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First-Order Fuzzy Logic

Abstract. This paper is an attempt to develop the many-valued first-order fuzzy logic. The set of its truth values is supposed to be either a finite chain or the interval $\langle 0, 1\rangle$ of reals. These are special cases of a residuated lattice $\langle L, \vee, \wedge, \otimes, \rightarrow, 1, 0\rangle$. It has been previously proved that the fuzzy propositional logic based on the same sets of truth values is semantically complete. In this paper the syntax and semantics of the first-order fuzzy logic is developed. Except for the basic connectives and quantifiers, its language may contain also additional n-ary connectives and quantifiers. Many propositions analogous to those in the classical logic are proved. The notion of the fuzzy theory in the first-order fuzzy logic is introduced and its canonical model is constructed. Finally, the extensions of Gödel's completeness theorems are proved which confirm that the first-order fuzzy logic is also semantically complete.

1. Introduction

The theory of fuzzy sets is now a quickly developing branch which gained its popularity because of its attractiveness and ease of applications. The connection between fuzzy sets and many-valued logic is very close. Any logic with more than two truth values is now usually called *fuzzy logic*. It should be stressed that the term "fuzzy logic" has, in fact, two meanings: *many-valued logic* and *linguistic logic* whose truth values are words of natural language. Linguistic logic has been introduced by L. A. Zadeh in [16]. In this paper we deal with many-valued logic whose truth values are taken either from the finite chain or from the interval of reals $\langle 0, 1\rangle$.

In [13] J. Pavelka introduced the propositional fuzzy logic based on the above mentioned truth sets (they are special cases of the so called residuated lattice) and he proved that it is semantically complete. A natural question arises if the first-order fuzzy logic being an extension of the propositional fuzzy logic is also semantically complete.

We have tried to develop the first-order fuzzy logic in this paper. Unlike Pavelka whose proof is based on the ultrafilter trick, we have tried to follow the classical method. We have proved the fuzzy extensions of Gödel's completeness theorems constructing the canonical model of the fuzzy theory (in the language of the first-order fuzzy logic). Since our proof does not use the results of Pavelka's proof it may also be considered to be a direct proof of the completeness of the fuzzy propositional logic. Fuzzy logic presented here is a direct generalization of the classical predicate calculus which we obtain when we replace the general finite chain of truth values by the Boolean lattice $\{0, 1\}$.

2. Truth values, operations and generalized functions

There are good reasons to suppose that truth values form a *complete, infinitely distributive* and *residuated lattice*

$$\mathscr{L} = \langle L, \vee, \wedge, \otimes, \rightarrow, \mathbf{1}, \mathbf{0}\rangle \tag{1}$$

where $\mathbf{1}$, $\mathbf{0}$ are the greatest and the smallest elements, respectively, and $\otimes$, $\rightarrow$ are additional binary operations of (*bold*) *multiplication* and *residuation*, respectively, having the properties:

a) $\langle L, \otimes, \mathbf{1}\rangle$ is a commutative monoid.

b) The operation $\otimes$ is isotone in both variables and $\rightarrow$ is antitone in the first and isotone in the second variable.

c) The adjuction property

$$\alpha \otimes \beta \leqslant \gamma \quad \text{iff} \quad \alpha \leqslant \beta \rightarrow \gamma$$

holds for every $\alpha, \beta, \gamma \in L$.

The reasons for using the residuated lattice are discused in [7, 12, 13]. When putting $L = \langle 0, 1\rangle$, $\vee = max$, $\wedge = min$, $\mathbf{0} = 0$, $\mathbf{1} = 1$ and

$$\begin{aligned} \alpha \otimes \beta &= 0 \vee (\alpha + \beta - 1) \\ \alpha \rightarrow \beta &= 1 \wedge (1 - \alpha + \beta) \end{aligned} \tag{2}$$

for every $\alpha, \beta \in \langle 0, 1\rangle$ we obtain the residuated lattice (1). Similarly, for $L = \{\mathbf{0} = a_n \leqslant \ldots \leqslant a_m = \mathbf{1}\}$ being a finite chain we put

$$\begin{aligned} a_k \otimes a_p &= a_{max(0,k+p-m)} \\ a_k \rightarrow a_p &= a_{min(m,m-k+p)} \end{aligned} \tag{3}$$

where $0 \leqslant k, p \leqslant m$. We again obtain the residuated lattice (1). This fact is very important since finite chain and the interval $\langle 0, 1\rangle$ have been used in all applications of fuzzy sets and are the most natural from the point of interpretation of results.

The choice of operations follows from the facts proved in [13]:

– If the operation $\rightarrow$ is not continuous on $\langle 0, 1\rangle \times \langle 0, 1\rangle$, then it is not possible to construct fuzzy logic with complete syntax.

– Every residuated lattice (1) based on $\langle 0, 1\rangle$ such that the operation $\rightarrow$ is continuous on $\langle 0, \mathbf{1}\rangle \times \langle 0, 1\rangle$ is isomorphis with the above lattice endowed with the operations (2).

We will frequently use the following symbols:

$$\begin{aligned} \alpha \leftrightarrow \beta &=_D (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha) \\ \neg \alpha &=_D \alpha \rightarrow \mathbf{0} \end{aligned}$$

($=_D$ stands for "being defined"). All through this paper we will confine ourselves to the assumption that the set of truth values forms the above defined residuated lattice either with $L = \langle 0, 1\rangle$ or L being a finite chain with the operations of multiplication and residuation defined by (2) or (3), respectively. Some of the properties introduced further hold, of course, for a general residuated lattice.

LEMMA 1. *Let $\alpha, \beta, \gamma \in L$ and $I, K \subseteq L$. Then*

(a) $\bigwedge_{\alpha\in K} \alpha \otimes \bigwedge_{\beta\in I} \beta \leqslant \bigwedge_{\alpha\in K} \bigwedge_{\beta\in I} (\alpha \otimes \beta)$

b) $\bigwedge_{\alpha,\beta\in K} (\alpha\leftrightarrow\beta) \leqslant \bigwedge_{\alpha\in K} \alpha \leftrightarrow \bigwedge_{\alpha\in K} \beta$

(c) $\bigwedge_{\alpha,\beta\in K} (\alpha\leftrightarrow\beta) \leqslant \bigvee_{\alpha\in K} \alpha \leftrightarrow \bigvee_{\beta\in K} \beta$

(d) $\neg(\neg\alpha) = \alpha$

(e) $\bigwedge_{\alpha\in K} (\alpha\otimes\beta) = \bigwedge_{\alpha\in K} \alpha\otimes\beta$

(f) $\alpha\wedge\beta = \neg(\neg\alpha\vee\neg\beta)$

(g) $\alpha\to\beta = \neg(\alpha\otimes\neg\beta)$

(h) $\alpha\to\beta = \neg\beta\to\neg\alpha$

The proof follows from the definition of the residuated lattice and, in some cases, from the assumption that L is a chain. □

Many other properties of operations in residuated lattices are proved in [7, 12, 13].

The four basic operations appear to be insufficient for applications of the fuzzy logic. The question arises whether we are able to introduce new operations which can enrich the given residuated lattice $\mathscr{L}$ in such a way that they will be the interpretation of new additional logical connectives (see Section 4). J. Pavelka in [13] has shown that any n-ary operation σ fulfilling the condition

$$(4)\qquad (\alpha_1\leftrightarrow\beta_1)^{k_1}\otimes \ldots \otimes(\alpha_n\leftrightarrow\beta_n)^{k_n} \leqslant \sigma(\alpha_1, \ldots, \alpha_n)\leftrightarrow\sigma(\beta_1, \ldots, \beta_n)$$

for some $k_1, \ldots, k_n$ where the exponentation is taken in the sense of the multiplication $\otimes$ can be used as the interpretation of n-ary logical connective. This condition called the *fitting condition* intuitively means that "roughly equal values of arguments imply roughly equal values of the operation σ in them". It can be verified that every basic operation $(\vee, \wedge, \otimes, \to)$ fulfils the fitting condition (4). Moreover, any operation derived from the operations fulfilling (4) also fulfils (4).

In the first-order fuzzy logic we will also introduce generalized operations Q,

$$Q: \mathscr{P}(L)\to L,$$

where $\mathscr{P}(L)$ is the power set of L. If $K \subseteq L$, then we will write $Q_{\alpha\in K}\, \alpha$ instead of $Q\{a; a \in K\}$ and Qa instead of $Q\{a\}$. We will call Q the *regular generalized operation* if it fulfils the conditions

(5.1) $\underset{a\in K}{Q}(a \otimes \beta) \leq \underset{a\in K}{Q} a \otimes \beta$

(5.2) $\underset{a\in K}{Q}\big((a \otimes \beta) \to 0\big) \to 0 \leq \big(\underset{a\in K}{Q}(a \to 0) \to 0\big) \otimes \beta,$

(5.3) $\bigwedge_{a\in K} a \leq \underset{a\in K}{Q} a \leq \bigvee_{a\in K} a, \ K \neq \emptyset.$

Put $\underset{a\in K}{\hat{Q}} a = \neg \underset{a\in K}{Q}(\neg a)$. It can be easily proved that $\hat{Q}$ is also a regular generalized operation. Moreover, $\neg \underset{a\in K}{\hat{Q}} a = \underset{a\in K}{Q}(\neg a)$ and $\hat{Q}(\neg a) = \neg Q a$. The operation Q is adjoined to $\hat{Q}$. If, moreover, $\underset{a\in K}{Q} a = \neg \underset{a\in K}{Q}(\neg a)$ holds, then Q and $\hat{Q}$ are *adjoint generalized operations*. The operations of supremum $\bigvee$ and infimum $\bigwedge$ are a pair of adjoint regular generalized operations.

In what follows we will suppose that the truth values form the algebra

(6) $\mathscr{L} = \langle L, \vee, \wedge, \otimes, \to, \{a; a \in L\}, \{\sigma_j; j \in Jop\}, \bigvee, \bigwedge, \{Q_j; j \in Jq\}\rangle$

where L is either the interval $\langle 0, 1\rangle$ or $(m+1)$-element chain, $\otimes$, $\to$ are the operations (2) or (3), respectively, $\{a; a \in L\}$ is a set of nullary operations, $\{\sigma_j; j \in Jop\}$ is a set of additional operations fulfilling (4) and $\{Q_j; j \in Jq\}$ is a set of additional generalized operations.

