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Sentential Logics and Maehara Interpolation Property*

Abstract. With each sentential logic C, identified with a structural consequence operation in a sentential language, the class $Matr^*(C)$ of factorial matrices which validate C is associated. The paper, which is a continuation of [2], concerns the connection between the purely syntactic property imposed on C, referred to as Maehara Interpolation Property (MIP), and three diagrammatic properties of the class $Matr^*(C)$: the Amalgamation Property (AP), the (deductive) Filter Extension Property (FEP) and Injections Transferable (IT). The main theorem of the paper (Theorem 2.2) is analogous to the Wroński's result for equational classes of algebras [13]. It reads that for a large class of logics the conjunction of (AP) and (FEP) is equivalent to (IT) and that the latter property is equivalent to (MIP).

§1. Preliminaries

A *sentential language* is an absolutely free algebra

$$\mathscr{S} = (S; \S_1, \S_2, \ldots)$$

freely generated by an infinite set $V(\mathscr{S}) = \{p, q, r, \ldots\}$ of *sentential variables* and endowed with countably many finitary operations $\S_1, \S_2, \ldots$, the *connectives of* $\mathscr{S}$. The members of S, the underlying set of the algebra $\mathscr{S}$, are called sentential *formulas*. If α is a formula, then $V(\alpha)$ denotes the set of sentential variables occurring in α. For any $X \subseteq S$ we set $V(X) = \bigcup\{V(\alpha)\colon \alpha \in X\}$. We shall often use the notation $X(p_1, \ldots, p_n)$ to indicate that X is a set of formulas such that $V(X) = \{p_1, \ldots, p_n\}$. The endomorphisms of $\mathscr{S}$ are customarily called *substitutions* in $\mathscr{S}$. If $X(p_1, \ldots, p_n)$ is a set of formulas of $\mathscr{S}$ and e is a substitution in $\mathscr{S}$ such that $ep_i = \alpha_i$, for $i = 1, \ldots, n$, then often the more explicit notation

$$X(\alpha_1, \ldots, \alpha_n)$$

is used to denote the set eX, the image of X under e. Thus $X(\alpha_1, \ldots, \alpha_n)$ denotes the set of formulas which result by the simultaneous substitution of α_i for p_i, $i = 1, \ldots, n$, in all formulas in X.

If V is a set of sentential variables of $\mathscr{S}$, then we let $\mathscr{S}_V$ denote the (rudimentary) sublanguage of $\mathscr{S}$ which consists of formulas containing only variables in V.

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By a *sentential logic* we understand a pair

$$(\mathscr{S}, C),$$

where $\mathscr{S}$ is a sentential language and C is a structural consequence on $\mathscr{S}$, that is, C satisfies the following conditions:

(a) $X \subseteq C(X)$
(b) if $X \subseteq Y$ then $C(X) \subseteq C(Y)$
(c) $C(C(X)) = C(X)$
(d) $eC(X) \subseteq C(eX)$,

for all sets $X, Y \subseteq S$ and for every substitution e of $\mathscr{S}$.

If no confusion is likely we shall identify a logic $(\mathscr{S}, C)$ with its consequence operation C. Moreover, if X a finite set of formulas, then instead of $C(X)$ we shall often write $C(a_1, \ldots, a_n)$, where $a_1, \ldots, a_n$ is a fixed arrangement of the elements of X. We also use $C(X, a)$ as an abbreviation for $C(X \cup \{a\})$.

Given a logic $(\mathscr{S}, C)$, the least infinite cardinal μ such that

$$C(X) = \bigcup \{C(Y)\colon\ Y \subseteq X \text{ and } \overline{\overline{Y}} < \mu\},$$

for all $X \subseteq S$, is called the cardinality of C and denoted *card*(C). C is *standard* if *card*$(C) = \aleph_0$.

A logic $(\mathscr{S}, C)$ is *equivalential* (see [8] and [1]) iff there exists a set $\Delta(p, q)$ of formulas of $\mathscr{S}$, hereafter referred to as a C-equivalence, such that the following conditions are fulfilled:

(i) $\Delta(p, p) \subseteq C(\emptyset)$
(ii) $\Delta(p, q) \subseteq C(\Delta(q, p))$
(iii) $\Delta(p, r) \subseteq C(\Delta(p, q) \cup \Delta(q, r))$
(iv) for each n-ary connective § of $\mathscr{S}$, $n \geqslant 0$, and any variables $p_1, \ldots, p_n, q_1, \ldots, q_n$,
$\Delta(\S(p_1 \ldots p_n), \S(q_1 \ldots q_n)) \subseteq C(\Delta(p_1, q_1) \cup \ldots \cup \Delta(p_n, q_n))$
(v) $q \in C(\Delta(p, q) \cup \{p\})$.

The class of equivalential logics includes a great many of the more important sentential logics, among others all implicative logics in the sense of Helena Rasiowa [9]. More information concerning equivalential logics can be found in [1] and [2].

A logic $(\mathscr{S}', C')$ is an *extension* of a logic $(\mathscr{S}, C)$ iff $\mathscr{S}$ is a subalgebra of $\mathscr{S}'$ (i.e. $V(\mathscr{S}) \subseteq V(\mathscr{S}')$ and both $\mathscr{S}$ and $\mathscr{S}'$ have the same stock of connectives) and $C(X) = C'(X) \cap S$, for all $X \subseteq S$. An extension $(\mathscr{S}', C')$ of a logic $(\mathscr{S}, C)$ is said to be *natural* when *card*(C) = *card*(C'). It is easy to see that any natural extension of an equivalential logic is again an equivalential logic. The following result is due to Shoesmith and Smiley [10, Lemma 2]: for every logic $(\mathscr{S}, C)$ and for every language $\mathscr{S}'$ including $\mathscr{S}$ as a subalgebra, there is a consequence C' on $\mathscr{S}'$ such that $(\mathscr{S}', C')$ is a natural extension of $(\mathscr{S}, C)$.

A (logical) *matrix* is a pair

$$\mathfrak{M} = (\mathscr{A}, D),$$

where $\mathscr{A}$ is an algebra, referred to as the algebra of $\mathfrak{M}$, and D is a subset of A called the set of designated elements of $\mathfrak{M}$. (In the sequel, algebras will be denoted by Script letters $\mathscr{A}$, $\mathscr{B}$, $\mathscr{S}$, etc., while their universes by the corresponding Italic characters A, B, S.) It will often be convenient to denote by $\mathscr{A}_{\mathfrak{M}}$ the algebra of the matrix $\mathfrak{M}$ and by $D_{\mathfrak{M}}$ the set of designated elements of $\mathfrak{M}$. Thus

$$\mathfrak{M} = (\mathscr{A}_{\mathfrak{M}}, D_{\mathfrak{M}}).$$

If the algebra of a matrix $\mathfrak{M}$ is similar to a sentential language $\mathscr{S}$, then $\mathfrak{M}$ is called a matrix for $\mathscr{S}$.

Given a class $\boldsymbol{K}$ of matrices for $\mathscr{S}$, we define the logic

$$Cn_{\boldsymbol{K}}$$

in $\mathscr{S}$ as follows: $\alpha \in Cn_{\boldsymbol{K}}(X)$ iff for every matrix $\mathfrak{M} = (\mathscr{A}, D)$ in $\boldsymbol{K}$ and every homomorphism v from $\mathscr{S}$ into $\mathscr{A}$, $v\alpha \in D$ whenever $vX \subseteq D$. (The homomorphisms from $\mathscr{S}$ into the algebra of a matrix $\mathfrak{M}$ are usually called *valuations* in $\mathfrak{M}$.) A class $\boldsymbol{K}$ of matrices is *strongly adequate* for a logic C iff $C = Cn_{\boldsymbol{K}}$.

