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An Example of a Quasiconvex Function that is not Polyconvex in Two Dimensions

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Abstract

We study the different notions of convexity for the function $f_\gamma(\xi) = |\xi|^2 (|\xi|^2 - 2\gamma \det \xi)$ where $\xi \in \mathbb{R}^{2\times 2}$, introduced by DACOROGNA & MARCELLINI. We show that f_γ is convex, polyconvex, quasiconvex, rank-one convex, if and only if $|\gamma| \leqq \frac{2}{3}\sqrt{2}$, 1, $1 + \varepsilon$ (for some $\varepsilon > 0$), $2/\sqrt{3}$, respectively.

§ 0. Introduction

Let $\mathbb{R}^{2\times 2}$ be the set of 2×2 real matrices endowed with the Euclidean norm

$$|\xi|^2 = \sum_{i,j=1}^{2} \xi_{ij}^2.$$

The main result of this paper is

Theorem 1. *Let* $\gamma \in \mathbb{R}$ *and let* $f_\gamma\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ *be defined as*

$$f_\gamma(\xi) = |\xi|^2(|\xi|^2 - 2\gamma \det \xi). \tag{0.1}$$

Then

$$f_\gamma \text{ is convex } \Leftrightarrow |\gamma| \leqq \tfrac{2}{3}\sqrt{2},$$

$$f_\gamma \text{ is polyconvex } \Leftrightarrow |\gamma| \leqq 1.$$

There exists an $\varepsilon > 0$ *such that*

$$f_\gamma \text{ is quasiconvex } \Leftrightarrow |\gamma| \leqq 1 + \varepsilon,$$

$$f_\gamma \text{ is rank-one convex } \Leftrightarrow |\gamma| \leqq \frac{2}{\sqrt{3}}.$$

The last result as well as the fact that if f_γ is polyconvex, then $|\gamma| \leqq 1$ have been established by DACOROGNA & MARCELLINI [1]. All the other results are new. The most interesting fact is the third one.

We now put our results into perspective and define the terminology used in Theorem 1 (as a general reference we use DACOROGNA [1]).

Definitions. Let $f: \mathbb{R}^{2\times 2} \to \mathbb{R}$. (i) f is said to be *polyconvex* if there exists a convex $g: \mathbb{R}^{2\times 2} \times \mathbb{R} \to \mathbb{R}$ such that

$$f(\xi) = g(\xi, \det \xi).$$

(ii) f is said to be *quasiconvex* if

$$\int_\Omega f(A + \nabla\varphi(x))\, dx \geqq f(A) \operatorname{meas} \Omega$$

for every bounded convex set $\Omega \subset \mathbb{R}^2$ (or equivalently for some $\Omega \subset \mathbb{R}^2$), for every $A \in \mathbb{R}^{2\times 2}$ and for every $\varphi \in W_0^{1,\infty}(\Omega; \mathbb{R}^2)$. ($W_0^{1,\infty}(\Omega, \mathbb{R}^2)$ is the space of Lipschitz continuous functions vanishing on $\partial\Omega$.)
(iii) f is said to be *rank-one convex* if

$$f(\lambda A + (1-\lambda) B) \leqq \lambda f(A) + (1-\lambda) f(B)$$

for every $\lambda \in [0, 1]$ and $A, B \in \mathbb{R}^{2\times 2}$ with $\det(A - B) = 0$.

It was well established in the work of MORREY [1], [2] and later of BALL [1] that, in general,

f is convex $\Rightarrow$ f is polyconvex $\Rightarrow$ f is quasiconvex $\Rightarrow$ f is rank-one convex.

The important notion with respect to minimisation problems is quasiconvexity, which is equivalent to the weak lower semicontinuity of the functional

$$I(u) = \int_\Omega f(\nabla u(x))\, dx.$$

It turns out that, in general, it is very hard to check whether or not a given function is quasiconvex. The conditions of rank-one convexity or polyconvexity are a lot easier, though still difficult, to verify.

In particular, there are very few examples of quasiconvex functions which are not polyconvex. One such example has been given recently by ŠVERÀK [1]. The question of finding rank-one convex functions which are not quasiconvex is still open, despite numerous attempts since the work of MORREY (for a survey on this question see BALL [2] or DACOROGNA [1]).

DACOROGNA & MARCELLINI [1] showed that (0.1) gives rise to a rank-one convex function which is not polyconvex if $1 < |\gamma| \leqq 2/\sqrt{3}$. They, however, were unable to settle the quasiconvexity of such an f_γ. Later a numerical computation (cf. DACOROGNA, DOUCHET, GANGBO & RAPPAZ [1]) seemed to indicate that f_γ is quasiconvex if $|\gamma| \leqq 2/\sqrt{3}$.

