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Further Existence Results for Classical Solutions of the Equations of a Second-Grade Fluid

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To our dear friend, Professor Christian Simader, on his 50th birthday

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1. Introduction

Over the last twenty years, there has been a remarkable interest in the study of a class of non-Newtonian fluids, known as fluids of grade n, from both theoretical and experimental points of view, see, *e.g.*, [7, 14, 15, 5, 9, 16, 19] and the references cited therein. For a general incompressible fluid of grade 2, the Cauchy stress $\boldsymbol{T}$ is given by

$$\boldsymbol{T} = -\tilde{p}\boldsymbol{I} + \mu \boldsymbol{A}_1 + \alpha_1 \boldsymbol{A}_2 + \alpha_2 \boldsymbol{A}_1^2, \tag{1.0}$$

where μ is the viscosity, α_1, α_2 are material coefficients (normal stress moduli), $\tilde{p}$ represents the pressure and $\boldsymbol{A}_1$, $\boldsymbol{A}_2$ are the first two Rivlin-Ericksen tensors [17, 20], defined by

$$\boldsymbol{A}_1 = \nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T,$$

$$\boldsymbol{A}_2 = \frac{d}{dt}\boldsymbol{A}_1 + \boldsymbol{A}_1 \nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T \boldsymbol{A}_1.$$

Here $\boldsymbol{v}$ is the velocity field and $\frac{d}{dt}$ denotes the material time derivative. The thermodynamical principles impose some restrictions on α_1 and α_2 [7]. In particular, the Clausius-Duhem inequality implies that

$$\mu \geqq 0, \quad \alpha_1 + \alpha_2 = 0$$

and the requirement that the free energy be a minimum in equilibrium implies that

$$\alpha_1 \geqq 0.$$

With these conditions on the stress moduli, the equations of motion for an incompressible homogeneous fluid of grade 2 are given by

$$\left.\begin{aligned} \frac{\partial}{\partial t}(\boldsymbol{v} - \alpha \Delta \boldsymbol{v}) - \nu \Delta \boldsymbol{v} &= \nabla p - \mathbf{curl}(\boldsymbol{v} - \alpha \Delta \boldsymbol{v}) \times \boldsymbol{v} \\ \nabla \cdot \boldsymbol{v} &= 0 \end{aligned}\right\} \quad \text{in } \Omega \times (0, T) \tag{1.1}$$

(cf. [7]) with the initial and boundary conditions

$$\begin{aligned} \boldsymbol{v}(x,t) &= \mathbf{0} \qquad \text{on } \partial\Omega \times (0,T), \\ \boldsymbol{v}(x,0) &= \boldsymbol{v}_0(x) \quad \text{in } \Omega. \end{aligned} \tag{1.2}$$

Here Ω is a bounded domain in $\boldsymbol{R}^3$ with smooth boundary $\partial\Omega$ and $\Omega \times (0,T)\,(T > 0)$ is a space-time region. The initial velocity is denoted by $\boldsymbol{v}_0(x)$ and $p = \tilde{p}/\rho$, $\alpha = \alpha_1/\rho$, $\nu = \mu/\rho$, where ρ is the (constant) density of the fluid. For simplicity, we assume that there are no external body forces acting on the fluid. Under the assumptions

$$\nu > 0, \quad \alpha > 0,$$

problem (1.1), (1.2) has been studied by several authors. In particular, CIORANESCU & EL HACÈNE [5] have given existence and uniqueness results for weak solutions global (in time) in two dimensions, and local (in time) in three dimensions. More recently, GALDI, GROBBELAAR & SAUER [10], by using a different approach, have shown for the general system (even without the restriction $\alpha_1 + \alpha_2 = 0$) existence and uniqueness of *classical* solutions for short time. Furthermore, if the size of the initial data is suitably restricted, their solutions exist for *all* time. However, for this latter result to hold, it is crucial to require that $\alpha_1\,(\equiv \alpha\rho)$ is "sufficiently large". It should be remarked that such a condition looks — in a sense — artificial, since one expects that global solutions should be favoured by a "large" viscosity and a "small" constant of elasticity. The main objective of the present paper is to remove such a restriction on α. Specifically, we show that if Ω is simply-connected and $\boldsymbol{v}_0$ is not too "large", problem (1.1), (1.2) admits one (and only one) *classical* solution, *global* in time, *for any* $\alpha > 0$.[1] For the general problem ($\alpha_1 + \alpha_2 \neq 0$) the steady case has been studied very recently by COSCIA & GALDI [6].

