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Univalent Minimizers of Polyconvex Functionals in Two Dimensions

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§1. Introduction

Let Ω be a bounded simply-connected domain in $\mathbb{R}^2$ of class $C^{2,\alpha}$ for some $\alpha > 0$. Let Ψ be a diffeomorphism of class $C^{2,\alpha}$ from $\partial\Omega$ onto the boundary of a convex domain $\mathcal{M} \subset \mathbb{R}^2$. Assume that Ψ is so oriented that its degree is one.

We consider the variational problem of minimizing the functional

$$\mathcal{W}(U) = \int_\Omega F(DU)dX \tag{1.1}$$

in the class

$$\mathcal{A} = \{U \in W^{1,2m}(\Omega; \mathbb{R}^2) : U = \Psi \text{ on } \partial\Omega\}.$$

A theorem of RADÓ [R] states that for the Dirichlet integral with $F(P) = |P|^2$ and $m = 1$, the minimizer is a homeomorphism from $\overline{\Omega}$ onto $\overline{\mathcal{M}}$. In this paper, we prove that minimizers for a class of integrands $F(\cdot)$ given in terms of $|P|$ and $\det P$ are homeomorphisms.

We consider

$$F(P) = G(|P|^2, \det P) \equiv (\mu + |P|^2)^m + \widetilde{G}(|P|^2, \det P) \tag{1.2}$$

for 2×2 matrices P. Our structural hypotheses are

(1.3) $\mu > 0, \quad m > 1,$

(1.4) $\widetilde{G} \in C^3([0, \infty) \times \mathbb{R})$ and $\widetilde{G} \geqq 0,$

(1.5) $\widetilde{G}(s^2, d)$ is a convex function of s and d for $(s, d) \in [0, \infty) \times \mathbb{R}.$

(1.6) $\partial_{s^2}\widetilde{G} \geqq 0,$

(1.7) $\partial_d\widetilde{G}(s^2, 0) = 0$ for $s \geqq 0,$

(1.8) there is a constant $C > 0$ so that

$$\lim_{|P|\to\infty} |P|^{2-2m} \cdot |D^2(F(P) - C|P|^{2m})| = 0.$$

Under these conditions minimizers are known to exist. Moreover, they are locally Lipschitz continuous in Ω.

Our principal result is

Theorem 6.1. *Assume F satisfies* (1.2)–(1.8) *and U is a minimizer for $\mathcal{W}(\cdot)$ in $\mathcal{A}$. Then U is a homeomorphism from $\overline{\Omega}$ onto $\overline{\mathcal{M}}$. Moreover,*

$$\underline{\lim}_{R\to 0} \fint_{B_R(X)} \det DU > 0 \textit{ for all } X \in \Omega. \tag{1.9}$$

This work is motivated by problems from two-dimensional nonlinear elasticity where one considers an image set $\mathcal{M}$ and seeks a univalent mapping $U: \overline{\Omega} \to \overline{\mathcal{M}}$ such that $U: \partial\Omega \to \partial\mathcal{M}$ is prescribed and U minimizes a stored-energy functional

$$\mathcal{E}(U) = \int_\Omega E(DU)dX. \tag{1.10}$$

One of the most elementary integrands considered is a perturbation of the neo-Hookean model

$$E(P) = \varepsilon(1 + |P|^2)^m + |P|^2 + H(\det P) \tag{1.11}$$

where $\varepsilon > 0$, $H \geqq 0$, H is convex, $\lim_{d\to 0^+} H(d) = \infty$ and $H(d) = \infty$ for $d \leqq 0$. If H blows up sufficiently rapidly at $d = 0$, BALL has shown that a univalent minimizer for $\mathcal{E}$ in $\mathcal{A}$ exists ([B]). To date, though, neither is there a regularity (i.e., a differentiability) theory for this solution, nor is it known to satisfy the Euler equation $\partial_P E(DU) = 0$.

For these reasons, one considers a sequence of approximate variational problems with

$$F_k(P) = \varepsilon(1 + |P|^2)^m + |P|^2 + H_k(\det P)$$

where $0 \leqq H_k(d) < \infty$ for all $d \in \mathbb{R}$, H_k is convex, $|H_k'(d)| + |dH_k''(d)| \leqq C(k) < \infty$ and $H_k \uparrow H$ as $k \to \infty$. It is not hard to show that given a sequence $\{U_k\} \subset \mathcal{A}$ of minimizers for $\mathcal{W}$ with $F = F_k$, there is a subsequence converging in $W^{1,2m}(\Omega; \mathbb{R}^2)$ whose limit is a minimizer for the elasticity problem (1.10), (1.11). Moreover, each U_k satisfies its corresponding Euler equation. In general, though, the U_k were not known to be one-to-one.

In Theorem 6.2, we apply Theorem 6.1 and results on null-Lagrangians to prove that the conclusion of Theorem 6.1 holds if $\widetilde{G}$ is replaced by $\widetilde{G} + \widetilde{H}(d)$, where $\widetilde{H}$ is smooth and convex, but $\widetilde{H}'(0)$ is not necessarily zero. An immediate consequence is:

Corollary. *If $\mathcal{M}$ is convex and $m > 2$, then minimizers for*

$$\mathcal{W}_k(U) = \int_\Omega F_k(DU)dX$$

are homeomorphisms. Moreover, (1.9) *holds.*

The main difficulty that we encounter in this paper is that minimizers for $\mathcal{W}(\cdot)$ are not known to be classical (although partial regularity is known). The paper is organized as follows. In §2, we collect the known properties of minimizers. In §3, we show that the Euler equation for $\mathcal{W}(\cdot)$ uncouples into elliptic equations for the components of U separately. Minimizers are shown in §4 to be one-to-one on a large subset of Ω. In §5, property (1.9) is established, and Theorems 6.1 and 6.2 are proved in §6.

§2. Preliminaries

A function $F(P)$ that can be expressed as a convex function of the minors of P is called *polyconvex*. In two dimensions, a function is *invariant to rotations* of the image and domain if

$$F(P) = F(QP) = F(PQ) \quad \text{for all } Q \in SO(2).$$

Such a function can be given in terms of $|P|$ and $\det P$. A class of functions both rotationally invariant and polyconvex are of the form $F(P) = \widehat{G}(|P|, \det P)$ when $\widehat{G}(\cdot, \cdot)$ is convex and increasing in its first variable. Functions F satisfying (1.2)–(1.6) are of this class. Existence results for minimizers of $\mathcal{W}$ in $\mathcal{A}$, assuming (1.2)–(1.6) can be found in Dacorogna [D]. The assumption $m > 1$ implies that $U \in C^{1-\frac{1}{m}}(\overline{\Omega}; \mathbb{R}^2)$ for minimizers. Condition (1.8) is a strong statement of convexity at infinity. Giaquinta & Modica [G-M] have shown that (1.8) implies minimizers are locally Lipschitz continuous

$$\|DU\|_{L^\infty(\Omega')} \leqq C(\Omega, \Omega', \mathcal{W}(U)) \quad \text{for all } \Omega' \Subset \Omega. \tag{2.1}$$

A fundamental result that follows from (1.2)–(1.8) is that a minimizer is classical (of class $C^{2,\alpha}$) on an open set $\Omega_0 \subset \Omega$ such that $|\Omega \setminus \Omega_0| = 0$ (due to Evans [E-1]). We need the following criterion. Denote the average of DU on $B_R(X)$ by

$$(DU)_{X,R} \equiv \fint_{B_R(X)} DU.$$

From §7 in [E-1] (or §4 in [E-2]), given $M < \infty$ there exists a constant $\varepsilon(M) > 0$ so that

$$\text{if } B_R(X) \subset \Omega, \quad |(DU)_{X,R}| < M \quad \text{and} \quad \fint_{B_R(X)} |DU - (DU)_{X,R}|^{2m} < \varepsilon, \quad \text{then } X \in \Omega_0. \tag{2.2}$$

Notation. Denote

$$\partial_{s^2} G \equiv G,_1\,, \quad \partial_d G \equiv G,_2\,.$$

From (1.2)–(1.7) there is a constant $\nu(\mu, m) > 0$ such that

$$G,_1 \geqq \nu, \qquad d \cdot G,_2 \geqq 0. \tag{2.3}$$

The second inequality follows from the facts that G is convex in d for s^2 fixed, and $G,_2\,(s^2, 0) = 0$.