Our result does not completely solve the problem, and in particular we have been unable to decide whether or not $1 + \varepsilon = 2/\sqrt{3}$. However, our theorem shows that the case $|\gamma| > 1$ gives rise to examples of quasiconvex functions which are not polyconvex. In addition, these examples do not satisfy

$$f(\xi) \geqq -\alpha(|\xi|^2 + 1), \tag{0.2}$$

where $\alpha > 0$. This answers a question raised by BALL [3]. It is therefore unreasonable to expect that quasiconvexity implies some lower bound, contrary to the case of polyconvex functions which, in the two-dimensional case considered here, necessarily satisfy (0.2).

Theorem 1 shows how these different notions of convexity arise in a particular example when one changes a single parameter. Perhaps more interesting, however, is the proof. Note that it is not even obvious that f_γ is quasiconvex at 0, i.e., that

$$\int_\Omega |\nabla\varphi(x)|^2 \left(|\nabla\varphi(x)|^2 - 2\gamma \det \nabla\varphi(x)\right) dx \geqq 0$$

for every $\varphi \in W_0^{1,\infty}(\Omega; \mathbb{R}^2)$ and for some $\gamma > 1$, since the integrand is not pointwise positive and can become arbitrarily negative. In addition to some algebraic computations, the key point is the following inequality: There exists an $\varepsilon = \varepsilon(\Omega) > 0$ such that

$$\int_\Omega \left(|\nabla\varphi(x)|^2 \pm 2\det \nabla\varphi(x)\right)^2 dx \geqq \varepsilon \int_\Omega |\nabla\varphi(x)|^4\, dx \tag{0.3}$$

for every $\varphi \in W_0^{1,\infty}(\Omega; \mathbb{R}^2)$. This inequality shows that the functional on the left-hand side of (0.3) is coercive in $W_0^{1,4}(\Omega; \mathbb{R}^2)$, even though the integrand is not coercive (not even up to a null Lagrangian which can here be at most quadratic).

§ 1. An Important Inequality

Before proceeding with the proof, we introduce some notations which will simplify the computations.

Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \mathbb{R}^{2\times 2}$; we define the matrices

$$\tilde{A} := \begin{pmatrix} A_{22} & -A_{21} \\ -A_{12} & A_{11} \end{pmatrix}, \quad A^+ := \tfrac{1}{2}(A + \tilde{A}), \quad A^- := \tfrac{1}{2}(A - \tilde{A}).$$

For $A, B \in \mathbb{R}^{2\times 2}$ we let $(A \cdot B) = \Sigma_{i,j=1}^2 A_{ij}B_{ij}$. We easily obtain the following properties.

(i) $\det(A + B) = \det A + (\tilde{A} \cdot B) + \det B$ and $2\det A = (\tilde{A} \cdot A)$,

(ii) $A = A^+ + A^-$ and $\tilde{A} = A^+ - A^-$,

(iii) $2\det A^+ = |A^+|^2$ and $2\det A^- = -|A^-|^2$,

(iv) $|A|^2 = |A^+|^2 + |A^-|^2$ and $2\det A = |A^+|^2 - |A^-|^2 = 2\det A^+ + 2\det A^-$,

(v) $(A \cdot B) = (A^+ \cdot B^+) + (A^- \cdot B^-)$ and $(A^+ \cdot B^-) = (A^- \cdot B^+) = 0$,
(vi) $|A|^2 - 2\det A = 2|A^-|^2$ and $|A|^2 + 2\det A = 2|A^+|^2$.

Remark 1.1. The decomposition (ii) of $\mathbb{R}^{2\times 2}$ corresponds to the splitting of matrices into a direct sum of a "conformal" matrix (i.e., one for which $(A^+)^{\mathsf{T}} A^+ = (\det A^+)\, I$) plus an "anticonformal" one (i.e., one for which $(A^-)^{\mathsf{T}} A^- = -(\det A^-)\, I$).

Remark 1.2. Observe that for every $\xi \in \mathbb{R}^{2\times 2}$,

$$f_\gamma(\xi) = (1-\gamma)|\xi^+|^4 + 2|\xi^+|^2|\xi^-|^2 + (1+\gamma)|\xi^-|^4. \tag{1.1}$$

We now establish the following result which will play a central role in the proof of Theorem 1.

Theorem 2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a sufficiently regular boundary. Then there exists an $\varepsilon = \varepsilon(\Omega) > 0$ such that*

$$\int_\Omega [|A + \nabla\varphi(x)|^2 \pm 2\det(A + \nabla\varphi(x))]^2\, dx \geqq (|A|^2 \pm 2\det A)^2 \operatorname{meas}\Omega + 4\varepsilon \int_\Omega |\nabla\varphi(x)|^4\, dx$$

or equivalently, in the above notation,

$$\int_\Omega |(A + \nabla\varphi(x))^\pm|^4\, dx \geqq |A^\pm|^4 \operatorname{meas}\Omega + \varepsilon \int_\Omega |\nabla\varphi(x)|^4\, dx \tag{1.2}$$

for every $A \in \mathbb{R}^{2\times 2}$ and every $\varphi \in W_0^{1,\infty}(\Omega; \mathbb{R}^2)$.