The method we follow here is based on an appropriate splitting of the original problem along with the use of the Schauder fixed-point theorem. The key requirement for this method to work is to obtain sufficiently strong *a priori* estimates on the higher-order derivatives of the solution. These are obtained by transforming the original problem into an equivalent one as follows. Using the elementary identity

$$\mathbf{curl}(\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v}\cdot\nabla\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{v} + (\nabla\cdot\boldsymbol{v})\boldsymbol{u} - (\nabla\cdot\boldsymbol{u})\boldsymbol{v} \tag{1.3}$$

with

$$\boldsymbol{u} = \mathbf{curl}(\boldsymbol{v} - \alpha\,\Delta\boldsymbol{v}) \tag{1.4}$$

and applying the **curl** operator to the first equation of (1.1) we find that

$$\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\nu}{\alpha}(\boldsymbol{u} - \mathbf{curl}\,\boldsymbol{v}) = \boldsymbol{u}\cdot\nabla\boldsymbol{v} - \boldsymbol{v}\cdot\nabla\boldsymbol{u} \quad \text{in } \Omega \times (0,T), \tag{1.5}$$

$$\nabla\cdot\boldsymbol{v} = 0 \quad \text{in } \Omega \times (0,T), \tag{1.6}$$

$$\begin{aligned} \boldsymbol{u}(x,0) &= \mathbf{curl}(\boldsymbol{v}_0 - \alpha\Delta\boldsymbol{v}_0) = \boldsymbol{u}_0 \quad \text{in } \Omega, \\ \boldsymbol{v}|_{\partial\Omega} &= 0 \quad \forall t \in [0,T]. \end{aligned} \tag{1.7}$$

[1] The main results of this paper still hold in two space dimensions (plane flow), and, in fact, their proofs are somewhat simpler. We also remark that for $\alpha = 0$ the problem is well-known since, in this case, equations (1.1) reduce to the Navier-Stokes equations.

It is clear that for classical solutions problems (1.1), (1.2) and (1.4)–(1.7) are equivalent. In fact, if $\boldsymbol{u}$ is a solution of (1.4)–(1.7), then from relation (1.3) and the hypothesis on Ω, it follows easily that there exists a function p such that (1.1) holds.

The Schauder theorem is used in the following manner. In a suitable class of functions $\boldsymbol{\varphi}$, we introduce the map

$$\Phi\colon \boldsymbol{\varphi} \mapsto \boldsymbol{u}$$

as the composition of the operator $\boldsymbol{\varphi} \mapsto \boldsymbol{v}$, defined by

$$\begin{aligned} \mathbf{curl}(\boldsymbol{v} - \alpha\,\Delta\boldsymbol{v}) &= \boldsymbol{\varphi} \quad \text{in } \Omega\times(0,T), \\ \nabla\cdot\boldsymbol{v} &= 0 \quad \text{in } \Omega\times(0,T), \\ \boldsymbol{v}\,|_{\partial\Omega} &= \mathbf{0} \end{aligned} \tag{1.8}$$

(where the time t is considered as a fixed parameter), with the operator $\boldsymbol{v} \mapsto \boldsymbol{u}$ defined by

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} + \frac{\nu}{\alpha}(\boldsymbol{u} - \mathbf{curl}\,\boldsymbol{v}) &= \boldsymbol{u}\cdot\nabla\boldsymbol{v} - \boldsymbol{v}\cdot\nabla\boldsymbol{u} \quad \text{in } \Omega\times(0,T), \\ \boldsymbol{u}(\cdot,0) &= \boldsymbol{u}_0 \quad \text{in } \Omega. \end{aligned} \tag{1.9}$$

The existence of a solution to the problem (1.4)–(1.7) (and hence to (1.1), (1.2)) is guaranteed as soon as we are able to show that $\Phi(\boldsymbol{\varphi})$ has a fixed point, which in turn will be achieved by making use of the Schauder theorem.