We use $|v|_{k,\beta;\mathcal{O}}$ to denote the norm of $v \in C^{k,\beta}(\overline{\mathcal{O}}; \mathbb{R}^s)$ and $\|v\|_{l,p;\mathcal{O}}$ to denote the norm of $v \in W^{l,p}(\mathcal{O}; \mathbb{R}^s)$.

§3. Elliptic Equations for the Components

Let $X = (x, y)$ and $U(X) = (u(x, y), v(x, y))$ be a minimizer. From the hypotheses (1.2)–(1.8) it follows directly that the first variation, $\partial_\varepsilon \mathcal{W}(U + \varepsilon\Phi)$, at $\varepsilon = 0$ exists for all $\Phi \in W_0^{1,2m}(\Omega; \mathbb{R}^2)$. (See GIAQUINTA [G].) Thus U is a weak solution to the equilibrium equations

$$\begin{aligned} (2G,_1 u_x)_x + (2G,_1 u_y)_y + (G,_2 v_y)_x - (G,_2 v_x)_y &= 0, \\ (2G,_1 v_x)_x + (2G,_1 v_y)_y - (G,_2 u_y)_x + (G,_2 u_x)_y &= 0 \end{aligned} \tag{3.1}$$

in Ω. Using (1.7), we first show that (3.1.1) can be rewritten as a homogeneous elliptic equation for u, and (3.1.2) as a homogeneous elliptic equation for v.

Lemma 3.1. *The components u and v satisfy the equations*

$$\begin{aligned} (a^1_{ij} u_{x_i})_{x_j} &= 0 \quad \textit{in } \Omega, \\ (a^2_{ij} v_{x_i})_{x_j} &= 0 \quad \textit{in } \Omega, \end{aligned} \tag{3.2}$$

where $[a^1_{ij}]$ and $[a^2_{ij}]$ are positive definite and the a^1_{ij} and a^2_{ij} are locally bounded.

Proof. We define g by

$$G,_2 = d \cdot g(s^2, d) = (u_x v_y - v_x u_y) g \quad \text{for } d \neq 0$$

with $g(s^2, 0) = G_{,22}(s^2, 0)$. From (1.4) and (1.7), the function g is of class C^1 for $s^2 \geqq 0$ and $d \in \mathbb{R}$. From (2.3) we see that $g \geqq 0$. Using this in (3.1), we have

$$\begin{aligned}
&([2G_{,1} + v_y^2 g]u_x + [-v_x v_y g]u_y)_x \\
&\qquad + ([-v_x v_y g]u_x + [2G_{,1} + v_x^2 g]u_y)_y = 0 \quad \text{in } \Omega, \\
&([2G_{,1} + u_y^2 g]v_x + [-u_x u_y g]v_y)_x \\
&\qquad + ([-u_x u_y g]v_x + [2G_{,1} + u_x^2 g]v_y)_y = 0 \quad \text{in } \Omega.
\end{aligned} \tag{3.3}$$

These can be viewed as linear equations in divergence form for u and v, respectively. Since DU is locally bounded, the coefficients are locally bounded. Moreover, since $G_{,1} \geqq \nu > 0$ and $g \geqq 0$, it follows that the coefficient matrices $[a_{ij}^1]$ and $[a_{ij}^2]$ are positive-definite. Thus the equations are uniformly elliptic with bounded measurable coefficients on compact subsets of Ω. □

Corollary 3.2. *A component of U cannot be constant on an open subset of Ω.*

Proof. Consider (3.2.1). From this equation it follows that there is a function φ such that

$$\varphi_{x_1} = a_{12}^1 u_{x_1} + a_{22}^1 u_{x_2}, \quad -\varphi_{x_2} = a_{11}^1 u_{x_1} + a_{21}^1 u_{x_2} \qquad \text{in } \Omega.$$

This is a first-order elliptic system for (φ, u) of the type analyzed in [B-J-S, Part II, Ch. 6, Eq. (1)]. For these systems, it is shown that a solution which is constant on an open subset of Ω is identically constant on all of Ω. Since u is not constant on $\partial\Omega$, the assertion holds for the component u.

The same argument applies to v as well. □

Definition. Let $V \in W^{1,p}(\Omega; \mathbb{R}^2)$ for some $p > 2$ and let A be a component of $\mathbb{R}^2 \setminus V(\partial\Omega)$. Then the *degree of V in A* is

$$\deg(V; A) = \int_\Omega \rho(V(X)) \det DV(X) dX$$

where ρ is any nonnegative element of $C_c(A)$ with $\int_A \rho dU = 1$.

It follows that $\deg(V; A)$ is integer-valued and depends on V only through $V|_{\partial\Omega}$.

The rotation or translation of a solution through either the x, y or u, v variables is again a solution to an equivalent problem. We use this together with the uncoupling of the system (3.2) to show that $U(\Omega) = \mathcal{M}$.

Lemma 3.3. $U(\Omega) = \mathcal{M}$.

Proof. By hypothesis, $\deg(U;\mathcal{M}) = 1$. Let $\widetilde{\mathcal{M}} = \{Y \in \mathcal{M} : U^{-1}(Y) \neq \emptyset\}$. By requiring the support of ρ to be in a sufficiently small neighborhood of an arbitrary point in $\mathcal{M}$, it follows from the definition of $\deg(U;\mathcal{M})$ that $\widetilde{\mathcal{M}}$ is dense in $\mathcal{M}$. The continuity of U and the fact that $U(\partial\Omega) = \partial\mathcal{M}$ then imply that $\widetilde{\mathcal{M}} = \mathcal{M}$. Hence, $\mathcal{M} \subset U(\Omega)$.1

Since $\mathcal{M}$ is convex, to prove that $U(\Omega) \subset \mathcal{M}$, it suffices to show that $U(\Omega) \subset \mathcal{L}$ where $\mathcal{L}$ is an arbitrary open half plane containing $\mathcal{M}$. By rotating and translating the u, v variables, we can assume that $\mathcal{L} = \{(u, v) : v < 0\}$ and $U(\partial\Omega) \subset \overline{\mathcal{L}}$. Thus we have $v \leqq 0$ and $v \not\equiv 0$ on $\partial\Omega$. We must show that $v < 0$ in Ω. From Lemma 3.1, v satisfies an equation of the form

$$(a_{ij}^2(X)v_{x_i})_{x_j} = 0 \quad \text{in } \Omega$$

where $(x, y) = (x_1, x_2)$, the a_{ij}^2 are locally bounded and $[a_{ij}^2]$ is positive-definite. Thus by the strong maximum principle, $v < 0$ in Ω. □

We remark that weak maximum principles for related polyconvex problems can be found in [F-H] and [L-1].

§4. The Structure of U on $\{X \in \Omega_0 : \det DU(X) \neq 0\}$

Let

$$\Omega_1 \equiv \{X \in \Omega_0 : \det DU(X) \neq 0\}.$$

In this section, we prove that U is one-to-one on Ω_1 (Theorem 4.5) and as a consequence that $\det DU > 0$ in Ω_1. It will be clearer if the ideas are outlined first.

