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# $L^2$-Decay for Navier-Stokes Flows in Unbounded Domains, with Application to Exterior Stationary Flows

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## 1. Introduction

The motion of a viscous incompressible fluid filling a domain $D \subset R^n$ is governed by the Navier-Stokes initial value problem:

$$\begin{aligned} \frac{\partial u}{\partial t} + u \cdot \nabla u &= \Delta u - \nabla p + f \qquad (x \in D,\ t > 0), \\ \nabla \cdot u &= 0 \qquad\qquad\qquad\quad (x \in D,\ t \geqq 0), \\ u|_S = 0; &\quad u|_{t=0} = a \end{aligned} \tag{NS}$$

for unknown velocity $u = (u_j)_{j=0}^n$ and pressure $p$. Here $S$ is the boundary of $D$, $x = (x_1, \ldots, x_n)$ is a point of $R^n$, $a$ and $f$ denote, respectively, the given initial velocity and external force; and $u \cdot \nabla u = \Sigma_j u_j \partial_j u_j$, $\nabla \cdot u = \Sigma_j \partial_j u_j$, $\nabla p = (\partial_j p)_{j=1}^n$, $\partial_j = \partial / \partial x_j$. The fluid density and the kinematic viscosity are normalized to be unity. It is known [16] that problem (NS) possesses at least one weak solution for an arbitary initial velocity $a$ in $L^2$. Uniqueness of weak solutions has only been proved for $n = 2$.

In this paper we study the existence of a weak solution, in an arbitrary unbounded domain, which goes to zero in $L^2$ as $t \to \infty$, with explicit rates. The $L^2$-decay problem for Navier-Stokes flows was first posed by LERAY [14] in case $D = R^3$. The first (affirmative) answer was given by KATO [13] in case $D = R^n$, $n = 3, 4$, through his study of strong solutions in general $L^p$ spaces. A different approach was then taken by SCHONBEK [20], which is based on the Fourier decomposition for the fluid velocity $u$; see also [12, 21, 23]. The idea of SCHONBEK was then applied by us [2, 3] to the case where $D$ is a half-space of $R^n$, $n \geqq 2$, or an exterior domain of $R^n$, $n \geqq 3$. In this paper we first show, in Sections 2 and 3, that the method developed in [2, 3, 12] can be modified so that it applies to the case of an arbitrary unbounded domain in $R^n$, $n \leqq 4$. The arguments developed in Sections 2 and 3 are then applied in Section 4 to the stability problem for exterior stationary flows in three dimensions.

To state our main results, we use the standard notation: $C^\infty_{0,\sigma}(D)$ denotes the set of smooth and compactly supported solenoidal vector fields on $D$. We denote by $H$ and $V$ the $L^2$- and $H^1$-closures of $C^\infty_{0,\sigma}(D)$. The orthogonal decomposition:

$$(L^2(D))^n = H \oplus H^\perp, \quad H^\perp = \{\nabla p;\, p \in L^2_{\mathrm{loc}}(D)\}$$

is well known [22, Chap. I]. We denote by $P$ the associated orthogonal projector onto $H$. With the bilinear form $(\nabla u, \nabla v)$ defined on $V \times V$, we associate a (unique) positive and self-adjoint operator $A$ in $H$ such that $D(A^{1/2}) = V$ and $\|A^{1/2}u\|_2 = \|\nabla u\|_2$, where $\|\cdot\|_r$ $(1 \leqq r \leqq \infty)$ is the usual $L^r$-norm. By $\hat{V}$ we denote the completion of $C^\infty_{0,\sigma}(D)$ in the norm $\|\nabla \cdot\|_2$, and by $\hat{V}^*$ its dual space. For simplicity in notation we assume that $f = Pf$, using the above orthogonal decomposition. Then the function

$$v(t) = e^{-tA}a + \int_0^t e^{-(t-s)A} f(s)\, ds$$

with $a \in H$ and $f \in L^2_{\mathrm{loc}}([0, \infty); H)$ satisfies the nonstationary Stokes system

$$\begin{aligned} \frac{\partial v}{\partial t} - \Delta v &= f - \nabla q \qquad (x \in D,\, t > 0), \\ \nabla \cdot v &= 0 \qquad (x \in D,\, t \geqq 0), \\ v|_S = 0; &\quad v|_{t=0} = a \end{aligned} \tag{S}$$

with an appropriate scalar distribution $q$; so the problem (NS) is formally transformed into the integral equation:

$$u(t) = e^{-tA}a + \int_0^t e^{-(t-s)A}[f(s) - P(u \cdot \nabla)\, u(s)]\, ds. \tag{I}$$

Given $a$ and $f$ as above, a weakly continuous function $u: [0, \infty) \to H$ is called a weak solution of (NS) (or equivalently of (I)) if it belongs to $L^\infty(0, T; H) \cap L^2(0, T; V)$ for all $T > 0$, and satisfies $u(0) = a$ and

$$\begin{aligned} (u(t), \phi(t)) &+ \int_s^t [(\nabla u, \nabla \phi) + (u \cdot \nabla u, \phi)]\, d\tau \\ &= (u(s), \phi(s)) + \int_s^t [(u, \phi') + (f, \phi)]\, d\tau \end{aligned} \tag{W}$$

for all $t \geqq s \geqq 0$, and all $\phi \in C([0, \infty); V) \cap C^1([0, \infty); H)$, where $(\cdot, \cdot)$ is the standard $L^2$-inner product and $\phi' = \partial\phi/\partial t$. The existence of a weak solution corresponding to arbitrary $a$ and $f$ is well known; see, e.g., [16]. All the weak solutions obtained so far satisfy the energy inequality

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u\|_2^2\, ds \leqq \|a\|_2^2 + 2\int_0^t (f, u)\, ds$$

for all $t \geqq 0$, and the equality sign holds in case $n = 2$. In Section 4 we shall deal with a more stringent form of the above energy inequality. Our main results are the following:

**Theorem 1.1.** *Let $n = 3, 4$, and let $D$ be an arbitrary $n$-dimensional unbounded domain for which the Poincaré inequality for functions in $C_0^\infty(D)$ may not hold. If $a \in H$, if $f \in L^2_{\mathrm{loc}}([0, \infty); H) \cap L^1(0, \infty; H) \cap L^1(0, \infty; \hat{V}^*)$, and if*

$$\int_0^\infty t \| f(t) \|_2 \, dt < +\infty,$$

*then there is a weak solution $u$ of* (NS) *such that*

(i) $\|u(t)\|_2 \to 0$ *as* $t \to \infty$.

(ii) *If* $\|e^{-tA}a\|_2 = O(t^{-\alpha})$ *for some* $\alpha > 0$, *then*

$$\|u(t)\|_2 = \begin{cases} O(t^{-\alpha}) & \text{if } \alpha < 1/2, \\ O(t^{\varepsilon - 1/2}) & \text{if } \alpha \geqq 1/2 \end{cases}$$

*where* $0 < \varepsilon < 1/2$ *in case* $n = 3$, *and* $\varepsilon = 0$ *in case* $n = 4$.

**Theorem 1.2.** *Let $D \subset R^2$ be an arbitrary unbounded domain for which the Poincaré inequality may not hold. Given $a \in H$ and $f$ as in Theorem 1.1, there is a unique weak solution $u$ such that*

(i) $\|u(t)\|_2 \to 0$ *as* $t \to \infty$.

(ii) *If* $\|e^{-tA}a\|_2 = O(t^{-\alpha})$ *for some* $\alpha > 0$, *then*

$$\|u(t)\|_2 = O((\log(t + e))^{-m/2})$$

*for all integers* $m \geqq 1$.

(iii) *If* $\|a\|_2$ *and* $\int_0^\infty \|f\|_2 \, ds$ *are small enough and, moreover,* $f \in L^2(0, \infty; \hat{V}^*)$, *then*

$$\|u(t)\|_2 = \begin{cases} O(t^{-\alpha}) & \text{if } \alpha < 1/2, \\ O(t^{\varepsilon - 1/2}) & \text{if } \alpha \geqq 1/2 \end{cases}$$

*where* $\varepsilon > 0$ *is arbitrary.*

(iv) *If* $a \in R(A^\alpha)$ *for some* $0 < \alpha \leqq 1/2$, *then assertion* (iii) *holds irrespectively of the size of* $\|a\|_2$ *and* $\int_0^\infty \|f\|_2 \, ds$.

When $D$ is the entire space $R^n$ or the half space $R^n_+$, $n \geqq 2$, and $f = 0$, it is known [2, 12, 21, 23] that there exists a weak solution $u$ satisfying

$$\|u(t)\|_2 = \begin{cases} O(t^{-\alpha}) & \text{for } \alpha < (n+2)/4, \\ O(t^{-(n+2)/4}) & \text{for } \alpha \geqq (n+2)/4 \end{cases}$$

provided that $\|e^{-tA}a\|_2 = O(t^{-\alpha})$ for some $\alpha > 0$. When $n \geqq 3$, $f = 0$, and $D$ is an exterior domain with smooth boundary, we have recently established in [3] the existence of a weak solution $u$ such that, under the assumption $\|e^{-tA}a\|_2 = O(t^{-\alpha})$,

$$\|u(t)\|_2 = \begin{cases} O(t^{-\alpha}) & \text{if } \alpha < n/4, \\ O(t^{\varepsilon - n/4}) & \text{if } \alpha \geqq n/4 \end{cases}$$

for any $0 < \varepsilon < 1/4$. All these results are deduced by essential use of various properties of the operator $A$ in general $L^r$ spaces. In our present case, however, the class of domains $D$ is so large that we cannot appeal to $L^r$-theories. So we restrict ourselves to the case of space dimensions $n \leqq 4$ and deduce our results by applying only the $L^2$-theory of the operator $A$. We note that Theorem 1.2 partially extends our previous result in [3] to the case of two-dimensional exterior domains (with nonsmooth boundaries).