The paper is organized as follows. In Section 2, we prove existence, uniqueness and *a priori* bounds for solutions to the linear auxiliary problems (1.8), (1.9). We also derive estimates of the solutions which hold for all times $t > 0$, provided the size of $\boldsymbol{u}_0$ is suitably restricted. The main results are stated and proved in Section 3. Taking advantage of the *a priori* estimates and standard compactness arguments, we apply the fixed-point theorem to show local existence in the time interval $[t_0, t_0 + T]$ ($t_0 \geqq 0$), where T depends solely on an upper bound for the initial data $\boldsymbol{u}(t_0)$ in the appropriate norm. Successively, we use this result together with the global estimates of Section 2 and a standard "bootstrap" extension argument to prove global existence of classical solutions to the original problem for small data.

2. Unique solvability of some auxiliary problems

To begin with, we introduce some standard notations. By Ω we denote a bounded domain of $\boldsymbol{R}^3$. Throughout the paper we assume that Ω is simply-connected, so that any irrotational vector function in Ω is the gradient of some scalar function[2].

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[2] Actually, some of our intermediate results hold under more general assumptions on Ω which, in particular, do not require simply-connectedness. However, such an assumption is crucial for Lemma 2.1, *e.g.*, and our main Theorem 3.2 to be valid.

We follow the usual convention (*cf.*, *e.g.*, [10]) of defining $W^{m,p}(\Omega)(H^m(\Omega)$, if $p=2)$ as the standard Sobolev space of order m, with the usual norm $\|\cdot\|_{m,p}(\|\cdot\|_m)$ and scalar product $(\cdot,\cdot)_{m,p}$ $((\cdot,\cdot)_m)$. In $H^0(\Omega)=L^2(\Omega)$ the norm is denoted by $\|\cdot\|_0$, and for simplicity, the scalar product by $(\cdot,\cdot)$. The space $H^{m-1/2}(\partial\Omega)$ equipped with norm $\|\cdot\|_{m-1/2,\partial\Omega}$ is the trace space associated with $H^m(\Omega)$. By $\boldsymbol{H}^m(\Omega)$ we denote the Sobolev space of the vector fields $\boldsymbol{v}=(v_1,v_2,v_3)$ such that $v_i\in H^m(\Omega)$, $i=1,2,3$. In the sequel this convention will be applied to the other function spaces and norms. For $m\geqq 0$, we consider the Hilbert spaces $\boldsymbol{V}_m=\{\boldsymbol{v}\in\boldsymbol{H}^m(\Omega):\nabla\cdot\boldsymbol{v}=0\}$, $\boldsymbol{X}_m=\{\boldsymbol{v}\in\boldsymbol{V}_m:\boldsymbol{v}\cdot\boldsymbol{n}=0 \text{ on } \partial\Omega\}$, where $\boldsymbol{n}$ denotes the unit outer normal vector. They are both closed subspaces of $\boldsymbol{H}^m(\Omega)$; the norms in $\boldsymbol{V}_m,\boldsymbol{X}_m$ are also denoted by $\|\cdot\|_m$. Moreover, for a Banach space Y with norm $\|\cdot\|_Y$ and the given time interval $I=(t_0,t_0+T)$, $t_0\geqq 0$, $T>0$, we recall the classical Banach spaces $(1\leqq p<\infty)$

$$L^p(I;Y)=\{v \text{ measurable}, v:t\in I\mapsto v(t)\in Y, \int_I\|v(t)\|_Y^p\,dt<\infty\}$$

and $W^{m,p}(I;Y)$, which is the space of functions such that the distributional time derivatives of order up to m are in $L^p(I;Y)$. For $p=\infty$, $L^\infty(I;Y)$ is the Banach space of the (measurable) essentially bounded functions defined on I with values in Y. We denote by $\|\cdot\|_{k,m,T}$ the usual norm in $W^{k,\infty}(I;H^m(\Omega))$, for $k\geqq 0$. In particular, for $k=0$ we write $\|\cdot\|_{0,m,T}=\|\cdot\|_{m,T}$. Finally, the space of functions of class C^m on I with values in Y is denoted by $C^m(I;Y)$.