By translating the solution if necessary assume there are two points X_1, $X_2 \in \Omega_1$ such that $U(X_1) = U(X_2) = 0$. Since these points are in $\Omega_1 \subset \Omega_0$, there are disjoint disks $B_{4\delta}(X_1)$ and $B_{4\delta}(X_2)$ on which U is of class $C^{2,\alpha}$. Set

$$e_\theta^1 = (\cos\theta, \sin\theta), \quad e_\theta^2 = (-\sin\theta, \cos\theta) \text{ for } 0 \leqq \theta \leqq \pi.$$

Let $v_\theta \equiv U \cdot e_\theta^2$ and note that

$$\{X_1, X_2\} \subset \{X : v_\theta(X) = 0\} \quad \text{for all } \theta. \tag{4.1}$$

If, in fact, $U \in C^2(\overline{\Omega};\mathbb{R}^2)$, then we have a continuity argument which shows that (4.1) leads to a contradiction. Since it is not known if U is globally regular, we approximate it by a function $\overline{U} \in C^{2,\alpha}(\overline{\Omega};\mathbb{R}^2)$ such that $\overline{U} = \Psi$ on $\partial\Omega$, and $\overline{U}$ is close to U, both in $W^{1,2m}(\Omega;\mathbb{R}^2)$ and in $C^2(\overline{B_{3\delta}(X_1)} \cup \overline{B_{3\delta}(X_2)};\mathbb{R}^2)$. Now $\overline{U}$ is not an equilibrium. To adjust for this, we set $\bar{u}_\theta \equiv \overline{U} \cdot e_\theta^1$ and define $\bar{v}_\theta$ as the solution to the *scalar* minimum problem:

$$\mathcal{W}_\theta(\bar{v}_\theta) = \inf_{z \in \mathcal{A}_\theta} \mathcal{W}_\theta(z)$$

where

$$\mathcal{W}_\theta(z) \equiv \mathcal{W}(ze_\theta^2 + \bar{u}_\theta e_\theta^1),$$
$$\mathcal{A}_\theta \equiv \{z \in W^{1,2m}(\Omega) : z = v_\theta \text{ on } \partial\Omega\}.$$

We prove that the minimizer exists, that it is unique and of class $C^{2,\alpha}(\overline{\Omega})$ and that it varies smoothly with θ. Our idea is to work with $\bar{v}_\theta$ in place of v_θ. Now, $\{X_1, X_2\}$ is not necessarily contained in the zero level set of $\bar{v}_\theta$ as in (4.1). Nevertheless, if δ is small enough and $\overline{U}$ is sufficiently close to U, it follows that $\{X : \bar{v}_\theta = 0\}$ intersects both $\overline{B_{2\delta}(X_1)}$ and $\overline{B_{2\delta}(X_2)}$ for all θ in a simple way. The continuity argument mentioned above is then adapted to derive a contradiction.

We begin by showing that the mapping: $\overline{U} \in C^{2,\alpha}(\overline{\Omega}; \mathbb{R}^2) \cap \mathcal{A} \to \bar{v}_\theta \in C^{2,\alpha}(\overline{\Omega})$ is well defined.

Lemma 4.1. *Let $\overline{U} \in C^{2,\alpha}(\overline{\Omega}; \mathbb{R}^2)$ such that $\overline{U} = \Psi$ on $\partial\Omega$. For each θ in $[0, \pi]$ there is a unique minimizer, $\bar{v}_\theta$, for $\mathcal{W}_\theta(\cdot)$ in $\mathcal{A}_\theta$. Moreover, $\bar{v}_\theta \in C^{2,\alpha}(\overline{\Omega})$ for all θ and $\bar{v}_0 = -\bar{v}_\pi$.*

Proof. For θ fixed, since $\{e_\theta^1, e_\theta^2\}$ is a positively oriented orthonormal basis of $\mathbb{R}^2$, we have

$$\mathcal{W}_\theta(z) = \int_\Omega G_\theta(\nabla z, X)\,dX$$

where

$$G_\theta(\nabla z, X) = G(z_x^2 + z_y^2 + \bar{u}_{\theta x}^2(X) + \bar{u}_{\theta y}^2(X), z_y\bar{u}_{\theta x}(X) - z_x\bar{u}_{\theta y}(X)) \tag{4.2}$$

and $\bar{u}_\theta \equiv \overline{U} \cdot e_\theta^1$. Since G is polyconvex, it is rank-1 convex. Hence G_θ is a convex function of ∇z. From (1.2)–(1.8), we have

$$G_\theta \geqq 0, \tag{4.3}$$

$$G_\theta \in C^{2,\alpha}(\mathbb{R}^2 \times \overline{\Omega}), \tag{4.4}$$

(4.5) there are constants $\Lambda = \Lambda(|\nabla \bar{u}_\theta|_{0;\Omega}) < \infty$ and

$\lambda = \lambda(|\nabla \bar{u}_\theta|_{0,\Omega}) > 0$ so that

$$\lambda(\mu + |p|^2)^{m-1}|\zeta|^2 \leqq \frac{\partial^2 G_\theta(p, X)}{\partial p_i \partial p_j}\zeta_i\zeta_j \leqq \Lambda(\mu + |p|^2)^{m-1}|\zeta|^2$$

for all $X \in \overline{\Omega}$, $p \in \mathbb{R}^2$, $\zeta \in \mathbb{R}^2$,

(4.6) there is a constant $\Lambda_1 = \Lambda_1(|\nabla \bar{u}_\theta|_{1;\Omega}) < \infty$ so that

$$\left|\frac{\partial^2 G_\theta}{\partial x_i \partial p_j}\right| \leqq \Lambda_1(\mu + |p|^2)^{m-1}$$

for all $X = (x_1, x_2) \in \overline{\Omega}$, $p \in \mathbb{R}^2$.

It follows from the standard theory for convex multiple integrals that $\mathcal{W}_\theta$ has a unique minimizer in $\mathcal{A}_\theta$ and that this function, $\bar{v}_\theta$, is the unique weak solution to the elliptic equilibrium problem

$$\partial_x G_{\theta_{p_1}}(\nabla \bar{v}_\theta, X) + \partial_y G_{\theta_{p_2}}(\nabla \bar{v}_\theta, X) = 0 \quad \text{in } \Omega, \tag{4.7}$$

$$\bar{v}_\theta = \Psi \cdot e^2_\theta \quad \text{on } \partial\Omega. \tag{4.8}$$

Under conditions (4.3)–(4.6), this problem has a classical solution in $C^{2,\alpha}(\overline{\Omega})$. (See GILBARG & TRUDINGER [G-T, Theorem 15.11].) Since a classical solution is also a weak solution, we have $\bar{v}_\theta \in C^{2,\alpha}(\overline{\Omega})$. This proves the first part of the lemma.

To prove that $-\bar{v}_0 = \bar{v}_\pi$, we note that by definition, $\bar{u}_\pi = \overline{U} \cdot e^1_\pi = -\overline{U} \cdot e^1_0 = -\bar{u}_0$. Thus, $G_0(-p, X) = G_\pi(p, X)$ and $\mathcal{A}_\pi = \{v : -v \in \mathcal{A}_0\}$. It follows that $-\bar{v}_0 \in \mathcal{A}_\pi$ and minimizes $\mathcal{W}_\pi(\cdot)$. □

Remark. It follows from the above proof that two minimizers for $\mathcal{W}(\cdot)$ which agree in one component must agree in the second component as well.

Lemma 4.2. $\bar{v}_\theta \in C([0, \pi]; C^2(\overline{\Omega}))$.