A matrix $\mathfrak{M}$ is said to *validate a logic* $(\mathscr{S}, C)$ (or, $\mathfrak{M}$ is a model for C), if $C(X) \subseteq Cn_{\mathfrak{M}}(X)$, for all $X \subseteq S$. (We use here the symbol $Cn_{\mathfrak{M}}$ instead of awkward $Cn_{\{\mathfrak{M}\}}$.) The symbol

$$Matr(C)$$

denotes the class of all matrices which validate the logic C. The members of $Matr(C)$ are also called C-matrices. $Matr(C)$ is the largest class of matrices strongly adequate for C. If C is standard, then $Matr(C)$ is a quasivariety (= an implicative class).

Since the notion of a matrix falls under the more general concept of an algebraic structure, we can apply to matrices all the model-theoretic operations which are performed on algebraic structures. For example, a mapping h is a homomorphism from a matrix $\mathfrak{M} = (\mathscr{A}, D)$ into $\mathfrak{N} = (\mathscr{B}, E)$ iff h is a homomorphism from the algebra $\mathscr{A}$ into $\mathscr{B}$ and $hD \subseteq E$. A homomorphism h from $\mathfrak{M}$ into $\mathfrak{N}$ is *strict* if moreover $h(A-D) \subseteq B-E$. A one-to-one strict homomorphism is called an *embedding* or, an *injection*. By an *isomorphism* we understand an onto embedding. The concept of a strict homomorphism is closely related to the concept of a strict congruence of a matrix. We shall say that a congruence Θ of the algebra $\mathscr{A}$ of a matrix $\mathfrak{M} = (\mathscr{A}, D)$ is strict iff for any $a, b \in A$, $a\Theta b$ entails $a \in D$ iff $b \in D$. In other words, the strict congruences of $\mathfrak{M}$ do not paste together the designated elements of $\mathfrak{M}$ with the undesignated ones. If Θ is a strict congruence of $\mathfrak{M} = (\mathscr{A}, D)$, then the canonical map from $\mathfrak{M}$ onto the quotient matrix $\mathfrak{M}/\Theta = (\mathscr{A}/\Theta, D/\Theta)$ is a strict homomorphism. Conversely, if h is a strict homomorphism from $\mathfrak{M}$ into $\mathfrak{N}$, then Θ_h, the kernel of h, is a strict congruence of $\mathfrak{M}$ and if h is "onto" then $\mathfrak{M}/\Theta_h$ is isomorphic with $\mathfrak{N}$. In every matrix $\mathfrak{M}$ there exists a greatest strict congruence, denoted by $\Theta_{\mathfrak{M}}$. If C is an equivalential logic and $\Delta(p, q)$ is a C-equivalence, then it is easy to prove that for every matrix $\mathfrak{M} = (\mathscr{A}, D)$ in $Matr(C)$, $a\Theta_{\mathfrak{M}}b$ iff for every $\lambda \in \Delta$, $\lambda_{\mathscr{A}}(a, b) \in D$. ($\lambda_{\mathscr{A}}$ is the polynomial over the algebra $\mathscr{A}$ corresponding to the formula λ. The polynomial $\lambda_{\mathscr{A}}$ will be also denoted by $\lambda_{\mathfrak{M}}$ whenever the matrix $\mathfrak{M}$ is fixed.) A matrix $\mathfrak{M}$ is called *simple* (or *factorial*), iff the greatest strict congruence of $\mathfrak{M}$ coincides with the identity relation in $\mathfrak{M}$. Every quotient matrix $\mathfrak{M}/\Theta_{\mathfrak{M}}$ is simple The class of simple C-matrices is denoted by

$$Matr^*(C).$$

For instance, if C is the classical logic then $Matr^*(C)$ may be identified with the class of Boolean algebras. If C is the intuitionistic logic then $Matr^*(C)$ may be identified with the class of Heyting algebras (see [1]). For every equivalential logic C the class $Matr^*(C)$ is closed under the formation of direct products and submatrices. The reader also easily verifies that if C' is a natural extension of C, then $Matr^*(C') = Matr^*(C)$. Our interest in simple matrices lies in the fact that for every matrix $\mathfrak{M}$, the logics induced by $\mathfrak{M}$ and $\mathfrak{M}/\Theta_{\mathfrak{M}}$, respectively, coincide.

Let $(\mathscr{S}, C)$ be a fixed equivalential logic and let $\Delta(p, q)$ be a C-equivalence. Then, given a matrix $\mathfrak{M} = (\mathscr{A}, D)$ in $Matr^*(C)$ whose cardinality does not exceed the cardinality of $\mathscr{S}$, we define the *description of* $\mathfrak{M}$, $DS(\mathfrak{M})$, in the language $\mathscr{S}$. Let $z\colon A \to V(S)$ be a fixed one-to-one mapping. Then we set

$$\begin{aligned} DS(\mathfrak{M}) = \{\Delta\big(\S(z_{a_1} \ldots z_{a_n}),\ z_{\S_{\mathscr{A}}(a_1,\ldots,a_n)}\big)\colon\ & n \in \omega, \\ & \S \text{ is an } n\text{-ary connective in } \mathscr{S} \text{ and } a_1, \ldots, a_n \in A\} \\ & \cup \{z_a\colon a \in D\}. \end{aligned}$$

$DS(\mathfrak{M})$ is a subset of S and every formula in $DS(\mathfrak{M})$ contains at most the variables z_a, $a \in A$.

The following two lemmas come from [2]:

LEMMA 1.1. *Let $(\mathscr{S}, C)$ and $\mathfrak{M} = (\mathscr{A}, D)$ be as above. Let $X = C\big(DS(\mathfrak{M})\big)$. Then the matrix $\mathfrak{M}$ is isomorphically embedded in $(\mathscr{S}, X)/\Theta_X$ by means of the mapping*

$$a \to [z_a],\ a \in A,$$

where Θ_X is the greatest strict congruence of $(\mathscr{S}, X)$ and $[\gamma]$ is the equivalence class of γ relative to Θ_X.

LEMMA 1.2. *Let $(\mathscr{S}, C)$ and $\mathfrak{M}$ be as above. Let $\alpha(z_{a_1}\dots z_{a_n})$ be a formula of $\mathscr{S}$ in the variables $z_{a_1}, \dots, z_{a_n}$, where $a_1, \dots, a_n \in A$, and let v be a valuation of $\mathscr{S}$ in $\mathfrak{M}$ such that $v(z_a) = a$, for all $a \in A$. Then*

$$\alpha \in C\big(DS(\mathfrak{M})\big) \quad iff \quad v\alpha \in D.$$

Let $(\mathscr{S}, C)$ be a sentential logic and let $\mathscr{A}$ be an algebra similar to $\mathscr{S}$. A subset ∇ of A is called a *deductive filter* on $\mathscr{A}$ (relative to C) if the matrix $(\mathscr{A}, \nabla)$ is a member of $Matr(C)$. The set

$$F_C(\mathscr{A})$$

of all deductive filters on $\mathscr{A}$ is a complete lattice under the set-theoretic inclusion. It should be stressed that the notion of a deductive filter is relativized to a given logic C — for that reason the members of $F_C(\mathscr{A})$ are also called C-filters on $\mathscr{A}$. Let us also note that for the sentential language $\mathscr{S}$, $F_C(\mathscr{S})$ coincides with the family of theories of C, i.e. subsets X of S such that $C(X) = X$. For instance, if C is the classical logic and $\mathscr{A}$ is a Boolean algebra, then $F_C(\mathscr{A})$ is the family of "usual" filters of $\mathscr{A}$.

If $\mathfrak{M} = (\mathscr{A}, D)$ is a matrix for $\mathscr{S}$, then we let

$$F_C(\mathfrak{M})$$

denote the lattice of all deductive filters on $\mathfrak{M}$, $F_C(\mathfrak{M}) = \{\nabla \in F_C(\mathscr{A}): D \subseteq \nabla\}$. $F_C(\mathfrak{M})$ is a complete sublattice of $F_C(\mathscr{A})$. The members of $F_C(\mathfrak{M})$ are referred to as deductive filters on $\mathfrak{M}$.