We prove Theorem 1.1 in Section 2, using a specific approximation scheme. Since the uniqueness of weak solutions remains open in case $n \geqq 3$, we first consider in Section 2 the decay problem for general weak solutions satisfying the energy inequality and show that the time-average $t^{-1}\int_0^t \|u\|_2\,ds$ of any such weak solution $u$ decays in the same way as stated in Theorem 1.1; see Theorem 2.1. It turns out that Theorem 1.1 immediately follows from Theorem 2.1.

Our proof of Theorems 1.1 and 2.1 does not work in the two-dimensional case. So in Section 3 we give a detailed proof of Theorem 1.2 which uses the spectral decomposition for the self-adjoint operator $A$. This approach was first suggested by SCHONBEK [20] and then systematically studied in [2, 3, 12]. It is also possible to prove Theorem 1.1 by using the spectral decomposition. However, we do not employ this method, since our argument in Section 2 provides Theorem 2.1, which is difficult to obtain by applying the spectral decomposition.

In both of Theorems 1.1 and 1.2, it is in general difficult to characterize completely the class of functions $a \in H$ satisfying the condition $\|e^{-tA}a\|_2 = O(t^{-\alpha})$. In Section 3 we show that this condition holds for $a$ in some $L^r$ spaces. This result is deduced from the fact that the range $R(A^\alpha)$ of the fractional power $A^\alpha$ remains invariant under the Navier-Stokes flow if $\alpha > 0$ satisfies an appropriate condition depending on the space dimension; this invariance property not only enables us to prove assertion (iv) of Theorem 1.2, but also implies the following

**Corollary 1.3.** (i) *If $n = 2$ and $a \in H \cap (L^r(D))^2$ for some $1 < r < 2$, then the corresponding weak solution $u$ satisfies*

$$\|u(t)\|_2 = O(t^{-(1/r-1/2)})$$

*provided that $f$ satisfies the assumption in Theorem 1.2.(iii).*
(ii) *If $n = 3$ and $a \in H \cap (L^r(D))^3$ for $6/5 \leqq r \leqq 3/2$, then there is a weak solution $u$ with $u(0) = a$ such that*

$$\|u(t)\|_2 = O(t^{-3(1/r-1/2)/2}).$$

(iii) *If $n = 4$ and $a \in H \cap (L^{4/3}(D))^4$, then there is a weak solution $u$ with $u(0) = a$ such that*

$$\|u(t)\|_2 = O(t^{-1/2}).$$

The problem of $L^2$-decay for Navier-Stokes flows is closely connected with the notion of *energy stability* in viscous fluid motions (see [6]). Indeed, Theorems 1.1 and 1.2 assert in particular that the trivial steady state $u = 0$ is globally asymptotically stable in this sense in arbitrary unbounded domains. In Section 4 we apply the method of proof of Theorems 1.1 and 1.2 to the stability problem for exterior stationary flows in three dimensions. We prove that an exterior stationary flow is globally asymptotically stable in the energy sense provided that the associated Reynolds number is small enough. See Theorem 4.2. This result improves and supplements the known results as given for instance in [6, 8, 9, 10, 15]. A novel feature of our result is that we deal with global $L^2$-norms of disturbances and deduce their explicit decay rates. However, we believe that our result in this section is not the optimal one.

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## 2. Proof of Theorem 1.1

First we deal with general weak solutions satisfying the energy inequality

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u\|_2^2\, ds \leqq \|a\|_2^2 + 2\int_0^t (f, u)\, ds \tag{E}$$

for all $t \geqq 0$.

**Theorem 2.1.** *Let the assumptions in Theorem 1.1 be satisfied. Let u be any weak solution satisfying the energy inequality* (E). *Then*
(i) $t^{-1}\int_0^t \|u\|_2\, ds \to 0$ *as* $t \to \infty$.
(ii) *If* $\|e^{-tA}a\|_2 = O(t^{-\alpha})$ *for some* $\alpha > 0$, *then*

$$\frac{1}{t}\int_0^t \|u\|_2\, ds = \begin{cases} O(t^{-\alpha}) & \text{if } \alpha < 1/2, \\ O(t^{\varepsilon - 1/2}) & \text{if } \alpha \geqq 1/2, \end{cases}$$

*where* $0 < \varepsilon < 1/2$ *in case* $n = 3$, *and* $\varepsilon = 0$ *in case* $n = 4$.

For the proof we prepare three lemmas.

**Lemma 2.2.** *Let* $L_w^p = L_w^p(R)$, $1 < p < \infty$, *denote the Banach space of measurable functions f on the real line R with norm*

$$\|f\|_{p,w} \equiv \sup_E |E|^{-1+1/p} \int_E |f|\, ds < \infty$$

*where* $|E|$ *is the Lebesgue measure of a measurable set E.*
(i) *If* $f \in L_w^p$, $g \in L_w^q$ *and* $1/p + 1/q = 1/r$, *then* $fg \in L_w^r$, *and*

$$\|fg\|_{r,w} \leqq C\|f\|_{p,w}\|g\|_{q,w}$$

*with* $C > 0$ *depending only on p and q.*
(ii) *If* $f \in L_w^p$, $g \in L_w^q$ *and* $1/p + 1/q = 1 + 1/r$, *then the convolution* $f * g$ *is in* $L_w^r$ *and there is a constant* $C > 0$ *depending only on p and q so that*

$$\|f * g\|_{r,w} \leqq C\|f\|_{p,w}\|g\|_{q,w}.$$

(iii) *If* $f \in L^p_w$ *and* $g \in L^1$, *then* $f * g \in L^p_w$, *and*

$$\| f * g \|_{p,w} \leqq \| f \|_{p,w} \| g \|_1 .$$

Lemma 2.2 (i) is the weak version of Hölder's inequality, while (ii) and (iii) are the weak versions of Young's inequality.

**Proof.** Statement (ii) is proved in [19, p. 32], and (iii) is easily deduced by using the definition of the norm $\| f \|_{p,w}$ given above. So we here prove only (i). First observe that $f$ is in $L^p_w$ if and only if

$$\| f \|^*_{p,w} \equiv \sup_{t>0} t \, |E(|f| > t)|^{1/p} < +\infty,$$

where $E(|f| > t) = \{s \in R; |f(s)| > t\}$, and that

$$\| f \|^*_{p,w} \leqq \| f \|_{p,w} \leqq \frac{p}{p-1} \| f \|^*_{p,w}$$

as shown for instance in [7, p. 585]. Applying the classical Young inequality:

$$|fg| \leqq \frac{r}{p} \varepsilon^{p/r} |f|^{p/r} + \frac{r}{q} \varepsilon^{-q/r} |q|^{q/r}$$

for any $\varepsilon > 0$, we get

$$E(|fg| > t) \subset E(|f| > c_1 \varepsilon^{-1} t^{r/p}) \cup E(|g| > c_2 \varepsilon t^{r/q})$$

with $c_1$ and $c_2$ depending only on $p$ and $q$. Direct calculation thus gives

$$(\| fg \|^*_{r,w})^r \leqq C_1 \varepsilon^p (\| f \|^*_{p,w})^p + C_2 \varepsilon^{-q} (\| q \|^*_{q,w})^q$$

for all $\varepsilon > 0$, where $C_1$ and $C_2$ depend only on $p$ and $q$. The result now follows by taking the minimum with respect to $\varepsilon > 0$.

**Lemma 2.3.** *Let* $f \geqq 0$ *be a measurable function on* $R$. *Suppose there exist constants* $M > 0$, $C > 0$ *and* $p > q > 1$ *so that* $0 \leqq f \leqq M$ *and*

$$\int_E f \, ds \leqq C(|E|^{1-1/p} + |E|^{1-1/q})$$

*for all measurable subsets* $E$. *Then there is another constant* $C' > 0$ *such that*

$$\int_E f \, ds \leqq C' |E|^{1-1/p}$$

*for all measurable* $E$.

**Proof.** Since $1 - 1/p > 1 - 1/q$, the result is obvious for $E$ with $|E| \geqq 1$. So we may assume $|E| < 1$. Then, since

$$\int_E f \, ds \leqq 2C |E|^{1-1/q},$$

Hölder's inequality yields, with $\theta = 1 - q/p$,

$$\int_E f\,ds \leqq M^\theta \int_E f^{1-\theta}\,ds \leqq M^\theta |E|^\theta \left[\int_E f\,ds\right]^{1-\theta}$$

$$\leqq M^\theta (2C)^{1-\theta} |E|^{1-1/p},$$

which completes the proof.

