ON THE NOETHER MAP

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ABSTRACT. For a wide class of Lagrangian systems we show rigorously that the conventional formulation of Noether's theorem provides a bijective map from the set of equivalence classes of Noether's symmetries onto the set of equivalence classes of conserved currents. We further discuss if Noether's theorem is generalized in a significant way by several formulations proposed in this decade.

1. INTRODUCTION

In a recent paper of Ibragimov [1] a method is proposed to relate conserved currents with symmetries for arbitrary systems of partial differential equations by means of a previously formulated generalization of Noether's theorem [2]. On the other hand, Candotti–Palmieri–Vitale [3, 4] and Rosen [5, 6] have generalized Noether's theorem in such a way that conserved currents appear in general to be unrelated with symmetries even for Lagrangian systems (see in particular [6] and Theorem 1 of [4]). Then the following questions naturally arise:

(1) Does Noether's theorem provide a deep relationship between symmetries and conservation laws for Lagrangian systems?

(2) To what extent is it possible to generalize Noether's theorem?

The present note is devoted to investigate these two questions. It is formulated in terms of algebraic methods as introduced by Gel'fand and Dikii in [7], and it uses several notions and results of a previous paper [8]. In order to keep the paper within usual language we adopt the formal term 'infinitesimal transformation' to describe the vector fields of first-order differential operators acting on the algebra of regular density functions.

Our analysis is based on the concept of an integrating factor of a system of partial differential equations, which is closely related to the conserved currents of the first kind [9]. It allows us to isolate that property of Lagrangian systems which is implicit in the conventional formulation of Noether's theorem [10]. This property is that integrating factors of Lagrangian systems define symmetries. It leads in a natural way to the Noether map between Noether symmetries and conservation laws. Our main task is to deduce rigorously under what conditions this map is bijective. The answer to this question must take into account two aspects. First, the algebraic

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nature of the discussion requires obtaining an algebraic characterization of the ideal of density functions which vanish on the solutions of the field equations; and, secondly, there are natural notions of equivalence in the sets of symmetries and conserved currents which suggest that it is more intrinsic to consider the Noether map in terms of equivalence classes. In this way, we are able to prove rigorously the bijective character of the Noether map for normal Lagrangian systems. Finally, we conclude that the above-mentioned generalizations of Noether's theorem do not improve in any sense the deep property of Lagrangian systems provided by the conventional formulation of Noether's theorem.

2. FORMULATION OF THE RESULTS

We follow the notation conventions of [8]. Let R be the algebra of C^∞ functions $F=F[x, u]$ depending upon n independent variables $x_i (i = 1, .., n)$ and derivatives of arbitrary order of m dependent variables u^r $(r = 1, ..., m)$. By a normal system of partial differential equations we shall mean a system of the form

$$\omega^r[x, u] \equiv \frac{\partial^{N_r+1} u^r}{\partial t^{N_r+1}} - \theta^r[x, \underline{u}] = 0, \qquad r = 1, ..., m, \tag{1}$$

where t denotes one of the coordinates x_i, N_r $(r = 1, ..., m)$ are non-negative integers, and $\underline{u}$ denotes the set of variables u^r_α such that the component of α corresponding to the coordinate t is less than or equal to N_r. Clearly, if $\theta^r \in R$ all the variables u^r_α may be written as C^∞ functions of the variables $[x, \underline{u}, \omega]$, where ω denotes the set of functions $D^\alpha\omega^r$ $(|\alpha| \geqslant 0, r = 1, ..., m)$. Then, every element of R may be expressed as a C^∞ function $F[x, \underline{u}, \omega]$. Let I be the ideal of functions $F \in R$ which vanish when the field equations (1) are satisfied. That is, $F \in I$ if and only if $F[x, \underline{u}, 0] = 0$. The following lemma establishes an important property of normal systems.