3. Language, terms and formulas of the first-order fuzzy logic

The *language* of fuzzy logic consists of:

a) Variables $x, y, \ldots$.

b) Constants $\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{r}, \ldots$.

c) Symbols for truth values $\boldsymbol{a}; a \in L$.

d) n-ary functional symbols $f^{(G)}$, $g^{(H)}, \ldots$ with various superscripts G, H, ...

e) n-ary predicate symbols p, q, ... and the designated identity symbol $=$.

f) Binary connectives $\vee^*$, $\wedge^*$, $\&^*$, $\Rightarrow^*$ and a set of n-ary connectives $\{\sigma_j; j \in Jop\}$ (the arity of σ_j is supposed to depend on the subscript j).

g) Symbols for the general $\forall$ and existential $\exists$ quantifiers and a set of generalized quantifiers $\{\mathrm{Q}_j; j \in Jq\}$.

h) Auxiliary symbols (,) etc.

The following are common recurrent definitions of terms and formulas.

Terms

a) A variable or a constant is a term without superscript.

b) If $f^{(G)}$ is an n-ary functional symbol and $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n$ are terms without superscript or with the same superscript G, then $f^{(G)}(\boldsymbol{a}_1, \dots, \boldsymbol{a}_n)$ is a term with the superscript G.

If the superscript is missing at the functional symbol, then the term is either without superscript or the superscript does not matter.

Formulas

a) The symbol $\boldsymbol{a}$, $a \in L$ for the truth value is a (atomic) formula.

b) If $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{a}_1, \dots, \boldsymbol{a}_n$ are terms with an arbitrary superscript and p is an n-ary predicate symbol, then $\boldsymbol{a} = \boldsymbol{b}$ and $p(\boldsymbol{a}_1, \dots, \boldsymbol{a}_n)$ are (atomic) formulas.

c) If $\varphi, \psi, \varphi_1, \dots, \varphi_n$ are formulas, then $\varphi \vee^* \psi$, $\varphi \wedge^* \psi$, $\varphi \,\&^*\, \psi$, $\varphi \Rightarrow^* \psi$, $\sigma_j(\varphi_1, \dots, \varphi_n)$ for $j \in Jop$, $(\forall x)\varphi$, $(\exists x)\varphi$, $(\mathrm{Q}_j x)\varphi$ for $j \in Jq$ are formulas.

The connectives $\vee^*$, $\wedge^*$, $\&^*$, $\Rightarrow^*$ will be called *disjunction, conjunction, bold conjunction* and *implication*, respectively. Let us introduce the following abbreviations of formulas:

$$\neg^* \varphi =_D \varphi \Rightarrow^* \mathbf{0} \quad \text{(negation)}$$

$$\varphi \Leftrightarrow^* \psi =_D (\varphi \Rightarrow^* \psi) \wedge^* (\psi \Rightarrow^* \varphi) \quad \text{(equivalence)}.$$

The set of all terms of a given language $\mathscr{J}$ will be denoted by $M_{\mathscr{J}}$ and the set of all formulas by $F_{\mathscr{J}}$.

Like in the classical logic we introduce the notions of free and bound variable, and substitutible term. If we substitute a term into a formula we must take care of its superscript: the term $\boldsymbol{b}$ is substitutible into the term $\boldsymbol{a}$ for the variable x if, moreover, $\boldsymbol{a}$ and $\boldsymbol{b}$ have either the same superscript or one of them is without superscript. If $\boldsymbol{a}$, $\boldsymbol{b}$ are terms with the same superscript and φ is a formula, then $\boldsymbol{a}_x[\boldsymbol{b}]$, $\varphi_x[\boldsymbol{b}]$ denote the term or the formula which we obtain by substitution of the term $\boldsymbol{b}$ for the varible x.

In the sequel, we will suppose that we are given some language $\mathscr{J}$ of the first-order fuzzy logic.

4. Semantics of the first-order fuzzy logic

4.1. Structures and truth valuations

A *fuzzy set* A in the universe U denoted by $A \underset{\sim}{\subseteq} U$ is a function

$$A: U \to L.$$

The element $Ax \in L$ for $x \in U$ is called the *grade of membership*. The function A is sometimes called the *membership function*, i.e. the membership function and the fuzzy set coincide. A fuzzy set is explicitly expressed by

$$\{Ax/x;\ x \in U\}.$$

A *fuzzy singleton* is a one-element fuzzy set $\{Ax/x\}$.

A *structure* for the language of the first-order fuzzy logic is a set together with fuzzy functions and fuzzy relations, i.e.

$$\mathscr{D} = \langle D, p_{\mathscr{D}}, \ldots, (g_{\mathscr{D}}, G), \ldots \rangle,$$

where D is a set, $p_{\mathscr{D}} \subseteq D^n$ are n-ary fuzzy relations and $g_{\mathscr{D}}$ are n-ary fuzzy functions with the domain $G \underset{\sim}{\subseteq} D$. The fuzzy function $g_{\mathscr{D}}$ is defined as follows. It is function $g_{\mathscr{D}}: D^n \to D$ fulfilling the condition

$$(7) \qquad Gg_{\mathscr{D}}(x_1, \ldots, x_n) \geqslant Gx_1 \wedge \ldots \wedge Gx_n, \qquad x_1, \ldots, x_n \in D.$$

The definition of a composite fuzzy function from fuzzy functions with the same domain is straightforward.

Interpretation of symbols of the language $\mathscr{J}$

a) Each constant is adjoined a fuzzy singleton in D. In the classical logic we usually suppose that every individual $d \in D$ is adjoined a name being a constant in the language $\mathscr{J}$. In the fuzzy logic we must deal with fuzzy singletons. Obviously, $\alpha \neq \beta$ follows from $\{\alpha/d\} \neq \{\beta/d\}$. Therefore, we must introduce names for all the nonempty fuzzy singletons in the language $\mathscr{J}$. They will be denoted by $\boldsymbol{d}_\alpha$, $\alpha \in L - \{0\}$.

b) Each n-ary functional symbol $g^{(G)}$ with the superscript G is adjoined an n-ary fuzzy function $d_{\mathscr{D}}$ with the domain $G \underset{\sim}{\subseteq} D$.

c) Each n-ary predicate symbol p is adjoined an n-ary fuzzy relation $p_{\mathscr{D}} \underset{\sim}{\subseteq} D^n$.

Interpretation of terms

Interpretation of terms are fuzzy singletons.

a) If $\boldsymbol{a}$ is $\boldsymbol{d}_\alpha$, $\alpha \in L - \{0\}$ where $\boldsymbol{d}_\alpha$ is the name of the fuzzy singleton $\{\alpha/d\} \underset{\sim}{\subseteq} D$, then the interpretation of the term $\boldsymbol{a}$ is this fuzzy singleton, i.e.

$$\mathscr{D}(\boldsymbol{a}) = \{\alpha/d\}.$$

If $\boldsymbol{a}$ is a constant without subscript α, then $\mathscr{D}(\boldsymbol{a}) = \{1/d\}$.

b) If $\boldsymbol{a} = g^{(G)}(\boldsymbol{b}_1^{(G)}, \ldots, \boldsymbol{b}_n^{(G)})$, then

$$\mathscr{D}(\boldsymbol{a}) = \{Gg_{\mathscr{D}}(\mathscr{D}(\boldsymbol{b}_1^{(G)}), \ldots, \mathscr{D}(\boldsymbol{b}_n^{(G)}))/g_{\mathscr{D}}(\mathscr{D}(\boldsymbol{b}_1^{(G)}), \ldots, \mathscr{D}(\boldsymbol{b}_n^{(G)}))\},$$

where $g_{\mathscr{D}}(\mathscr{D}(\boldsymbol{b}_1^{(G)}), \ldots, \mathscr{D}(\boldsymbol{b}_n^{(G)}))$ denotes the composite fuzzy function.

We will often write d_α instead of $\{\alpha/d\}$ and $d \in_\alpha D$ instead of $d_\alpha \underset{\sim}{\subseteq} D$.

Interpretation of formulas

Assume that the structure $\mathscr{D}$ for the language $\mathscr{J}$ of the first-order fuzzy logic is given. We define the truth valuation $\mathscr{D}(\chi)$ of the formulas $\chi \in F_{\mathscr{J}}$ in the structure $\mathscr{D}$ as follows. Let $\varphi, \psi, \varphi_1, \ldots, \varphi_n \in F_{\mathscr{J}}$.

a) $\qquad \chi =_D \boldsymbol{a}, \; a \in L$

$$\mathscr{D}(\chi) = a$$

b) $\chi =_D \boldsymbol{a} = \boldsymbol{b}$

$$\mathscr{D}(\chi) = \begin{cases} \mathbf{1} \text{ if } \mathscr{D}(a) = \mathscr{D}(\boldsymbol{b}) \text{ i.e. if } \mathscr{D}(\boldsymbol{a}) = d_\alpha \text{ and} \\ \mathscr{D}(\boldsymbol{b}) = d'_\beta \text{ then } \alpha = \beta \text{ and } d = d' \\ \mathbf{0} \text{ otherwise} \end{cases}$$

c) $\chi =_D \varphi \vee^* \psi$

$\mathscr{D}(\chi) = \mathscr{D}(\varphi) \vee \mathscr{D}(\psi)$.

d) $\chi =_D \varphi \wedge^* \psi$

$\mathscr{D}(\chi) = \mathscr{D}(\varphi) \wedge \mathscr{D}(\psi)$.

e) $\chi =_D \varphi \,\&^*\, \psi$

$\mathscr{D}(\chi) = \mathscr{D}(\varphi) \otimes \mathscr{D}(\psi)$.

f) $\chi =_D \varphi \Rightarrow^* \psi$

$\mathscr{D}(\chi) = \mathscr{D}(\varphi) \to \mathscr{D}(\psi)$.

g) $\chi =_D \sigma_j(\varphi_1, \ldots, \varphi_n),\ j \in Jop$

$\mathscr{D}(\chi) = \sigma_j(\mathscr{D}(\varphi_1), \ldots, \mathscr{D}(\varphi\ ))$.

h) $\chi =_D (\exists x)\varphi$

$\mathscr{D}(\chi) = \bigvee_{d \in_\alpha D} \mathscr{D}(\varphi_x[\boldsymbol{d}_\alpha])$.

i) $\chi =_D (\forall x)\varphi$

$\mathscr{D}(\chi) = \bigwedge_{d \in_\alpha D} \mathscr{D}(\varphi_x[\boldsymbol{d}_\alpha])$.

j) $\chi =_D (\mathsf{Q}_j x)\varphi,\ j \in Jq$

$\mathscr{D}(\chi) = \underset{d \in_\alpha D}{Q_j} \mathscr{D}(\varphi_x[\boldsymbol{d}_\alpha])$.

REMARK. All through this paper Q denotes the quantifier and Q the generalized operation being its interpretation in the structure.