Remark 1.3. In the proof of Theorem 1, we require (1.2) only for $A = 0$.

Proof of Theorem 2. We shall prove (1.2) only for the minus sign, the proof being identical for the plus sign. Adapting an idea of ŠVERÀK [1], we also prove first the result for $A = 0$.
Step 1. Let $\varphi = (\varphi_1, \varphi_2)$. Then denoting $\partial\varphi_j/\partial x_i$ by $\partial_i\varphi_j$, $i, j \in \{1,2\}$, we have

$$\int_\Omega 4|\nabla^-\varphi(x)|^4\, dx = \int_\Omega [(\partial_1\varphi_1(x) - \partial_2\varphi_2(x))^2 + (\partial_2\varphi_1(x) + \partial_1\varphi_2(x))^2]^2\, dx$$
$$\geqq \int_\Omega [(\partial_1\varphi_1(x) - \partial_2\varphi_2(x))^4 + (\partial_2\varphi_1(x) + \partial_1\varphi_2(x))^4]\, dx.$$

It will be sufficient to prove that there exists a $\beta = \beta(\Omega) > 0$ such that

$$\|\varphi_1\|_{W^{1,4}}^4 + \|\varphi_2\|_{W^{1,4}}^4 \leqq \beta \int_\Omega [(\partial_1\varphi_1(x) - \partial_2\varphi_2(x))^4 + (\partial_2\varphi_1(x) + \partial_1\varphi_2(x))^4]\, dx \tag{1.3}$$

for every $\varphi_1, \varphi_2 \in W_0^{1,4}(\Omega)$ to get (1.2) with $A = 0$. But (1.3) expresses the regularity of solutions of the inhomogeneous Cauchy-Riemann equations.

Indeed, for

$$\partial_1\varphi_1 - \partial_2\varphi_2 = a \quad \text{in } \Omega,$$
$$\partial_2\varphi_1 + \partial_1\varphi_2 = b \quad \text{in } \Omega,$$
$$\varphi_1 = \varphi_2 = 0 \quad \text{on } \partial\Omega,$$

the classical regularity results for elliptic equations give (1.3) (see for example SIMADER [1]). This establishes (1.2) for $A = 0$.

Step 2. Since $\varphi = 0$ on $\partial\Omega$, we have

$$\begin{aligned}\int_\Omega |A^- + \nabla^-\varphi(x)|^4\,dx &= \int_\Omega [|A^-|^2 + 2(A^- \cdot \nabla^-\varphi(x)) + |\nabla^-\varphi(x)|^2]^2\,dx \\ &= |A^-|^4 \operatorname{meas}\Omega + \tfrac{1}{3}\int_\Omega |\nabla^-\varphi(x)|^4\,dx \\ &\quad + 2\int_\Omega [|A^-|^2\,|\nabla^-\varphi(x)|^2 - (A^-\cdot\nabla^-\varphi(x))^2]\,dx \\ &\quad + \int_\Omega [6(A^-\cdot\nabla^-\varphi(x))^2 + 4(A^-\cdot\nabla^-\varphi(x))\,|\nabla^-\varphi(x)|^2 \\ &\quad + \tfrac{2}{3}\,|\nabla^-\varphi(x)|^4]\,dx.\end{aligned}$$

Observing that the last integral is positive, while the second is positive by the Cauchy-Schwarz inequality, and using Step 1 for the first integral, we establish the result.

§ 2. Proof of the Theorem

Proof of Theorem 1. The fourth part as well as the fact that if f_γ is polyconvex, then $|\gamma| \leqq 1$ have been established by DACOROGNA & MARCELLINI [1] (see also DACOROGNA [1]). To be complete we shall however prove these two facts again.

We divide the proof into four parts, each dealing with the corresponding convexity condition.

Part I. We prove that f_γ is convex if and only if $|\gamma| \leqq \frac{2}{3}\sqrt{2}$. We divide the proof into two steps.