The following two lemmas can be found in [3]:

LEMMA 1.3. *Given an equivalential logic C, let $\mathfrak{M}$ and $\mathfrak{N} = (\mathscr{B}, E)$ be matrices in $Matr(C)$ and let h be a homomorphism from $\mathfrak{M}$ onto $\mathfrak{N}$. Then or every $\nabla \in F_C(\mathfrak{M})$, if $\overleftarrow{h}(E) \subseteq \nabla$, then $h\nabla \in F_C(\mathfrak{N})$.*

(In fact, Lemma 1.3 is valid for a much broader class than the class of equivalential logics.) $\overleftarrow{h}(E)$ is the preimage of E by h.

The above lemma is the C-filter counterpart of a fairly obvious algebraic fact: if $h\colon \mathscr{A} \to \mathscr{B}$ is a homomorphism from an algebra $\mathscr{A}$ onto $\mathscr{B}$ and Φ is a congruence of $\mathscr{A}$ such that $ker(h) \subseteq \Phi$ then $\{(ha, hb): (a, b) \in \Phi\}$ is a congruence of $\mathscr{B}$.

LEMMA 1.4. *Let C be an equivalential logic. Then for every matrix $\mathfrak{M}$ in $Matr(C)$, the lattices $F_C(\mathfrak{M})$ and $F_C(\mathfrak{M}/\Theta_{\mathfrak{M}})$ are isomorphic. The isomorphism is established by means of the mapping ψ, where*

$$\psi(\nabla) = \nabla/\Theta_{\mathfrak{M}},$$

for all $\nabla \in F_C(\mathfrak{M})$.

We admit the following notation:

$f\colon \mathfrak{M} \mapsto \mathfrak{N}$ (or $\mathfrak{M} \overset{f}{\mapsto} \mathfrak{N}$) whenever f is a homomorphism from a matrix $\mathfrak{M}$ into $\mathfrak{N}$;

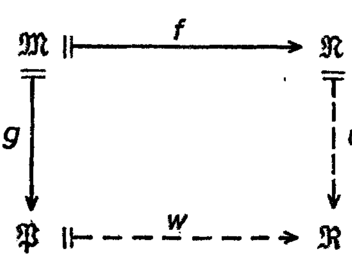
$f\colon \mathfrak{M} \Vdash\!\!\rightarrow \mathfrak{N}$ (or $\mathfrak{M} \overset{f}{\Vdash\!\!\longrightarrow} \mathfrak{N}$) whenever f is an embedding

of $\mathfrak{M}$ into $\mathfrak{N}$;

$f\colon \mathfrak{M} \vdash\!\!\twoheadrightarrow \mathfrak{N}$ (or $\mathfrak{M} \overset{f}{\vdash\!\!\twoheadrightarrow} \mathfrak{N}$) whenever f is a homomorphism

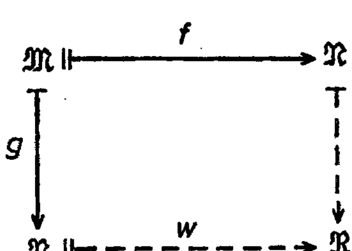
from $\mathfrak{M}$ onto $\mathfrak{N}$.

Let K be a class of similar matrices closed under isomorphisms. We shall say that K has the *amalgamation property* (AP, for short) if for any matrices $\mathfrak{M}$, $\mathfrak{N}$ and $\mathfrak{P}$ in K the following condition is fulfilled: for any embeddings $f\colon \mathfrak{M} \Vdash\!\!\rightarrow \mathfrak{N}$, $g\colon \mathfrak{M} \Vdash\!\!\rightarrow \mathfrak{P}$, there exist a matrix $\mathfrak{R}$ in K and embeddings $u\colon \mathfrak{N} \Vdash\!\!\rightarrow \mathfrak{R}$, $w\colon \mathfrak{P} \Vdash\!\!\rightarrow \mathfrak{R}$ such that $u \circ f = w \circ g$. Graphically:



The system $\langle \mathfrak{R}, u, w\rangle$ is then referred to as a common extension of the matrices $\mathfrak{N}$, $\mathfrak{P}$ over $\mathfrak{M}$ in K.

The second diagrammatic property we are interested in is that of injections transferable. Let K be as above. K is said to have *injections transferable* (IT, for short), if for any matrices $\mathfrak{M}$, $\mathfrak{N}$ and $\mathfrak{P}$ in K the following condition is fulfilled: for every embedding $f\colon \mathfrak{M} \Vdash\!\!\rightarrow \mathfrak{N}$ and every homomorphism $g\colon \mathfrak{M} \vdash\!\!\rightarrow \mathfrak{P}$, there exist a matrix $\mathfrak{R}$ in K, a homomorphism $u\colon \mathfrak{N} \vdash\!\!\rightarrow \mathfrak{R}$ and an embedding $w\colon \mathfrak{P} \Vdash\!\!\rightarrow \mathfrak{R}$ such that $u \circ f = w \circ g$. This property can graphically visualized by means of the following diagram:



Let C be a logic in a language $\mathscr{S}$ and let K be a subclass of $Matr(C)$. K is said to have the (deductive) *filter extension property* (FEP, for short) with respect to C if for each matrix $\mathfrak{N}$ in K, every deductive filter ∇_0

on a submatrix $\mathfrak{M} = (\mathscr{A}_{\mathfrak{M}}, D_{\mathfrak{M}})$ of $\mathfrak{N}$ can be extended to a deductive filter on $\mathfrak{N}$, i.e., there exists a deductive filter $\nabla \in F_C(\mathfrak{N})$ such that $\nabla_0 = \nabla \cap A_{\mathfrak{M}}$.

We are mainly interested in the case when the whole class $Matr(C)$ has the filter extension property relative to C. We shall show that if C is equivalential and the narrower class $Matr^*(C)$ of simple C-matrices has (FEP), then $Matr(C)$ has (FEP). Moreover in the case of equivalential logics there is another property closely connected with (FEP). The following lemma makes these remarks precise.

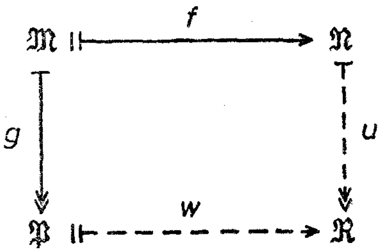
LEMMA 1.5. *Let C be an equivalential logic. The following assertions are equivalent:*

(i) *the class $Matr(C)$ has* (FEP) *relative to C*

(ii) *the class $Matr^*(C)$ has* (FEP)

(iii) *for any three matrices $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{P}$ in $Matr^*(C)$ and any mappings f and g such that $f\colon \mathfrak{M} \Vdash\!\!\rightarrow \mathfrak{N}$ and $g\colon \mathfrak{M} \Vdash\!\!\twoheadrightarrow \mathfrak{P}$ there exists a matrix $\mathfrak{R}$ in $Matr^*(C)$ and mappings $u\colon \mathfrak{N} \Vdash\!\!\twoheadrightarrow \mathfrak{R}$, $w\colon \mathfrak{P} \Vdash\!\!\rightarrow \mathfrak{R}$ such that $u \circ f = w \circ g$.*

The property expressed by condition (iii) can be visualized by means of the following diagram:



PROOF of Lemma 1.5. (i) $\Rightarrow$ (ii). This is trivial because (ii) is a particular case of (i).

(ii) $\Rightarrow$ (i). Let $\mathfrak{M} = (\mathscr{A}_{\mathfrak{M}}, D_{\mathfrak{M}})$ be a submatrix of $\mathfrak{N} = (\mathscr{A}_{\mathfrak{N}}, D_{\mathfrak{N}})$, where both $\mathfrak{M}$ an $\mathfrak{N}$ belong to $Matr(C)$. In view of Lemma 1.4, the lattices $F_C(\mathfrak{M})$ and $F_C(\mathfrak{M}/\Theta_{\mathfrak{M}})$ are isomorphic. Similarly, the lattice $F_C(\mathfrak{N})$ is isomorphic with $F_C(\mathfrak{N}/\Theta_{\mathfrak{N}})$.