The purpose of this section is to study the unique solvability of certain auxiliary problems related to (1.4)–(1.7). We begin by considering the linear problem

$$\begin{aligned}\mathbf{curl}(\boldsymbol{v}-\alpha\Delta\boldsymbol{v})&=\boldsymbol{\varphi}\quad\text{in }\Omega_T,\\ \nabla\cdot\boldsymbol{v}&=\boldsymbol{0}\quad\text{in }\Omega_T,\\ \boldsymbol{v}|_{\partial\Omega}&=\boldsymbol{0}\end{aligned}\tag{2.1}$$

for a known function $\boldsymbol{\varphi}$, where $\Omega_T=\Omega\times I$ (the time t is a fixed parameter).

We have

Lemma 2.1. *Let* $\boldsymbol{\varphi}\in W^{k,\infty}(I;\boldsymbol{V}_m)$, $k,m\geqq 0$. *Then there exists a unique vector field* $\boldsymbol{\psi}\in W^{k,\infty}(I;\boldsymbol{X}_{m+1})$ *such that*

$$\mathbf{curl}\,\boldsymbol{\psi}=\boldsymbol{\varphi},\tag{2.2}$$

$$\|\boldsymbol{\psi}\|_{k,m+1,T}\leqq C\|\boldsymbol{\varphi}\|_{k,m,T}\tag{2.3}$$

for a constant $C=C(m,\Omega)$.

Proof. The existence of a unique vector potential $\boldsymbol{\psi}\in\boldsymbol{X}_1$ satisfying $\mathbf{curl}\,\boldsymbol{\psi}=\boldsymbol{\varphi}$ (for $\boldsymbol{\varphi}\in\boldsymbol{V}_0$) and the inequality $\|\boldsymbol{\psi}\|_1\leqq C\|\boldsymbol{\varphi}\|_0$ is proved, for instance, in [12]. The regularity result (2.3) for $k=0$ follows easily from [8, Proposition 1.4, page 41]. Finally, differentiating (2.2) k times with respect to t and using uniqueness, one easily sees that $\boldsymbol{\psi}\in W^{k,\infty}(I;\boldsymbol{X}_{m+1})$ and that the estimate (2.3) holds for any $k\geqq 0$. □

Lemma 2.2. *For any* $\varphi \in W^{k,\infty}(I; V_m)$, $k, m \geqq 0$, *there exists a unique* $v \in W^{k,\infty}(I; V_{m+3})$ *satisfying both* (2.1) *and the estimate*

$$\|v\|_{k,m+3,T} \leqq C\|\varphi\|_{k,m,T}. \tag{2.4}$$

Proof. By Lemma 2.1 we can write $\varphi = \mathbf{curl}\,\psi$ for some $\psi \in W^{k,\infty}(I; X_{m+1})$. Let us consider the following Stokes-like problem

$$\begin{aligned} v - \alpha\Delta v + \nabla\pi &= \psi \quad \text{in } \Omega_T, \\ \nabla\cdot v &= 0 \quad \text{in } \Omega_T, \\ v|_{\partial\Omega} &= 0, \end{aligned} \tag{2.5}$$

where π is a pressure term associated with the irrotational vector field $v - \alpha\,\Delta v - \psi$. From known existence and uniqueness results [4, 2] we find that the field v exists and satisfies the estimate

$$\|v\|_{k,m+3,T} \leqq C\|\psi\|_{k,m+1,T}.$$

From Lemma 2.1 it is clear that v satisfies (2.1) and inequality (2.4). It is also clear, in view of the assumption on Ω and of the uniqueness of problem (2.5), that v is uniquely determined. □

Our next goal is to show solvability and obtain appropriate *a priori* estimates for certain initial-value problems. To this end we need a preliminary lemma on inequalities involving norms in $H^m(\Omega)$.