Step 1. We first show that if $|\gamma| \leqq \frac{2}{3}\sqrt{2}$, then f_γ is convex. This is equivalent to showing that

$$4\psi_\gamma(A, B) := \sum_{i,j,k,l=1}^{2} \frac{\partial^2 f_\gamma}{\partial\xi_{ij}\,\partial\xi_{kl}}(A)\,B_{ij}B_{kl} \geqq 0 \quad \text{for every } A, B \in \mathbb{R}^{2\times 2}. \tag{2.1}$$

A direct computation gives

$$\begin{aligned}\psi_\gamma(A, B) &= 2(1-\gamma)\,(A^+\cdot B^+)^2 + 4(A^+\cdot B^+)\,(A^-\cdot B^-) + 2(1+\gamma)\,(A^-\cdot B^-)^2 \\ &\quad + (1-\gamma)\,|A^+|^2\,|B^+|^2 + |A^+|^2\,|B^-|^2 + |A^-|^2\,|B^+|^2 \\ &\quad + (1+\gamma)\,|A^-|^2\,|B^-|^2.\end{aligned} \tag{2.2}$$

Since $|\gamma| \leqq \frac{2}{3}\sqrt{2} < 1$, we may rewrite (2.2) as

$$\begin{aligned}\psi_\gamma(A, B) = &[2(1-\gamma)(A^+\cdot B^+)^2 \\ &+ 4\sqrt{1-\gamma^2}\,(A^+\cdot B^+)(A^-\cdot B^-) + 2(1+\gamma)(A^-\cdot B^-)^2] \\ &+ [|A^+|^2|B^-|^2 + |A^-|^2|B^+|^2 + 2(A^+\cdot B^+)(A^-\cdot B^-)] \\ &+ [(1-\gamma)|A^+|^2|B^+|^2 \\ &+ 2(1-2\sqrt{1-\gamma^2})(A^+\cdot B^+)(A^-\cdot B^-) + (1+\gamma)|A^-|^2|B^-|^2].\end{aligned}$$

Observing that the first bracketed term is positive, while the second and third are positive by the Cauchy-Schwarz inequality for every γ such that $(1-\gamma)(1+\gamma) - (1-2\sqrt{1-\gamma^2})^2 \geqq 0$, i.e., such that $|\gamma| \leqq \frac{2}{3}\sqrt{2}$, we obtain the result.

Step 2. We now show that if f_γ is convex, then $|\gamma| \leqq \frac{2}{3}\sqrt{2}$. We assume therefore that $1 \geqq |\gamma| > \frac{2}{3}\sqrt{2}$ (since by Part III below, the convexity of f_γ implies the polyconvexity of f_γ, which implies that $|\gamma| \leqq 1$). We then show that f_γ is not convex, which is equivalent (cf. (2.1)) to showing that there exist $A_\gamma, B_\gamma \in \mathbb{R}^{2\times 2}$ such that $\psi_\gamma(A_\gamma, B_\gamma) < 0$. This is easily done by choosing

$$B_\gamma = \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} \quad \text{with } b \text{ any root of}$$

$$b^2 - \frac{4\gamma}{4-3\gamma^2}b + 1 = 0 \tag{2.3}$$

$$A_\gamma = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \quad \text{with } a \text{ defined below.}$$

A direct computation gives

$$\psi_\gamma(A_\gamma, B_\gamma) = a^2(3b^2 - 3\gamma b + 1) - a(3\gamma b^2 - 4b + 3\gamma) + (b^2 - 3\gamma b + 3).$$

To show that we can choose a so that $\psi_\gamma(A_\gamma, B_\gamma) < 0$, it is sufficient to show that

$$\Delta = (3\gamma b^2 - 4b + 3\gamma)^2 - 4(3b^2 - 3\gamma b + 1)(b^2 - 3\gamma b + 3) > 0$$

for $1 \geqq |\gamma| > \frac{2}{3}\sqrt{2}$ and b satisfying (2.3). It turns out that in fact

$$\Delta = 12\gamma^2 b^2 \frac{9\gamma^2 - 8}{4 - 3\gamma^2},$$

which is strictly positive if $1 \geqq |\gamma| > \frac{2}{3}\sqrt{2}$. This concludes the proof of Part I.

Part II. We now prove that f_γ is rank-one convex if and only if $|\gamma| \leqq 2/\sqrt{3}$. We divide the proof into two steps.

Step 1. We first show that if $|\gamma| \leqq 2/\sqrt{3}$, then f_γ is rank-one convex. This is equivalent to showing (see, for example, DACOROGNA [1]) that the Legendre-

Hadamard condition holds, i.e.,

$$4\psi_\gamma(A, B) \geqq 0 \quad \text{for every } A, B \in \mathbb{R}^{2\times 2} \text{ with } \det B = 0. \tag{2.4}$$

Using (2.2) and the fact that $\det B = 0$ if and only if $|B^+|^2 = |B^-|^2$, we immediatly obtain

$$\begin{aligned}\psi_\gamma(A, B) &= 2(1-\gamma)(A^+\cdot B^+)^2 + 4(A^+\cdot B^+)(A^-\cdot B^-) + 2(1+\gamma)(A^-\cdot B^-)^2\\ &\quad + (2-\gamma)|A^+|^2|B^+|^2 + (2+\gamma)|A^-|^2|B^-|^2.\\ &= [(4-3\gamma)(A^+\cdot B^+)^2 + 4(A^+\cdot B^+)(A^-\cdot B^-) + (4+3\gamma)(A^-\cdot B^-)^2]\\ &\quad + [(2-\gamma)(|A^+|^2|B^+|^2 - (A^+\cdot B^+)^2)\\ &\quad + (2+\gamma)(|A^-|^2|B^-|^2 - (A^-\cdot B^-)^2)].\end{aligned}$$