Now let $\nabla_0 \in F_C(\mathfrak{M})$. Then, by Lemma 1.4, $\nabla_0/\Theta_{\mathfrak{M}} \in F_C(\mathfrak{M}/\Theta_{\mathfrak{M}})$. Since C is equivalential, the mapping φ, where

$$\varphi([a]_{\Theta_{\mathfrak{M}}}) = [a]_{\Theta_{\mathfrak{N}}},$$

for all $a \in A_{\mathfrak{M}}$, is an embedding of $\mathfrak{M}/\Theta_{\mathfrak{M}}$ into $\mathfrak{N}/\Theta_{\mathfrak{N}}$. As the class $Matr^*(C)$ has (FEP), there is a deductive filter $\nabla^* \in F_C(\mathfrak{N}/\Theta_{\mathfrak{N}})$ such that

(1) $\nabla^* \cap \varphi(A_{\mathfrak{M}}/\Theta_{\mathfrak{M}}) = \varphi(\nabla_0/\Theta_{\mathfrak{M}})$.

Furthermore, there exists a deductive filter $\nabla \in F_C(\mathfrak{N})$ such that

(2) $\nabla^* = \{[a]_{\Theta_{\mathfrak{N}}}: a \in \nabla\}$.

(1) and (2) yield

$$\nabla \cap A_{\mathfrak{M}} = \nabla_0.$$

Indeed, let $a \in A_{\mathfrak{M}}$. If $a \in \nabla_0$ then $[a]_{\Theta_{\mathfrak{M}}} \in \nabla_0/\Theta_{\mathfrak{M}}$ whence $[a]_{\Theta_{\mathfrak{N}}} = \varphi([a]_{\Theta_{\mathfrak{M}}}) \in \varphi(\nabla_0/\Theta_{\mathfrak{M}})$. Hence, by (1), $[a]_{\Theta_{\mathfrak{N}}} \in \nabla^*$. Thus, by (2), there is $b \in \nabla$ such that $[a]_{\Theta_{\mathfrak{N}}} = [b]_{\Theta_{\mathfrak{N}}}$. Hence $\Delta_{\mathfrak{N}}(a, b) \subseteq D_{\mathfrak{N}} \subseteq \nabla$, where Δ is a fixed C-equivalence. Thus $\{b\} \cup \Delta_{\mathfrak{N}}(a, b) \subseteq \nabla$ whence, by clause (v) of the definition of an equivalential logic, $a \in \nabla$. Conversely, let $a \in \nabla \cap A_{\mathfrak{M}}$. Then $[a]_{\Theta_{\mathfrak{N}}} \in \nabla^*$ and clearly $[a]_{\Theta_{\mathfrak{N}}} \in \varphi(A_{\mathfrak{M}}/\Theta_{\mathfrak{M}})$. Thus (1) yields $[a]_{\Theta_{\mathfrak{N}}} \in \varphi(\nabla_0/\Theta_{\mathfrak{M}})$, i.e.,

(3) $\varphi([a]_{\Theta_{\mathfrak{M}}}) \in \varphi(\nabla_0/\Theta_{\mathfrak{M}})$.

Since φ is an isomorphism between the algebras $\mathscr{A}_{\mathfrak{M}}/\Theta_{\mathfrak{M}}$ and $\varphi(\mathscr{A}_{\mathfrak{M}}/\Theta_{\mathfrak{M}})$, it follows from (3) that $[a]_{\Theta_{\mathfrak{M}}} \in \nabla_0/\Theta_{\mathfrak{M}}$ whence $a \in \nabla$.

Thus ∇ is an extension of ∇_0 onto $\mathfrak{N}$.

(ii)$\Rightarrow$(iii). Assume (ii) and let $\mathfrak{M} = (\mathscr{A}_{\mathfrak{M}}, D_{\mathfrak{M}})$, $\mathfrak{N} = (\mathscr{A}_{\mathfrak{N}}, D_{\mathfrak{N}})$, $\mathfrak{P} = (\mathscr{A}_{\mathfrak{P}}, D_{\mathfrak{P}})$ and f, g be as in (iii). Define

$$\nabla_0 = g(D_{\mathfrak{P}}).$$

Then g is a strict homomorphism from $(\mathscr{A}_{\mathfrak{M}}, \nabla_0)$ onto $\mathfrak{P}$ whence

$$\nabla_0 \in F_C(\mathfrak{M}).$$

Since f is an embedding, the matrix $f(\mathfrak{M}) =_{df} (f(\mathscr{A}_{\mathfrak{M}}), f(D_{\mathfrak{M}}))$ is isomorphic with $\mathfrak{M}$. Clearly, $f(\mathfrak{M})$ is a submatrix of $\mathfrak{N}$ and $f(\nabla_0) \in F_C(f(\mathfrak{M}))$. In view of (ii), there exists a deductive filter $\nabla \in F_C(\mathfrak{N})$ such that

(4) $\nabla \cap f(A_{\mathfrak{M}}) = f(\nabla_0)$.

Let

$$\mathfrak{R} =_{df} (\mathscr{A}_{\mathfrak{N}}/\Theta_{\nabla}, \nabla/\Theta_{\nabla}),$$

where Θ_{∇} is the greatest strict congruence of $(\mathscr{A}_{\mathfrak{N}}, \nabla)$. Clearly $\mathfrak{R}$ is a member of $Matr^*(C)$. Since C is equivalential, it follows that the mapping u defined as follows:

$$u(b) = [b]_{\nabla},$$

for all $b \in A_{\mathfrak{N}}$, is a homomorphism from $\mathfrak{N}$ onto $\mathfrak{R}$. ($[b]_{\nabla}$ is the equivalence class of b with respect to Θ_{∇}.)

On the other hand, the mapping w defined by means of the formula

$$w(g(a)) =_{df} [f(a)]_{\nabla},$$

for all $a \in A_{\mathfrak{M}}$, is an embedding of $\mathfrak{P}$ into $\mathfrak{R}$. Indeed, if $c, d \in A_{\mathfrak{P}}$, then there exist $a, b \in A_{\mathfrak{M}}$ such that $c = g(a)$ and $d = g(b)$ whence $c = d$ iff $g(a) = g(b)$ iff $\Lambda_{\mathscr{A}_{\mathfrak{P}}}(g(a), g(b)) \subseteq D_{\mathfrak{P}}$ iff $\Lambda_{\mathscr{A}_{\mathfrak{M}}}(a, b) \subseteq \nabla_0$ iff $\Lambda_{f(\mathscr{A}_{\mathfrak{M}})}(f(a), f(b)) \subseteq f(\nabla_0)$ iff (by (4)) $\Lambda_{\mathscr{A}_{\mathfrak{N}}}(f(a), f(b)) \subseteq \nabla$ iff $[f(a)]_{\nabla} = [f(b)]_{\nabla}$. Thus w is well-defined and one-to-one. We also have that $c = g(a) \in D_{\mathfrak{P}}$ iff $a \in \nabla_0$ iff (by (4)) $f(a) \in \nabla$ iff $[f(a)]_{\nabla} \in \nabla/\Theta_{\nabla}$. Hence g is also an embedding of $\mathfrak{P}$ into $\mathfrak{R}$.

It follows from the definitions of u and w that $u \circ f = w \circ g$. Thus (iii) has been proved.

(iii) ⇒ (ii). Let $\mathfrak{M} = (\mathscr{A}_{\mathfrak{M}}, D_{\mathfrak{M}})$ be a submatrix of $\mathfrak{N} = (\mathscr{A}_{\mathfrak{N}}, D_{\mathfrak{N}})$ where both $\mathfrak{M}$ and $\mathfrak{N}$ belong to $Matr^*(C)$. Clearly, the identity map f is an embedding of $\mathfrak{M}$ into $\mathfrak{N}$.