The symbol $\underset{d \in_\alpha D}{Q} \mathscr{D}(\varphi_x[\boldsymbol{d}_\alpha])$ denotes the operation Q realized on the set $K \subseteq L$ being determined by the values $\mathscr{D}(\varphi_x[d_\alpha]) \in L$ after substituting all the $\boldsymbol{d}_\alpha$'s for all the d's $\in D$ and $\alpha \in L - \{\mathbf{0}\}$.

Obviously,

$$\mathscr{D}(\varphi \Leftrightarrow^* \psi) = \mathscr{D}(\varphi) \leftrightarrow \mathscr{D}(\psi)$$
$$\mathscr{D}(\neg^* \varphi) = \neg \mathscr{D}(\varphi) = \mathscr{D}(\varphi) \to \mathbf{0}.$$

If the operation Q_j, $j \in Jq$ being adjoined to the quantifier Q is regular, then we can enrich the language $\mathscr{J}$ by the symbol $\hat{\mathsf{Q}}_j$ to which we adjoin the operation $\hat{Q}_j$. Then we will call Q_j and $\hat{\mathsf{Q}}_j$ *adjoint quantifiers*. The quantifiers Q_j whose interpretation is the regular operation Q_j will be called *regular quantifiers*.

Clearly, $\forall$ and $\exists$ are adjoint regular quantifiers. In the sequel, we will suppose that the language $\mathscr{J}$ contains only pairs of adjoint regular quantifiers.

LEMMA 2. *Let $\mathscr{D}$ be a structure for the language $\mathscr{J}$, $\boldsymbol{a}^{(G)}(x)$ a term with superscript G and a free variable x, and $\varphi(x)$ be a formula with a free variable x. Let $\boldsymbol{b}^{(G)}$ be a term without free variables such that $\mathscr{D}(\boldsymbol{b}^{(G)}) = \{a/d\}$ is a fuzzy singleton with the name $\boldsymbol{d}_a$. Then*

$$\mathscr{D}(\boldsymbol{a}_x^{(G)}[\boldsymbol{b}^{(G)}]) = \mathscr{D}(\boldsymbol{a}_x^{(G)}[\boldsymbol{d}_a])$$
$$\mathscr{D}(\varphi_x[\boldsymbol{b}^{(G)}]) = \mathscr{D}(\varphi_x[\boldsymbol{d}_a])$$

holds true.

The proof proceeds by induction on the complexity of the term and formula. □

It can also be shown that the truth valuation behaves well with respect to the algebra of formulas.

4.2. Operation of semantic consequence

The operation of semantic consequence may now be introduced. Let $X \underset{\sim}{\subseteq} F_{\mathscr{J}}$ be a fuzzy set of formulas. Then the fuzzy set of *semantic consequences* of the fuzzy set X is

$(\mathscr{C}^{sem}X)\varphi = \bigwedge\{\mathscr{D}(\varphi);\ X \leqslant \mathscr{D}$ for any structure $\mathscr{D}$ for the language $\mathscr{J}\}$.

If $(\mathscr{C}^{sem}X)\varphi \geqslant a$, then will write $X \models_a \varphi$. The formula $\varphi(x_1, \dots, x_n)$ is *a-true* in the structure $\mathscr{D}$ if

$$a = \bigwedge_{\substack{d_i \in a_i D \\ i=1,\dots,n}} \mathscr{D}(\varphi_{x_1\dots x_n}[\boldsymbol{d}_{1_{a_1}}, \dots, \boldsymbol{d}_{n_{a_n}}]\ ).$$

Then we write $\mathscr{D} \models_a \varphi$. The formula $\varphi \in F_{\mathscr{J}}$ is an *a-tautology* if

$$a = (\mathscr{C}^{sem}\emptyset)\varphi$$

and we write $\models_a\varphi$. If $\models_1\varphi$, then we simply write $\models\varphi$ and say that φ is a *tautology*. An element $a \in L$ for which

$$a = \bigwedge \{\beta;\ \mathscr{D} \models_\beta \varphi \text{ in any structure } \mathscr{D} \text{ of the language } \mathscr{J}\}$$

holds will be called the *degree of validity* of the formula φ (or, equivalently we will say that φ is a-valid). It can be immediately seen that the formula φ is an a-tautology iff it is a-valid.

LEMMA 3. *Let $\varphi, \psi \in F_{\mathscr{J}}$. Then*

(a) $\models \varphi \Rightarrow^* \psi$ *iff* $\mathscr{D}(\varphi) \leqslant \mathscr{D}(\psi)$

(b) $\models \varphi \Leftrightarrow^* \psi$ *iff* $\mathscr{D}(\varphi) = \mathscr{D}(\psi)$

hold in any structure $\mathscr{D}$.

It follows from Lemma 3 that $\models\varphi\Leftrightarrow^*\psi$ iff $\models\varphi\Rightarrow^*\psi$ as well as $\models\varphi\Rightarrow^*\psi$.

LEMMA 4. *The following formulas are tautologies* ($\mathsf{Q}, \hat{\mathsf{Q}}$ *are adjoint regular quantifiers*):

(a) $\models\varphi_x[\boldsymbol{a}]\Rightarrow^*(\exists x)\varphi$

(b) $\models(\forall x)\varphi\Rightarrow^*\varphi_x[\boldsymbol{a}]$

(c) $\models\neg^*\neg^*\varphi\Leftrightarrow^*\varphi$

(d) $\models(\mathsf{Q}x)\varphi\Leftrightarrow^*\neg^*(\hat{\mathsf{Q}}x)\neg^*\varphi$

(e) $\models(\forall x)(\varphi\Leftrightarrow^*\psi)\Rightarrow^*\big((\mathsf{Q}x)\varphi\Leftrightarrow^*(\mathsf{Q}x)\psi\big)$

(f) $\models(\mathsf{Q}x)\neg^*\varphi\Leftrightarrow^*\neg^*(\hat{\mathsf{Q}}x)\varphi$

(g) $\models(\hat{\mathsf{Q}}x)\infty^*\varphi\Leftrightarrow^*\neg^*(\mathsf{Q}x)\varphi$

(h) $\models\neg^*(\varphi\Rightarrow^*\psi)\Leftrightarrow^*(\varphi\ \&^*\ \neg^*\psi)$

(i) $\models(\varphi\wedge^*\psi)\Leftrightarrow^*\neg^*(\neg^*\varphi\vee^*\neg^*\psi)$

(j) $\models(\varphi\Rightarrow^*\psi)\Leftrightarrow^*\neg^*(\varphi\&^*\ \neg^*\psi)$

(k) $\models(\varphi\ \&^*\psi)\Rightarrow^*\big((\psi\Rightarrow^*\chi)\Rightarrow^*(\varphi\ \&^*\ \chi)\big)$

(l) $\models(\neg^*\varphi\Rightarrow^*\ \psi)\Leftrightarrow^*(\neg^*\psi\Rightarrow^*\varphi)$

(m) $\models\varphi\ \&^*\ \psi\Rightarrow^*\varphi$

(n) $\models\varphi\ \&^*\ \psi\Rightarrow^*\psi$

The proof follows from Lemma 1 and the definitions. □

If $\varphi(x_1, \ldots, x_n)$ is a formula, then the formula $(\forall x_1)\ \ldots\ (\forall x_n)\varphi(x_1, \ldots$ $\ldots, x_n)$ is its *closure*.

THEOREM 1. *Let* φ' *be a closure of the formula* φ. *Then*

$$\mathscr{D}\models_a\varphi \quad \textit{iff} \quad \mathscr{D}\models_a\varphi'$$

COROLLARY. $\models_a\varphi$ *iff* $\models_a\varphi'$.

THEOREM 2. *Let* φ *be a tautology of the fuzzy propositional calculus. Then the formula resulting from* φ *if we replace some variables of* φ *by formulas of the language of the first-order logic is a tautology of the first--order fuzzy logic.*

LEMMA 5 (AXIOMS OF IDENTITY). *The following fromulas are tautologies*:

(a) $\models x = x$

(b) $\models x_1 = y_1\Rightarrow^*\big(x_2 = y_2\Rightarrow^*\big(\ldots\Rightarrow^*\big(f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)\big)\ldots\big)$

(c) $\models x_1 = y_1\Rightarrow^*\big(x_2 = y_2\Rightarrow^*\big(\ldots\ \Rightarrow^*\big(p(x_1, \ldots, x_n)\Rightarrow^* p(y_1, \ldots, y_n)\big)\ldots\big)$.

PROOF. We will demonstrate only (b). Then

$$(8)\qquad \mathscr{D}(\boldsymbol{d}_{1_{a_1}} = \boldsymbol{d}'_{1_{a'_1}})\to\Big(\mathscr{D}(\boldsymbol{d}_{2_{a_2}} = \boldsymbol{d}'_{2_{a'_2}})\to\big(\ldots \to\big(\mathscr{D}\big(f(\boldsymbol{d}_{1_{a_1}}, \ldots, \boldsymbol{d}_{n_{a_n}}) = f(\boldsymbol{d}'_{1_{a'_1}}, \ldots, \boldsymbol{d}'_{n_{a'_n}})\big)\big)\ldots\big)\Big) = 1$$

must hold for arbitrary $d_i \in_{\alpha_i} D$, $d'_i \in'_{\alpha_i} D$, $i = 1, \dots, n$ in an arbitrary structure $\mathcal{D}$. If $\mathcal{D}(\boldsymbol{d}_{i_{\alpha_i}} = \boldsymbol{d}'_{i_{\alpha'_i}}) = \mathbf{0}$ holds for some $\boldsymbol{d}_i$, $\boldsymbol{d}'_i$, then (8) holds, too, since $\mathbf{0} \to \beta = \mathbf{1}$ holds for every $\beta \in L$. If $\mathcal{D}(\boldsymbol{d}_{i_{\alpha_i}} = \boldsymbol{d}'_{i_{\alpha'_i}}) = \mathbf{1}$, $i = 1, \dots$ $\dots, n$, then $d_{i_{\alpha_i}} = d'_{i_{\alpha'_i}}$ and $f_{\mathcal{D}}(d_{1_{\alpha_1}}, \dots, d_{n_{\alpha_n}}) = f_{\mathcal{D}}(d'_{1_{\alpha_1}}, \dots, d'_{n_{\alpha_n}})$ which again follows from (8). □

5. Syntax of the first-order fuzzy logic

5.1. Rules of inference

The rules of inference have generally been defined in [13]. Every *n-ary rule of inference* r is a pair $\langle r^{syn}, r^{sem} \rangle$ where r^{syn} is a *syntactical part* of the rule r being a partial n-ary operation on $F_{\mathcal{J}}$ and r^{sem} a *semantical part* of r being an n-ary semicontinuous operation on L, i.e. the operation preserving all the non-empty joins in L in each variable. Then the operation r^{sem} is isotone in each variable. A fuzzy set $X \lesssim F_{\mathcal{J}}$ is *closed* with respect to $r = \langle r^{syn}, r^{sem} \rangle$ if

$$X r^{syn}(\varphi_1, \dots, \varphi_n) \geqslant r^{sem}(X\varphi_1, \dots, \varphi_n)$$

holds for every $\varphi_i \in F_{\mathcal{J}}$ for which r^{syn} is defined. The rule of inference is *X-sound* if

$$\mathcal{D}(r^{syn}(\varphi_1, \dots, \varphi_n)) \geqslant r^{sem}\big(\mathcal{D}(\varphi_1), \dots, \mathcal{D}(\varphi_n)\big) \tag{9}$$

holds in any structure $\mathcal{D}$ such that $X\varphi \leqslant \mathcal{D}(\varphi)$ for all $\varphi \in F_{\mathcal{J}}$. The rule is *sound* if it is Ø-sound. The rules of inference are written in the following way:

$$r: \frac{\varphi_1, \dots, \varphi_n}{r^{syn}(\varphi_1, \dots, \varphi_n)} \left(\frac{(a_1, \dots, a_n}{r^{sem}(a_1, \dots, a_n)} \right)$$

where $a_i \in L$ are valuations of the formulas φ_i, $i = 1, \dots, n$.