Proof. We first derive a number of *a priori* estimates. From (4.2) we have

$$\nabla_p G_\theta \cdot p = 2G_{,1}\,|p|^2 + G_{,2}\cdot(p_2\bar{u}_{\theta_x} - p_1\bar{u}_{\theta_y})$$

where $G_{,1}$ and $G_{,2}$ are evaluated at $s^2 = |p|^2 + |\nabla \bar{u}_\theta|^2$ and $d = p_2\bar{u}_{\theta_x} - p_1\bar{u}_{\theta_y}$. Thus, from (2.3),

$$\nabla_p G_\theta \cdot p \geqq \nu|p|^2.$$

Let $\varphi = \max(\bar{v}_\theta, \max_{\partial\Omega} v_\theta) - \max_{\partial\Omega} v_\theta$. Using φ as a test function in (4.7), we have

$$\nu \int_{\{\bar{v}_\theta > \max_{\partial\Omega} v_\theta\}} |\nabla \bar{v}_\theta|^2 dX \leqq \int_\Omega \nabla_p G_\theta \cdot \nabla\varphi dX = 0.$$

Thus,

$$\bar{v}_\theta \leqq \max_{\partial\Omega} v_\theta \leqq |\Psi|_{0;\partial\Omega}.$$

A similar statement holds for $-\bar{v}_\theta$, and we find that

$$|\bar{v}_\theta|_{0;\Omega} \leqq |\Psi|_{0;\partial\Omega}. \tag{4.9}$$

Now G_θ depends upon θ only through $\bar{u}_\theta$, for which we have $\bar{u}_\theta \in C([0,\pi]; C^{2,\alpha}(\overline{\Omega}))$ and $|\bar{u}_\theta|_{2,\alpha;\Omega} \leqq |\overline{U}|_{2,\alpha;\Omega}$. Thus, using (4.4)–(4.6), we apply the *a priori* estimates from [G-T, Theorems 13.2, 14.1, 15.9] to conclude that

$$|\bar{v}_\theta|_{2,\alpha;\Omega} \leqq C(|\overline{U}|_{2,\alpha;\Omega},\ |\bar{v}_\theta|_{0;\Omega}).$$

Since (4.7), (4.8) has a unique solution, it follows from compactness, elliptic estimates, and the smooth dependence of G_θ on θ that $\bar{v}_\theta \in C([0,\pi]; C^2(\overline{\Omega}))$. □

Assume that $0 \in \mathcal{M}$. Since $\mathcal{M}$ is convex for each θ, the line through e^1_θ intersects $\partial\mathcal{M}$ at two points. As Ψ is a diffeomorphism of $\partial\Omega$ onto $\partial\mathcal{M}$, there are exactly two points $\{X^1_\theta, X^2_\theta\} \subset \partial\Omega$ such that $\bar{v}_\theta(X^1_\theta) = \bar{v}_\theta(X^2_\theta) = 0$. Set

$$N_\theta \equiv \{X \in \overline{\Omega} : \bar{v}_\theta = 0\}.$$

Lemma 4.3. *Assume that* $0 \in \mathcal{M}$ *and that* $\overline{U}$ *is as in Lemma 4.1. Then for each* θ, N_θ *is a connected imbedded* C^1 *curve with endpoints* $\{X^1_\theta, X^2_\theta\}$. *Moreover,* N_θ *varies continuously with respect to* θ *in the sense that given* $\kappa > 0$, *there exists* $\eta > 0$ *so that if* $|\theta - \theta'| < \eta$, *then*

$$N_{\theta'} \subset \{X \in \overline{\Omega} : \operatorname{dist}(X, N_\theta) < \kappa\}.$$

Proof. As in the derivation of (3.3.2), it follows from (4.2) and (4.7) that $\bar{v}_\theta$ satisfies the Dirichlet problem

$$\begin{aligned}([2G,_1 + \bar{u}^2_{\theta_y} g]\bar{v}_{\theta_x} + [-\bar{u}_{\theta_x}\bar{u}_{\theta_y} g]\bar{v}_{\theta_y})_x \\ + ([-\bar{u}_{\theta_x}\bar{u}_{\theta_y} g]\bar{v}_{\theta_x} + [2G,_1 + \bar{u}^2_{\theta_x} g]\bar{v}_{\theta_y})_y = 0 \quad \text{in } \Omega,\end{aligned} \tag{4.10}$$

$$\bar{v}_\theta = \Psi \cdot e^2_\theta \quad \text{on } \partial\Omega \tag{4.11}$$

where $G_{,1}$ and g are evaluated at

$$s^2 = |\nabla \bar{v}_\theta|^2 + |\nabla \bar{u}_\theta|^2, \quad d = \bar{u}_{\theta_x}\bar{v}_{\theta_y} - \bar{u}_{\theta_y}\bar{v}_{\theta_x}.$$

Note that (4.10) is of the form

$$(4.12) \qquad (a_{ij}(X)\bar{v}_{\theta_{x_i}})_{x_j} = 0 \quad \text{in } \Omega$$

where the equation is elliptic and the $a_{ij}(X)$ are of class $C^1(\overline{\Omega})$. We use (4.11)–(4.12) to study the zero set of $\bar{v}_\theta$, namely N_θ.

We have $N_\theta \cap \partial\Omega = \{X_\theta^1, X_\theta^2\}$. Let $\partial\Psi/\partial s$ denote the derivative of Ψ with respect to the arc length of $\partial\Omega$. Consider $\partial\Psi/\partial s$ evaluated at X_θ^1. Since Ψ is a diffeomorphism $\partial\Psi/\partial s \neq 0$, and since $\mathcal{M}$ is convex, the tangent line to $\partial\mathcal{M}$ determined at $\Psi(X_\theta^1)$ by $\partial\Psi/\partial s$ cannot pass through $0 \in \mathcal{M}$. Thus, $\frac{\partial\Psi}{\partial s}(X_\theta^1)$ is not parallel to e_θ^1. Hence $\frac{\partial \bar{v}_\theta}{\partial s}(X_\theta^1) = e_\theta^2 \cdot \frac{\partial\Psi}{\partial s}(X_\theta^1) \neq 0$. Similarly, $\frac{\partial \bar{v}_\theta}{\partial s}(X_\theta^2) \neq 0$. Thus, N_θ is a C^1 curve in neighborhoods of X_θ^1 and X_θ^2, respectively, intersecting $\partial\Omega$ at these points nontangentially.

Since $\bar{v}_\theta$ satisfies the elliptic problem, it follows from HARTMAN & WINTNER [H-W] that $\bar{v}_\theta$ has at most isolated critical points in $N_\theta \cap \Omega$. Moreover, in a neighborhood of any such point, N_θ is made up of $2k$ ($k \geqq 2$) arcs meeting at this point. If a critical point exists, then since $N_\theta \cap \partial\Omega$ consists of just two points, it follows that N_θ must contain a closed loop, that is, some component $\mathcal{D}$ of $\{\bar{v}_\theta \neq 0\}$ has $\partial\mathcal{D} \subset N_\theta$. Since $\bar{v}_\theta$ satifies (4.12) on $\mathcal{D}$, it follows from the maximum principle that $\bar{v}_\theta \equiv 0$ on $\mathcal{D}$, which is a contradiction. Thus, $\nabla \bar{v}_\theta \neq 0$ on N_θ. Hence, N_θ is an imbedded C^1 curve. It must be connected since if not, then as before, it would contain a closed loop.

The last assertion of the lemma follows from the fact that the mapping $\theta \to \bar{v}_\theta$ is of class $C([0, \pi]; C^2(\overline{\Omega}))$. □

We now show that we can choose $\overline{U}$ sufficiently close to U to ensure that $\bar{v}_\theta$ is close to $U \cdot e_\theta^2$ for all θ.