Lemma 2.3. *Let* $m \geqq 0$. *If* $v \in X_{m+2}$ *and* $u \in H^m(\Omega)$, *then*[3]

$$|(v\cdot\nabla u, u)_m| \leqq C_1\|v\|_{m+2}\|u\|_m^2. \tag{2.6}$$

If $v \in H^{m+3}(\Omega)$ *and* $u \in H^m(\Omega)$, *then*

$$\|u\cdot\nabla v\|_m \leqq C_2\|v\|_{m+3}\|u\|_m. \tag{2.7}$$

If $v \in H^{m+2}(\Omega)$ *and* $u \in H^{m+1}(\Omega)$, then

$$\|v\cdot\nabla u\|_m \leqq C_3\|v\|_{m+2}\|u\|_{m+1}. \tag{2.8}$$

Proof. Inequality (2.6) is proved as in [10]. Concerning (2.7), we notice that

$$\|u\cdot\nabla v\|_m \leqq C\|v\|_{C^{m+1}}\|u\|_m$$

and so, by the Sobolev embedding theorem [1], we deduce (2.7). Inequality (2.8) is proved in the same way. □

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[3] Notice that if $m = 0$, the trilinear form $(v\cdot\nabla u, u)$ is identically zero.

We now consider the unique solvability of the initial-value problem

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} + \frac{v}{\alpha}(\boldsymbol{u} - \mathbf{curl}\,\boldsymbol{v}) &= \boldsymbol{u}\cdot\nabla\boldsymbol{v} - \boldsymbol{v}\cdot\nabla\boldsymbol{u} \quad \text{in } \Omega_T, \\ \boldsymbol{u}(\cdot, t_0) &= \boldsymbol{u}(t_0) \quad \text{in } \Omega. \end{aligned} \tag{2.9}$$

Specifically, we have

Lemma 2.4. *Assume that* $\boldsymbol{v} \in L^\infty(I;\boldsymbol{X}_{m+3})$, $m \geqq 1$, *with* $\|\boldsymbol{v}\|_{m+3,T} \leqq M$, *and that* $\boldsymbol{u}(t_0) \in \boldsymbol{H}^m(\Omega)$. *Then there exists a unique solution* $\boldsymbol{u}$ *to* (2.9) *such that*

$$\boldsymbol{u} \in L^\infty(I;\boldsymbol{H}^m(\Omega)) \cap W^{1,\infty}(I;\boldsymbol{H}^{m-1}(\Omega))$$

$$\|\boldsymbol{u}\|_{m,T} + \left\|\frac{d\boldsymbol{u}}{dt}\right\|_{m-1,T} \leqq C, \tag{2.10}$$

with $C = C(\Omega, m, M, T, v, \alpha, \|\boldsymbol{u}(t_0)\|_m)$. *Moreover, if* $\nabla\cdot\boldsymbol{u}(t_0) = 0$, *then* $\nabla\cdot\boldsymbol{u} = 0$ *in* Ω_T.

Proof. Let us derive an *a priori* estimate for the solution of problem (2.9). To this end, we apply the derivative operator D^k (k is a multi-index) to both sides of $(2.9)_1$, take the scalar product in $L^2(\Omega)$ with $D^k\boldsymbol{u}$ and sum over k, with $0 \leqq |k| \leqq m$. We thus obtain

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}\|_m^2 + \frac{v}{\alpha}\|\boldsymbol{u}\|_m^2 = \frac{v}{\alpha}(\mathbf{curl}\,\boldsymbol{v}, \boldsymbol{u})_m + (\boldsymbol{u}\cdot\nabla\boldsymbol{v}, \boldsymbol{u})_m - (\boldsymbol{v}\cdot\nabla\boldsymbol{u}, \boldsymbol{u})_m. \tag{2.11}$$