Thus

$$\psi_\gamma(A, B) \geqq 0 \quad \text{for every } |\gamma| \leqq \frac{2}{\sqrt{3}},$$

since the second bracketed term is positive by the Cauchy-Schwarz inequality for every $|\gamma| \leqq 2$ and the first is positive for every γ such that $(4-3\gamma)(4+3\gamma) - 4 \geqq 0$, i.e., $|\gamma| \leqq 2/\sqrt{3}$. The proof of Step 1 is complete.

Step 2. We now prove that if f_γ is rank-one convex, then $|\gamma| \leqq 2/\sqrt{3}$. We assume therefore that $|\gamma| > 2/\sqrt{3}$. We then show that f_γ is not rank-one convex, which is equivalent (cf. (2.4)) to showing that there exist $A_\gamma, B_\gamma \in \mathbb{R}^{2\times 2}$ with $\det B_\gamma = 0$ such that $\psi_\gamma(A_\gamma, B_\gamma) < 0$. This is easily done. Choose

$$B_\gamma = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \quad A_\gamma = \begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}$$

with a defined below. A direct computation gives

$$\psi_\gamma(A_\gamma, B_\gamma) = 3a^2 - 3\gamma a + 1.$$

To show that we can choose a so that $\psi_\gamma(A_\gamma, B_\gamma) < 0$, it is sufficient to show that

$$\Delta = 9\gamma^2 - 12 > 0 \quad \text{for } |\gamma| > \frac{2}{\sqrt{3}},$$

which is trivial. This concludes the proof of Part II.

We now turn our attention to the polyconvexity and quasiconvexity of f_γ. Observing that $f_\gamma(Q\xi) = f_{-\gamma}(\xi)$ for every $(\gamma, \xi, Q) \in \mathbb{R}\times\mathbb{R}^{2\times 2}\times\mathbb{R}^{2\times 2}$ with $Q^{\mathrm{T}}Q = I$ and $\det Q = -1$, we easily see that f_γ is polyconvex if and only if $f_{-\gamma}$ is polyconvex and that f_γ is quasiconvex if and only if $f_{-\gamma}$ is quasiconvex. Hence, we may asume without loss of generality that $\gamma \geqq 0$.

Part III. We prove that f_γ is polyconvex $\Leftrightarrow$ $0 \leqq \gamma \leqq 1$. We do this in two steps.

Step 1. We first show that if f_γ is polyconvex, then $0 \leqq \gamma \leqq 1$. Observe that if f_γ is polyconvex, then f_γ must be bounded from below by a quasiaffine function, that is,

$$f_\gamma(\xi) \geqq c_0 + c_1 \det \xi + (\xi_0 \cdot \xi) \tag{2.5}$$

for some $c_0, c_1 \in \mathbb{R}$, some $\xi_0 \in \mathbb{R}^{2\times 2}$ and for every $\xi \in \mathbb{R}^{2\times 2}$. Combining (1.1) and (2.5) (and choosing $\xi^- = 0$) we immediately obtain

$$(1-\gamma)\,|\xi^+|^4 \geqq c_0 + \frac{c_1}{2}\,|\xi^+|^2 + (\xi_0 \cdot \xi^+) \quad \text{for every } \xi \in \mathbb{R}^{2\times 2},$$

which is possible only if $1 - \gamma \geqq 0$. This concludes the proof of Step 1.
Step 2. We now prove that if $0 \leqq \gamma \leqq 1$, then f_γ is polyconvex. Observe that

$$f_\gamma(A+B) - f_\gamma(A) = \sigma_\gamma(A, B) + \left(\sum_{i,j=1}^{2} \frac{\partial f_\gamma}{\partial \xi_{ij}}(A) B_{ij} - 2\gamma\,|A|^2 \det B \right) \tag{2.6}$$

where (cf. (2.1))

$$\sigma_\gamma(A, B) := f_\gamma(B) + \sum_{i,j=1}^{2} \frac{\partial f_\gamma}{\partial \xi_{ij}}(B) A_{ij} + 2\psi_\gamma(A, B) + 2\gamma\,|A|^2 \det B. \tag{2.7}$$

If we show that $\sigma_\gamma(A, B) \geqq 0$ for every $A, B \in \mathbb{R}^{2\times 2}$ and every $\gamma \in [0, 1]$ then, since the second term in (2.6) is a null Lagrangian, we should have (cf., for example, DACOROGNA [1]) that f_γ is polyconvex whenever $\gamma \in [0, 1]$. Observe that σ_γ is affine in γ and therefore

$$\sigma_\gamma(A, B) \geqq \min\{\sigma_0(A, B), \sigma_1(A, B)\} \quad \text{for every } \gamma \in [0, 1].$$