LEMMA 6. *Assume that the following rules of inference are given:*
(a) *Modus ponens*

$$r_{MP}: \frac{\varphi, \varphi \Rightarrow^* \psi}{\psi} \left(\frac{\alpha, \beta}{\alpha \otimes \beta} \right).$$

(b) *φ, a-lifting rules*

$$r_{R\varphi,a}: \frac{\psi}{\varphi \Rightarrow^* \psi} \left(\frac{\beta}{a \to \beta} \right)$$

where $a = (\mathcal{C}^{syn} X)\varphi$.
(c) *Generalization*

$$r_G: \frac{\varphi}{(\mathsf{Q}x)\varphi} \left(\frac{a}{a} \right).$$

(d) *Generalization of the consequent*

$$r_{GC}: \frac{\varphi \Rightarrow^* \psi}{\varphi \Rightarrow^* (\mathrm{Q}x)\psi} \left(\frac{\alpha}{\alpha}\right)$$

on the assumption that x is not free in φ.

(e) *Generalization of the antecedent*

$$r_{GA}: \frac{\varphi \Rightarrow^* \psi}{(\mathrm{Q}x)\varphi \Rightarrow^* \psi} \left(\frac{\alpha}{\alpha}\right)$$

on the assumption that x is not free in ψ.

(f) *Distributivity of the quantifier*

$$r_{DQ}: \frac{(\mathrm{Q}x)(\varphi \;\&^* \psi)}{(\mathrm{Q}x)\varphi \,\&^* \psi} \left(\frac{\alpha}{\alpha}\right)$$

on the assumption that x is not free in ψ. The quantifiers are supposed to be regular. Then all the rules are sound except for the rule (b) *which is X-sound.*

PROOF. (a) has been proved in [13]. We will demonstrate only (b) and (d).

(b) This rule is a generalization of the rule $r_2\alpha$ from [13] where its semicontinuity was proved. Soundness:

$$\mathscr{D}\left(r^{syn}_{R\varphi,\alpha}(\psi)\right) = \mathscr{D}(\varphi \Rightarrow^* \psi) = \mathscr{D}(\varphi) \to \mathscr{D}(\psi) = \alpha' \to \beta \geqslant \alpha \to \beta = r^{sem}_{R\varphi,\alpha}\left(\mathscr{D}(\psi)\right),$$

where $\beta = \mathscr{D}(\psi)$ and $\alpha = (\mathscr{C}^{syn}X)\varphi \leqslant \mathscr{D}(\varphi) = \alpha'$ for every $\mathscr{D}$ fulfilling the above assumption.

(d) Semicontinuity is obviously fulfilled. Soundness:

$$\mathscr{D}\left(r^{syn}(\varphi \Rightarrow^* \psi)\right) = \mathscr{D}\left(\varphi \Rightarrow^* (\mathrm{Q}x)\psi\right) = \mathscr{D}(\varphi) \to \underset{d \in_a D}{\mathrm{Q}} \mathscr{D}(\psi_x[\boldsymbol{d}_a]) \geqslant \bigwedge_{d \in_a D} (\mathscr{D}(\varphi) \to \mathscr{D}(\psi\ [\boldsymbol{d}_a]) = r^{sem}\left(\mathscr{D}(\varphi \Rightarrow^* \psi)\right). \quad \square$$

Obviously, the rule (f) remains sound even if we interchange the formulas φ and ψ. Every rule $r_{Ra,\alpha}$, $\alpha \in L$ is sound.

5.2. Operation of syntactic consequence

We will work with fuzzy sets of *logical axioms* $A_L \subsim F_{\mathscr{J}}$ and *special axioms* $X \subsim F_{\mathscr{J}}$. Let the set $\mathscr{R}$ of X-sound rules of inference be given. Then the pair $\langle A_L, \mathscr{R}\rangle$ will be called a *syntax of fuzzy logic*. The operation of *syntactic consequences* from a fuzzy set of formulas X is defined as follows:

$$(\mathscr{C}^{syn}X)\varphi = \bigwedge \{U\varphi;\ A_L, X \subseteq U \text{ and } U \subseteq F_{\mathscr{J}} \text{ and } U \textit{ is closed with respect to all the rules } r \in \mathscr{R}\}$$

where $\varphi \in F_{\mathscr{J}}$.

A *proof* ω of the formula φ from the fuzzy set X is a sequence of formulas

$$\omega =_D \varphi_0, \varphi_1, \ldots, \varphi_n = \varphi$$

such that each φ_i, $0 \leqslant i \leqslant n$ is either a logical or a special axiom from X or it is a formula r^{syn} is $(\varphi_{i_1}, \ldots, \varphi_{i_n})$ where $i_1, \ldots, i_n < i$ and r is an n-ary X-sound rule of inference. Every proof has a *value*

$$Val(\omega) = \begin{array}{l} A_L\varphi_n \text{ if } \varphi_n \text{ is a logical axiom} \\ X\varphi_n \text{ if } \varphi_n \text{ is a special axiom} \\ r^{sem}(Val(\omega_{(i_1)}), \ldots, Val(\omega_{(i_n)}) \text{ if } \varphi_n \text{ was} \\ \text{derived from } \varphi_{i_1}, \ldots, \varphi_{i_n} \text{ using some } n\text{-ary} \\ \text{rule of inference } r, \end{array}$$

where $\omega_{(i_j)}$ denotes the proof $\omega_{(i_j)} =_D \varphi_0, \varphi_1, \ldots, \varphi_{i_j}$.
We will usually write down the proofs more extensively:

$$\omega =_D \varphi_0[a_0; r_0],\ \varphi_1[a_1; r_1], \ldots, \varphi_n = \varphi[a_n;\ r_n],$$

where $a_i = Val(\omega_{(i)})$ and r_i is a rule of inference by means of which the formula φ_i was derived from some previous formulas (if such a rule does exist).

THEOREM 3.

$$(\mathscr{C}^{syn}X)_\varphi = \bigvee\{Val\ (\omega);\ \omega \textit{ is a proof of } \varphi \textit{ from the fuzzy set } X \underset{\sim}{\subseteq} F_{\mathscr{J}}\}.$$

For proof see [13]. □

The syntax $\langle A_L, \mathscr{R}\rangle$ is X-sound if

$$A_L \subseteq \mathscr{C}^{sem}\emptyset \text{ and every rule } r \in \mathscr{R} \text{ is } X\text{-sound}.$$

The *syntax is sound* if all the rules $r \in \mathscr{R}$ are sound.

We will write $X \vdash_a \varphi$ instead of $(\mathscr{C}^{syn}X)\varphi \geqslant a)$.

The syntax of the first-order fuzzy logic will be the following. The fuzzy set A_L of logical axioms consists of the formulas:

a) $\boldsymbol{a}$, $\boldsymbol{a}$ &* $\boldsymbol{\beta}$, $\boldsymbol{a} \Rightarrow^* \boldsymbol{\beta}$ with the degrees of membership a, $a \otimes \beta$, $a \rightarrow \beta$

b) All the tautologies Σ, Σ^Δ from [13] with the degrees of membership equal to $\mathbf{1}$.

c) All the tautologies from Lemmas 4, 5 and tautologies of the form $\varphi \square \psi \Leftrightarrow^* \psi \square \varphi$ where $\square$ is any of the connectives $\vee^*$, $\wedge^*$, &* with the degree of membership equal to $\mathbf{1}$.

d) The other formulas with the degree of membership $\mathbf{0}$.

The set of rules of inference is the set

$$\mathscr{R} = \{r_{MP},\ \{r_{Ra,a};\ a \in L\},\ r_G, r_{GC}, r_{GA}, r_{DQ}\}.$$

6. Fuzzy theories of the first order

6.1. Properties of fuzzy theories

The *theory* $\mathscr{T}$ in the language $\mathscr{J}$ of the first-order fuzzy logic (shortly — the *fuzzy theory*) is the three-tuple

$$\mathscr{T} = \langle A_L, A_S, \mathscr{R}\rangle$$

where $\langle A_L, \mathscr{R}\rangle$ is the syntax of the first-order fuzzy logic and $A_S \subseteq F_{\mathscr{J}}$ is a fuzzy set of special axioms.

Let $\mathscr{D}$ be a structure for the language $\mathscr{J}$. Then $\mathscr{D}$ is a *model* of the theory $\mathscr{T}$ if

$$A_S\varphi \leqslant \mathscr{D}(\varphi)$$

holds for every $\varphi \in F_{\mathscr{J}}$ and we will write $\mathscr{D} \models \mathscr{T}$. Obviously, $A_L\varphi \leqslant \mathscr{D}(\varphi)$ and

$$(\mathscr{C}^{syn} A_S)\varphi = \bigwedge \{\mathscr{D}(\varphi);\ \mathscr{D} \text{ is a model of the theory } \mathscr{T}\}$$

holds for every $\varphi \in F_{\mathscr{J}}$. If $(\mathscr{C}^{sem} A_S)\varphi = a$, then φ is *a-true* (true in the degree a) and we write $\mathscr{T} \models_a \varphi$.