By using the Schwarz and Cauchy inequalities along with Lemma 2.3, we find that

$$\begin{aligned} |(\mathbf{curl}\,\boldsymbol{v}, \boldsymbol{u})_m| &\leqq \tfrac{1}{2}(\|\boldsymbol{v}\|_{m+1}^2 + \|\boldsymbol{u}\|_m^2), \\ |(\boldsymbol{u}\cdot\nabla\boldsymbol{v}, \boldsymbol{u})_m| &\leqq \|\boldsymbol{u}\cdot\nabla\boldsymbol{v}\|_m \|\boldsymbol{u}\|_m \leqq C_2\|\boldsymbol{v}\|_{m+3}\|\boldsymbol{u}\|_m^2, \\ |(\boldsymbol{v}\cdot\nabla\boldsymbol{u}, \boldsymbol{u})_m| &\leqq C_1\|\boldsymbol{v}\|_{m+2}\|\boldsymbol{u}\|_m^2. \end{aligned} \tag{2.12}$$

Thus, collecting (2.11), (2.12) we conclude that

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}\|_m^2 + \frac{v}{2\alpha}\|\boldsymbol{u}\|_m^2 \leqq \frac{v}{2\alpha}\|\boldsymbol{v}\|_{m+1}^2 + (C_1 + C_2)\|\boldsymbol{v}\|_{m+3}\|\boldsymbol{u}\|_m^2. \tag{2.13}$$

Integrating this inequality over I, with the help of Gronwall's lemma we find that

$$\|\boldsymbol{u}\|_m \leqq D_1, \tag{2.14}$$

with $D_1 = D_1(\Omega, m, M, T, v, \alpha, \|\boldsymbol{u}(t_0)\|_m)$. Moreover, from (2.9), with (2.14) and the estimates of Lemma 2.3, it also follows that

$$\begin{aligned} \left\|\frac{d\boldsymbol{u}}{dt}\right\|_{m-1,T} &\leqq \frac{v}{\alpha}(\|\boldsymbol{u}\|_{m-1,T} + \|\boldsymbol{v}\|_{m,T}) \\ &\quad + C(\|\boldsymbol{v}\|_{m+2,T}\|\boldsymbol{u}\|_{m-1,T} + \|\boldsymbol{v}\|_{m+1,T}\|\boldsymbol{u}\|_{m,T}) \\ &\leqq D_2 \end{aligned} \tag{2.15}$$

with $D_2 = D_2(\Omega, m, M, T, v, \alpha, \|\boldsymbol{u}(t_0)\|_m)$.

With estimates (2.14) and (2.15) in hand, we can show by using the Galerkin method [18] that there exists a solution to problem (2.9), satisfying the regularity properties stated in the lemma, for all $T > 0$. The uniqueness of the solution follows easily from Gronwall's lemma.

Let us finally show that $\boldsymbol{u}$ is solenoidal in Ω_T, provided that $\boldsymbol{u}(t_0)$ has the same property. In fact, taking the divergence on both sides of $(2.9)_1$ and using the general identity (1.3) we get

$$\frac{\partial \zeta}{\partial t} + \frac{\nu}{\alpha}\zeta = -\nabla\cdot(\boldsymbol{v}\cdot\nabla\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{v}) = -\nabla\cdot(\zeta\boldsymbol{v})$$

with $\zeta = \nabla\cdot\boldsymbol{u}$.