LEMMA 1. *The ideal I is generated by the functions $D^\alpha\omega^r$ $(|\alpha| \geqslant 0, r = 1, ..., m)$.*
Proof. Evidently $D^\alpha\omega^r \in I$. On the other hand, from a fundamental theorem of calculus we have

$$F[x, \underline{u}, \omega] = F[x, \underline{u}, 0] + \sum_{r,\alpha} D^\alpha\omega^r \int_0^1 \frac{\partial F}{\partial D^\alpha\omega^r} \circ \gamma(\tau)\mathrm{d}\tau$$

where γ is the curve $\gamma(\tau) = [x, \underline{u}, \tau\omega]$. The conclusion follows at once. □

Given $F, G \in R$ we write $F \overset{\circ}{=} G$ when $F - G \in I$. A vector function $\vec{A} = (A_1, ..., A_n)(A_i \in R)$ is said to be a conserved current of (1) if $\vec{D} \cdot \vec{A} \overset{\circ}{=} 0$. Clearly, if $\vec{A} \overset{\circ}{=} 0$ or $\vec{D} \cdot \vec{A} \equiv 0$, then $\vec{A}$ is a conserved current. Two conserved currents $\vec{A}$ and $\vec{A}'$ are said to be equivalent if $\vec{A} - \vec{A}' = \vec{B} + \vec{C}$ with $\vec{B} \overset{\circ}{=} 0$ and $\vec{D} \cdot \vec{C} \equiv 0$. From Lemma 1 every conserved current of (1) must satisfy an equation of the form $\vec{D} \cdot \vec{A} = \Sigma_{r,\alpha}\lambda^r_\alpha \cdot D^\alpha\omega^r$ where $\lambda^r_\alpha \in R$ and the sum extends to a finite number of terms.

An m-component function $\lambda = (\lambda^1, ..., \lambda^m)$ is said to be an integrating factor of (1) if the following system of equations is satisfied

$$\frac{\delta}{\delta u^r}\left(\sum_s \lambda^s \omega^s\right) = 0, \qquad r = 1, \dots, m. \tag{2}$$

Since the kernel of the variational derivative coincides with the range of the divergence operator [8], every integrating factor determines, up to a divergenceless term, a conserved current by the equation

$$\vec{D} \cdot \vec{A}_\lambda = \sum_r \lambda^r \omega^r. \tag{3}$$

These conserved currents associated with integrating factors are called first kind conserved currents [9].

The next lemma describes an identity which can be deduced in a way similar to the way we deduced identity (6) in [8]. It will be used in the proof of Lemma 3.

LEMMA 2. *Given* λ^r_α, $\psi^r \in R$ $(|\alpha| \geqslant 0, r = 1, \dots, m)$ *then we have*

$$\sum_{r,\,\alpha} \lambda^r_\alpha \cdot D^\alpha \psi^r = \sum_{r,\,\alpha} D^\alpha(\{\lambda^r_\alpha\}\psi^r) \tag{4}$$

where

$$\{\lambda^r_\alpha\} \equiv \sum_\beta (-1)^{|\beta|} \binom{\alpha+\beta}{\beta} D^\beta \lambda^r_{\alpha+\beta}. \tag{5}$$

Now, we are ready to prove

LEMMA 3. *For normal systems*: (i) *Every conserved current is equivalent to some first kind conserved current associated with an integrating factor of the form* $\lambda = \lambda[x, \underline{u}]$. (ii) *Two first kind conserved currents* $\vec{A}_\lambda$ *and* $\vec{A}_{\lambda'}$ *are equivalent if and only if* $\lambda \stackrel{\circ}{=} \lambda'$ (*i.e.* $\lambda^r \stackrel{\circ}{=} \lambda'^r$ *for all* r).

Proof. (i) We will use the following alternative notation for the derivatives of the field

$$u^r_\alpha = u^r_{l,a} = \frac{\partial^l}{\partial t^l} u^r_a$$

where a is a symbol with $n-1$ components which indicates the partial derivatives with respect to the coordinates different from t.

Let $\vec{A} = \vec{A}[x, \underline{u}, \omega]$ be a conserved current and let $\vec{B}[x, \underline{u}] \equiv \vec{A}[x, \underline{u}, 0]$. Clearly, $\vec{B}$ is a conserved current equivalent to $\vec{A}$. Then

$$\vec{D} \cdot \vec{B} - \sum_{r,\,a} \frac{\partial B_0}{\partial u^r_{N_r,a}} \cdot D^a \left(\frac{\partial^{N_r+1}}{\partial t^{N_r+1}} u^r - \theta^r\right) \stackrel{\circ}{=} 0, \tag{6}$$

where B_0 denotes the t-component of $\vec{B}$. Let us observe that the left-hand side of (6) depends only on the variables $[x, \underline{u}]$. Then it must vanish identically in order to be an element of the ideal I. Therefore, from the identity (4) we deduce

$$\vec{D}\cdot\vec{B} = \sum_{r,a} D^a(\{\rho_a^r\}\omega^r) = \sum_r \{\rho^r\}\omega^r + \vec{D}\cdot\vec{C} \tag{7}$$

where $\rho_a^r \equiv \partial B_0/\partial u_{N_r,a}^r$, and $\vec{C}$ is determined, up to divergenceless term, by

$$\vec{D}\cdot\vec{C} = \sum_{r,\,|a|\geqslant 1} D^a(\{\rho_a^r\}\omega^r).$$

Clearly, $\lambda^r \equiv \{\rho^r\} = \lambda^r[x, \underline{u}]$, and $\vec{A}_\lambda \equiv \vec{B} - \vec{C}$ is equivalent to $\vec{B}$. Therefore, $\vec{A}$ is equivalent to $\vec{A}_\lambda$.