**Lemma 2.4.** *Let $n = 3, 4$. Then for all $v \in V$ and $w \in H^1(R^n)$ with $\nabla \cdot w = 0$,*

$$\| e^{-tA} P(w \cdot \nabla) v \|_2 \leqq C t^{-1/2} (\| w \|_2 \, \| v \|_2)^{1-n/4} \, (\| \nabla w \|_2 \, \| \nabla v \|_2)^{n/4}.$$

**Proof.** Let $\phi \in C_{0,\sigma}^\infty(D)$. Since

$$\| \nabla e^{-tA} \phi \|_2 = \| A^{1/2} e^{-tA} \phi \|_2 \leqq t^{-1/2} \| \phi \|_2,$$

direct calculation gives

$$|(e^{-tA} P(w \cdot \nabla) v, \phi)| = |(v, w \cdot \nabla e^{-tA} \phi)| \leqq \| v \|_4 \, \| w \|_4 \, \| \nabla e^{-tA} \phi \|_2$$

$$\leqq t^{-1/2} \| v \|_4 \, \| w \|_4 \, \| \phi \|_2.$$

The result follows by applying the Sobolev inequality

$$\| f \|_4 \leqq C \| f \|_2^{1-n/4} \, \| \nabla f \|_2^{n/4}.$$

**Proof of Theorem 2.1.** First observe that the energy inequality (E) gives

$$\| u(t) \|_2^2 + 2 \int_0^t \| \nabla u \|_2^2 \, ds \leqq \| a \|_2^2 + \int_0^t \| f \|_2 \, (1 + \| u \|_2^2) \, ds.$$

Applying Gronwall's Lemma yields

$$\| u(t) \|_2^2 + 2 \int_0^t \| \nabla u \|_2^2 \, ds \leqq \left( \| a \|_2^2 + \int_0^\infty \| f \|_2 \, ds \right) \exp \left( \int_0^\infty \| f \|_2 \, ds \right).$$

Thus, $\| u \|_2 \in L^\infty$ and $\| \nabla u \|_2^2 \in L^1$. Now, substituting $\phi(\tau) = e^{-(t-\tau)A} \psi$ with $\psi \in C_{0,\sigma}^\infty(D)$ into (W) with $s = 0$, we obtain

$$(u(t), \psi) = (e^{-tA} a, \psi) - \int_0^t (u \cdot \nabla u(\tau), e^{-(t-s)A} \psi) \, ds + \int_0^t (f, e^{-(t-s)A} \psi) \, ds.$$

We apply Lemma 2.4 to estimate the nonlinear term to get

$$\| u(t) \|_2 \leqq \| e^{-tA} a \|_2 + C \int_0^t (t-s)^{-1/2} \, (\| u \|_2^{1/2} \, \| \nabla u \|_2^{3/2} + \| f \|_{\hat{V}^*}) \, ds \tag{2.1}$$

when $n = 3$, and

$$\| u(t) \|_2 \leqq \| e^{-tA} a \|_2 + C \int_0^t (t-s)^{-1/2} \, (\| \nabla u \|_2^2 + \| f \|_{\hat{V}^*}) \, ds \tag{2.2}$$

when $n = 4$. From now on we regard $\|u\|_2$ and $\|\nabla u\|_2$ as defined to be zero on the negative real axis. Since $\|\nabla u\|_2^2 \in L^1$, it follows from Lemma 2.2 that

$$\frac{1}{t}\int_0^t \|u\|_2\, ds \leqq \frac{1}{t}\int_0^t \|e^{-sA}a\|_2\, ds + C(t^{-\beta} + t^{-1/2}) \tag{2.3}$$

where $\beta = 1/4$ when $n = 3$, and $\beta = 1/2$ when $n = 4$. This proves (i) as well as (ii) with $n = 4$. To prove (ii) with $n = 3$, we systematically apply Lemmas 2.2 and 2.3. First observe that (2.3) shows the result for $\alpha < 1/4$. When $\alpha \geqq 1/4$, (2.1) shows that $\|u\|_2$ is bounded from above by a function belonging to $L_w^{1/\alpha} + L_w^4 + L_w^2$. Since $\|u\|_2 \in L^\infty$, Lemma 2.3 implies that $\|u\|_2 \in L_w^4$, and so $\|u\|_2^{1/2} \in L_w^8$. Thus, by Lemma 2.2 (i), $\|u\|_2^{1/2}\|\nabla u\|_2^{3/2} \in L_w^p$ with $1/p = 1/8 + 3/4$. Since $1/2 + 1/p = 1 + 1/4 + 1/8$, Lemma 2.2 (ii) implies

$$\frac{1}{t}\int_0^t \|u\|_2\, ds \leqq C(t^{-\alpha} + t^{-1/q} + t^{-1/2})$$

with $1/q = 1/4 + 1/8$, and this shows the result for $\alpha < 1/q$. When $\alpha \geqq 1/q$, (2.1) shows that $\|u\|_2$ is bounded from above by a function in $L_w^{1/\alpha} + L_w^q + L_w^2$, so $\|u\|_2 \in L_w^q$ by Lemma 2.3. Thus, the same argument as above gives

$$\int_0^t (t-s)^{-1/2}\, \|u\|_2^{1/2}\, \|\nabla u\|_2^{3/2}\, ds \in L_w^r$$

with $1/r = 1/4 + 1/8 + 1/16$. Hence

$$\frac{1}{t}\int_0^t \|u\|_2\, ds \leqq C(t^{-\alpha} + t^{-1/r} + t^{-1/2}),$$

and this proves the result for $\alpha < 1/r$. Repeating these processes eventually yields the desired result. The proof is complete.

**Proof of Theorem 1.1.** We first construct approximate solutions of (NS), for $n = 3, 4$, by solving

$$u_k(t) = e^{-tA}a_k - \int_0^t e^{-(t-s)A}(P(\bar{u}_k\cdot\nabla)\, u_k - f_k)\,(s)\, ds, \qquad k = 1, 2, \ldots, \tag{IE}$$

where $a_k = (I + k^{-1}A)^{-1}a$, $f_k = (I + k^{-1}A)^{-1}f$, and $\bar{u}_k = J_{1/k}\tilde{u}_k$ is the spatial mollification of the zero-extension $\tilde{u}_k$ of $u_k$. The unique solvability of (IE) in the space $C([0, T]; V)$ as well as the fact that $u_k$ satisfies, in $L^2(0, T; D(A)) \cap W^{1,2}(0, T; H)$, the problem

$$\frac{du_k}{dt} + Au_k + P(\bar{u}_k\cdot\nabla)\, u_k = f_k, \qquad \text{a.e. } t > 0;\ u_k(0) = a_k \tag{2.4}$$

can be shown as in [2, 18]. From (2.4) we get

$$\|u_k(t)\|_2^2 + 2\int_s^t \|\nabla u_k\|_2^2\, d\tau = \|u_k(s)\|_2^2 + 2\int_s^t (f_k, u_k)\, d\tau \tag{2.5}$$

for $t \geqq s \geqq 0$. Upon taking $s = 0$ and using $\| a_k \| \leqq \| a \|_2$, we obtain from (2.5) that

$$u_k \text{ is bounded in } L^\infty(0, T; H) \cap L^2(0, T; V).$$

Hence we may assume that a subsequence of $u_k$ converges weakly-star in $L^\infty(0, T; H)$ and weakly in $L^2(0, T; V)$. Moreover, a standard argument ([22, Chap. III]) can be applied to show that if we define $v_k(t) = u_k(t)$ for $t \in (0, T)$ and $v_k(t) = 0$ otherwise, then the fractional derivatives $D_t^\gamma v_k$, defined via the Fourier transform of $v_k(t)$ in $t$, remain bounded in $L^2(R; H)$ provided $0 < \gamma < 1/4$. We thus conclude that a subsequence, again denoted $u_k$, converges in $L^2_{\text{loc}}([0, T] \times D)$ to a function $u$, and it is readily seen (cf. [16]) that the limit function $u$ is a weak solution of (NS). Notice that the above argument implies that

$$\| u(t) \|_2 \leqq \liminf_{k \to \infty} \| u_k(t) \|_2$$

for a.e. $t > 0$, so we need only show that $u_k(t)$ decays in $L^2$ as indicated in Theorem 1.1 *uniformly in k.*

Now the energy equality (2.5) implies that

$$\| u_k(t) \|_2^2 + 2 \int_s^t \| \nabla u_k \|_2^2 \, d\tau \leqq \| u_k(s) \|_2^2 + \int_s^t \| f \|_2 \, (1 + \| u_k \|_2^2) \, d\tau,$$

so that, by Gronwall's Lemma,

$$\| u_k(t) \|_2^2 + 2 \int_s^t \| \nabla u_k \|_2^2 \, d\tau \leqq C \left( \| u_k(s) \|_2^2 + \int_s^t \| f \|_2 \, d\tau \right),$$

where $C = \exp \left( \int_0^\infty \| f \|_2 \, ds \right)$. Hence $\int_0^\infty \| \nabla u_k \|_2^2 \, ds$ is bounded uniformly in $k$, and

$$\| u_k(t) \|_2 \leqq C \left[ \| u_k(t) \|_2 + \left( \int_s^t \| f \|_2 \, d\tau \right)^{1/2} \right].$$

Integrating this in $s \in (0, t)$ gives

$$\begin{aligned} \| u_k(t) \|_2 &\leqq C t^{-1} \int_0^t \| u_k \|_2 \, ds + C t^{-1/2} \left[ \int_0^t \tau \, \| f(\tau) \|_2 \, d\tau \right]^{1/2} \\ &\leqq C t^{-1} \int_0^t \| u_k \|_2 \, ds + C t^{-1/2}. \end{aligned} \tag{2.6}$$

On the other hand, applying Lemma 2.3 to (IE) gives

$$\| u_k(s) \|_2 \leqq \| e^{-sA} a \|_2 + \int_0^s (s - \tau)^{-1/2} \, (\| \nabla u_k \|_2^2 + \| f \|_{\hat{V}^*}) \, d\tau \tag{2.7}$$

when $n = 4$, and

$$\| u_k(s) \|_2 \leqq \| e^{-sA} a \|_2 + \int_0^s (s - \tau)^{-1/2} \, (\| u_k \|_2^{1/2} \, \| \nabla u_k \|_2^{3/2} + \| f \|_{\hat{V}^*}) \, d\tau \tag{2.8}$$

when $n = 3$. Note that we have used $\| e^{-tA} a_k \|_2 \leqq \| e^{-tA} a \|_2$, $\| \bar{u}_k \|_2 \leqq \| u_k \|_2$, and $\| \nabla \bar{u}_k \|_2 \leqq \| \nabla u_k \|_2$. Since $\| u_k \|_2 \in L^\infty$ and $\| \nabla u_k \|_2^2 \in L^1$ uniformly in $k$, we get the desired result by combining (2.6) with (2.7) or (2.8) and applying the reasoning in the proof of Theorem 2.1. This completes the proof of Theorem 1.1.