If $(\mathscr{C}^{syn} A_S)\varphi = a$, then φ is just an *a-theorem* of the theory $\mathscr{T}$ and we write $\mathscr{T} \vdash_a \varphi$ (we also say that φ is *provable in the degree a*). If $\mathscr{T} \vdash_1 \varphi$, then φ is a *theorem* of $\mathscr{T}$ and we simply write $\mathscr{T} \vdash \varphi$.

LEMMA 7. (a) $\mathscr{C}^{syn} A_S \subseteq \mathscr{C}^{sem} A_S$

b) $\mathscr{T} \vdash_a \varphi$ *follows from* $\mathscr{D} \models_\beta \varphi$, $a \leqslant \beta$

holds for every model $\mathscr{D}$ *of the theory* $\mathscr{T}$.

THEOREM 4 (VALIDITY THEOREM). *If* $\mathscr{T} \vdash_a \varphi$, *and* $\mathscr{T} \models_\beta \varphi$ *then* $a \leqslant \beta$.

Hence, if $\mathscr{T} \vdash_a \varphi$, then $\mathscr{D}(\varphi) \geqslant a$ holds for every model of the theory $\mathscr{T}$.

The element $\beta = \bigvee \{\beta';\ \beta' = a_1 \otimes a_2 \text{ and } \mathscr{T} \vdash_{a_1} \varphi,\ \mathscr{T} \vdash_{a_2} \neg^*\varphi,\ \varphi \in F_{\mathscr{J}}\}$ is a *degree of contradictoriness* of the theory $\mathscr{T}$ (or, respectively, $\mathscr{T}$ is *β-contradictory*). The element $\gamma = \neg\beta$ is the *degree of its consistency*. If $\mathscr{T}$ is consistent, then $\mathscr{T} \vdash \varphi$ follows from $\mathscr{T} \vdash_0 \neg^*\varphi$. In general, however, it may turn out that $\mathscr{T} \vdash_{a_1} \varphi$ and $\mathscr{T} \vdash_{a_2} \neg^*\varphi$ and $a_1 \neq \mathbf{0}$, $a_2 \neq \mathbf{0}$. It means that a formula as well as its negation are provable simultaneously to a certain degree. Such a situation is rather common in practice.

LEMMA 8. *Let* $\mathscr{T}$ *be a β-contradictory theorey. Then*

$$\beta \leqslant \bigvee \{\gamma;\ \mathscr{T} \vdash_\gamma \varphi \mathbin{\&} \neg^*\varphi,\ \varphi \in F_{\mathscr{J}}\}.$$

Conversely, if $\mathscr{T} \vdash_a \varphi \mathbin{\&^*} \neg^*\varphi$ *for some* $a > 0$ *and* $\varphi \in F_{\mathscr{J}}$, *then* $\mathscr{T}$ *is contradictory to some degree.*

PROOF. Let $\mathscr{T}$ be β-contradictory. Then there is a formula φ such that $\mathscr{T} \vdash_{a_1} \varphi$ and $\mathscr{T} \vdash_{a_2} \neg^*\varphi$ and $\beta' = a_1 \otimes a_2 \leqslant \beta$. Let ω_φ, $\omega_{\neg\varphi}$ be proofs of φ and $\neg^*\varphi$, $Val(\omega_\varphi) = a_1'$, $Val\ (\omega_{\neg\varphi}) = a_2'$. Then

$$\begin{aligned}(10)\qquad \omega =_D\ & \omega_\varphi[a_a'],\ \omega_{\neg\varphi}[a_2'],\ \varphi \Rightarrow^*(\neg^*\varphi \Rightarrow^* \varphi \mathbin{\&^*} \neg^*\varphi)\ [\mathbf{1}],\\ & \neg^*\varphi \Rightarrow^*(\varphi \mathbin{\&^*} \neg^*\varphi)\ [a_1';\ \vdash_{MP}],\ \varphi \mathbin{\&^*} \neg^*\varphi[a_1' \otimes a_2';\ r_{MP}]\end{aligned}$$

is a proof with the value $a_1' \otimes a_2'$ and $\bigvee \{Val(\omega);\ \omega \text{ is the proof (10)}\} = \beta'$. Thence $\mathscr{T} \vdash_\gamma \varphi \mathbin{\&^*} \neg^*\varphi$ where $\beta' \leqslant \gamma$ and $\beta = \bigvee_{\varphi \in F_{\mathscr{J}}} \beta'$

Let $\mathcal{T} \vdash_a \varphi$ &* $\neg^* \varphi$ and $a > \mathbf{0}$ and $\mathcal{T}$ be consistent. Then $\mathcal{D} \models_\beta \varphi$ &* $\neg^*\varphi$ holds for every model $\mathcal{D}$ where $\mathbf{0} < a \leqslant \beta$. But $\mathcal{D}(\varphi^*\& \ \neg^*\varphi) = \mathcal{D}(\varphi) \otimes \otimes \neg \mathcal{D}(\varphi) = \mathbf{0}$ in any model – a contradiction. □

LEMMA 9. *Let a theory $\mathcal{T}$ have a model $\mathcal{D}$. Then $\mathcal{T}$ is consistent.*

PROOF. If $\mathcal{D} \models_a \varphi$, then $\mathcal{D} \models_{\neg a} \neg\varphi$ and, by the validity theorem, $\mathcal{T} \vdash_\beta \varphi$, $\mathcal{T} \vdash_\gamma \neg^*\varphi$ where $\beta \leqslant a$ and $\gamma \leqslant \neg a$ which follows from $\beta \otimes \gamma \leqslant 0$. □

Since A_S is a fuzzy set, i.e. the function A_S: $F_{\mathcal{J}} \to L$, every formula $\varphi \in F_{\mathcal{J}}$ is adjoined a degree of membership. If $A_S\varphi = \mathbf{0}$, then $\mathcal{T} \vdash_0 \varphi$ might become like in the worst case. It means that every formula is "in some way" provable, at least in the degree $\mathbf{0}$.

A theory $\mathcal{T}$ is complete if it is consistent and $\mathcal{T} \vdash_a \varphi$ follows from $\mathcal{T} \vdash_{\neg a} \neg^*\varphi$. The notions of extension, conservative and simple extension of a theory can be defined quite analogously to how it is done in the classical logic.

LEMMA 10. *Let a theory $\mathcal{T}'$ be an extension of $\mathcal{T}$, $\mathcal{J}(\mathcal{T}) \subseteq \mathcal{J}(\mathcal{T}')$ and let $\mathcal{T}'$ have a model A'. Then restriction of the model $\mathcal{D}'$ to $\mathcal{J}(\mathcal{T})$ is a model $\mathcal{D}$ of the theory $\mathcal{T}$ and for every formula $\varphi \in F_{\mathcal{J}(\mathcal{T})}$*

$$\mathcal{D}'(\varphi) = \mathcal{D}(\varphi)$$

holds.

If $B \subseteq F_{\mathcal{J}(\mathcal{T})}$ is a fuzzy set of formulas and $\mathcal{T}$ a theory, then $\mathcal{T}' = A_L$, $A_S \cup B$, $\mathcal{R}\rangle$ is an extension of the theory $\mathcal{T}$. We will write $\mathcal{T}' = \mathcal{T} \cup B$.

LEMMA 11. *Let $\mathcal{T}$ be a consistent theory and let $\mathcal{T} \vdash_a \varphi$. Then $\mathcal{T}' = \mathcal{T} \cup \{\beta / \neg\varphi\}$ is a consistent theory iff $\beta \leqslant \neg a$.*

PROOF. $\mathcal{T} \vdash_a \varphi$ and $\beta < a \to \mathbf{0}$ follow from $\mathcal{T}' \vdash_{\beta'} \neg\varphi$ and $\mathcal{T}' \vdash_{a'} \varphi$ where $a \leqslant a'$, $\beta \leqslant \beta'$, i.e. $\mathcal{T}'$ is contradictory at least in the degree $a' \otimes \otimes \beta' > \mathbf{0}$.

Conversely, let $\beta \leqslant \neg a$ and $\mathcal{T} \vdash_{\beta'} \neg^*\varphi$. If $\beta \leqslant \beta'$, then $\mathcal{T}' = \mathcal{T}$. If $\beta' \leqslant \beta \leqslant \neg a$, then no proof of $\neg^*\varphi$ can exceed the value β and so $\mathcal{T}'$ is not contradictory. Assume $\mathcal{T} \vdash_\varepsilon \neg^*\psi$ and let ψ be derived from $\neg^*\varphi$ by means of the proof

$$\omega_\psi =_D \dots \neg^*\varphi[\hat{\beta}], {}_2\neg^*\varphi \Rightarrow^* \psi[\delta], \psi[\hat{\beta} \otimes \delta; r_{MP}]$$

where $\hat{\beta} \leqslant \mathrm{B}'$. Since $\mathcal{T}$ is consistent, $\hat{\beta} \otimes \delta \otimes \hat{\varepsilon} = \mathbf{0}$ holds for the value $\hat{\varepsilon}$ of a proof of $\neg^*\psi$. Let ω_ψ be a proof causing contradictoriness of $\mathcal{T}'$, i.e. $\beta \otimes \delta \otimes \varepsilon > \mathbf{0}$. Using the proof

$$\omega_\varphi =_D \dots \neg^*\psi[\hat{\varepsilon}], (\neg^*\varphi \Rightarrow^* \psi) \Rightarrow^* (\neg^*\psi \Rightarrow^* \varphi) \ [\mathbf{1}], \ \neg^*\varphi \Rightarrow^* \psi \ [\delta]$$
$$\neg^*\psi \Rightarrow^* \varphi[\delta; r_{MP}], \varphi[\hat{\varepsilon} \otimes \delta; r_{MP}]$$

we obtain $\varepsilon \otimes \delta \otimes \beta \leqslant a \otimes \beta = \mathbf{0}$ – a contradiction.

Let $\psi =_D \;\; \Rightarrow^* \chi$ be derived by means of the proof

$$\omega'_\varphi =_D \dots \neg^*\varphi[\hat{\beta}],\ \neg^*\varphi \Rightarrow^* \chi\ [\delta],\ \chi[\hat{\beta}\otimes\delta; r_{MP}],\ \mu \Rightarrow^* \chi[\alpha \to \hat{\beta}\otimes\delta; r_{R\mu,\alpha}].$$

Then there is a proof of μ &* $\neg^*\chi$ with the value at least $\hat{\varepsilon}$. Using the proof

$$\dots \mu \&^* \neg^*\chi[\hat{\varepsilon}],\ \mu \&^* \neg^*\chi \Rightarrow^* \neg^*\chi\ [\mathbf{1}],\ \neg^*\chi[\hat{\varepsilon}; r_{MP}]$$

we obtain $\hat{\varepsilon}\otimes\delta \leqslant \alpha$, i.e. the proof is converted to the previous case. □

THEOREM 5 (COMPLETION THEOREM). *Every consistent theory $\mathscr{T}$ can be extended to a complete theory $\mathscr{T}$ which is a simple extension of $\mathscr{T}$.*

PROOF. The construction of a maximal consistent theory is analoguous to that in the classical logic.