(ii) Let $\vec{A}_\lambda$ and $\vec{A}_{\lambda'}$ be equivalent first kind conserved currents. Then

$$\vec{A} \equiv \vec{A}_\lambda - \vec{A}_{\lambda'} = \sum_{r,\alpha} \vec{\lambda}_\alpha^r\cdot D^\alpha\omega^r + \vec{B}$$

where $\vec{D}\vec{B} \equiv 0$. If we apply to A the divergence operator we get

$$\sum_r (\lambda^r - \lambda'^r)\omega^r = \sum_{r,\alpha} (\vec{D}\vec{\lambda}_\alpha^r\cdot D^\alpha\omega^r + \vec{\lambda}_\alpha^r\cdot\vec{D}D^\alpha\omega^r). \tag{8}$$

This equation is an identity of the form $\Sigma_{r,\alpha}\eta_\alpha^r[x, \underline{u}, \omega]D^\alpha\omega^r = \Sigma_{r,\alpha}\eta_\alpha'^r[x, \underline{u}, \omega]D^\alpha\omega^r$ and it implies $\eta_\alpha^r \stackrel{\circ}{=} \eta_\alpha'^r$ for all r, α. Then, from (5) it follows that $\{\eta_\alpha^r\} \stackrel{\circ}{=} \{\eta_\alpha'^r\}$ for all r, α. But (8) yields $\{\eta^r\} = \lambda^r - \lambda'^r$ and $\{\eta'^r\} = 0$. Therefore $\lambda \stackrel{\circ}{=} \lambda'$.

On the other hand, if $\lambda \stackrel{\circ}{=} \lambda'$, then $\vec{A}_\lambda - \vec{A}_{\lambda'}$ is a first kind conserved current with integrating factor $\eta = \lambda - \lambda' \stackrel{\circ}{=} 0$. From part (i) of this theorem we have that there will be a first kind conserved current $\vec{A}_\rho$ equivalent to $\vec{A}_\eta$ such that $\rho = \rho[x, \underline{u}]$. As we have seen, this implies $\rho \stackrel{\circ}{=} \eta \stackrel{\circ}{=} 0$. But $\rho[x, \underline{u}] \stackrel{\circ}{=} 0$ if and only if $\rho \equiv 0$. Therefore $\vec{D}\cdot\vec{A}_\rho \equiv 0$, and this means that $\vec{A}_\lambda$ and $\vec{A}_{\lambda'}$ are equivalent.

We now turn our attention to normal Lagrangian systems. Let us suppose that (1) is a Lagrangian system. That is, there is $L \in R$ such that $\omega^r = \delta L/\delta u^r$. Given $\xi_i, \eta^r \in R$ let us consider the infinitesimal transformation

$$x_i' = x_i + \epsilon\xi_i[x, u(x)], \qquad u'^r(x') = u^r(x) + \epsilon\eta^r[x, u(x)] \tag{9}$$

which induces infinitesimal total variations of the fields of the form

$$u'^r(x) = u^r(x) + \epsilon\rho^r[x, u(x)], \qquad \rho^r \equiv \eta^r - \vec{\xi}\cdot\vec{D}u^r. \tag{10}$$

In the usual context of Noether's theorem [10] it is proved that the transformed field $u' = u'(x)$ satisfies to first order in the parameter ϵ the Euler–Lagrange equations associated with the Lagrangian function $L' = L + \epsilon\delta L$ where

$$\delta L = \vec{D}(\vec{\xi}L) + \sum_{r,\,\alpha} D^\alpha \rho^r \cdot \frac{\partial L}{\partial u^r_\alpha} = \vec{D} \cdot \vec{J} + \sum_r \rho^r \frac{\delta L}{\delta u^r} \tag{11}$$

and $\vec{J}$ is determined up to a divergenceless term by [8]

$$\vec{D} \cdot \vec{J} = \vec{D}(\vec{\xi}L) + \sum_{r,\,|\alpha| \geqslant 1} D^\alpha \left(\rho^r \frac{\delta L}{\delta u^r_\alpha} \right) . \tag{12}$$