Now if ∇_0 is a deductive filter in $F_C(\mathfrak{M})$, then the factorial matrix

$$\mathfrak{P} =_{df} (\mathscr{A}_{\mathfrak{M}}/\Theta_{\nabla_0}, \nabla_0/\Theta_{\nabla_0}),$$

where Θ_{∇_0} is the greatest strict congruence of $(A_{\mathfrak{M}}, \nabla_0)$, belongs to $Matr^*(C)$ and the mapping g defined as follows:

$$g(a) = [a]_{\nabla_0},$$

for all $a \in A_{\mathfrak{M}}$, is a homomorphism from $\mathfrak{M}$ onto $\mathfrak{P}$.

According to (iii), there exists a matrix

$$\mathfrak{R} = (\mathscr{A}_{\mathfrak{R}}, D_{\mathfrak{R}})$$

in $Matr^*(C)$ and mappings

$$u\colon \mathfrak{N} \twoheadrightarrow \mathfrak{R}, \qquad w\colon \mathfrak{P} \rightarrowtail \mathfrak{R}$$

such that $u \circ f = w \circ g$. Let

$$\nabla =_{df} \{b \in A_{\mathfrak{N}}\colon u(b) \in D_{\mathfrak{R}}\}.$$

Clearly $D_{\mathfrak{N}} \subseteq \nabla$ and the mapping u is a strict homomorphism from $(\mathscr{A}_{\mathfrak{N}}, \nabla)$ onto $\mathfrak{R}$. Consequently $\nabla \in F_C(\mathfrak{N})$.

We claim that $\nabla \cap A_{\mathfrak{M}} = \nabla_0$. Let $a \in \nabla_0$. Then $g(a) \in \nabla_0/\Theta_{\nabla_0}$ whence $w \circ g(a) = u \circ f(a) = u(a) \in D_{\mathfrak{R}}$. Thus $a \in \nabla$. Conversely, let $a \in \nabla \cap A_{\mathfrak{M}}$. Then, by the definition of ∇, $u(a) \in D_{\mathfrak{R}}$. Since $u(a) = u \circ f(a) = w \circ g(a)$ we thus have that $w \circ g(a) \in D_{\mathfrak{R}}$. But w is an embedding of $\mathfrak{P}$ into $\mathfrak{R}$ whence $g(a) \in \nabla_0/\Theta_{\nabla_0}$, i.e., $[a]_{\nabla_0} \in \nabla_0/\Theta_{\nabla_0}$. Thus $a \in \nabla_0$.

This completes the proof of (ii). ∎

In a quite analogous manner one proves the following lemmas.

LEMMA 1.6. *Let C be an equivalential logic. The following two assertions are equivalent:*

(i) *the class $Matr(C)$ has the amalgamation property*

(ii) *the class $Matr^*(C)$ of simple C-matrices has the amalgamation property.*

LEMMA 1.7. *Let C be as above. The class $Matr(C)$ has injections transferable if so has the class $Matr^*(C)$.*

We omit the proofs.

Finally, let us note that logics C for which the class $Matr(C)$ is known to have (AP) or (IT) are relatively rare. For instance, there are only seven nontrivial equational classes of Heyting algebras with the amalgamation property. We refer to [6] and [2] for more information. On the other hand, there are many logics C for which the class $Matr(C)$ (or $Matr^*(C)$) has the filter extension property. Partial justification of this fact can be carried out as follows. There are many sentential logics which admit deduction theorems. (By the deduction theorem for C with respect to a binary connective $\to$ we understand the following statement: for any set X of formulas and for any formulas α and β, $\beta \in C(X, \alpha)$ iff $\alpha \to \beta \in C(X)$.) For instance, every intermediate logic, i.e., every axiomatic strengthening of the intuitionistic logic admits deduction theorem with respect to the implication connective. It can be easily shown that if a logic C admits deduction theorem, then the class $Matr(C)$ has the filter extension property. (In fact, $Matr(C)$ has (FEP) under much weaker assumptions. More information can be found e.g., in [4].)

§2. Maehara Interpolation Property

Let C be a logic in a language $\mathscr{S}$. C is said to have the *Maehara interpolation property* (MIP, for short) if for any sets of formulas $X, Y \subseteq S$ and for any formula α of $\mathscr{S}$ the following condition is satisfied: if $V(X \cup \{\alpha\}) \cap V(Y) \neq \emptyset$ and $\alpha \in C(X \cup Y)$, then there exists a set of formulas Z such that $V(Z) \subseteq V(X \cup \{\alpha\}) \cap V(Y)$, $Z \subseteq C(Y)$ and $\alpha \in C(X \cup Z)$.

Let us note that if C is standard, then we may demand the above set Z to be finite. Moreover, if C has conjunction connective, that is, there exists a binary connective $\wedge$ in $\mathscr{S}$ such that $C(\alpha \wedge \beta) = C(\alpha, \beta)$, for all formulas α, β, then forming the conjunction of the members of Z we may even demand Z to consist of a single formula. In this case (MIP) reduces to the following property:

(MIP)* for all sets $X, Y \subseteq S$ and for every formula α of $\mathscr{S}$, if $V(X \cup \{\alpha\}) \cap V(Y) \neq \emptyset$ and $\alpha \in C(X \cup Y)$, then there exists a formula δ such that $V(\delta) \subseteq V(X \cup \{\alpha\}) \cap V(Y)$, $\delta \in C(Y)$ and $\alpha \in C(X, \delta)$.

(MIP)* is the exact formulation of a lemma proved originally by Maehara for the intuitionistic sentential logic (see [5] and [11]). The significance of (MIP)* in the more general context of proof theory is described in detail in Takeuti's book [11]. Some similar forms of the interpolation property in the domain of equational logic have been investigated, e.g., in Pigozzi [7].

The intuitionistic logic is clearly standard and is endowed with conjunction. Thus, since in view of the Maehara's result it has (MIP)*, we see that it has also (MIP).

We now turn attention to another purely syntactic property imposed on sentential logics. We call it property (I). This property as well as some consequences of it were examined in [2]. Let C be a logic in a sentential language $\mathscr{S}$. Then C is said to have *property* (I) if for all $X, Y \subseteq S$ and all $\alpha \in S$, if $\alpha \in C(X \cup Y)$ and $V(\alpha) \subseteq V(X)$ and $V(X) \cap V(Y) \neq \emptyset$ then $\exists Z \subseteq S \;\; V(Z) \subseteq V(X) \cap V(Y) \;\&\; Z \subseteq C(Y) \;\&\; \alpha \in C(X \cup Z)$. It is a fairly trivial matter to show that the Maehara interpolation property (MIP) implies property (I). As the referee pointed out, for a wide class of logics the (MIP) and (I) properties are actually equivalent.

LEMMA 2.1. *Let* $(\mathscr{S}, C)$ *be a logic such that* $C(\emptyset) \neq \emptyset$. *Then* (MIP) *is equivalent to* (I).

PROOF. To see that (I) implies (MIP) we argue as follows. Assume $V(X \cup \{\alpha\}) \cap V(Y) \neq \emptyset$ and $\alpha \in C(X \cup Y)$. Let X' be any set of formulas such that $V(X') = V(X \cup \{\alpha\})$ and $C(X') = C(X)$. (X' can be obtained from X by adjoining theorems of C that contain the variables in α.) Then $V(X') \cap V(Y) \neq \emptyset$ and $\alpha \in S_{V(X')} \cap C(X' \cup Y)$. Thus by (I) $\alpha \in C\big(X' \cup \cup \big(S_{V(X')} \cap S_{V(Y)} \cap C(Y)\big)\big) = C\big(X \cup \big(S_{V(X \cup \{\alpha\})} \cap S_{V(Y)} \cap C(Y)\big)\big)$. Thus there exists a Z such that $V(Z) \subseteq V(X \cup \{\alpha\}) \cap V(Y)$, $Z \subseteq C(Y)$ and $\alpha \in C(X \cup Z)$. So (MIP) holds.

The purpose of the present paper is to determine semantic equivalents of the purely syntactic Maehara interpolation property. The basic result of the paper, Theorem 2.2, states that given an equivalential logic C, (MIP) is equivalent to Injections Transferable for the class $Matr^*(C)$ of simple C-matrices.