Since σ_0 is trivially positive and σ_1 is positive by Lemma 3 below, we have indeed established the second statement of the theorem.
Part IV. We now proceed to establish the third statement of the theorem. We do this in two steps.
Step 1. We prove that if f_γ is quasiconvex, then $0 \leqq \gamma \leqq 1 + \varepsilon$ for some $\varepsilon > 0$, as follows. Let

$$I_\gamma(A, \varphi) := \int_\Omega [f_\gamma(A + \nabla\varphi(x)) - f_\gamma(A)]\, dx$$

for every $A \in \mathbb{R}^{2\times 2}$ and every $\varphi \in W_0^{1,\infty}(\Omega; \mathbb{R}^2)$. In view of Step 2 below, the fourth part of Theorem 1 and the general fact that if f_γ is quasiconvex, then f_γ is rank-one convex, it will be sufficient, in order to obtain the result, to prove that $I_\gamma(A, \varphi) \geqq 0$ implies $I_\beta(A, \varphi) \geqq 0$ provided $0 \leqq \beta \leqq \gamma$. This is easily done by observing that the two following cases imply the result.

Case 1. $\int_\Omega [|A + \nabla\varphi(x)|^2 \det(A + \nabla\varphi(x)) - |A|^2 \det A]\, dx \leqq 0.$

Then, since $\beta \geqq 0$, it is clear that $I_\beta(A, \varphi) \geqq 0$.

Case 2. $\int_\Omega [|A + \nabla\varphi(x)|^2 \det(A + \nabla\varphi(x)) - |A|^2 \det A]\, dx \geqq 0.$

We then have

$$I_\beta(A, \varphi) - I_\gamma(A, \varphi) = 2(\gamma - \beta) \int_\Omega [|A + \nabla\varphi(x)|^2 \det (A + \nabla\varphi(x)) - |A|^2 \det A]\, dx \geqq 0.$$

Step 2. We finally obtain that if $0 \leqq \gamma \leqq 1 + \varepsilon$ for some $\varepsilon > 0$, then f_γ is quasiconvex. Since this is the most interesting part of the theorem, we shall prove an intermediate result (Step $\tilde{2}$), which is unnecessary but might clarify the argument.

Step $\tilde{2}$. We prove the quasiconvexity of f_γ at 0 for $\gamma = 1 + \alpha$ with $\alpha > 0$ small enough. We have to prove that

$$\int_\Omega f_\gamma(\nabla\varphi(x))\, dx \geqq 0$$

for every $\varphi \in W_0^{1,\infty}(\Omega; \mathbb{R}^2)$ and for some $\alpha > 0$. Recall that by (1.1) we have

$$f_\gamma(\xi) = -\alpha|\xi^+|^4 + 2|\xi^+|^2|\xi^-|^2 + (2+\alpha)|\xi^-|^4 \geqq 2|\xi^-|^4 - \alpha|\xi|^4$$

for every $\xi \in \mathbb{R}^{2\times 2}$. By Theorem 2, we immediately obtain

$$\int_\Omega f_\gamma(\nabla\varphi(x))\, dx \geqq (2\varepsilon - \alpha) \int_\Omega |\nabla\varphi(x)|^4\, dx. \tag{2.8}$$

Therefore choosing $0 < \alpha < 2\varepsilon$, we have the result that f_γ is quasiconvex at 0.

Step 2. We now prove that if $0 \leqq \gamma \leqq 1 + \varepsilon$ for some $\varepsilon > 0$, then f_γ is quasiconvex. We write $\gamma = 1 + \alpha$ and assume that $\alpha > 0$, the case $0 \leqq \gamma \leqq 1$ having been dealt with in Part III. We want to show that

$$\int_\Omega [f_\gamma(A + \nabla\varphi(x)) - f_\gamma(A)]\, dx \geqq 0 \tag{2.9}$$

for every $A \in \mathbb{R}^{2\times 2}$, $\varphi \in W_0^{1,\infty}(\Omega; \mathbb{R}^2)$ and for some $\alpha > 0$. By (2.6) this is equivalent to showing that

$$\int_\Omega \sigma_\gamma(A, \nabla\varphi(x))\, dx \geqq 0.$$

Keeping in mind that $\gamma = 1 + \alpha$ and using (2.7) we rewrite σ_γ as

$$\sigma_\gamma(A, B) = \sigma_{\varepsilon,\alpha}(A, B) + \varepsilon f_1(B) - 2\alpha|B|^2 \det B - \frac{\varepsilon^2}{2}|B|^4$$

where $f_1(B) = |B|^2(|B|^2 - 2\det B)$ and

$$\sigma_{\varepsilon,\alpha}(A, B) := (1-\varepsilon) f_1(B) + \frac{\varepsilon^2}{2}|B|^4 + \sum_{i,j=1}^{2} \frac{\partial f_\gamma}{\partial \xi_{ij}}(B) A_{ij} + (2\psi_\gamma(A, B) + 2\gamma|A|^2 \det B). \tag{2.10}$$