Theorem 2.1 (i) can be slightly improved. Indeed, one can prove

**Corollary 2.5.** *Let* $n = 3, 4$. *Then, for every weak solution* $u$ *satisfying* (E) *there holds*

$$\lim_{t \to \infty} \int_t^{t+1} \| u \|_2 \, ds = 0.$$

For simplicity we treat only the case $n = 3$. Inserting $\phi(\tau) = e^{-(t-\tau)A} \psi$, $\psi \in C_{0,\sigma}^\infty(D)$, into (W) gives

$$(u(t), \psi) = (e^{-(t-s)A} u(s), \psi) - \int_s^t (u \cdot \nabla u(\tau), e^{-(t-\tau)A} \psi) \, d\tau$$
$$+ \int_s^t (f, e^{-(t-\tau)A} \psi) \, d\tau$$

for $0 \leqq s \leqq t$. So we get, as in the proof of Theorem 2.1,

$$\| u(t) \|_2 \leqq \| e^{-(t-s)A} u(s) \|_2 + C \int_s^t (t - \tau)^{-1/2} (\| u \|_2^{1/2} \| \nabla u \|_2^{3/2} + \| f \|_{\hat{p}^*}) \, d\tau$$

with $C$ independent of $s \geqq 0$. We regard every function of $\tau$ as defined to be zero for $\tau < s$. Since $\| \nabla u \|_2^{3/2} \| u \|_2^{1/2} \in L^{4/3} \subset L_w^{4/3}$, it follows from the weak Young inequality that the integral on the right-hand side belongs to $L_w^4 + L_w^2$ and the norm is bounded by

$$C \left[ \int_s^\infty \| \nabla u \|_2^2 \, d\tau \right]^{3/4} + C \int_s^\infty \| f \|_{\hat{p}^*} \, d\tau$$

with $C$ independent of $s > 0$. For an arbitrary $\varepsilon > 0$ we fix $s > 0$ so that the above quantity is less than $\varepsilon$. Then, by the definition of the $L_w^p$-norm we get

$$\int_t^{t+1} \| u \|_2 \, d\tau \leqq \int_t^{t+1} \| e^{-(\tau-s)A} u(s) \|_2 \, d\tau + \varepsilon$$

for all $t > s$. Since $\lim_{t \to \infty} \| e^{-tA} a \|_2 = 0$ because of the injectivity of $A$, application of the Bounded Convergence Theorem gives

$$\limsup_{t \to \infty} \int_t^{t+1} \| u \|_2 \, d\tau \leqq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this proves the corollary.

## 3. Proof of Theorem 1.2

In this section we first prove Theorem 1.2, using the spectral decomposition of the positive self-adjoint operator $A$. To prove assertion (iv) of Theorem 1.2, we establish the invariance of some of the ranges $R(A^\alpha)$ of the fractional powers $A^\alpha$. As a by-product we obtain Corollary 1.3. The argument below originates from those of [12, 21, 23] and, as in Section 2, relies on the following

**Lemma 3.1.** (i) *Let* $A = \int_0^\infty \lambda \, dE_\lambda$ *be the spectral decomposition of* $A$. *If* $n = 2$, *then for* $v \in V$,

$$\|E_\lambda P(v \cdot \nabla) v\|_2 \leqq C\lambda^{1/2} \|v\|_2 \|\nabla v\|_2, \quad \lambda > 0.$$

(ii) *Under the same assumption as in* (i),

$$\|e^{-tA} P(v \cdot \nabla) u\|_2 \leqq Ct^{-1/2} \|v\|_2 \|\nabla v\|_2, \quad t > 0.$$

**Proof.** By the definition of $P$ and $E_\lambda$ we easily see that, for $\phi \in C_{0,\sigma}^\infty(D)$,

$$|(E_\lambda P(v \cdot \nabla) v, \phi)| = |(v, v \cdot \nabla E_\lambda \phi)| \leqq \|v\|_4^2 \|\nabla E_\lambda \phi\|_2$$
$$= \|v\|_4^2 \|A^{1/2} E_\lambda \phi\|_2 \leqq \lambda^{1/2} \|v\|_4^2 \|\phi\|_2.$$

Applying the Sobolev inequality

$$\|f\|_4 \leqq C \|f\|_2^{1/2} \|\nabla f\|_2^{1/2}$$

to the last term yields the desired estimate. This proves (i). Assertion (ii) is proved in the same way as in Lemma 2.4. The proof is complete.

**Proof of Theorem 1.2.** The standard theory of the two-dimensional Navier-Stokes equations as given in [16, 22] asserts the existence of a unique weak solution $u$ satisfying the energy equality

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u\|_2^2 \, ds = \|a\|_2^2 + 2\int_0^t (f, u) \, ds$$

for all $t \geqq 0$. Hence we have

$$\frac{d}{dt} \|u\|_2^2 + 2\|\nabla u\|_2^2 = 2(f, u) \leqq 2\|f\|_2 \|u\|_2. \tag{3.1}$$

Using the estimate $2\|f\|_2 \|u\|_2 \leqq \|f\|_2 (1 + \|u\|_2^2)$ and Gronwall's Lemma, we easily see that

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u\|_2^2 \, ds \leqq \left(\|a\|_2^2 + \int_0^\infty \|f\|_2 \, ds\right) \exp\left(\int_0^\infty \|f\|_2 \, ds\right), \tag{3.2}$$

and therefore $\|u\|_2 \in L^\infty$ and $\|\nabla u\|_2^2 \in L^1$. Now, since $\|\nabla u\|_2 = \|A^{1/2} u\|_2$, using the estimate

$$2\|A^{1/2} u\|_2^2 \geqq 2\int_0^\infty \lambda \, d\|E_\lambda u\|_2^2 \geqq 2\rho \int_0^\infty d\|E_\lambda u\|_2^2 = 2\rho(\|u\|_2^2 - \|E_\rho u\|_2^2),$$

and $\|E_\rho u\|_2 \leqq \|u\|_2$, we see from (3.1) that

$$\frac{d}{dt} \|u\|_2 + \rho\|u\|_2 \leqq \rho\|E_\rho u\|_2 + \|f\|_2 \tag{3.3}$$

for all $\rho > 0$. To deal with the right-hand side we substitute $\phi(\tau) = e^{-(t-\tau)A}E_\rho\psi$, $\psi \in C_{0,\sigma}^\infty(D)$ into (W) and get

$$(E_\rho u(t), \psi) = (E_\rho e^{-tA}a, \psi) + \int_0^t (E_\rho e^{-(t-s)A}f, \psi)\, ds$$
$$- \int_0^t (E_\rho e^{-(t-s)A}P(u\cdot\nabla u), \psi)\, ds,$$

so that, by duality and Lemma 3.1(i),

$$\| E_\rho u(t)\|_2 \leqq \| e^{-tA}a\|_2 + C\rho^{1/2} \int_0^t (\|u\|_2 \|\nabla u\|_2 + \|f\|_{\hat{V}^*})\, ds.$$

Combining this with (3.3) and applying Hölder's inequality gives

$$\frac{d}{dt}\|u\|_2 + \rho\|u\|_2 \leqq C\rho\left[\|e^{-tA}a\|_2 + \rho^{1/2}\left(\int_0^t \|u\|_2^2\, ds\right)^{1/2} + \rho^{1/2}\right] + \|f\|_2 \quad (3.4)$$

since $\|\nabla u\|_2^2 \in L^1$ by (3.2). In (3.4) we take $\rho = 2/(t+e)\log(t+e)$ and then multiply both sides by $(\log(t+e))^2$ to obtain

$$\frac{d}{dt}((\log(t+e))^2\|u\|_2) \leqq 2C(t+e)^{-1}\log(t+e)\,[\|e^{-tA}a\|_2 + C(\log(t+e))^{-1/2}]$$
$$+ C(\log(t+e))^2(t+e)^{-1}\|f\|_2(t+e).$$

Since $\|e^{-tA}a\|_2 \leqq \|a\|_2$, since $\|e^{-tA}a\|_2 \to 0$ as $t\to\infty$ because $A$ is injective, and since $\int_0^\infty \|f\|_2(t+e)\,dt$ is finite by assumption, we obtain

$$\|u(t)\|_2 \leqq (\log(t+e))^{-2}\left[\|a\|_2 + C\int_0^t (s+e)^{-1}\log(s+e)\|e^{-sA}a\|_2\, ds\right]$$
$$+ C(\log(t+e))^{-1/2} + C(\log(t+e))^{-2} \to 0 \quad \text{as } t\to\infty.$$