THEOREM 2.2. *Let C be an equivalential logic in a sentential language $\mathscr{S}$. The following assertions are equivalent*:

(i) *C has the Maehara interpolation property* (MIP)

(ii) *the class $Matr^*(C)$ has the amalgamation property* (AP) *and the* (*deductive*) *filter extension property* (FEP) *with respect to C*

(iii) *the class $Matr^*(C)$ has injections transferable* (IT).

PROOF. (i) $\Rightarrow$ (ii). Since (MIP) implies (I), applying Theorem 2 of [2] we obtain that the class $Matr^*(C)$ has the amalgamation property.

We shall now check that (MIP) implies that the class $Matr^*(C)$ has the filter extension property (relative to C).

Assume (MIP) and let $\mathfrak{M} = (\mathscr{A}_{\mathfrak{M}}, D_{\mathfrak{M}})$ be a submatrix of $\mathfrak{N} = (\mathscr{A}_{\mathfrak{N}}, D_{\mathfrak{N}})$, where $\mathfrak{N} \in Matr^*(C)$ and let $\nabla_0 \in F_C(\mathfrak{M})$. Since C is equivalential, $\mathfrak{M}$ is also a member of $Matr^*(C)$.

Let us assume first that the matrix $\mathfrak{N}$ has a cardinality not exceeding the cardinality of $\mathscr{S}$. Let $DS(\mathfrak{M})$ and $DS(\mathfrak{N})$ be the descriptions of $\mathfrak{M}$ and $\mathfrak{N}$ in $\mathscr{S}$ in the variables $\{z_a^1\colon a \in A_{\mathfrak{M}}\}$ and $\{z_a^2\colon a \in A_{\mathfrak{N}}\}$, respectively. We assume that $z_a^1 = z_a^2$, for all $a \in A_{\mathfrak{M}}$. Then $DS(\mathfrak{M}) \subseteq DS(\mathfrak{N})$.

CLAIM. *Let β be a formula in variables from $\{z_a^1\colon a \in A_{\mathfrak{M}}\}$ such that $\beta \in C(DS(\mathfrak{N}))$. Then $\beta \in C(DS(\mathfrak{M}))$.*

Indeed, if $\beta \in C(DS(\mathfrak{N}))$, then (MIP) implies that there is a set Z of formulas of $\mathscr{S}$ such that

$$V(Z) \subseteq \{z_a^1\colon a \in A_{\mathfrak{M}}\},$$

(5) $$Z \subseteq C(DS(\mathfrak{N}))$$

and

(6) $$\beta \in C(DS(\mathfrak{M}) \cup Z).$$

Let v be a characteristic valuation of $\mathscr{S}$ in $\mathfrak{N}$, i.e. $vz_a^2 = a$, for all $a \in A_{\mathfrak{N}}$. Then, in view of (5),

(7) $$v(Z) \subseteq D_{\mathfrak{N}}.$$

Since $V(Z) \subseteq \{z_a^1\colon a \in A_{\mathfrak{M}}\}$ and $\mathfrak{M}$ is a submatrix of $\mathfrak{N}$, (7) yields

(8) $$v(Z) \subseteq D_{\mathfrak{M}}.$$

But v is at the same time a characteristic valuation in $\mathfrak{M}$. Recalling once more Lemma 1.2, on the strength of (8) we get

$$Z \subseteq C(DS(\mathfrak{M})).$$

Hence $C(DS(\mathfrak{M}) \cup Z) = C(DS(\mathfrak{M}))$, whence, by (6),

$$\beta \in C(DS(\mathfrak{M})).$$

This proves our claim.

Let v be as above and let

$$\nabla =_{df} v(S_{V(DS(\mathfrak{N}))} \cap C(DS(N) \cup \{z_a\colon a \in \nabla_0\})).$$

We shall show that ∇ is the required extension of ∇_0, i.e.,

(a) $$\nabla \in F_C(\mathfrak{N})$$

and

(b) $$\nabla \cap A_{\mathfrak{M}} = \nabla_0.$$

As to (a) notice that $D_{\mathfrak{N}} \subseteq \nabla$. Indeed, if $a \in D_{\mathfrak{N}}$ then $z_a^2 \in DS(\mathfrak{N}) \subseteq$ $\subseteq S_{V(DS(\mathfrak{N}))} \cap C(DS(\mathfrak{N}))$. Since $vz_a^2 = a$, it follows from above that $a \in$ $\in v(S_{V(DS(\mathfrak{N}))} \cap C(DS(\mathfrak{N})))$.

In order to show that ∇ is a deductive filter with respect to C, we shall make use of Lemma 1.3.

The matrix

$$\mathfrak{P} =_{df} \left(\mathscr{S}_{V(DS(\mathfrak{N}))}, S_{V(DS(\mathfrak{N}))} \cap C(\emptyset)\right),$$

being a submatrix of $(\mathscr{S}, C(\emptyset))$, is also a member of $Matr(C)$. Moreover v, restricted to $\mathscr{S}_{V(DS(\mathfrak{N}))}$, is a homomorphism from $\mathfrak{P}$ onto $\mathfrak{N}$. Let

$$X =_{df} S_{V(DS(\mathfrak{N}))} \cap C\left(DS(\mathfrak{N}) \cup \{z_a\colon a \in \nabla_0\}\right).$$

Then clearly $X \in F_C(\mathfrak{P})$. We shall check that

$$S_{V(DS(\mathfrak{N}))} \cap \overleftarrow{v}(D_{\mathfrak{N}}) \subseteq X.$$

Let $\beta \in \overleftarrow{v}(D_{\mathfrak{N}})$, i.e. $v\beta \in D_{\mathfrak{N}}$. Since $\beta \in S_{V(DS(\mathfrak{N}))}$ we apply Lemma 1.2 which yields $\beta \in C\left(DS(\mathfrak{N})\right)$ whence $\beta \in X$.

Thus Lemma 1.3 gives that

$$\nabla = v(X) \in F_C(\mathfrak{N}).$$

As to (b) notice that from the definition of ∇ and the properties of v it follows that $\nabla_0 \subseteq \nabla \cap A_{\mathfrak{M}}$.

Let $b \in \nabla \cap A_{\mathfrak{M}}$. Hence $b = va$, where

(9) $\quad a \in S_{V(DS(\mathfrak{N}))} \cap C\left(DS(\mathfrak{N}) \cup \{z_a\colon a \in \nabla_0\}\right).$

Since also $b \in A_{\mathfrak{M}} \subseteq A_{\mathfrak{N}}$, we have that $b = vz_b$. Thus $b = va = vz_b$ in $\mathscr{A}_{\mathfrak{N}}$. Whence

(10) $\quad v\left(\varDelta(a, z_b)\right) = \varDelta_{\mathfrak{N}}(va, vz_b) \subseteq D_{\mathfrak{N}}.$

Since every formula in $\varDelta(a, z_b)$ is built-up by means of variables from $V\left(DS(\mathfrak{N})\right)$, Lemma 1.2 and (10) yield

(11) $\quad \varDelta(a, z_b) \subseteq C\left(DS(\mathfrak{N})\right) \subseteq C\left(DS(\mathfrak{N}) \cup \{z_a\colon a \in \nabla_0\}\right).$

9), (11) and the fact that $\varDelta$ is a C-equivalence give

$$z_b \in C\left(\varDelta(a, z_b) \cup \{a\}\right) \subseteq C\left(DS(\mathfrak{N}) \cup \{z_a\colon a \in \nabla_0\}\right).$$

Applying (MIP) we obtain

$$z_b \in C\left((S_{\{z_a\colon a \in \nabla_0\} \cup \{z_b\}} \cap C\left(DS(\mathfrak{N})\right) \cup \{z_a\colon a \in \nabla_0\}\right).$$

Since $S_{\{z_a\colon a \in \nabla_0\} \cup \{z_b\}} \subseteq S_{V(DS(\mathfrak{M}))}$ we thus have

$$z_b \in C\left((S_{V(DS(\mathfrak{M}))} \cap C\left(DS(\mathfrak{N})\right) \cup \{z_a\colon a \in \nabla_0\}\right).$$

In view of our Claim

$$S_{V(DS(\mathfrak{M}))} \cap C\left(DS(\mathfrak{N})\right) \subseteq C\left(DS(\mathfrak{M})\right).$$

Consequently

(12) $\quad z_b \in C\left(DS(\mathfrak{M}) \cup \{z_a\colon a \in \nabla_0\}\right).$

Since $(\mathscr{A}_{\mathfrak{M}}, \nabla_0) \in Matr(C)$ and

$$v\big(DS(\mathfrak{M}) \cup \{z_a\colon a \in \nabla_0\}\big) \subseteq \nabla_0,$$

(12) entails that

$$b = vz_b \in \nabla_0.$$

This proves (b).