By Lemma 3 below, $\sigma_{\varepsilon,\alpha} \geqq 0$ for ε and α small enough. Therefore (2.9) and the quasiconvexity of f_γ are implied by

$$\int_\Omega \left[\varepsilon f_1(\nabla\varphi(x)) - 2\alpha|\nabla\varphi(x)|^2 \det \nabla\varphi(x) - \frac{\varepsilon^2}{2}|\nabla\varphi(x)|^4\right] dx \geqq 0$$

for every $\varphi \in W_0^{1,\infty}(\Omega; \mathbb{R}^2)$. This follows from (2.8) (with $\alpha = 0$) and Theorem 2 by choosing $\alpha \leqq \varepsilon^2/2$ and using the fact that $-2|\nabla\varphi|^2 \det \nabla\varphi \geqq -|\nabla\varphi|^4$. Therefore the proof of the theorem will be complete once Lemma 3 is established.

Lemma 3. *For every ε, $\alpha \geqq 0$ sufficiently small, $\sigma_{\varepsilon,\alpha}$ given by* (2.10) *satisfy*

$$\sigma_{\varepsilon,\alpha}(A, B) \geqq 0 \quad \textit{for every } A, B \in \mathbb{R}^{2\times 2}.$$

In particular, the result holds for $\varepsilon = \alpha = 0$, i.e., $\sigma_1(A, B) \geqq 0$ in (2.7).

Proof. The idea of the proof is to establish that for every B fixed, there exists an $A = A(B)$ that minimizes $\sigma_{\varepsilon,\alpha}$, and then to show that $\sigma_{\varepsilon,\alpha} \geqq 0$ at every stationary point of $\sigma_{\varepsilon,\alpha}(A, B)$ for sufficiently small ε and α.
Step 1. In the following we shall always assume that $B \neq 0$; otherwise the result is trivial. We now fix B, $B \neq 0$. To show that $\sigma_{\varepsilon,\alpha}$ has a minimum, we need to show that the quadratic term in (2.10) is always positive, i.e., that

$$\min_{|A|=1} \{\psi_\gamma(A, B) + \gamma|A|^2 \det B\} > 0. \tag{2.11}$$

If this holds, then for fixed $B \neq 0$, $\sigma_{\varepsilon,\alpha}$ attains its minimum. Our aim will then be to show that this minimum is positive.

To prove (2.11), observe that the minimum is always attained. Let μ be the corresponding Lagrange multiplier. We therefore have

$$\frac{\partial\psi_\gamma}{\partial A}(A, B) + 2\gamma(\det B)A = \mu A, \tag{2.12}$$

that is,

$$2|B|^2A - \gamma|B|^2\tilde{A} + 4(A\cdot B)B - 2\gamma(\tilde{A}\cdot B)B - 2\gamma(A\cdot B)\tilde{B} = \mu A. \tag{2.13}$$

By multiplying (2.13) by A, we see that (2.11) is equivalent to showing that $\mu > 0$. Multiplying (2.13) first by B and then by $\tilde{B}$ we find that

$$(A\cdot B)(\mu + 4\gamma \det B - 6|B|^2) + (\tilde{A}\cdot B)(3\gamma|B|^2) = 0,$$

$$(A\cdot B)(3\gamma|B|^2 - 8 \det B) + (\tilde{A}\cdot B)(\mu + 4\gamma \det B - 2|B|^2) = 0.$$

Thus two cases can happen, either $(A\cdot B) = (\tilde{A}\cdot B) = 0$ and therefore (2.11) is satisfied for $\alpha = \gamma - 1 > 0$ small enough, or

$$(\mu + 4\gamma \det B - 6|B|^2)(\mu + 4\gamma \det B - 2|B|^2) - 3\gamma|B|^2(3\gamma|B|^2 - 8 \det B) = 0,$$

i.e.,

$$\mu^2 - 8\mu(|B|^2 - \gamma \det B) + [(12 - 9\gamma^2)|B|^4 - 8\gamma|B|^2 \det B + 16\gamma^2(\det B)^2] = 0.$$