Lemma 4.4. *Let U be a minimizer, K a compact subset of Ω_0 and $\varepsilon > 0$. Then there is a function $\overline{U} \in C^{2,\alpha}(\overline{\Omega}; \mathbb{R}^2)$ with $\overline{U} = \Psi$ on $\partial\Omega$ such that if we denote*

$$(4.13) \qquad W_\theta = \bar{v}_\theta e_\theta^2 + \bar{u}_\theta e_\theta^1 \quad \textit{for } 0 \leqq \theta \leqq \pi,$$

then

$$|U - W_\theta|_{2;K} < \varepsilon,$$

$$\|U - W_\theta\|_{1,2m;\Omega} < \varepsilon.$$

Proof. Let Ω', Ω'' be open sets such that $K \subset \Omega'' \Subset \Omega' \Subset \Omega_0$. For each integer n (using a partition of unity) we can define

$$U_n \equiv \Upsilon_n + \Phi_n \quad \text{where} \quad \Phi_n \in C^{2,\alpha}(\overline{\Omega}; \mathbb{R}^2) \cap \mathcal{A},$$

$$\operatorname{supp} \Upsilon_n \Subset \Omega, \quad U_n = U \text{ on } \Omega', \quad \|U - U_n\|_{1,2m;\Omega} < \frac{1}{n}.$$

Let $(\Upsilon_n)_\zeta$ be a mollification of Υ_n. Fix $\zeta = \zeta(n)$ sufficiently small so that $\overline{U}_n \equiv (\Upsilon_n)_\zeta + \Phi_n \in \mathcal{A}$,

$$|U - \overline{U}_n|_{2,\frac{\alpha}{2};\Omega''} < \frac{1}{n}, \tag{4.14}$$

$$\|U - \overline{U}_n\|_{1,2m;\Omega} < \frac{1}{n}. \tag{4.15}$$

This is possible since $\Upsilon_n \in C^{2,\alpha}(\overline{\Omega}'; \mathbb{R}^2)$. For n sufficiently large, $\overline{U}_n$ satisfies the lemma. To see this, let W^n_θ be defined via $\overline{U}_n$ as in (4.13). It suffices to show that

$$\lim_{n\to\infty} |U - W^n_\theta|_{2;K} = \lim_{n\to\infty} \|U - W^n_\theta\|_{1,2m;\Omega} = 0,$$

where the limits are to be uniform in θ. To this end, let $\{\theta_{n_i}\} \subset [0,\pi]$, $\lim_{i\to\infty} \theta_{n_i} = \theta_0$ and $W_i \equiv W^{n_i}_{\theta_{n_i}}$. Assume first that there is a $\delta_0 > 0$ so that

$$\|U - W_i\|_{1,2m;\Omega} \geqq \delta_0 \quad \text{for all } i. \tag{4.16}$$

From (4.15) it follows that $\{\overline{U}_n\}$ is a minimizing sequence for $\mathcal{W}(\cdot)$ in $\mathcal{A}$. By construction $\{W_i\} \subset \mathcal{A}$ and $\mathcal{W}(W_i) \leqq \mathcal{W}(\overline{U}_{n_i})$. Hence, $\{W_i\}$ is a minimizing sequence as well. It follows from EVANS & GARIEPY [E-G, § 1] that a subsequence (still denoted $\{W_i\}$) converges in $W^{1,2m}(\Omega)$ to a minimizer Z. Now $\lim_{i\to\infty} e^1_{\theta_{n_i}} = e^1_{\theta_0}$ and by definition

$$Z \cdot e^1_{\theta_0} = \lim_{i\to\infty} W_i \cdot e^1_{\theta_{n_i}} = \lim_{i\to\infty} \overline{U}_{n_i} \cdot e^1_{\theta_{n_i}} = U \cdot e^1_{\theta_0}$$

where the limits are taken in $W^{1,2m}$. Thus Z and U agree in one component. It follows from the remark following Lemma 4.1 that $Z \equiv U$. Hence (4.16) is not possible.

Assume next that

$$|U - W_i|_{2;K} \geqq \delta_0 \quad \text{for all } i. \tag{4.17}$$

From (4.14), $\bar{u}_i \equiv \overline{U}_{n_i} \cdot e^1_{\theta_{n_i}} = W_i \cdot e^1_{\theta_{n_i}}$ converges to $U \cdot e^1_{\theta_0}$ as $i \to \infty$ in $C^{2,\frac{\alpha}{2}}(\overline{\Omega}'')$, and $\bar{v}_i \equiv W_i \cdot e^2_{\theta_{n_i}}$ satisfies (4.7). From *a priori* interior estimates, we find that for $K \Subset \Omega''' \Subset \Omega''$,

$$|\bar{v}_i|_{2,\frac{\alpha}{2},\Omega'''} \leqq C(|\bar{u}_i|_{2,\frac{\alpha}{2};\Omega''}, |\bar{v}_i|_{0;\Omega''}).$$

Using (4.9) we have

$$|\bar{v}_i|_{0;\Omega} \leqq |\Psi|_{0;\partial\Omega}.$$

Hence $\{|\bar{v}_i|_{2,\frac{\alpha}{2};\Omega'''}\}$ is uniformly bounded. From the first part of this proof, we can assume that $\{\bar{v}_i\}$ converges to $U \cdot e^2_{\theta_0}$ pointwise. Thus from compactness a subsequence $\{\bar{v}_{i'}\}$ converges to $U \cdot e^2_{\theta_0}$ in $C^2(K)$. This contradicts (4.17). □

Now using our approximations, we show that U is one-to-one in Ω_1.

Theorem 4.5. *Let U be a minimizer and $\{X_1, X_2\} \subset \Omega_1$ such that $U(X_1) = U(X_2)$. Then $X_1 = X_2$.*

Proof. Assume that $X_1 \neq X_2$ and without loss of generality that $U(X_1) = 0$. Choose ε_0, $\delta > 0$ so that

(4.18) 1) $10\delta < |X_1 - X_2|$.
2) $U : \overline{B_{4\delta}(X_i)} \to U(\overline{B_{4\delta}(X_i)})$ is a C^2 diffeomorphism for $i = 1, 2$.
3) For each $\nu \in \mathbb{R}^2$ with $|\nu| = 1$ and each function W such that $|U - W|_{2;\overline{B_{3\delta}(X_i)}} < \varepsilon_0$ we have $0 \in W(B_\delta(X_i))$ and the set $\{X \in \overline{B_{2\delta}}(X_i) : \nu \cdot W(X) = 0\}$ is a connected C^1 curve for $i = 1, 2$.

Note that U is of class C^2 in a neighborhood of $\{X_1, X_2\}$ and the $\det DU(X_i) \neq 0$ for $i = 1, 2$. Thus, (4.18) can be verified for sufficiently small ε_0 and δ by applying the inverse function theorem.

We apply Lemma 4.4 to fix $\overline{U}$ by setting $K = \overline{B_{3\delta}(X_1)} \cup \overline{B_{3\delta}(X_2)}$ and $\varepsilon = \varepsilon_0$. We obtain W_θ from (4.13), which, by Lemma 4.2, is of class $C([0, \pi]; C^2(\overline{\Omega}; \mathbb{R}^2))$ and satisfies (4.18.3) for each θ.

Let $\Psi = (\psi_1, \psi_2)$. By Lemma 3.3 we have $0 \in \mathcal{M}$. Since $\mathcal{M}$ is convex, there are unique points $\{X_0^1, X_0^2\} \subset \partial\Omega$ so that

$$\Psi(X_0^1) = (\psi_1(X_0^1), 0), \quad \Psi(X_0^2) = (\psi_1(X_0^2), 0)$$

with $\psi_1(X_0^1) > 0$ and $\psi_1(X_0^2) < 0$. We write $\partial\Omega \setminus \{X_0^1, X_0^2\} = \Gamma_1 \cup \Gamma_2$ where each Γ_i is a connected curve. We can assume that $\psi_2(X) > 0$ for $X \in \Gamma_1$ and $\psi_2(X) < 0$ for $X \in \Gamma_2$. Thus, Ψ determines a homeomorphism

$$\widetilde{X}(\theta) : [0, \pi] \to \overline{\Gamma}_1$$

such that $\widetilde{X}(0) = X_0^1$ and $\widetilde{X}(\pi) = X_0^2$.

For $\theta \in [0, \pi]$, we have $N_\theta = \{\bar{v}_\theta = 0\}$ and $\bar{v}_\theta = W_\theta \cdot e^2_\theta$. By Lemma 4.3, N_θ is a connected non-self-intersecting C^1 curve with two endpoints, one in $\Gamma_1 \cup \{X_0^1\}$ and one in $\Gamma_2 \cup \{X_0^2\}$. We define an orientation for N_θ such that its initial point is $\widetilde{X}(\theta)$.