Let $\overline{\mathscr{T}} \vdash_a \varphi$ and $\overline{\mathscr{T}} \vdash_{a'} \neg^*\varphi$ where $\alpha' < \neg\alpha$ and put $\overline{\mathscr{T}}' = \overline{\mathscr{T}} \cup \{\neg\alpha/\neg^*\varphi\}$. Then, by Lemma 11, $\overline{\mathscr{T}}'$ is consistent but $\overline{\mathscr{T}}' = \overline{\mathscr{T}} \cup (A \cup \{\neg\alpha/\neg^*\varphi\})$, i.e. $A \subseteq A \cup \{\neg\alpha/\neg^*\varphi\}$ — a contradiction.

LEMMA 12. *Let φ be a formula of the theory $\mathscr{T}$ and let $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n$ be terms substitutible in φ for the variables $x_1, \dots, x_n$. Then $\mathscr{T} \vdash_a \varphi$ and $\mathscr{T} \vdash_{a'} \varphi_{x_1 \dots x_n}[\boldsymbol{a}_1, \dots, \boldsymbol{a}_n]$ implies $\alpha \leqslant \alpha'$.*

PROOF. The proposition is obtained using the proof

$$(11) \qquad \omega =_D \omega_\varphi[\alpha'],\ (\forall x)\varphi\ [\alpha';\ r_G],\ (\forall x)\varphi \Rightarrow^* \varphi_x[\boldsymbol{a}]\ [\mathbf{1}],\ \varphi_x[\boldsymbol{a}]\ [\alpha'; r_{MP}]$$

where ω_φ is a proof of φ. □

THEOREM 6 (ON CONSTANTS). *Let $\mathscr{T} = \langle A_L, A_S, \mathscr{R}\rangle$ be the theory in the language $\mathscr{J}$. Let us extend $\mathscr{J}$ by new constants $\boldsymbol{e} \in C$, i.e. $\mathscr{J}' = \mathscr{J} \cup C$ and let $\mathscr{T}'$ be a theory $\mathscr{T}' = \langle A'_L, A_S, \mathscr{R}\rangle$ in the language $\mathscr{J}'$ such that $A'_S\varphi = A_S\varphi$ for $\varphi \in F_{\mathscr{J}}$* and *$A'_S\varphi = \mathbf{0}$ for $\varphi \notin F_{\mathscr{J}}$. Then for every formula $\varphi \in F_{\mathscr{J}}$*

$$\mathscr{T} \vdash_a \varphi_{x_1 \dots x_n}[e_1, \dots, e_n] \quad \textit{iff} \quad \mathscr{T} \vdash_a \varphi$$

holds true where $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n \in C$.

PROOF. Let $\mathscr{T}' \vdash_a \varphi_{x_1 \dots x_n}[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$. Using the same idea as in the classical logic we obtain $\mathscr{T} \vdash_{\beta'} \varphi\ (y_1, \dots, y_n)$ where y_i are new variables and $\alpha \leqslant \beta'$. Hence, by Lemma 12, $\beta' \leqslant \beta$ and, again by Lemma 12.

$$\mathscr{T} \vdash_\beta \varphi \text{ follows from } \mathscr{T}' \vdash_\gamma \varphi_{x_1 \dots x_n}[e_1, \dots, e_n],\ \beta \leqslant \gamma$$

but β has been derived from α and therefore $\gamma \leqslant \alpha \leqslant \beta \leqslant \gamma$. □

COROLLARY. *The theory $\mathscr{T}'$ is a conservative extension of $\mathscr{T}$.*

The proof of the following lemma is based on the same idea as the proof of Theorem 6.

LEMMA 13. *Let $\mathcal{T}$, $\mathcal{T}'$ be theories and φ, φ' formulas. If for any $\alpha \in L$*

$$\mathcal{T} \vdash_\alpha \varphi \quad \textit{follows from} \quad \mathcal{T}' \vdash_\beta \varphi'$$

holds for some β, $\alpha \leqslant \beta$, and at the same time, for any $\gamma \in L$

$$\mathcal{T}' \vdash_\gamma \varphi' \quad \textit{follows from} \quad \mathcal{T} \vdash_\delta \varphi$$

holds for some δ, $\gamma \leqslant \delta$, then

$$\mathcal{T} \vdash_\alpha \varphi \quad \textit{iff} \quad \mathcal{T}' \vdash_\alpha \varphi'.$$

THEOREM 7 (CLOSURE THEOREM). *Let φ be a formula of the theory $\mathcal{T}$ and φ' be its closure. Then*

$$\mathcal{T} \vdash_\alpha \varphi \quad \textit{iff} \quad \mathcal{T} \vdash_\alpha \varphi'.$$

PROOF. It proceeds analogously to that in the classical logic using Lemmas 12, 13. □

6.2 Canonical model of the fuzzy theory

The next step to the proof of the completeness theorem is a construction of the *canonical model* $\mathcal{D}_0$ of the fuzzy theory $\mathcal{T}$. Like in the classical logic, we must add *axioms of Henkin type* (Henkin axioms) into the theory.

If the Henkin axiom $(\exists x)\varphi \Rightarrow^* \varphi_x[\boldsymbol{r}]$ is true in the degree 1, then

$$\bigvee_{d \in_\alpha D} \mathcal{D}(\varphi_x[\boldsymbol{d}_\alpha]) \leqslant \mathcal{D}(\varphi_x[\boldsymbol{r}])$$

must hold in any structure $\mathcal{D}$. This is possible only in the case that $\boldsymbol{r}$ is a term with the interpretation $\mathcal{D}(\boldsymbol{r}) = d'_\beta$ such that $\bigvee_{d \in_\alpha D} \mathcal{D}(\varphi_x[\boldsymbol{d}_\alpha]) = \mathcal{D}(\varphi_x[\boldsymbol{d}'_\beta])$. Analogously for $\varphi_x[\boldsymbol{r}] \Rightarrow^* (\forall x)\varphi$. The Henkin axiom ensures the existence of the fuzzy singleton d'_β with the property mentioned above. We will introduce Henkin axioms generally for every regular quantifier Q:

(12) $\quad (\mathsf{Q}x)\varphi \Rightarrow^* \varphi_x[\boldsymbol{r}]$

(13) $\quad \varphi_x[\boldsymbol{r}] \Rightarrow^* (\mathsf{Q}x)\varphi$

where $\boldsymbol{r}$ is a special constant. The theory $\mathcal{T}$ is Henkin if it contains formulas (12) and (13) as special axioms with the degree of membership **1**. From Theorem 4 it follows that there must exist β and $d_0 \in D$ in every model $\mathcal{D}$ of the Henkin theory $\mathcal{T}$ such that

$$\mathop{\mathsf{Q}}_{d \in_\alpha D} \mathcal{D}(\varphi_x[\boldsymbol{d}_\alpha]) = \mathcal{D}(\varphi_x[\boldsymbol{d}_{0\beta}]).$$

LEMMA 14. *Let $\mathcal{T}$ be a Henkin theory and $\boldsymbol{r}$ a special constant for the formula φ and a regular quantifier Q. Then*

$$\mathcal{T} \vdash_\alpha (\mathsf{Q}x)\varphi \quad \textit{iff} \quad \mathcal{T} \vdash_\alpha \varphi_x[\boldsymbol{r}].$$

Canonical model of the theory $\mathcal{T}$.

Assume $\mathscr{T}$ is a complete Henkin theory. Let $\boldsymbol{a}$, $\boldsymbol{b}$ be terms without free variables. We define the relation

$$\boldsymbol{a} \approx \boldsymbol{b} \quad iff \quad \mathscr{T} \vdash \boldsymbol{a} = \boldsymbol{b}.$$

LEMMA 15. *The relation $\approx$ is an equivalence.*

The support of the canonical model will be the factor set $D_0 = \boldsymbol{M}_v | \approx$. Its elements are equivalence classes $|\boldsymbol{a}|$, $\boldsymbol{a} \in \boldsymbol{M}_v$. Now, we will define the structure of $\mathscr{D}_0$:

a) Functions $(f_{\mathscr{D}_0}, G)$

$$(14) \qquad f_{\mathscr{D}_0}(|\boldsymbol{a}_1^{(G)}|, \ldots, |\boldsymbol{a}_n^{(G)}|) = |f^{(G)}(\boldsymbol{a}_1^{(G)}, \ldots, \boldsymbol{a}_n^{(G)})|.$$

The fuzzy set $G \subseteq D_0$ will be defined as follows:

$$G|\boldsymbol{a}| = \begin{cases} \mathbf{1} & \text{if there is a term } f^{(G)}(\boldsymbol{a}_1^{(G)}, \ldots, \boldsymbol{a}_n^{(G)}) \\ & \text{such that } \mathscr{T} \vdash f^{(G)}(\boldsymbol{a}_1^{(G)}, \ldots, \boldsymbol{a})_n^{(G)} = \boldsymbol{a} \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

The superscript of $\boldsymbol{a}$ need not be G.

b) Predicates $p_{\mathscr{D}_0}$

$$(15) \qquad p_{\mathscr{D}_0}(|\boldsymbol{a}_1|, \ldots, |\boldsymbol{a}_n|) = a \quad iff \quad \mathscr{T} \vdash_a p(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n).$$

Like in the classical logic we can demonstrate that the values of $f_{\mathscr{D}_0}$ and $p_{\mathscr{D}_0}$ do not depend on the choice of representatives of the equivalence classes $|\boldsymbol{a}_i|$, $i = 1, \ldots, n$.

We define

$$\mathscr{D}_0 = \langle D_0, (f_{\mathscr{D}_0}, G), \ldots, p_{\mathscr{D}_0}, \ldots \rangle$$

and call $\mathscr{D}_0$ a *canonical structure* for $\mathscr{T}$. Now, we must show that

$$(16) \qquad \mathscr{D}_0 \models_a \varphi \quad \text{iff} \quad \mathscr{T} \vdash_a \varphi$$

holds for every formula $\varphi \in F_{\mathscr{J}(\mathscr{T})}$. The proof proceeds by induction on the complexity of the formula.