An explicit choice of $\vec{J}$ may be written in terms of the notation conventions (11)–(13) of [8] in the following form

$$\vec{J} = \xi_i L + \sum_{r,k \geqslant 0} [ii_1 .. i_k] D^{i_1 \cdots i_k} \left(\rho^r \frac{\delta L}{\delta u^r_{ii_1 .. i_k}} \right) . \tag{13}$$

From (11) it follows that δL is a divergence if and only if ρ is an integrating factor. In this case, L and L' determine the same system of Euler–Langrange equations, and then (9) defines a symmetry of the Lagrangian system. These symmetries associated with integrating factors are called Noether symmetries. Moreover if $\delta L = \vec{D} \cdot \vec{K}$, identity (11) implies that $\vec{K} - \vec{J}$ is a conserved current and it is precisely a first kind conserved current $\vec{A}_\rho$ associated with the integrating factor ρ.

Let us denote by S_ρ an infinitesimal symmetry $u'(x) = u(x) + \epsilon\rho[x, u(x)]$ of (1). If ρ' is a m-component function such that $\rho \triangleq \rho'$, then ρ' and ρ determine the same total variations on the solutions of (1) and therefore $S_{\rho'}$ is also an infinitesimal symmetry of (1). Consequently, it is natural to define that two infinitesimal symmetries S_ρ and $S_{\rho'}$ are equivalent if and only if $\rho \triangleq \rho'$. In this way, if we call the correspondence $S_\rho \to \vec{A}_\rho$ between Noether symmetries and first kind conserved currents of Lagrangian systems the Noether map, the following result follows at once from Lemma 3.

THEOREM. *For normal Lagrangian systems the Noether map induces a bijective correspondence from the set of equivalence classes of Noether symmetries onto the set of equivalence classes of conserved currents.*

We shall end our analysis with some comments:

(a) The result of the theorem cannot be generalized to all the Lagrangian systems. It is well-known [11] that Noether currents associated with gauge transformations are equivalent to the zero current. It does not contradict our result since equations of motion of gauge covariant theories are not normal systems.

(b) As we have seen, the Noether map between symmetries and conserved currents is a consequence of the dual role played by integrating factors in Lagrangian systems. For arbitrary systems of partial differential equations integrating factors determine conserved currents by means of (3), but in general they do not define symmetries. This means that the fundamental property of Lagrangian systems which is implicit in Noether's Theorem is that integrating factors give rise to symmetries. This fact is the origin of the relationship between conservation laws and invariance properties in the Lagrangian formalism. Notwithstanding, several generalizations of Noether's Theorem have been proposed in which conserved currents appear unconnected with symmetries (see [4] and [6]). We disagree with this point of view since it lacks the basic content of Noether's Theorem.

(c) In [1] Ibragimov gives a way to derive all the conserved currents of arbitrary systems of partial differential equations from symmetry groups. Ibragimov's argument (see the proof of Theorem 2 of [1]) is that every conserved current $\vec{A}$ may be obtained from the infinitesimal symmetry

$$\xi_i = A_i/L, \qquad \eta^r = \vec{\xi} \cdot \vec{D}u^r, \tag{14}$$

where L is a weak Lagrangian such that $L \not\approx 0$. But this choice of η^r corresponds to a null total variation of the field (see (10)). That is, it generates the identity transformation $u' = u$. Clearly, this result cannot be interpreted as a group theoretical foundation of all the conserved currents. Moreover, it follows at once that the infinitesimal transformations which lead to a given conserved current $\vec{A}$ in Ibragimov's approach are given by (see formula (6) of [1])

$$\xi_i = A_i/L - \sum_{r,k \geqslant 0} [ii_1 .. i_k] D^{i_1 .. i_k} \quad \rho^r \frac{\delta L}{\delta u^r_{ii_1 .. i_k}} \tag{15}$$

with $\rho^r \equiv \eta^r - \vec{\xi} \cdot \vec{D}u^r$ arbitrary. Therefore, all the infinitesimal transformations $u'^r(x) = u^r(x) + \epsilon\rho^r[x, u(x)]$ may be associated with $\vec{A}$. This means that the method of Ibragimov does not provide any intrinsic relationship between conserved currents and symmetries.

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