The proof of (i) is complete. We next prove (ii). Since $\|e^{-tA}a\|_2 \leqq C(t+e)^{-\alpha}$ by assumption, the proof of (i) shows that

$$\|u(t)\|_2 \leqq C[(\log(t+e))^{-2} + (\log(t+e))^{-1/2}].$$

This proves (ii) for $m = 1$. From this we obtain (see [21, 23])

$$\int_0^t \|u\|_2^2\, ds \leqq C\int_0^t (\log(s+e))^{-1}\, ds \leqq C(t+e)(\log(t+e))^{-1},$$

so that, as in the proof of (i),

$$\frac{d}{dt}((\log(t+e))^2\|u\|_2) \leqq C(t+e)^{-1}\log(t+e)[(t+e)^{-\alpha} + (\log(t+e))^{-1}]$$
$$+ (\log(t+e))^2(t+e)^{-1}\|f\|_2(t+e).$$

Integrating this gives

$$\|u(t)\|_2 \leqq C(\log(t+e))^{-1}.$$

This proves (ii) for $m = 2$. Now suppose (ii) is true for some $m \geqq 2$. Taking $\rho = (m+1)/(t+e)\log(t+e)$ in (3.4) and then multiplying by $(\log(t+e))^{m+1}$ we obtain

$$\frac{d}{dt}((\log(t+e))^{m+1}\|u\|_2) \leqq C(t+e)^{-1}\log(t+e))^m \times[(t+e)^{-\alpha} + (\log(t+e))^{-(m+1)/2}] + C(\log(t+e))^{m+1}(t+e)^{-1}\|f\|_2(t+e),$$

since $\int_0^t (\log(s+e))^{-m}\,ds \leqq C_m(t+e)(\log(t+e))^{-m}$ ([21, 23]). Hence

$$\|u(t)\|_2 \leqq C(\log(t+e))^{-(m+1)/2},$$

and this completes the proof of (ii).

We next prove (iii), following [12, pp. 142, 143]. Substituting $\phi(\tau) = e^{-(t-\tau)A}\psi$, $\psi \in C_{0,\sigma}^\infty(D)$ into (W) and applying Lemma 3.1(ii) gives

$$\|u(t)\|_2 \leqq \|e^{-tA}a\|_2 + C\int_0^t (t-s)^{-1/2}(\|u\|_2\|\nabla u\|_2 + \|f\|_{\hat{V}^*})\,ds. \tag{3.5}$$

Assume first that $0 < \alpha < 1/2$ and choose $q > 2$ so that $q\alpha < 1$. Since $1 + 1/q = 1/2 + (q+2)/2q$, the Hardy-Littlewood-Sobolev inequality [19] applied to (3.5) gives

$$\left[\int_0^t \|u\|_2^q\,ds\right]^{1/q} \leqq C_1(t+1)^{1/q-\alpha} + C\left[\int_0^t (\|u\|_2\|\nabla u\|_2)^{2q/(q+2)}\,ds\right]^{(q+2)/2q} + C\left[\int_0^t \|f\|_{\hat{V}^*}^{2q/(q+2)}\,ds\right]^{(q+2)/2q}.$$

Notice that the last term is bounded in $t$ by the assumption $f \in L^p(0, \infty; \hat{V}^*)$ for $p = 1, 2$. Since $(q+2)/2q = 1/2 + 1/q$, Hölder's inequality and (3.2) together yield

$$\left[\int_0^t \|u\|_2^q\,ds\right]^{1/q} \leqq C_1(t+1)^{1/q-\alpha} + C_2\left[\int_0^t \|u\|_2^q\,ds\right]^{1/q} + C_3$$

where $C_1$ and $C_2$ are constants and $C_2 = C_2(a, f)$ is the square root of the right-hand side of (3.2). Here we assume that $\|a\|_2$ and $\int_0^\infty \|f\|_2\,dt$ are so small that $C_2 \leqq 1/2$ to obtain from the above

$$\left[\int_0^t \|u\|_2^q\,ds\right]^{1/q} \leqq C(t+1)^{1/q-\alpha} + C \leqq C(t+1)^{1/q-\alpha}$$

because $1/q - \alpha > 0$. Inserting this into (3.4) gives

$$\frac{d}{dt}\|u\|_2 + \rho\|u\|_2 \leqq C\rho\left[(t+1)^{-\alpha} + \rho^{1/2}(t+1)^{1/2-1/q}\left(\int_0^t \|u\|_2^q\,ds\right)^{1/q} + \rho^{1/2}\right] + \|f\|_2$$
$$\leqq C\rho((t+1)^{-\alpha} + \rho^{1/2}(t+1))^{1/2-\alpha} + \|f\|_2.$$

Taking $\rho = m/(t+1)$, multiplying by $(t+1)^m$, and then proceeding as in the proof of (ii), we obtain $\|u(t)\|_2 = O((t+1)^{-\alpha})$, and this proves (iii) in case $\alpha < 1/2$. If $\alpha \geqq 1/2$, then $q\alpha > 1$ for all $q > 2$, and so $\int_0^\infty \|e^{-tA}a\|_2^q \, dt < +\infty$. The foregoing argument thus gives

$$\left[\int_0^t \|u\|_2^q \, ds\right]^{1/q} \leqq C_1 + C_2 \left[\int_0^t \|u\|_2^q \, ds\right]^{1/q} + C_3,$$

so that $[\int_0^t \|u\|_2^q \, ds]^{1/q} \leqq C$ if $C_2 \leqq 1/2$. Inserting this into (3.4) and repeating the same argument as in the case $\alpha < 1/2$, we obtain

$$\|u(t)\|_2 \leqq C((t+1)^{-\alpha} + (t+1)^{-1/q}).$$

Since $q > 2$ is arbitrary, the proof of (iii) is now complete.

To prove (iv), we need only show that the (unique) weak solution $u(t)$ belongs to $R(A^\alpha)$ for all $t > 0$ provided $a \in R(A^\alpha)$, because it then follows that

$$\|e^{-tA}u(s)\|_2 = O(t^{-\alpha})$$

for any fixed $s \geqq 0$. Thus the proof of (iii) applies if we choose $s \geqq 0$ as the initial time so that $\|u(s)\|_2$ and $\int_s^\infty \|f\|_2 \, d\tau$ are small enough. To this end we use

**Proposition 3.2.** *Let* $n = 2$ *and* $0 < \alpha \leqq 1/2$. *If* $v \in V$, *then for all* $\lambda > 0$,

$$\|(\lambda + A)^{-\alpha} P(v \cdot \nabla) v\|_2 \leqq C \|v\|_2^{2\alpha} \|\nabla v\|_2^{2-2\alpha}$$

*with* $C$ *depending only on* $\alpha$. *Moreover, we have*

$$\|(\lambda + A)^{-\alpha} f\|_2 \leqq \|f\|_{V^*}^{2\alpha} \|f\|_2^{1-2\alpha}.$$

We continue the proof of Theorem 1.2(iv), admitting Proposition 3.2 for a moment. Let

$$u_\lambda(t) = (\lambda + A)^{-\alpha} u(t), \quad a_\lambda = (\lambda + A)^{-\alpha} a.$$

Since $a \in R(A^\alpha)$ by assumption, $\|a_\lambda\|_2$ is uniformly bounded in $\lambda > 0$. On the other hand, inserting $\phi(\tau) = (\lambda + A)^{-\alpha} e^{-(t-\tau)A}\psi$, $\psi \in C_{0,\sigma}^\infty(D)$ into (W) and applying Proposition 3.2 (i), we obtain

$$\begin{aligned}\|u_\lambda(t)\|_2 &\leqq \|a_\lambda\|_2 + \int_0^t \|(\lambda + A)^{-\alpha} P(u \cdot \nabla) u(s)\|_2 \, ds + \int_0^t \|(\lambda + A)^{-\alpha} f\|_2 \, ds \\ &\leqq C_1 + C_2 \int_0^t \|u\|_2^{2\alpha} \|\nabla u\|_2^{2-2\alpha} \, ds + \int_0^t \|f\|_{V^*}^{2\alpha} \|f\|_2^{1-2\alpha} \, ds \\ &\leqq C_1 + C \left[\int_0^t \|u\|_2^2 \, ds\right]^\alpha + C_3.\end{aligned}$$

This shows that $u_\lambda(t)$ remains bounded in $H$ for any fixed $t > 0$. Hence we may assume that $u_\lambda(t) \to w(t)$ as $\lambda \to 0$ weakly in $H$ and therefore, for any

$\phi \in D(A^{\alpha})$,

$$\big(u(t), \phi\big) = \big(u_{\lambda}(t), (\lambda + A)^{\alpha} \phi\big) \to \big(w(t), A^{\alpha}\phi\big).$$

Hence $u(t) \in R(A^{\alpha})$ for all $t \geqq 0$, and this proves (iv). The proof is complete.

**Proof of Proposition 3.2.** First we obtain the estimate

$$\|\phi\|_p \leqq C\|A^{\alpha}\phi\|_2 \tag{3.6}$$

for $1/p = 1/2 - \alpha$, and $0 \leqq \alpha < 1/2$. This is deduced from the following: (i) The family $\{D^{\alpha};\ 0 \leqq \alpha \leqq 1/2\}$ of the completion $D^{\alpha}$ of $D(A^{\alpha})$ in the norm $\|A^{\alpha}\cdot\|_2$ forms a complex interpolation family (see, e.g., [17]); (ii) $D^{1/2} \subset BMO$ with continuous injection (see, e.g., [7, Prop. 3.4]); and (iii) $[L^2, BMO]_{\theta} = L^{2/(1-\theta)}$, $0 \leqq \theta < 1$, where the bracket denotes the complex interpolation (see [11]).