Fix θ. From (4.18.3) with $\nu = e_\theta^2$ and $W = W_\theta$, we find that $N_\theta \cap B_\delta(X_i) \neq \emptyset$ and that $N_\theta \cap \overline{B_{2\delta}(X_i)}$ is connected for each i. Thus, starting at $\widetilde{X}(\theta)$ and moving along the curve N_θ, one enters each $\overline{B_{2\delta}(X_1)}$ and $\overline{B_{2\delta}(X_2)}$ once and only once. Moreover, due to (4.18.1) the disks are passed through in a well-defined consecutive order. Finally, using the stability of $\theta \to N_\theta$ proved in Lemma 4.3, we prove (in Lemma 4.6 below) that the order remains the same for all θ' near θ.

Since a locally constant function defined on a connected set is globally constant, we see that the order remains for all $\theta \in [0, \pi]$. Without loss of generality, we may assume that each directed curve N_θ first intersects $\overline{B_{2\delta}(X_1)}$ and then intersects $\overline{B_{2\delta}(X_2)}$.

From Lemma 4.2, we have $\bar{v}_\pi = -\bar{v}_0$. Thus, N_π is just N_0 with the reverse orientation. This would mean that N_π first intersects $\overline{B_{2\delta}(X_2)}$ and then intersects $\overline{B_{2\delta}(X_1)}$, a contradiction; thus, $X_1 = X_2$. □

Lemma 4.6. *For $\theta \in [0, \pi]$ and all θ' sufficiently close to θ, the curves N_θ and $N_{\theta'}$ intersect $\{\overline{B_{2\delta}(X_1)}, \overline{B_{2\delta}(X_2)}\}$ in the same order.*

Proof. Assume that N_θ intersects $\overline{B_{2\delta}(X_1)}$ first. Let $X(t)$ for $0 < t \leqq \bar{t}$ be a parametrization of N_θ. Let $0 < t_1 < t_2 < \bar{t}$ be such that

$$Q_2 \equiv \{X(t) : t_1 < t < t_2\} \subset B_\delta(X_1),$$

and set

$$Q_1 \equiv \{X(t) : 0 \leqq t \leqq t_1\},$$

$$Q_3 \equiv \{X(t) : t_2 \leqq t \leqq \bar{t}\},$$

$$\kappa = \frac{1}{4} \min\left(\operatorname{dist}(Q_1, Q_3), \operatorname{dist}(Q_1, \overline{B_{2\delta}(X_2)}), \delta \right) > 0.$$

Let $\eta = \eta(\frac{\kappa}{4})$ be as in Lemma 4.3. Thus, for $|\theta - \theta'| < \eta$, $N_{\theta'}$ is in a $\frac{\kappa}{4}$-neighborhood of N_θ. Finally, we set

$$\mathcal{P}_\kappa = \{X : \operatorname{dist}(X, Q_1) < \kappa\}.$$

We assume that θ' is so close to θ that

$$|\theta - \theta'| < \eta, \quad |\widetilde{X}(\theta) - \widetilde{X}(\theta')| < \kappa,$$

where $\widetilde{X}(\cdot)$ is the homeomorphism from $[0, \pi]$ into $\partial\Omega$ defined in the proof of Theorem 4.5.

Let $N_{\theta'}$ be given by $Y(\tau)$ for $0 \leqq \tau \leqq \bar{\tau}$. We have $|Y(0) - X(0)| = |\widetilde{X}(\theta') - \widetilde{X}(\theta)| < \kappa$. Thus, $Y(0) \in \mathcal{P}_\kappa$. We shall show at the first τ^* for which $Y(\tau^*) \in \partial\mathcal{P}_\kappa$, one has $Y(\tau^*) \in \overline{B_{2\delta}(X_1)}$ as well. By construction, $\{Y(\tau) : 0 \leqq \tau \leqq \tau^*\} \subset \overline{\mathcal{P}_\kappa}$ and $\overline{\mathcal{P}_\kappa} \cap \overline{B_{2\delta}(X_2)} = \emptyset$. Thus, the lemma will be proved.

By the choice of η, there exists $X^* \in N_\theta$ such that $|X^* - Y(\tau^*)| \leqq \frac{\kappa}{4}$. Now X^* must belong to exactly one of Q_1, Q_2, Q_3. Since $Y(\tau^*) \in \partial\mathcal{P}_\kappa$, it follows that $X^* \notin Q_1$. If $X^* \in Q_3$, then since $Y(\tau^*) \in \partial P_\kappa$, there exists $X^{**} \in Q_1$ such that $|X^{**} - Y(\tau^*)| = \kappa$. Hence

$$\operatorname{dist}(Q_1, Q_3) \leqq |X^* - X^{**}| \leqq 2\kappa \leqq \tfrac{1}{2} \operatorname{dist}(Q_1, Q_3)$$

which is impossible. Thus, $X^* \in Q_2 \subset B_\delta(X_1)$. By the choice of κ, $Y(\tau^*) \in \overline{B_{2\delta}(X_1)}$. □

Remark. Our proof of Theorem 4.5 was inspired by an argument of PAYNE [P].

Corollary 4.7. $\operatorname{Det} DU(X) > 0$ *in* Ω_1.

Proof. If not, there are disks $B_\delta \subset \Omega_1$ and $B_\varepsilon \subset \mathcal{M}$ so that $\det DU < 0$ on B_δ and $B_\varepsilon \subset U(B_\delta)$. From Theorem 4.5, we have $\Omega_1 \cap U^{-1}(B_\varepsilon) \subset B_\delta$. Let $\rho \geqq 0$ be as in the definition of $\deg(U; \mathcal{M})$ with $\operatorname{supp} \rho \subset B_\varepsilon$ and $\int_{B_\varepsilon} \rho dU = 1$. Recalling that $\det DU = 0$ almost everywhere on $\Omega \setminus \Omega_1$, we find that

$$1 = \int_\Omega \rho(U(X)) \det DU(X) dX = \int_{B_\delta} \rho(U(X)) \det DU(X) dX < 0$$

which is impossible. □

§5. Positivity of the Jacobian

From §4, we have $\det DU \geqq 0$ almost everywhere in Ω. In this section, we prove that

$$\lim_{R \to 0} \fint_{B_R(X)} \det DU > 0 \quad \text{for all } X \in \Omega. \tag{5.1}$$

Definition. We say that V is a *relative minimizer* in a bounded open set $\mathcal{O}$ if $V \in W^{1,2m}(\mathcal{O})$ and

$$\int_\mathcal{O} F(DV) dX \leqq \int_\mathcal{O} F(D(V + \Phi)) dX$$

for all $\Phi \in W_0^{1,2m}(\mathcal{O}; \mathbb{R}^2)$.

We first use this to show

Lemma 5.1. *If V is a relative minimizer on $\mathcal{O}$ and* $\det DV = 0$ *almost everywhere on $\mathcal{O}$, then $V \in C^{2,\alpha}(\mathcal{O})$.*

Proof. The function V satisfies the system (3.1) in $\mathcal{O}$. But since $G_{,2}(|DV|^2, 0) \equiv 0$ in $\mathcal{O}$ by (1.7), we see that (3.1) reduces to the Euler equation for the functional

$$J(\Phi; \mathcal{O}) \equiv \int_{\mathcal{O}} G(|D\Phi|^2, 0)\, dX.$$

Since F satisfies (1.8) and the function $s \to G(s^2, 0)$ is strictly convex, a result of UHLENBECK implies that a local minimizer is classical (see [U] and [G-M]). Thus, V is of class $C^{2,\alpha}$ on $\mathcal{O}$. □

Next we use the partial regularity theory for minimizers to show that if $X \in \Omega \setminus \Omega_0$, then $⨍_{B_R(X)} \det DU$ is bounded away from zero.