If the matrix $\mathfrak{N}$ has a cardinality greater than the cardinality of $\mathscr{S}$, we form a natural extension $(\mathscr{S}', C')$ of $(\mathscr{S}, C)$ such that the cardinality of $\mathscr{S}'$ is equal or greater than the cardinality of $\mathfrak{N}$. Then $Matr^*(C) = Matr^*(C')$ and, as is easy to check, the logic $(\mathscr{S}', C')$ has also (MIP). Further we argue as above, taking the logic $(\mathscr{S}', C')$ instead of $(\mathscr{S}, C)$ as the point of departure.

Thus (ii) has been proved.

(ii) ⇒ (iii). Assume (ii). Let $\mathfrak{M}, \mathfrak{N}, \mathfrak{P}$ be matrices in $Matr^*(C)$ and let f, g be mappings such that

$$f\colon \mathfrak{M} \Vdash\!\!\to \mathfrak{N} \quad\text{and}\quad g\colon \mathfrak{M} \vdash\!\!\to \mathfrak{P}.$$

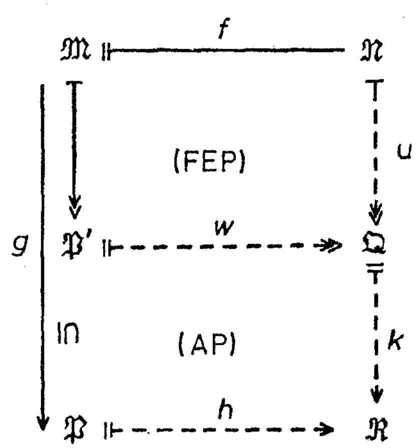
There exists a submatrix $\mathfrak{P}'$ of $\mathfrak{P}$ such that g is a homomorphism from $\mathfrak{M}$ onto $\mathfrak{P}'$. Clearly $\mathfrak{P}'$ is also a member of $Matr^*(C)$. Since the class $Matr^*(C)$ has (FEP), there exists a matrix $\mathfrak{Q}$ in $Matr^*(C)$ and mappings

$$u\colon \mathfrak{N} \vdash\!\!\twoheadrightarrow \mathfrak{Q}, \quad w\colon \mathfrak{P} \Vdash\!\!\to \mathfrak{Q}$$

such that

(13) $\quad u \circ f = w \circ g$

(See the diagram below.)



Thus w is an embedding of $\mathfrak{P}'$ into $\mathfrak{Q}$ and $\mathfrak{P}'$ is a submatrix of $\mathfrak{P}$. Since $Matr^*(C)$ has the amalgamation property, there exists a matrix $\mathfrak{R}$ in $Matr^*(C)$ and mappings h, k such that

$$h\colon \mathfrak{P} \Vdash\!\!\to \mathfrak{R}, \quad k\colon \mathfrak{Q} \Vdash\!\!\to \mathfrak{R}$$

and

(14) $k \circ w = h \circ id,$

where id is the identity map from $\mathfrak{P}'$ into $\mathfrak{P}$. (13) and (14) imply that $k \circ u$ is a homomorphism from $\mathfrak{N}$ into $\mathfrak{R}$ and

$$h \circ g = (k \circ u) \circ f.$$

Thus $Matr^*(C)$ has (IT).

(iii)$\Rightarrow$(i). Assume that $Matr^*(C)$ has injections transferable.

Let $a \in C(X \cup Y)$, where $V(X \cup \{a\}) \cap V(Y) \neq \emptyset$. Consider the rudimentary sublanguages $\mathscr{S}_{V(X)}, \mathscr{S}_{V(Y)}$ and $\mathscr{S}_0 =_{df} \mathscr{S}_{V(X \cup \{a\}) \cap V(Y)}$.

Let

$$\mathfrak{M} =_{df} (\mathscr{S}_0, S_0 \cap C(Y))$$

$$\mathfrak{N} =_{df} (\mathscr{S}_{V(Y)}, S_{V(Y)} \cap C(Y))$$

$$\mathfrak{P} =_{df} \big(\mathscr{S}_{V(X \cup \{a\})}, S_{V(X \cup \{a\})} \cap C(X \cup S_0 \cap C(Y))\big).$$

Since $\mathfrak{M}$ is a submatrix of $\mathfrak{N}$ and C is equivalential, it follows that the mapping f defined as follows

$$f([\gamma]_{\Theta_{\mathfrak{M}}}) = [\gamma]_{\Theta_{\mathfrak{N}}},$$

is an embedding of $\mathfrak{M}/\Theta_{\mathfrak{M}}$ into $\mathfrak{N}/\Theta_{\mathfrak{N}}$. On the other hand, as is easy to check, the mapping g, where

$$g([\gamma]_{\Theta_{\mathfrak{M}}}) = [\gamma]_{\Theta_{\mathfrak{P}}}$$

and $\gamma \in S_0$, is a homomorphism from $\mathfrak{M}/\Theta_{\mathfrak{M}}$ into $\mathfrak{P}/\Theta_{\mathfrak{P}}$.

According to (IT)-property, there exists a matrix $\mathfrak{R} = (\mathscr{A}_{\mathfrak{R}}, D_{\mathfrak{R}})$ in $Matr^*(C)$ and homomorphisms

$$u\colon \mathfrak{N}/\Theta_{\mathfrak{N}} \mapsto \mathfrak{R}, \quad w\colon \mathfrak{P}/\Theta_{\mathfrak{P}} \mapsto \mathfrak{R}$$

such that

(15) $u \circ f = w \circ g.$

Since (15) holds true, the following valuation v of $\mathscr{S}$ into $\mathfrak{R}$ is well--defined:

$$vp = \begin{cases} u([p]_{\Theta_{\mathfrak{N}}}) & \text{if} \quad p \in V(Y) \\ w([p]_{\Theta_{\mathfrak{P}}}) & \text{if} \quad p \in V(X \cup \{a\}) \\ \text{arbitrary} & \text{if} \quad p \notin V(X \cup \{a\}) \cup V(Y), \end{cases}$$

Moreover, the definitions of $\mathfrak{N}, \mathfrak{P}$ and v yield

(16) $X \cup Y \subseteq \overleftarrow{v}(D_{\mathfrak{R}}).$

Since $\mathfrak{R}$ is a model for C, $a \in C(X \cup Y)$ and (16) imply that $va \in D_{\mathfrak{R}}$. In turn, $a \in S_{V(X \cup \{a\})}$ implies that $va = w([a]_{\Theta_{\mathfrak{P}}})$. Thus $w([a]_{\Theta_{\mathfrak{P}}}) \in D_{\mathfrak{R}}$.

Since w is an embedding of $\mathfrak{P}/\Theta_{\mathfrak{P}}$ into $\mathfrak{R}$, we obtain that $[\alpha]_{\Theta_{\mathfrak{P}}}$ is a designated element in $\mathfrak{P}/\Theta_{\mathfrak{P}}$. Then the definition of $\mathfrak{P}$ gives

(17) $\quad \alpha \in C\big(X \cup S_0 \cap C(Y)\big).$

It follows from (17) that there exists a set $Z \subseteq C(Y)$ such that $V(Z) \subseteq V(X \cup \{\alpha\}) \cap V(Y)$ and $\alpha \in C(X \cup Z)$. Thus C has (MIP).