Now, estimate (3.6) implies that

$$\begin{aligned}
|((\lambda + A)^{-\alpha} P(v\cdot\nabla)v, \phi)| &= |(v\cdot\nabla v,\ (\lambda + A)^{-\alpha}\phi)| \\
&\leqq \|\nabla v\|_2\, \|v\|_{1/\alpha}\, \|(\lambda + A)^{-\alpha}\phi\|_p \\
&\leqq C\|\nabla v\|_2^{2-2\alpha}\, \|v\|_2^{2\alpha}\, \|A^{\alpha}(\lambda + A)^{-\alpha}\phi\|_2 \\
&\leqq C\|\nabla v\|_2^{2-2\alpha}\, \|v\|_2^{2\alpha}\, \|\phi\|_2
\end{aligned}$$

for $\alpha < 1/2$, and

$$\begin{aligned}
&|((\lambda + A)^{-1/2} P(v\cdot\nabla)v, \phi)| \\
&\quad = |(v, v\cdot\nabla(\lambda + A)^{-1/2}\phi)| \leqq \|v\|_4^2\, \|\nabla(\lambda + A)^{-1/2}\phi\|_2 \\
&\quad = C\|v\|_4^2\, \|A^{1/2}(\lambda + A)^{-1/2}\phi\|_2 \leqq C\|v\|_2\, \|\nabla v\|_2\, \|\phi\|_2
\end{aligned}$$

for $\alpha = 1/2$. This proves the first assertion. The second assertion easily follows from the moment inequality

$$\|(\lambda + A)^{-\alpha} f\|_2 \leqq \|f\|_2^{1-2\alpha}\, \|(\lambda + A)^{-1/2} f\|_2^{2\alpha}$$

and the estimate

$$\|(\lambda + A)^{-1/2} f\|_2 \leqq \|f\|_{\dot{V}^*}.$$

This completes the proof of Proposition 3.2.

Using the estimates

$$\|\phi\|_p \leqq C\|A^{\alpha}\phi\|_2$$

where $1/p = 1/2 - 2\alpha/3$, $1/4 \leqq \alpha \leqq 1/2$ if $n = 3$; and $p = 4$, $\alpha = 1/2$ if $n = 4$, we can also prove

**Proposition 3.3.** *Let $v \in V$ and $w \in H^1(R^n)$ with $\nabla\cdot w = 0$.*
(i) *If $n = 3$ and $1/4 \leqq \alpha \leqq 1/2$, then for all $\lambda > 0$,*

$$\|(\lambda + A)^{-\alpha} P(w\cdot\nabla)v\|_2 \leqq C\|w\|_2^{2\alpha - 1/2}\, \|\nabla w\|_2^{3/2-2\alpha}\, \|\nabla v\|_2$$

*with $C$ depending only on $\alpha$.*

(ii) *If* $n = 4$, *then for all* $\lambda > 0$,

$$\|(\lambda + A)^{-1/2} P(w \cdot \nabla) v\|_2 \leqq C \|\nabla w\|_2 \|\nabla v\|_2$$

*with* $C$ *independent of* $v$ *and* $w$.
(iii) $R(A^\alpha)$ *is invariant under the Navier-Stokes flow provided either* $n = 3$, $1/4 \leqq \alpha \leqq 1/2$; *or* $n = 4$, $\alpha = 1/2$.

Finally, Corollary 1.3 is immediately obtained from Theorems 1.1, 1.2 and the following

**Corollary 3.4.** *The inclusion*

$$H \cap (L^r(D))^n \subset R(A^\alpha), \qquad \alpha = n(1/r - 1/2)/2$$

*holds for* $1 < r < 2$ *if* $n = 2$, *for* $6/5 \leqq r \leqq 3/2$ *if* $n = 3$, *and for* $r = 4/3$ *if* $n = 4$. *More precisely, to each* $a \in H \cap (L^r(D))^n$ *there corresponds a unique* $b \in D(A^\alpha)$ *such that* $a = A^\alpha b$ *and* $\|b\|_2 \leqq C\|a\|_r$ *with* $C > 0$ *independent of* $a$.

**Proof.** We consider only the case $n = 2$; the other cases are treated similarly. The function $a_\lambda = (\lambda + A)^{-\alpha} a$ satisfies

$$|(a_\lambda, \phi)| = |(a, (\lambda + A)^{-\alpha} \phi)| \leqq \|a\|_r \|(\lambda + A)^{-\alpha} \phi\|_{r'}$$
$$\leqq C\|a\|_r \|A^\alpha(\lambda + A)^{-\alpha} \phi\|_2 \leqq C\|a\|_r \|\phi\|_2$$

for all $\phi \in H$, where $1/r' = 1 - 1/r$. This shows that $a_\lambda$ is bounded in $H$ for $\lambda > 0$; thus a subsequence converges weakly in $H$ as $\lambda \to 0$ to a function $b \in H$ with $\|b\|_2 \leqq C\|a\|_r$. But then,

$$(a, \phi) = (a_\lambda, (\lambda + A)^\alpha \phi) \to (b, A^\alpha \phi)$$

for all $\phi \in D(A^\alpha)$, so $a = A^\alpha b$ and $b$ is thus determined uniquely. The proof is complete.

*Remark.* When $n \geqq 3$ and $D$ is an exterior domain of $R^n$ with smooth boundary, the $L^r$-theory of the operator $A$ as developed in [3] gives the estimate

$$\|\phi\|_p \leqq C \|A^\alpha \phi\|_r$$

for $1 < r < n$, $1 < p < \infty$ and $1/p = 1/r - 2\alpha/n$. This implies that $R(A^\alpha)$ is invariant under the Navier-Stokes flow (with $f = 0$) provided $\alpha < n/4$, and that the inclusion of Corollary 3.4 holds for all $1 < r < 2$. More generally, the $L^r$-theory implies that

$$L^r_\sigma \cap L^q_\sigma \subset R(A^\alpha_q)$$

whenever $n' < q < \infty$, $1 < r < \infty$, and $1/r = 1/q + 2\alpha/n$. Here $1/n' = 1 - 1/n$, $L^r_\sigma$ denotes the $L^r$-closure of $C^\infty_{0,\sigma}(D)$, and $A_q$ is the operator $A$ regarded as a closed linear operator in $L^q_\sigma$. We can further show that the space $L^r \cap H$ is invariant under the Navier-Stokes flow provided $1 < r \leqq n'$. See [3, Secs. 4, 5] for the details.

## 4. Energy Stability of Exterior Stationary Flows

This section deals with the problem

$$\begin{aligned} &\frac{\partial v}{\partial t} + v \cdot \nabla v = \Delta v - \nabla p + f_0 \qquad (x \in D,\ t > 0), \\ &\nabla \cdot v = 0 \qquad (x \in D,\ t > 0), \\ &v|_S = v^*, \quad v \to v^\infty \quad \text{as } |x| \to \infty, \\ &v|_{t=0} = v_0 \end{aligned} \tag{4.1}$$

in an exterior domain $D$ in $R^3$ with smooth boundary $S$. Hence $v^* = v^*(x)$ is a given smooth vector field on $S$, $v^\infty$ is a given constant vector, and $f_0 = f_0(x)$ is a given external force.

Under some assumptions on $v^*$, $v^\infty$ and $f_0$, FINN [4, 5] and BABENKO [1] proved the existence of a stationary solution $w$ to problem (4.1) satisfying

$$w - v^\infty \in L^3(D), \quad \nabla w \in L^3(D), \quad C_0 \equiv \sup_D |x| \cdot |w(x) - v^\infty| < +\infty. \tag{4.2}$$

In this section we study the stability of these stationary solutions with respect to $L^2$ disturbances. Given a stationary solution $w$ and disturbances $f$ and $a = v_0 - w$, the time-evolution of $u = v - w$ is governed by

$$\begin{aligned} &\frac{\partial u}{\partial t} + u \cdot \nabla u = \Delta u - w \cdot \nabla u - u \cdot \nabla w - \nabla q + f \qquad (x \in D,\ t > 0), \\ &\nabla \cdot u = 0 \qquad (x \in D,\ t > 0), \\ &u|_S = 0, \quad u \to 0 \quad \text{as } |x| \to \infty, \\ &u|_{t=0} = a. \end{aligned} \tag{P}$$

The problem (P) is formally transformed into the integral equation

$$u(t) = e^{-tL} a - \int_0^t e^{-(t-s)L} (P(u \cdot \nabla) u - f)(s)\, ds \tag{4.3}$$

where

$$L = A + B, \quad Bu = P(w \cdot \nabla) u + P(u \cdot \nabla) w,$$

and $\{e^{-tL}; t \geqq 0\}$ is the analytic semigroup in $H$ generated by $-L$. (See [18].)

Given $a \in H$, a weakly continuous function $u: [0, \infty) \to H$ is called a weak solution of (P) if $u \in L^\infty(0, T; H) \cap L^2(0, T; V)$ for all $T > 0$, $u(0) = a$, and $u$ satisfies

$$\begin{aligned} &(u(t), \phi(t)) + \int_s^t [(\nabla u, \nabla \phi) + (w \cdot \nabla u + u \cdot \nabla w + u \cdot \nabla u, \phi)]\, d\tau \\ &\qquad = (u(s), \phi(s)) + \int_s^t (u, \phi')\, d\tau + \int_s^t (f, \phi)\, d\tau \end{aligned} \tag{4.4}$$

for all $t \geqq s \geqq 0$ and all $\phi \in C^1([0, \infty); H) \cap C^0([0, \infty); V)$.

