\Delta_{m-1}/\Delta_{m-2}$ should read $\Delta_{m-1}(n)/\Delta_{m-2}(n-1)$ and Δ_j should read $\Delta_j(n)$.

These changes also apply to the Preprint LPTHE–79/11 (end p. 4, beginning p. 5).