Theorem 5.2. *Let $\Omega' \Subset \Omega$. There exists $\sigma(\Omega') > 0$ such that if $B_R(X) \subset \Omega'$ and*

$$⨍_{B_R(X)} \det DU \leqq \sigma,$$

then $X \in \Omega_0$.

Proof. We give an indirect argument. Assume there is a sequence $\{B_{R_n}(X_n)\}$ such that $B_{R_n}(X_n) \subset \Omega'$, $⨍_{B_{R_n}(X_n)} \det DU \leqq \frac{1}{n}$ and $X_n \notin \Omega_0$. Consider the normalized sequence of functions

$$U_n(X) = \frac{1}{R_n}[U(R_n X + X_n) - U(X_n)] \quad \text{for } X \in B_1.$$

By hypothesis, $\|DU\|_{\infty;\Omega'} \leqq M < \infty$. Thus, $\|DU_n\|_{\infty;B_1} \leqq M$ for all n. Each U_n is a relative minimizer in B_1. It follows from [E-G, Theorem 2] that for a subsequence, $U_n \rightharpoonup \widetilde{U}$ in $W^{1,2m}_{\text{loc}}(B_1; \mathbb{R}^2)$ where $\widetilde{U}$ is a relative minimizer. Moreover, $\det D\widetilde{U} \equiv 0$. To see this, note that

$$\lim_{n\to\infty} ⨍_{B_1} \det DU_n = \lim_{n\to\infty} ⨍_{B_{R_n}(X_n)} \det DU = 0.$$

Since $\det DU_n \geqq 0$ almost everywhere, we have $\det DU_n \to 0$ in $L^1(B_1)$. But $DU_n \rightharpoonup D\widetilde{U}$ in $L^2_{\text{loc}}(B_1)$, and hence $\det D\widetilde{U} = 0$ almost everywhere in B_1. Thus, from Lemma 5.1, $\widetilde{U} \in C^{2,\alpha}(B_{1/2}; \mathbb{R}^2)$.

We now invoke EVANS' partial regularity criterion. Let $\varepsilon(M)$ be as in (2.2). Since $\widetilde{U}$ is of class C^1, we may choose $0 < \tau < \frac{1}{2}$ such that

$$\fint_{B_\tau(0)} |D\widetilde{U} - (D\widetilde{U})_{0,\tau}|^{2m} < \varepsilon.$$

Since $U_n \to \widetilde{U}$ in $W^{1,2m}(B_\tau; \mathbb{R}^2)$, we see that

$$\fint_{B_\tau(0)} |DU_n - (DU_n)_{0,\tau}|^{2m} < \varepsilon$$

for n large enough. Thus,

$$\fint_{B_{\tau R_n}(X_n)} |DU - (DU)_{X_n,\tau R_n}|^{2m} < \varepsilon,$$

and it follows from (2.2) that $X_n \in \Omega_0$. This is a contradiction. □

It follows from Theorem 5.2 that (5.1) holds for all X in $\Omega \setminus \Omega_0$. To prove (5.1) for all X in Ω_0, we need the following lemma:

Lemma 5.3. *Let f and g be nonconstant and harmonic on an open connected set $\mathbb{O}$. Assume there is a function k such that*

$$\nabla f = k(x, y)\nabla g \tag{5.2}$$

for all (x, y) such that $\nabla g \neq 0$. Then $k \equiv$ constant and $\nabla f = k\nabla g$ in $\mathbb{O}$.

Proof. Since the zeroes of ∇g are isolated points in $\mathbb{O}$, it is sufficient to prove the lemma by assuming that $\nabla g \neq 0$ in $\mathbb{O}$. Taking the divergence of (5.2), we have $\Delta f = \nabla k \cdot \nabla g + k\Delta g$. Thus,

$$k_x g_x + k_y g_y = 0 \quad \text{in } \mathbb{O}.$$

Now taking the curl of (5.2), we obtain

$$k_y g_x - k_x g_y = 0 \quad \text{in } \mathbb{O}.$$

These two equations imply that $\nabla k \equiv 0$ in $\mathbb{O}$. □

Theorem 5.4. Det $DU > 0$ *in* Ω_0.

Proof. Assume that $0 \in \Omega_0$ and that $\det DU(0) = 0$. By a translation and rotation of the u, v variables, we can also assume that $u(0) = v(0) = 0$ and $\nabla v(0) = 0$. Now U is a classical solution to (3.1) in some disk $B_R(0) = B_R$.

From Lemma 3.1, u and v satisfy

$$(a^1_{ij}(X)u_{x_i})_{x_j} = 0 \quad \text{in } B_R,$$

$$(a^2_{ij}(X)v_{x_i})_{x_j} = 0 \quad \text{in } B_R.$$

By Corollary 3.2, neither u nor v can vanish identically on B_R. We split our argument into two parts.

Case I. Assume that $\nabla u(0) = 0$. Thus, $DU(0) = 0$, and it follows directly from (3.3) that $a^1_{ij}(0) = a^2_{ij}(0) = 2G,_1(0,0)\delta_{ij}$. We apply a result from [H-W] stating that there are positive integers l and n so that

$$\begin{aligned} u(X) &= f_l(X) + o(|X|^l), \\ \nabla u(X) &= \nabla f_l(X) + o(|X|^{l-1}), \\ v(X) &= g_n(X) + o(|X|^n), \\ \nabla v(X) &= \nabla g_n(X) + o(|X|^{n-1}) \end{aligned}$$

in B_R where f_l and g_n are nontrivial homogeneous harmonic polynomials of order l and n, respectively. We have

$$\det DU = \frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(f_l,g_n)}{\partial(x,y)} + o(|X|^{n+l-2}).$$

Since $\frac{\partial(u,v)}{\partial(x,y)} \geqq 0$ (by Corollary 4.7) and $\frac{\partial(f_l,g_n)}{\partial(x,y)}$ is a homogeneous polynomial of order $l+n-2$ in B_R, it follows that $\frac{\partial(f_l,g_n)}{\partial(x,y)} \geqq 0$.

Suppose that $l \neq n$. The components of ∇f_l and ∇g_n are of the form $(ar^{l-1}\cos(l-1)\theta + br^{l-1}\sin(l-1)\theta)$ and $(cr^{n-1}\cos(n-1)\theta + dr^{n-1}\sin(n-1)\theta)$, respectively. The integral of the product of such functions over $B_R(0)$ is zero. Thus, $\frac{\partial(f_l,g_n)}{\partial(x,y)} \equiv 0$. This implies that $\nabla f_l = k(x,y)\nabla g_n$ for (x,y) such that $\nabla g_n \neq 0$. From Lemma 5.3, k is constant. Thus, $f_l = kg_n$, which is impossible since f_l and g_n are of different order.

Assume now that $l = n$. From [H-W]

$$\begin{pmatrix} f_l \\ g_l \end{pmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{pmatrix} r^l\cos(l\theta) \\ r^l\sin(l\theta) \end{pmatrix}. \tag{5.3}$$

Let $C = [c_{ij}]$. We have $\frac{\partial(f_l,g_l)}{\partial(x,y)} = \det C \cdot l^2|X|^{2l-2}$. Thus $\det C \geqq 0$. If $\det C = 0$, we choose $A \in SO(2)$ such that $AC = \begin{bmatrix} b_{11} & b_{12} \\ 0 & 0 \end{bmatrix}$. If we set $\widetilde{U} = (\widetilde{u}, \widetilde{v})$ so that

$$\begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix},$$

then $\widetilde{U}$ is another solution with leading terms $\widetilde{f}_l$ and $\widetilde{g}_n$ such that the order of $\widetilde{f}_l$ equals l, and the order of $\widetilde{g}_n$ is greater than l. This is just as in the previous case and leads to a contradiction in the same manner.