For the existence of weak solutions, we already have the following result:

**Theorem 4.1** ([18]). *Given $a \in H$ and $f \in L^2_{\mathrm{loc}}([0, \infty); H)$, problem* (P) *possesses a weak solution $u$ which, moreover, satisfies the energy inequality in the form*

$$\| u(t) \|_2^2 + 2 \int_s^t \| \nabla u \|_2^2 \, d\tau + 2 \int_s^t (u \cdot \nabla w, u) \, d\tau \leqq \| u(s) \|_2^2 + 2 \int_s^t (f, u) \, d\tau \quad \text{(SE)}$$

*for $s = 0$, a.e. $s > 0$, and all $t \geqq s$.*

In this section we shall prove

**Theorem 4.2.** *Let $C_0 < 1/2$, $a \in H$, and suppose that $f$ satisfies the assumption of Theorem 1.1. Then any weak solution $u$ of problem* (P) *satisfying the energy inequality* (SE) *has the following decay properties:*
(i) $\| u(t) \|_2 \to 0$ as $t \to \infty$.
(ii) *If $\| e^{-tL} a \|_2 = O(t^{-\alpha})$ for some $\alpha > 0$, then*

$$\| u(t) \|_2 = O\left( (\log t)^{\varepsilon - 1/2} \right).$$

(iii) *If $v^\infty = 0$ and $\| e^{-tL} a \|_2 = O(t^{-\alpha})$ for some $\alpha > 0$, then*

$$\| u(t) \|_2 = \begin{cases} O(t^{-\alpha}) & \text{if } \alpha < 1/2, \\ O(t^{\varepsilon - 1/2}) & \text{if } \alpha \geqq 1/2 \end{cases}$$

*where $\varepsilon > 0$ is arbitrary.*

Theorems 4.1 and 4.2 mean that an exterior stationary flow $w$ is globally asymptotically stable in energy sense provided $C_0 < 1/2$. The energy inequality (SE) was first deduced by LERAY [14] in the case $D = R^3$, $w = 0$ and $f = 0$. Part (i) of Theorem 3.2 was proved in [18]. HEYWOOD [8, 9] and GALDI & RIONERO [6, p. 41] discuss the decay of local $L^2$-norms of strong solutions in case $C_0 < 1/2$. Contrary to these works, our Theorem 4.2 deals with global $L^2$-norms of weak solutions satisfying (SE). The decay properties of strong solutions in other function spaces are discussed in detail by HEYWOOD [10] and MASUDA [15]. For various problems related to the stability of fluid motions, we refer to [6] and references therein.

We begin the proof of Theorem 4.2 by establishing the following

**Lemma 4.3.** *Let $C_0 < 1/2$, and let $L^*$ denote the adjoint operator of $L$. Then*
(i) *$\{e^{-tL}; t \geqq 0\}$ and $\{e^{-tL^*}; t \geqq 0\}$ are contraction semigroups in $H$.*
(ii) *$\| e^{-tL} a \|_2 \to 0$ as $t \to \infty$.*
(iii) *The estimate*

$$\| E_\rho e^{-tL} P(u \cdot \nabla) u \|_2 \leqq C(\rho^{1/2} + \rho^{1/4}) \| u \|_2^{1/2} \| \nabla u \|_2^{3/2}$$

*holds for all $u \in V$ and $\rho > 0$, where $E_\rho$ is the spectral measure associated with $A$.*

(iv) *If* $v^\infty = 0$, *then for all* $u \in V$ *and* $t > 0$,

$$\| e^{-tL} P(u \cdot \nabla) u \|_2 \leqq C t^{-1/2} \| u \|_2^{1/2} \| \nabla u \|_2^{3/2}.$$

**Proof.** As shown in [15], we have the estimate

$$|(\phi \cdot \nabla w, \phi)| \leqq 2C_0 \| \nabla \phi \|_2^2. \tag{4.5}$$

Since $(w \cdot \nabla \phi, \phi) = 0$, (4.5) implies that

$$(L\phi, \phi) = (\phi, L^* \phi) \geqq (1 - 2C_0) \| \nabla \phi \|_2^2 \tag{4.6}$$

for all $\phi \in D(L) = D(L^*) = D(A)$. Hence, we get (i) if $C_0 \leqq 1/2$. To show (ii), suppose first that $a \in R(L)$ and hence $a = Lb$ for some $b \in D(L)$. Then $e^{-tL} a = -v'(t)$ with $v(t) = e^{-tL} b$. Since

$$v''(t) + Lv'(t) = 0,$$

we get

$$\| v'(t) \|_2^2 + 2(1 - 2C_0) \int_s^t \| \nabla v' \|_2^2 \, d\tau \leqq \| v'(s) \|_2^2 \tag{4.7}$$

for all $t \geqq s > 0$. This implies that $\| \nabla v' \|_2$ is in $L^2$ on the interval $[1, \infty)$. Next, from the Sobolev inequality

$$\| f \|_6 \leqq \frac{2}{\sqrt{3}} \| \nabla f \|_2 \tag{4.8}$$

we see that $\| Bv \|_2 \leqq (\| w \|_\infty + C \| \nabla w \|_3) \| \nabla v \|_2$. Direct calculation thus gives

$$\begin{aligned} (v', v') &= -(v', Av) - (v', Bv) \leqq \| \nabla v' \|_2 \| \nabla v \|_2 + C \| v' \|_2 \| \nabla v \|_2 \\ &\leqq \| \nabla v' \|_2^2 + \| \nabla v \|_2^2 + \tfrac{1}{2} \| v' \|_2^2 + C \| \nabla v \|_2^2, \end{aligned}$$

so that

$$\| v' \|_2^2 \leqq C (\| \nabla v \|_2^2 + \| \nabla v' \|_2^2)$$

and hence $\| v' \|_2 \in L^2$ on $[1, \infty)$. From (4.7) it follows that

$$(t - 1) \| v'(t) \|_2^2 \leqq \int_1^t \| v' \|_2^2 \, ds \leqq \int_1^\infty \| v' \|_2^2 \, ds < +\infty,$$

and we conclude that

$$\| e^{-tL} a \|_2 = \| v'(t) \|_2 \to 0$$

as $t \to \infty$. This proves (ii) in case $a \in R(L)$. To complete the proof, it suffices to show that $R(L)$ is dense in $H$. But this follows from the equation $N(L^*) = 0$, which is a consequence of the assumption that $C_0 < 1/2$. The proof of (ii) is complete.

To prove (iii), observe that (4.6) gives

$$\| \nabla e^{-tL^*} E_\rho \phi \|_2 \leqq C \| L^* e^{-tL^*} E_\rho \phi \|_2^{1/2} \| E_\rho \phi \|_2^{1/2} \leqq C \| L^* E_\rho \phi \|_2^{1/2} \| \phi \|_2^{1/2}.$$

By the definition of $L$ we easily see that

$$\| L^* E_\rho \phi \|_2 \leqq \| A E_\rho \phi \|_2 + C(\| w \|_\infty + \| \nabla w \|_3) \| \nabla E_\rho \phi \|_2.$$

The Hölder and Sobolev inequalities then imply that

$$\begin{aligned}|(e^{-tL}P(u\cdot\nabla)\,u,\,E_\rho\phi)| &= |(u,\,u\cdot\nabla e^{-tL^*}E_\rho\phi)|\\ &\leqq C\,\|u\|_4^2\,\|\nabla e^{-tL^*}E_\rho\phi\|_2\\ &\leqq C(\rho^{1/2}+\rho^{1/4})\,\|u\|_2^{1/2}\,\|\nabla u\|_2^{3/2}\,\|\phi\|_2.\end{aligned}$$

This proves (iii). To prove (iv), consider the problem

$$\frac{dv}{dt}+e^{i\theta}L^*v=0,\ \ v(0)=a$$

in the complexification of the Hilbert space $H$. The standard argument then gives

$$\|v(t)\|_2^2+2\,\mathrm{Re}\int_s^t(e^{i\theta}L^*v,\,v)\,d\tau=\|v(s)\|_2^2.$$

Estimating each term of

$$\mathrm{Re}\,(e^{i\theta}L^*v,\,v)=\mathrm{Re}\,(e^{i\theta}Av,\,v)+\mathrm{Re}\,(e^{i\theta}v,\,w\cdot\nabla v)+\mathrm{Re}\,(e^{i\theta}v,\,v\cdot\nabla w)$$

by (4.5) and the Sobolev inequality (4.8), we see that

$$\|v(t)\|_2^2+2\left[\cos\theta-2\left(C_0+\frac{|\sin\theta|}{\sqrt{3}}\,\|w\|_3\right)\right]\int_s^t\|\nabla v\|_2^2\,d\tau\leqq\|v(s)\|_2^2.$$

Hence

$$\|v(t)\|_2\leqq\|a\|_2$$

provided $|\theta|$ is small enough. This means that $\{e^{-tL^*};\,t\geqq 0\}$ is a bounded analytic semigroup in $H$ [19], so we have

$$\|L^*e^{-tL^*}a\|_2\leqq Ct^{-1}\|a\|_2.$$

Thus (4.6) yields

$$\|\nabla e^{-tL^*}\phi\|_2^2\leqq C\,\|L^*e^{-tL^*}\phi\|_2\,\|e^{-tL^*}\phi\|_2\leqq Ct^{-1}\|\phi\|_2^2.$$

We therefore obtain

$$\begin{aligned}|(e^{-tL}P(u\cdot\nabla)\,u,\,\phi)| &= |(u,\,u\cdot\nabla e^{-tL^*}\phi)|\\ &\leqq Ct^{-1/2}\|u\|_2^{1/2}\,\|\nabla u\|_2^{3/2}\,\|\phi\|_2,\end{aligned}$$

which proves (iv).