This implies that for $\alpha = \gamma - 1 > 0$ small enough the roots of the last equation are positive if they are real. Therefore, the proof of (2.11) is complete.
Step 2. We compute the stationary points of $\sigma_{\varepsilon,\alpha}$ for B fixed:

$$\frac{\partial\sigma_{\varepsilon,\alpha}}{\partial A}(A, B) = 4|B|^2B - 4\gamma(\det B)B - 2\gamma|B|^2\tilde{B} + 4|B|^2A - 2\gamma|B|^2\tilde{A}$$
$$+ 8(A\cdot B)B - 4\gamma(A\cdot B)\tilde{B} - 4\gamma(\tilde{A}\cdot B)B = 0. \tag{2.14}$$

We now multiply (2.14) first by A, then by B and finally by $\tilde{B}$ to get

$$\begin{aligned} 2(A\cdot B)|B|^2 - 2\gamma(A\cdot B)\det B - \gamma(\tilde{A}\cdot B)|B|^2 \\ = 2|A|^2|B|^2 - 2\gamma|B|^2\det A + 4(A\cdot B)^2 - 4\gamma(A\cdot B)(\tilde{A}\cdot B) \\ + 4|B|^2(A\cdot B) - 4\gamma(A\cdot B)\det B - 2\gamma(\tilde{A}\cdot B)|B|^2, \end{aligned} \quad (2.15)$$

$$\begin{aligned} -\gamma(\tilde{A}\cdot B)|B|^2 = -\tfrac{2}{3}|B|^4 + \tfrac{4}{3}\gamma|B|^2\det B - 2(A\cdot B)|B|^2 \\ + \tfrac{4}{3}\gamma(A\cdot B)\det B, \end{aligned} \quad (2.16)$$

$$\begin{aligned} 2(\tilde{A}\cdot B)(|B|^2 - 2\gamma\det B) = (A\cdot B)(3\gamma|B|^2 - 8\det B) + \gamma|B|^4 \\ - 4|B|^2\det B + 4\gamma(\det B)^2. \end{aligned} \quad (2.17)$$

We now will use (2.14) to (2.17) to show that $\sigma_{\varepsilon,\alpha} \geqq 0$ at a stationary point provided $\alpha = \gamma - 1$ and ε are small enough. Combining (2.16) and (2.17) we find that

$$\begin{aligned} (A\cdot B)[3(1-6\alpha-3\alpha^2)|B|^4 - 8(1+\alpha)|B|^2\det B + 16(1+\alpha)^2(\det B)^2] \\ = -|B|^2[(1-6\alpha-3\alpha^2)|B|^4 - 4(1+\alpha)|B|^2\det B + 4(1+\alpha)^2(\det B)^2]. \end{aligned} \quad (2.18)$$

We now use (2.15), (2.16) and (2.18) to compute $\sigma_{\varepsilon,\alpha}$ at the stationary point to get

$$\begin{aligned} \frac{3\sigma_{\varepsilon,\alpha}}{|B|^2}[3(1-6\alpha-3\alpha^2)|B|^4 - 8(1+\alpha)|B|^2\det B + 16(1+\alpha)^2(\det B)^2] \\ = [\tfrac{3}{2}\varepsilon^2|B|^2 + (1-3\varepsilon)|B|^2 - 2(1-3\varepsilon-2\alpha)\det B] \\ \times[3(1-6\alpha-3\alpha^2)|B|^4 - 8(1+\alpha)|B|^2\det B + 16(1+\alpha)^2(\det B)^2] \\ + 2(1+\alpha)\det B[(1-6\alpha-3\alpha^2)|B|^4 - 4(1+\alpha)|B|^2\det B \\ + 4(1+\alpha)^2(\det B)^2]. \end{aligned} \quad (2.19)$$

Therefore the positivity of $\sigma_{\varepsilon,\alpha}$ at the stationary point is equivalent to the positivity of the right-hand side of (2.19), denoted by $\tau_{\varepsilon,\alpha}$. Since α will be chosen very small even with respect to ε we shall group all the terms depending on α together; we find

$$\begin{aligned} \tau_{\varepsilon,\alpha}(A,B) &= [(1-3\varepsilon)|B|^2 + \tfrac{3}{2}\varepsilon^2|B|^2 - 2(1-3\varepsilon)\det B] \\ &\quad\times[3|B|^4 - 8|B|^2\det B + 16(\det B)^2] \\ &\quad + 2\det B[|B|^4 - 4|B|^2\det B + 4(\det B)^2] + O(\alpha, |B|^6) \\ &= [|B|^2 - 2\det B][3(1-3\varepsilon)|B|^4 - 6(1-4\varepsilon)|B|^2\det B \\ &\quad + 12(1-4\varepsilon)(\det B)^2] \\ &\quad + \tfrac{3}{2}\varepsilon^2|B|^2[2|B|^4 + (|B|^2 - 4\det B)^2] + O(\alpha, |B|^6). \end{aligned}$$

Therefore choosing ε small enough and α sufficiently small compared to $\frac{3}{2}\varepsilon^2$ we have indeed established Lemma 3.

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