1. *Interpretation of terms*

$$\mathscr{D}_0(\boldsymbol{a}^{(G)}) = \{G|\boldsymbol{a}^{(G)}| / |\boldsymbol{a}^{(G)}|\}.$$

By induction on the length of $\boldsymbol{a}$ we obtain

$$\begin{aligned} \mathscr{D}_0(\boldsymbol{a}^{(G)}) &= \mathscr{D}_0(f^{(G)}(\boldsymbol{a}_1^{(G)}, \ldots, \boldsymbol{a}_n^{(G)}) \\ &= \{G|f^{(G)}(\boldsymbol{a}_1^{(G)}, \ldots, \boldsymbol{a}_n^{(G)})| / |f^{(G)}(\boldsymbol{a}_1^{(G)}, \ldots, \boldsymbol{a}_n^{(G)})|\} \\ &= \{G|\boldsymbol{a}^{(G)}| / |\boldsymbol{a}^{(G)}|\}. \end{aligned}$$

2. *Interpretation of formulas*

a) $\varphi =_D \boldsymbol{a}$, $a \in L$.

Then $\mathscr{D}_0(\varphi) = a$ and we conclude that $\mathscr{T} \vdash_\beta \boldsymbol{a}$, $a \leqslant \beta$. By the validity theorem we have $\beta \leqslant a$, i.e. $\beta = a$.

b) $\varphi =_D \boldsymbol{a} = \boldsymbol{b}$.
Then

$$\mathscr{D}_0(\boldsymbol{a} = \boldsymbol{b}) = \mathbf{1} \text{ iff } \mathscr{D}_0(\boldsymbol{a}) = \mathscr{D}_0(\boldsymbol{b}) \text{ iff } a_a = a_a \text{ and } |\boldsymbol{a}| = |\boldsymbol{b}| \text{ iff } a_a = a_b \text{ and } \mathscr{T} \vdash \boldsymbol{a} = \boldsymbol{b}.$$

If superscripts of both terms are different, e.g. $\boldsymbol{a}^{(G)}$ and $\boldsymbol{b}^{(H)}$, then $a_a = G|\boldsymbol{a}^{(G)}| = G|\boldsymbol{b}^{(H)}| = 1$ but $H|\boldsymbol{b}^{(H)}| = \mathbf{1}$ which implies $a_a = a_b$.

c) $\varphi =_D p(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$.
Then

$$\mathscr{D}_0\big(p(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)\big) = a \text{ iff } p_{\mathscr{D}_0}\big(\mathscr{D}_0(\boldsymbol{a}_1), \ldots, \mathscr{D}_0(\boldsymbol{a}_n)\big) = a \text{ iff } p_{\mathscr{D}_0}(|\boldsymbol{a}_1|, \ldots, |\boldsymbol{a}_n|) = a \text{ iff } \mathscr{T} \vdash_a p(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n).$$

d) $\varphi =_D \psi \vee^* \chi$.
Then

$$\mathscr{D}_0(\varphi) = a \text{ iff } \mathscr{D}_0(\psi) \vee \mathscr{D}_0(\chi) = a \text{ iff } \mathscr{D}_0(\psi) = \beta$$

and $\mathscr{D}_0(\chi) = \gamma$ and $a = \beta \vee \gamma$. Assume $\mathscr{T} \vdash_\beta \psi$, $\mathscr{T} \vdash_\gamma \chi$. Let ω_ψ and ω_χ be proofs of ψ and χ, respectively, $Val(\omega_\psi) = \beta'$ $Val(\omega_\chi) = \gamma'$. Using the proofs

$$\omega_\varphi =_D \omega_\psi[\beta'],\ \psi \Rightarrow^* (\psi \vee^* \chi)\ [\mathbf{1}],\ \psi \vee^* \chi\ [\beta';\ r_{MP}]$$
$$\omega'_\varphi =_D \omega_\chi[\gamma'],\ \chi \Rightarrow^* \chi \vee^* \psi\ [\mathbf{1}],\ \chi \vee^* \psi\ [\gamma';\ r_{MP}],$$
$$\chi \vee^* \psi \Rightarrow^* \psi \vee^* \chi [\mathbf{1}],\ \psi \vee^* \chi\ [\gamma';\ r_{MP}]$$

we get $(\mathscr{C}^{syn} A_S)\varphi \geqslant \bigvee_{\omega_\varphi, \omega'_\varphi} (\beta' \vee \gamma')$ and $\mathscr{T} \vdash_a \varphi$ for $a = \beta \vee \gamma \leqslant \delta$. By the validity theorem we have $\delta \leqslant a$, i.e. $\delta = a$.

e) $\varphi =_D \psi \mathbin{\&} \chi$.

Analogously to what we have done in d), we use the proof

$$\omega_\varphi =_D \omega_\psi[\beta'],\ \omega_\chi[\gamma'],\ \psi \Rightarrow^* \big(\chi \Rightarrow^* (\psi \mathbin{\&^*} \chi)\big)[\mathbf{1}],\ \chi \Rightarrow^* (\psi \mathbin{\&^*} \chi)[\beta';\ r_{MP}],$$
$$\psi \mathbin{\&^*} \chi [\beta' \otimes \gamma';\ r_{MP}]$$

and conclude that $\mathscr{T} \vdash_\delta \varphi$, $\beta \otimes \gamma \leqslant \delta$. Hence, we obtain $\delta = a$.

f) $\varphi =_D \psi \Rightarrow^* \chi$.

First, suppose that $\chi =_D \mathbf{0}$. Then $\mathscr{D}_0(\varphi) = a$ iff $\mathscr{D}_0(\psi) = \beta$ and $\beta \to 0 = \neg\beta = a$. Since $\mathscr{T}$ is complete, we have $\mathscr{T} \vdash_a \neg^*\psi$.

For arbitrary χ we have $\mathscr{D}_0(\varphi) = a$ iff $\mathscr{D}_0(\psi) \to \mathscr{D}_0(\chi) = a$ iff $\mathscr{D}\ (\psi) = \beta$ and $\mathscr{D}_0(\chi) = \gamma$ where $\beta \to \gamma = a$. From the completeness of $\mathscr{T}$ and the inductive assumption we prove that

$$\mathscr{T} \vdash_{\delta'} \neg^*(\psi \mathbin{\&^*} \neg^*\chi)$$

where $\delta' = \neg(\beta \otimes \neg\gamma) = a$. If ω' is a proof of $\neg^*(\psi \mathbin{\&^*} \neg^*\chi)$, $Val(\omega') = \delta''$, then using the proof

$$\omega =_D \omega'[\delta''],\ \neg^*(\psi \mathbin{\&^*} \neg^*\chi) \Rightarrow^* (\psi \Rightarrow^* \chi)[\mathbf{1}],\ \psi \Rightarrow^* \chi [\delta'';\ r_{MP}]$$

we obtain $\mathscr{T} \vdash_\delta \varphi$, $a = \delta' \leqslant \delta$. Thence, $a = \delta$ by the validity theorem.

g) $\varphi =_D \psi \wedge^* \chi$.

This formula can be proved using f) and Lemmas 1 (f) and 4(i).

h) $\varphi =_D \sigma_j(\psi_1, \ldots, \psi_n)$, $j \in Jop$.

Then

$$\mathscr{D}_0(\varphi) = a \quad \text{iff} \quad \mathscr{D}_0(\psi_i) = \beta_i,\ i = 1, \ldots, n \text{ and}$$

$\sigma_j(\beta_1, \ldots, \beta_n) = a$. From f), g) and Lemma 3 it follows that $\mathscr{T} \vdash \psi_i \Leftrightarrow^* \beta_i$, $i = 1, \ldots, n$. Let ω_i' be a proof of $\psi_i \Leftrightarrow^* \beta_i$, $Val(\omega_i') = \gamma_i$. Using the tautology $\varphi \Rightarrow^* (\psi \Rightarrow^* (\varphi \ \&^* \ \psi))$ we can write down a proof of the formula $(\psi_1 \Leftrightarrow^* \beta_1)^{k_1} \&^* \ldots \&^* (\psi_n \Leftrightarrow^* \beta_n)^{k_n}$ with the value $\gamma_1^{k_1} \otimes \ldots \otimes \gamma_n^{k_n}$. Then

$$\mathscr{T} \vdash (\psi_1 \Leftrightarrow^* \beta_1)^{k_1} \ \&^* \ldots \&^* (\psi_n \Leftrightarrow^* \beta_n)^{k_n}.$$

Let ω'' be a proof of this formula, $Val(\omega'') = \gamma$. Using the proof

$$\begin{aligned}
&\omega =_D \omega''[\gamma],\ \big((\psi_1 \Leftrightarrow^* \beta_1)^{k_1} \ \&^* \ldots \&^* (\psi_n \Leftrightarrow^* \boldsymbol{\beta}_n)^{k_n} \Rightarrow^* (\sigma_j(\psi_1, \ldots, \psi_n) \\
&\quad \Leftrightarrow^* \boldsymbol{\sigma}_j(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_n)\big)\ [\mathbf{1}],\ \sigma_j(\psi_1, \ldots, \psi_n) \Leftrightarrow^* \boldsymbol{\sigma}_j(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_n)[\gamma],\ \boldsymbol{\sigma}_j(\boldsymbol{\beta}_1 t, \ldots \\
&\quad \ldots, \boldsymbol{\beta}_j) \Leftrightarrow^* \boldsymbol{\sigma}_j(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_n)\ [\mathbf{1}],\ \big(\boldsymbol{\sigma}_j(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_n) \Leftrightarrow^* (\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_n)\big) \Rightarrow^* \big(\boldsymbol{\sigma}_j(\boldsymbol{\beta}_1, \\
&\quad \ldots, \boldsymbol{\beta}_n) \Rightarrow^* \boldsymbol{\sigma}_j(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_n)\big)\ [\mathbf{1}],\ \boldsymbol{\sigma}_j(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_n) \Rightarrow^* \boldsymbol{\sigma}_j(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_n)\ [\mathbf{1}; r_{MP}], \\
&\quad \boldsymbol{\sigma}_j(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_n)\ [\sigma_j(\beta_1, \ldots, \beta_n)],\ \boldsymbol{\sigma}_j(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_n)\ [\sigma_j(\beta_1, \ldots, \beta_n); r_{MP}], \\
&\quad \big(\sigma_j(\psi_1, \ldots, \psi_n) \Leftrightarrow^* \boldsymbol{\sigma}_j(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_n)\big) \Rightarrow^* \big(\boldsymbol{\sigma}_j(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_n) \Rightarrow^* \sigma_j(\psi_1, \ldots \psi_n)\big) \\
&\quad [\mathbf{1}],\ \boldsymbol{\sigma}_j(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_n) \Rightarrow^* \sigma_j(\psi_1, \ldots, \psi_n)\ [\gamma; r_{MP}], \\
&\quad \sigma_j(\psi_1, \ldots, \psi_n)\ [\gamma \otimes \sigma_j(\beta_1, \ldots, \beta_n);\ r_{MP}]
\end{aligned}$$

we obtain $\mathscr{T} \vdash_\delta \sigma_j(\psi_1, \ldots, \psi_n)$, $\sigma_j(\beta_1, \ldots, \beta_n) \leqslant \delta$ where $\boldsymbol{\sigma}_j(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_n)$ denotes a formula corresponding to the truth value $\sigma_j(\beta_1, \ldots, \beta_n)$. But $\delta \leqslant a = \sigma_j(\beta_1, \ldots, \beta_n)$ and we conclude $\delta = a$.

i) $\varphi =_D (\mathrm{Q}x)\psi$, where Q is a regular quantifier.