**Proof of Theorem 4.2.** Since $(Lu,\,u)=\|\nabla u\|_2^2+(u\cdot\nabla w,\,u)$, inequality (4.6) and the energy inequality (SE) yield

$$\|u(t)\|_2^2+2(1-2C_0)\int_s^t\|\nabla u\|_2^2\,d\tau\leqq\|u(s)\|_2^2+2\int_s^t(f,\,u)\,d\tau \tag{4.9}$$

for a.e. $s>0$ and all $t\geqq s$. Thus, if $C_0<1/2$, then as in Section 2, we see that $\|u\|_2\in L^\infty$, $\|\nabla u\|_2^2\in L^1$. As in Section 3 we get

$$\|u(t)\|_2^2+g(t,\,s)\leqq\|u(s)\|_2^2+h(t,\,s) \tag{4.10}$$

for a.e. $s > 0$ and all $t \geqq s$, where

$$g(t, s) = C \int_s^t \rho \| u \|_2^2 \, d\tau, \quad h(t, s) = C \left( \int_s^t \rho \| E_\rho u \|_2^2 \, d\tau + \int_s^t \| f \|_2 \, d\tau \right),$$

and the function $\rho(\tau) > 0$ is to be fixed later. Since

$$\frac{\partial g}{\partial \tau} = -C\rho(\tau) \| u(\tau) \|_2^2 \leqq -C\rho(\tau) \left[ \| u(t) \|_2^2 + g(t, \tau) - h(t, \tau) \right],$$

we may assume the existence of a function $F(\tau) > 0$ with $F' = \rho F$ to obtain

$$F(t) \| u(t) \|_2^2 \leqq F(s) \| u(s) \|_2^2 - \int_s^t F \frac{\partial h}{\partial \tau} \, d\tau$$

by Gronwall's Lemma. Letting $s \to 0$, we thus have

$$\begin{aligned} \| u(t) \|_2^2 \leqq {} & CF(t)^{-1} \| a \|_2^2 + CF(t)^{-1} \int_0^t F'(\tau) \| E_\rho u \|_2^2 \, d\tau \\ & + CF(t)^{-1} \int_0^t F(\tau) (1 + \tau)^{-1} \| f \|_2 (1 + \tau) \, d\tau. \end{aligned} \tag{4.11}$$

Next, in (4.4) we take $\phi(\tau) = e^{-(t-\tau)L^*} E_\lambda \psi$, $\psi \in C_{0,\sigma}^\infty(D)$, and set $s = 0$, to obtain

$$\begin{aligned} |(u(t), E_\lambda \psi)| \leqq {} & \| \psi \|_2 \left[ \| e^{-tL} a \|_2 + \int_0^t \| E_\lambda e^{-(t-\tau)L} P(u \cdot \nabla) u \|_2 \, d\tau \right] \\ & + \| \psi \|_2 \int_0^t \| E_\lambda e^{-(t-\tau)L} f \|_2 \, d\tau. \end{aligned}$$

Applying Lemma 4.3 and Hölder's inequality yields

$$\| E_\lambda u(s) \|_2 \leqq \| e^{-sL} a \|_2 + C(\lambda^{1/2} + \lambda^{1/4}) \left[ \left( \int_0^s \| u \|_2^2 \, d\tau \right)^{1/4} + \int_0^s \| f \|_{\tilde{v}^*} \, d\tau \right]. \tag{4.12}$$

Here we set $\rho(\tau) = 2/(\tau + e) \log(\tau + e)$ and $F = (\log(\tau + e))^2$. Assertions (i) and (ii) follow from (4.11) and (4.12), by repeating the argument in the proof of (i) and (ii) of Theorem 1.2.

We finally prove (iii). As in Section 2, applying Gronwall's Lemma to (4.9) gives

$$\| u(t) \|_2^2 + 2(1 - 2C_0) \int_s^t \| \nabla u \|_2^2 \, d\tau \leqq C \left( \| u(s) \|_2^2 + \int_s^t \| f \|_2 \, d\tau \right)$$

for a.e. $s \in (0, t)$. Thus we get $\| u \|_2 \in L^\infty$, $\| \nabla u \|_2 \in L^2$, and

$$\begin{aligned} \| u(t) \|_2 & \leqq Ct^{-1} \int_0^t \| u \|_2 \, ds + Ct^{-1} \int_0^t \left[ \int_s^t \| f \|_2 \, d\tau \right]^{1/2} ds \\ & \leqq Ct^{-1} \int_0^t \| u \|_2 \, ds + Ct^{-1/2} \left[ \int_0^t s \| f(s) \|_2 \, ds \right]^{1/2} \\ & \leqq Ct^{-1} \int_0^t \| u \|_2 \, ds + Ct^{-1/2}. \end{aligned} \tag{4.13}$$

On the other hand, taking $\phi(\tau) = e^{-(t-\tau)L^*}\psi$, $\psi \in C_{0,\sigma}^\infty(D)$, in (4.4) and then applying Lemma 4.3 (iv) yields

$$\|u(s)\|_2 \leqq \|e^{-sL}a\|_2 + C\int_0^s (s-\tau)^{-1/2}(\|u\|_2^{1/2}\|\nabla u\|_2^{3/2} + \|f\|_{\dot{V}^*})\,d\tau. \quad (4.14)$$

The result now follows from (4.13) and (4.14) in the same way as in the proof of Theorem 1.1 (ii). The proof is complete.

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## References

1. BABENKO, K. I., On stationary solutions of the problem of flow past a body of viscous incompressible fluid. *Math. USSR-Sb.* **20**, 1–25 (1973).
2. BORCHERS, W., & MIYAKAWA, T., $L^2$ decay for the Navier-Stokes flow in halfspaces. *Math. Ann.* **282**, 139–155 (1988).
3. BORCHERS, W., & MIYAKAWA, T., Algebraic $L^2$ decay for Navier-Stokes flows in exterior domains. *Acta Math.* **165**, 189–227 (1990).
4. FINN, R., On the exterior stationary problem for the Navier-Stokes equations, and associated perturbation problems. *Arch. Rational Mech. Anal.* **19**, 363–406 (1965).
5. FINN, R., Mathematical questions relating to viscous fluid flow in an exterior domain. *Rocky Mountain J. Math.* **3**, 107–140 (1973).
6. GALDI, G. P., & RIONERO, S., *Weighted Energy Methods in Fluid Dynamics and Elasticity.* Lecture Notes in Math. **1134**, Berlin–Heidelberg–New York: Springer-Verlag 1985.
7. GIGA, Y., & MIYAKAWA, T., Navier-Stokes flow in $R^3$ with measures as initial vorticity and Morrey spaces. *Commun. in Partial Diff. Eq.* **14**, 577–618 (1989).
8. HEYWOOD, J. G., On stationary solutions of the Navier-Stokes equations as limits of nonstationary solutions. *Arch. Rational Mech. Anal.* **37**, 48–60 (1970).
9. HEYWOOD, J. G., The exterior nonstationary problem for the Navier-Stokes equations. *Acta Math.* **129**, 11–34 (1972).
10. HEYWOOD, J. G., The Navier-Stokes equations: On the existence, regularity and decay of solutions. *Indiana Univ. Math. J.* **29**, 639–681 (1980).
11. JANSON, S., & JONES, P. W., Interpolation between $H^p$ spaces: the complex method. *J. Funct. Anal.* **48**, 58–80 (1982).
12. KAJIKIYA, R., & MIYAKAWA, T., On $L^2$ decay of weak solutions of the Navier-Stokes equations in $R^m$. *Math. Z.* **192**, 135–148 (1986).
13. KATO, T., Strong $L^p$ solutions of the Navier-Stokes equation in $R^m$, with applications to weak solutions. *Math. Z.* **187**, 471–480 (1984).
14. LERAY, J., Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Math.* **63**, 193–248 (1934).
15. MASUDA, K., On the stability of incompressible viscous fluid motions past objects. *J. Math. Soc. Japan* **27**, 294–327 (1975).
16. MASUDA, K., Weak solutions of the Navier-Stokes equations. *Tôhoku Math. J.* **36**, 623–646 (1984).
17. MIYAKAWA, T., On nonstationary solutions of the Navier-Stokes equations in an exterior domain. *Hiroshima Math. J.* **12**, 115–140 (1982).

18. MIYAKAWA, T., & SOHR, H., On energy inequality, smoothness and large time behavior in $L^2$ for weak solutions of the Navier-Stokes equations in exterior domains. *Math. Z.* **199**, 455–478 (1988).
19. REED, M., & SIMON, B., *Methods of Modern Mathematical Physics*, Vol. II; *Fourier Analysis, Self-Adjointness.* New York: Academic Press 1975.
20. SCHONBEK, M. E., $L^2$ decay for weak solutions of the Navier-Stokes equations. *Arch. Rational Mech. Anal.* **88**, 209–222 (1985).
21. SCHONBEK, M. E., Large time behaviour of solutions to the Navier-Stokes equations. *Commun. in Partial Diff. Eq.* **11**, 733–763 (1986).
22. TEMAM, R., *Navier-Stokes Equations.* Amsterdam: North-Holland 1977.
23. WIEGNER, M., Decay results for weak solutions of the Navier-Stokes equations in $R^n$. *J. London Math. Soc.* **35**, 303–313 (1987).
24. BORCHERS, W., & MIYAKAWA, T., On large time behavior of the total kinetic energy for weak solutions of the Navier-Stokes equations in unbounded domains. *Proc. of a symposium on the Navier-Stokes equations*, Oberwolfach 1988; to appear in *Lecture Notes in Math.*, Berlin–Heidelberg–New York: Springer-Verlag.

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