If $\det C > 0$, then we have

$$\det DU = \det C \cdot l^2 |X|^{2l-2} + o(|X|^{2l-2}) \text{ in } B_R$$

where $l > 1$. Thus, $\det DU > 0$ in $B_r(0) \setminus \{0\}$ for r sufficiently small. Hence, $B_r \setminus \{0\} \subset \Omega_1$. By (5.3), (f_l, g_l) is an l-to-one map from $B_r \setminus \{0\}$ onto a punctured neighborhood of the origin with $l > 1$. Also,

$$r^{-l}|(f_l, g_l)| \geqq c > 0 \quad \text{and} \quad r^{-l}|(f_l - u, g_l - v)| = o(1) \quad \text{on} \quad \partial B_r \text{ as } r \to 0.$$

Thus, by degree theory for r sufficiently small, U is an l-to-one map on $B_r \setminus \{0\} \subset \Omega_1$. This contradicts Theorem 4.5.

Case II. Assume that $\nabla u(0) \neq 0$. Let L be a 2×2 matrix with $\det L > 0$. Set $X = LY$ and $\widehat{U}(Y) = U(LY)$. This can be done so that equation (3.2.2) transforms to $(\widehat{a}_{ij} \hat{v}_{y_i})_{y_j} = 0$ with $\widehat{a}_{ij}(0) = \delta_{ij}$. From [H-W],

$$\begin{aligned} \hat{u}(Y) &= \widehat{f}_1(Y) \quad + o(|Y|), \\ \nabla \hat{u}(Y) &= \nabla \widehat{f}_1(Y) + o(1), \\ \hat{v}(Y) &= \widehat{g}_n(Y) \quad + o(|Y|^n), \\ \nabla \hat{v}(Y) &= \nabla \widehat{g}_n(Y) + o(|Y|^{n-1}), \end{aligned}$$

where $n > 1$ since $\nabla v(0) = 0$. Again, $\widehat{g}_n$ is a homogeneous harmonic polynomial of order n and $\widehat{f}_1$ is linear (hence harmonic). We then arrive at a contradiction, just as in Case I. □

Remark. Our proof is similar to the one of LEWY [L-2] where he proves the nonvanishing of the Jacobian of a homeomorphism whose coordinate functions are harmonic.

Corollary 5.5. *For all X in Ω,* $\displaystyle\underline{\lim}_{R \to 0} \fint_{B_R(X)} \det DU > 0.$

Proof. If $X \in \Omega \setminus \Omega_0$, the assertion follows from Theorem 5.2. If $X \in \Omega_0$, this is just Theorem 5.4. □

We remark that by Corollary 5.5, we have $\Omega_0 = \Omega_1$.

§6. Univalent Minimizers

In this section, we apply the results of §§3, 4, and 5 to prove:

Theorem 6.1. *A minimizer U for $\mathcal{W}(\cdot)$ in $\mathcal{A}$ is a homeomorphism from $\overline{\Omega}$ onto $\overline{M}$ that is locally Lipschitz continuous in Ω and satisfies*

$$\varliminf_{R\to 0} \fint_{B_R(X)} \det DU > 0 \quad \textit{for all} \quad X \in \Omega. \tag{6.1}$$

Proof. By Lemma 3.3, U is a continuous mapping of $\overline{\Omega}$ onto $\overline{\mathcal{M}}$ and $U(\Omega) = \mathcal{M}$. Since $U(\partial\Omega) = \Psi(\partial\Omega) = \partial\mathcal{M}$, it suffices to prove that U is one-to-one from Ω onto $\mathcal{M}$.

Let $Y \in \mathcal{M}$. It follows from BALL [B, Theorem 1] and Corollary 5.5 that $U^{-1}(Y)$ is a compact connected subset of Ω. Suppose $U^{-1}(Y)$ contains more than one point. First, we point out that $U^{-1}(Y) \cap \Omega_0 = \emptyset$. Indeed, if not, let $X \in U^{-1}(Y) \cap \Omega_0 = U^{-1}(Y) \cap \Omega_1$. Since U is locally one-to-one near $X \in \Omega_1$, we deduce that $U^{-1}(Y)$ admits a separation, which is a contradiction.

Second, we remark that a connected set in $\mathbb{R}^2$ with more than one element has positive one-dimensional Hausdorff measure. Thus, there is a constant $l > 0$ such that $\mathcal{H}^1(U^{-1}(Y)) > l$.

For each $\varepsilon > 0$ we cover $U^{-1}(Y)$ by a finite family of disks with centers in $U^{-1}(Y)$, $\{B_\varepsilon(X_i)\}$, such that

$$\sum_i \chi_{B_\varepsilon(X_i)}(X) \leqq C, \tag{6.2}$$

where C is independent of ε. Note that

$$\sum_i |B_\varepsilon(X_i)| \geqq \tfrac{\pi}{2} l\varepsilon \tag{6.3}$$

for all ε sufficiently small.

Now, since U is locally Lipschitz continuous and $\operatorname{dist}(U^{-1}(Y), \partial\Omega) > 0$, we have

$$\|DU\|_{\infty;B_\varepsilon(X_i)} \leqq M < \infty$$

for all i and for all ε sufficiently small. Thus $U(B_\varepsilon(X_i)) \subset B_{M\varepsilon}(Y)$ for all i. From this and Corollary 5.5, it follows that

$$\int_{\cup_i B_\varepsilon(X_i)} \det DU\, dX = |U(\cup_i B_\varepsilon(X_i))| \leqq \pi M^2 \varepsilon^2.$$

On the other hand, since each $X_i \in \Omega \setminus \Omega_0$, it follows from Theorem 5.2 that there exists $\sigma > 0$ (independent of ε for ε small enough) so that

$$\int_{B_\varepsilon(X_i)} \det DU\, dX \geqq \sigma |B_\varepsilon(X_i)|.$$

Hence, using (6.2) and (6.3), we have

$$\pi M^2\varepsilon^2 \geqq \int_{\cup B_\varepsilon(X_i)} \det DU dX \geqq \frac{1}{C}\sum_i \int_{B_\varepsilon(X_i)} \det DU dX$$
$$\geqq \frac{\sigma}{C}\sum_i |B_\varepsilon(X_i)| \geqq \frac{\pi\sigma l}{2C}\varepsilon,$$

for all ε sufficiently small. This is a contradiction. □

We now show that we can adjust $F(P)$ so that we can treat approximate energies from nonlinear elasticity. Let $H \in C^3(\mathbb{R})$, $H \geqq 0$, H be convex and assume that

$$\lim_{d\to\pm\infty} |d|^{1-m}(|H'(d)| + |dH''(d)|) = 0.$$

Let $F(P)$ satisfy (1.2)–(1.8) and set

$$\widehat{F}(P) = F(P) + H(\det P).$$

Theorem 6.2. *Assume that $m > 2$. A minimizer for*

$$\widehat{\mathcal{W}}(V) = \int_\Omega \widehat{F}(DU)dX$$

in $\mathcal{A}$ is a homeomorphism from $\overline{\Omega}$ onto $\overline{\mathcal{M}}$ that is locally Lipschitz continuous in Ω and satisfies (6.1).

Proof. Consider the functional

$$\mathcal{N}(V) = \int_\Omega \det DV dX \quad \text{for} \quad v \in \mathcal{A}.$$

Then $\mathcal{N}(V)$ is constant on $\mathcal{A}$ (cf. [D]). Thus for $V \in \mathcal{A}$, the functional

$$\hat{\widehat{\mathcal{W}}}(V) = \int_\Omega (\widehat{F}(DV) - H'(0)\det DV)dX$$

differs from $\widehat{\mathcal{W}}(V)$ by a constant. Moreover, since $m > 2$, its integrand satisfies (1.2)–(1.8). As a result, $\hat{\widehat{\mathcal{W}}}(\cdot)$ and $\widehat{\mathcal{W}}(\cdot)$ have the same set of minimizers in $\mathcal{A}$, and we can apply Theorem 6.1 to any minimizer for $\hat{\widehat{\mathcal{W}}}$. □

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