This completes the proof of Theorem 2.2. ∎

REMARK. Note that the proof of (ii)⇒(iii) in Theorem 2.2 does not require C to be equivalential. It can be easily shown that the equivalence (ii)⇔(iii) is valid for any class K of matrices closed under the formation of submatrices and direct products.

We close this section with three remarks and one open problem.

Note 1. It is well-known that each intermediate logic C can be assigned in a one-to-one manner the equational class $Alg^*(C)$ consisting of Heyting algebras which validate the theses of C. Without loss of generality we may identify the classes $Matr^*(C)$ and $Alg^*(C)$ (see [1]). Since every intermediate logic C has (FEP), Theorem 2.2 implies that the equational class $Alg^*(C)$ has (IT) iff it has the amalgamation property. Consequently Theorem 2.2, Theorem 3 of [2] and Lemma 2.1 yield

COROLLARY 2.3. *Let C be an intermediate logic. The following assertions are equivalent*

(i) *for any formulas α and β, if $V(\alpha) \cap V(\beta) \neq \emptyset$ and $\beta \in C(\alpha)$, then there exists a formula γ such that $V(\gamma) \subseteq V(\alpha) \cap V(\beta)$ and $\gamma \in C(\alpha)$, $\beta \in C(\gamma)$*

(ii) *for any formulas α and β, if $V(\alpha) \cap V(\beta) \neq \emptyset$ and $\alpha \to \beta \in C(\emptyset)$, then there exists a formula γ such that $V(\gamma) \subseteq V(\alpha) \cap V(\beta)$ and $\alpha \to \gamma \in C(\emptyset)$, $\gamma \to \beta \in C(\emptyset)$, where $\to$ is the implication connective of C*

(iii) *C has the Maehara interpolation property*

(iv) *the equational class $Alg^*(C)$ of Heyting algebras has the amalgamation property*

(v) *the equational class $Alg^*(C)$ has injections transferable.*

In the light of the results obtained by Maximova [6] there are exactly seven nontrivial intermediate logics with the Maehara interpolation property. Let us also note that the equivalence of clauses (iii)-(v) of Corollary 2.3 can be also easily derived from the main theorem of Wroński [13].

Note 2. Let $\overrightarrow{Kr}$ be the logic (i.e. the consequence operation) in the modal language defined as follows: the Kripke's system Kr constitutes the set of axioms of $\overrightarrow{Kr}$ and Modus Ponens for the material implication is the only primitive rule of inference of $\overrightarrow{Kr}$. Let $\Box^n\alpha$ be an abbreviation for $\Box \ldots \Box\alpha$, with the box $\Box$ preceding α n-times. Thus in particular

$\Box^0 a = a$. The logic $\overrightarrow{Kr}$ is equivalential and the infinite set $\{\Box^n(p \leftrightarrow q)\colon n \in \omega\}$ is known to be a $\overrightarrow{Kr}$-equivalence. The class $Matr^*(\overrightarrow{Kr})$ consists of all matrices of the form $(\mathscr{A}, D)$, where $\mathscr{A}$ is a modal algebra and D is a Boolean filter in $\mathscr{A}$ with the following property: if $\Box^n(a \leftrightarrow b) \in D$ for all $n \in \omega$, then $a = b$.

If C is axiomatic strengthening of $\overrightarrow{Kr}$, then clearly the deduction theorem holds for C relative to material implication. Consequently the class $Matr^*(C)$ has (FEP). Thus Theorem 2.2 and Theorem 1 of [2] yield

COROLLARY 2.4. *Let C be an axiomatic strengthening of $\overrightarrow{Kr}$. Then the following assertions are equivalent*:

(i) *for any formulas α and β, if $V(\alpha) \cap V(\beta) \neq \emptyset$ and $\beta \in C(\alpha)$, then there exists a formula γ such that $V(\gamma) \subseteq V(\alpha) \cap V(\beta)$ and $\gamma \in C(\alpha)$, $\beta \in C(\gamma)$*

(ii) *for any formulas α and β, if $V(\alpha) \cap V(\beta) \neq \emptyset$ and $\alpha \to \beta \in C(\emptyset)$, then there exists a formula γ such that $V(\gamma) \subseteq V(\alpha) \cap V(\beta)$ and $\alpha \to \gamma \in C(\emptyset)$, $\gamma \to \beta \in C(\emptyset)$, where $\to$ is the material implication*

(iii) *C has the Maehara interpolation property*

(iv) *the class $Matr^*(C)$ has the amalgamation property*

(v) *the class $Matr^*(C)$ has injections transferable.*

Note 3. Let $\overrightarrow{Kr}^{\Box}$ be the strengthening of the logic $\overrightarrow{Kr}$ obtained by adjoining the rule of prefixing of $\Box$, $\alpha/\Box\alpha$, to the set of rules of inference of $\overrightarrow{Kr}$. The logic $\overrightarrow{Kr}^{\Box}$, in contradistinction to $\overrightarrow{Kr}$, is implicative in the sense of Rasiowa [8] and $\overrightarrow{Kr}^{\Box}(\emptyset) = \overrightarrow{Kr}(\emptyset)$.

For every axiomatic strengthening C of $\overrightarrow{Kr}^{\Box}$, the class $Matr^*(C)$ can be identified with the equational class $Alg^*(C)$ of modal algebras $\mathscr{A}$ (with the unit of $\mathscr{A}$ as the only designated element) which validate the theses of C. The equational class $Alg^*(C)$ is known to have (FEP) relative to C (see [4]). Thus Theorem 2.2 and Theorem 3 of [2] yield

COROLLARY 2.5. *Let C be an axiomatic strengthening of $\overrightarrow{Kr}^{\Box}$. Then the following assertions are equivalent*:

(i) *for every formulas α and β, if $V(\alpha) \cap V(\beta) \neq \emptyset$ and $\beta \in C(\alpha)$, then there exists a formula γ such that $V(\gamma) \subseteq V(\alpha) \cap V(\beta)$ and $\gamma \in C(\alpha)$, $\beta \in C(\gamma)$*

(ii) *C has the Maehara interpolation property*

(iii) *the equational class $Alg^*(C)$ of modal algebras has the amalgamation property*

(iv) *the equational class $Alg^*(C)$ has injections transferable.*

Finally, we pose a problem which seems to be a natural outgrowth of the results of this section and the well-known Banaschewski Theorem for equational classes of algebras (see [12], p. 397).

PROBLEM. Let C be a standard sentential logic. We shall say that the class $Matr^*(C)$ is *residually small* if $Matr^*(C)$ contains only a set of $Matr^*(C)$-subdirectly irreducible matrices, i.e., the matrices subdirectly irreducible *in* the class $Matr^*(C)$ do not form a proper class. (We refer the reader to [1] for some basic facts concerning subdirectly irreducible matrices.) Call a matrix $\mathfrak{M}$ a *C-injective* if $\mathfrak{M} \in Matr^*(C)$ and for any matrices $\mathfrak{N}$ and $\mathfrak{P}$ in $Matr^*(C)$, if $\mathfrak{N}$ is a submatrix of $\mathfrak{P}$ and $f\colon \mathfrak{N} \mapsto \mathfrak{M}$ is a homomorphism, then there exists an extension of f to a homomorphism $g\colon \mathfrak{P} \mapsto \mathfrak{M}$. The class $Matr^*(C)$ is *injectively complete* (or, "has enough injectives") if every matrix in $Matr^*(C)$ is embeddable in a C-injective. Let us notice that if $Matr^*(C)$ is injectively complete, then the C-injectives constitute the class which is strongly adequate for C. Is it true that the conjunction of (AP), (FEP) and residuall smallness is equivalent to the injective completeness of the class $Matr^*(C)$, for every standard equivalential logic C?

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