Then

$$\mathscr{D}_0(\varphi) = a \text{ iff } \underset{|a| \in_a D_0}{\mathrm{Q}} \mathscr{D}_0(\psi_x[\boldsymbol{a}]) = a \text{ iff } \mathscr{D}_0(\psi_x[\boldsymbol{r}]) = a$$

because $\mathscr{T}$ is a Henkin theory. It follows from Lemma 14 that

$$\begin{aligned}
\mathscr{T} \vdash_a (\mathrm{Q}x)\psi \text{ iff } \mathscr{T} \vdash_a \psi_x[\boldsymbol{r}] \text{ iff } \mathscr{D}_0(\psi_x[\boldsymbol{r}]) = a \\
\text{iff } \mathscr{D}_0\big((\mathrm{Q}x)\psi\big) = a.
\end{aligned}$$

THEOREM 8. *A complete Henkin theory $\mathscr{T}$ having only regular quantifiers has a canonical model $\mathscr{D}_0$ such that*

$$\mathscr{T} \vdash_a \varphi \quad \text{iff} \quad \mathscr{D}_0 \models_a \varphi$$

holds for every formula $\varphi \in F_{\mathscr{J}(\mathscr{T})}$.

So far, we have not employed the rules of inference $r_{R\varphi(a)}$. They can be ommitted from the syntax of the fuzzy logic. If however, we have these rules at disposal we can omit the presumption of the completeness of the theory $\mathcal{T}$.

THEOREM 9. *A consistent Henkin theory $\mathcal{T}$ having only regular quantifiers has a canonical model $\mathcal{D}_0$ such that*

$$\mathcal{T} \vdash_a \varphi \quad \text{iff} \quad \mathcal{D}_0 \models_a \varphi$$

holds for every formula $\varphi \in F_{\mathcal{J}(\mathcal{T})}$.

PROOF. The completeness presumption was used only for $\varphi =_D \psi \Rightarrow^* \lambda$. The proposition is obtained using the proof

(17) $$\omega =_D \omega_\chi [\gamma'], \ \psi \Rightarrow^* \chi [\beta \to \gamma'; \ r_{R\psi\beta}]$$

where ω_χ is a proof of χ with the value γ'. □

6.3 Completeness theorem

In this section, we will prove two theorems which are extensions of the classical Gödel's theorems on the completeness of the predicate calculus. The quantifiers of the language $\mathcal{J}(\mathcal{T})$ are supposed to be regular.

THEOREM 10. *Let $\mathcal{T}$ be a consistent theory, C a set of special constants for closed formulas $(Qx)\varphi$ and let A_H be the fuzzy set of all Henkin axioms χ_H χ_H (of the form (12), (13)), $A_H \chi_H = \mathbf{1}$. Then the theory $\mathcal{T}' = \mathcal{T} \cup A_H$ in the language $\mathcal{J}(\mathcal{T}') = \mathcal{J}(\mathcal{T}) \cup C$ is a consistent extension of the theory $\mathcal{T}$.*

PROOF. Analogously to the classical proof, we successively construct sets $C_1, C_2, \ldots$ of special constants of a given level and successively extend the theory $\mathcal{T}$ by Henkin axioms χ_H.

a) Let $\mathcal{T}_i \cup \{\mathbf{1}/((Qx)\varphi \Rightarrow^* \varphi_x[\boldsymbol{r}])$ be contradictory to some degree for some φ and a special constant $\boldsymbol{r}$. Then

$$\mathcal{T}_i \vdash_a \neg^*((Qx)\varphi \Rightarrow^* \varphi_x[\boldsymbol{r}]), \text{ for some } a > \mathbf{0}.$$

Using the theorem on constants and the proof

$$\begin{aligned}\omega =_D\ & \omega'[a'], \ \neg^*((Qx)\varphi \Rightarrow^* \varphi) \Rightarrow^* (Qx)\varphi \ \&^* \ \neg^*\varphi)[\mathbf{1}], \ (Qx)\varphi \ \&^* \ \neg^*\varphi[a'; \\ & r_{MP}] \ (\hat{Q}x)((Qx)\varphi \ \&^* \ \neg^*\varphi)[a'; r_G], \ (Qx)\varphi \ \&^* \ (\hat{Q}x)\neg^*\varphi[a'; \ r_{DQ}], \\ & (Qx)\varphi \ \&^* \ (\hat{Q}x)\neg^*\varphi \Rightarrow^* ((\hat{Q}x)\neg\varphi \Rightarrow^* \neg(Qx)\varphi) \Rightarrow^* ((Qx)\varphi \ \&^* \ \neg^*(Qx)\varphi) \\ & [\mathbf{1}], \ ((\hat{Q}x)\neg^*\varphi \Rightarrow^* \neg(Qx)\varphi) \Rightarrow^* ((Qx)\varphi \ \&^* \ \neg^*(Qx)\varphi), \ [a'; \ r_{MP}], \\ & (\hat{Q}x)\neg^*\varphi \Rightarrow^* \neg^*(Qx)\varphi \ 1[\mathbf{1}], \ (Qx)\varphi \ \&^* \ \neg^*(Qx)\varphi[a'; r_{MP}]\end{aligned}$$

we conclude that $\mathcal{T}_i \vdash_\beta (Qx)\varphi \ \&^* \ \neg^*(Qx)\varphi$, $a \leqslant \beta$. Then $\mathcal{T}_i$ is contradictory to some degree (cf. Lemma 8) – a contradiction.

b) Analogously, let $\mathcal{T}_i \vdash_a \neg^*(\varphi_x[\boldsymbol{r}] \Rightarrow^* (Qx)\varphi)$, $a \rangle \geqslant 0$ and ω' be a proof of $\neg^*(\varphi \Rightarrow^* (Qx)\varphi)$ with the value a'. Using the proof

$$\begin{aligned}\omega =_D\ & \omega'[a'], \ \neg^*(\varphi \Rightarrow^* (Qx)\varphi) \Rightarrow^* \varphi \ \&^* \ \neg^*(Qx)\varphi \ [\mathbf{1}], \ \varphi \ \&^* \ \neg^*(Qx)\varphi \\ & [a'; \ r_{MP}], \ (Qx)(\varphi \ \&^* \ \neg^*(Qx)\varphi)[a'; \ r_G], \\ & (Qx)\varphi \ \&^* \ \neg^*(Qx)\varphi \ [a'; r_{DQ}]\end{aligned}$$

we obtain $\mathscr{T}_i \vdash_\beta (\mathrm{Q}x)\varphi \,\&^* \urcorner^*(\mathrm{Q}x)\varphi$, $\alpha \leqslant \beta$, which is again a contradiction. □

THEOREM 11 (COMPLETENESS THEOREM II). *A theory $\mathscr{T}$ is consistent iff it has a model. A consistent theory $\mathscr{T}$ has a model $\mathscr{D}_0$ in which*

$$\mathscr{T} \vdash_\alpha \varphi \quad \text{iff} \quad \mathscr{D}_0 \models_\alpha \varphi$$

holds for every formula $\varphi \in F_{\mathscr{J}(\mathscr{T})}$.

The proof proceeds analogously to the one in the classical logic using Lemmas 9, 10 and Theorems 8, 10. □

THEOREM 12 (COMPLETENESS THEOREM I). *A formula φ is α-theorem of the theory $\mathscr{T}$ iff it is true in the degree α in it:*

$$\mathscr{T} \vdash_\alpha \varphi \quad \text{iff} \quad \mathscr{T} \models_\alpha \varphi.$$

PROOF. From the validity theorem we obtain

$$\mathscr{T}_\alpha\!\vdash \varphi \text{ follows from } \mathscr{T} \models_\beta \varphi,\ \alpha \leqslant \beta.$$

Let $\mathscr{T} \models_\beta \varphi$ and $\mathscr{T} \vdash_\alpha \varphi$. Since the theory $\mathscr{T}$ is consistent, it has a model $\mathscr{D}_0$ for which $\mathscr{D}_0 \models_\alpha \varphi$ holds. Then $\beta \leqslant \alpha$, i.e. $\alpha = \beta$. □

7. Discussion

In this paper, we have developed the first-order fuzzy logic based on the truth set being either the interval $\langle 0, 1\rangle$ or a finite chain. We have developed its syntax and semantics and we have proved the completeness theorems which are fuzzy extensions of the classical Gödel's theorems (i.e. the classical ones are special cases of our theorems).

The first-order fuzzy logic has many important applications. First of all are, of course, the applications in the fuzzy set theory where it can elucidate many questions which have been doubtful up till now. But it can also show some properties of the classical predicate calculus from the different point of view.

The fuzzy logic is a special case of the continuous logic presented in [2] where the detailed relation between formal theory and its model is not studied. This paper solves this gap for the case when the degrees of truth form a rather special structure which, however, ensures the completeness property. Our concept of the model is stronger than that in [2] and, thus, we can expect that most theorems presented there will hold true in a stronger form.

What should be stressed are the additional logical connectives and quantifiers. They include all the connectives considered in any other systems of fuzzy logic (as far as the author is familiar with them).

We have left some questions open, e.g. the introduction of a new functional symbol or a predicate symbol. However, we can hardly expect any surprising results.

The kind of fuzzy function used in this paper proved to be the most natural among various fuzzy functions. It has interesting properties e.g. from the categorical point of view (cf. e.g. [8]). It is doubtful if the use of another type of fuzzy function would lead to the same result.

The completeness theorems hold for regular quantifiers which are more special cases of generalized operations than those considered in [2]. It is not clear for the present whether the presumption of regularity of quantifiers can be abandoned.

Acknowledgment

The author wishes to express his thanks to A. Pultr, J. Ivánek and J. Malík for their helpful criticism which led to considerable improvements in this paper.

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Received February 11, 1985
Revised February 11, 1986