Now, from the equality

$$\nabla\cdot(\zeta\boldsymbol{v}) = (\boldsymbol{v}\cdot\nabla)\zeta + \zeta(\nabla\cdot\boldsymbol{v})$$

it follows that

$$\frac{\partial \zeta}{\partial t} + \frac{\nu}{\alpha}\zeta = -(\boldsymbol{v}\cdot\nabla)\zeta, \quad \zeta(\cdot, t_0) = 0,$$

and we easily prove that the unique solution of this homogeneous transport equation is $\zeta(t) \equiv 0$ for all $t \in I$. The proof of the lemma is therefore completed. □

The last result of this section concerns *a priori* estimates for solutions to problem (1.4)–(1.7) which for convenience we rewrite:

$$\left.\begin{aligned} &\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\nu}{\alpha}(\boldsymbol{u} - \mathbf{curl}\,\boldsymbol{v}) = \boldsymbol{u}\cdot\nabla\boldsymbol{v} - \boldsymbol{v}\cdot\nabla\boldsymbol{u} \\ &\mathbf{curl}(\boldsymbol{v} - \alpha\Delta\boldsymbol{v}) = \boldsymbol{u} \\ &\nabla\cdot\boldsymbol{v} = 0 \end{aligned}\right\} \text{ in } \Omega\times(0,T),$$
$$\begin{aligned} &\boldsymbol{v}|_{\partial\Omega} = \mathbf{0}, \quad t \in [0,T], \\ &\boldsymbol{u}(x,0) = \mathbf{curl}(\boldsymbol{v}_0 - \alpha\Delta\boldsymbol{v}_0) \equiv \boldsymbol{u}_0. \end{aligned} \tag{2.16}$$

In this analysis we often need a simple result on a differential inequality, which we prove here in the form of a lemma.

Lemma 2.5. *Let $y(t)$ be a smooth positive function in $[0,T]$ satisfying the inequality*

$$y'(t) + (k_1 - k_2 y^\rho(t))y(t) \leqq F(t) \quad \textit{for all } t \in [0,T], \tag{2.17}$$

where $k_1 > 0$, $k_2 \in \boldsymbol{R}$, $\rho \geqq 0$ and

$$\int_0^T F(t)\,dt < \infty.$$

Moreover, let $\varepsilon > 0$ *be such that* $k_1 - k_2\varepsilon^\rho = k > 0$. *If*

$$\int_0^T F(t)\,dt < \frac{\varepsilon}{2}, \quad y(0) < \frac{\varepsilon}{2},$$

then it follows that $y(t) < \varepsilon$ *for all* $t \in [0, T]$.

Proof. Assume for contradiction that for some $\bar{t}$ we have

$$y(\bar{t}) = \varepsilon \quad \text{and} \quad y(t) < \varepsilon \quad \forall t \in [0, \bar{t}).$$

Since $k_1 - k_2\varepsilon^\rho = k > 0$, we obtain the inequality

$$y'(t) + ky(t) \leqq F(t) \quad \forall t \in [0, \bar{t}],$$

which when integrated over $[0, \bar{t}]$ gives

$$y(\bar{t}) \leqq y(0) + \int_0^{\bar{t}} F(s)\,ds;$$

hence

$$y(\bar{t}) \leqq y(0) + \delta < \frac{\varepsilon}{2} + \delta < \varepsilon$$

for δ small enough. □

Lemma 2.6. *Assume that* $(\boldsymbol{v}, \boldsymbol{u})$ *is a solution to* (2.16) *and that* $\boldsymbol{u}_0 \in V_m$, $m \geqq 1$, *with*

$$\boldsymbol{u} \in L^\infty(0, T; \boldsymbol{H}^m(\Omega)) \cap W^{1,\infty}(0, T; \boldsymbol{H}^{m-1}(\Omega)),$$
$$\boldsymbol{v} \in L^\infty(0, T; \boldsymbol{X}_{m+3}).$$

Then there exists $\delta = \delta(\Omega, m, \nu, \alpha) > 0$ *such that if*

$$\|\boldsymbol{u}_0\|_m \leqq \delta, \tag{2.18}$$

then

$$\|\boldsymbol{u}\|_{m,T}^2 + \int_0^T \|\boldsymbol{u}(s)\|_m^2\,ds \leqq \delta_1 \|\boldsymbol{u}_0\|_m^2, \quad \left\|\frac{d\boldsymbol{u}}{dt}\right\|_{m-1,T} \leqq \delta_2, \tag{2.19}$$

where δ_1 *and* δ_2 *depend only on* Ω, m, ν, α *and* δ.

Proof. We replace $\boldsymbol{u}$ in $(2.16)_1$ by $\mathbf{curl}(\boldsymbol{v} - \alpha\Delta\boldsymbol{v})$ and use the identity (1.3) to obtain

$$\frac{\partial}{\partial t}\mathbf{curl}(\boldsymbol{v} - \alpha\Delta\boldsymbol{v}) + \frac{\nu}{\alpha}(\mathbf{curl}(\boldsymbol{v} - \alpha\Delta\boldsymbol{v}) - \mathbf{curl}\,\boldsymbol{v}) = \mathbf{curl}(\mathbf{curl}(\alpha\Delta\boldsymbol{v} - \boldsymbol{v}) \times \boldsymbol{v}). \tag{2.20}$$

Setting $\boldsymbol{\omega} = \operatorname{curl}\boldsymbol{v}$ and eliminating the **curl** on both sides of (2.20), we find that there exists a scalar field p such that

$$\frac{\partial}{\partial t}(\boldsymbol{v} - \alpha\Delta\boldsymbol{v}) - \nu\Delta\boldsymbol{v} = (\alpha\Delta\omega - \omega) \times \boldsymbol{v} + \nabla p. \tag{2.21}$$

Since $\nabla \cdot \boldsymbol{v} = 0$, $\boldsymbol{v}|_{\partial\Omega} = \mathbf{0}$, multiplying (2.21) by $\boldsymbol{v}$ and integrating by parts over Ω, we get

$$\frac{d}{dt}\int_\Omega (|\boldsymbol{v}|^2 + \alpha|\nabla\boldsymbol{v}|^2)\,dx + 2\nu\int_\Omega |\nabla\boldsymbol{v}|^2\,dx = 0.$$

Now we set

$$\alpha_0 = \min\{1,\alpha\}, \quad \alpha_1 = \max\{1,\alpha\}$$

and integrate over $[0,t]$ to obtain

$$\alpha_0 \|\boldsymbol{v}\|_1^2 + 2\nu\int_0^t\int_\Omega (|\nabla\boldsymbol{v}|^2\ dx)(s)\,ds \leqq \alpha_1 \|\boldsymbol{v}_0\|_1^2 \quad \forall t \in [0,T].$$

From the Poincaré inequality (with constant γ) and with $\nu_0 = \min\{\nu, \nu\gamma\}$ we obtain

$$\alpha_0 \|\boldsymbol{v}\|_{1,T}^2 + \nu_0\int_0^T \|\boldsymbol{v}(s)\|_1^2\,ds \leqq 2\alpha_1 \|\boldsymbol{v}_0\|_1^2.$$

If we take $\beta = 2\alpha_1/\min\{\nu_0,\alpha_0\}$, then this implies that

$$\|\boldsymbol{v}\|_{1,T}^2 + \int_0^T \|\boldsymbol{v}(s)\|_1^2\,ds \leqq \beta\|\boldsymbol{v}_0\|_1^2. \tag{2.22}$$

Now let us prove an energy estimate on $\boldsymbol{u}$. Multiplying $(2.16)_1$ by $\boldsymbol{u}$ and integrating by parts over Ω, we easily find that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\boldsymbol{u}|^2\,dx + \frac{\nu}{\alpha}\int_\Omega |\boldsymbol{u}|^2\,dx = \frac{\nu}{\alpha}\int_\Omega \mathbf{curl}\,\boldsymbol{v}\cdot\boldsymbol{u}\,dx + \int_\Omega \boldsymbol{u}\cdot\nabla\boldsymbol{v}\cdot\boldsymbol{u}\,dx$$

since it is clear that $\int_\Omega \boldsymbol{v}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,dx = 0$. Using the Cauchy inequality, the estimates of Lemma 2.3 for $m = 0$, and the estimate (2.4) of Lemma 2.2 (with $\varphi = \boldsymbol{u}$), we get

$$\frac{d}{dt}\|\boldsymbol{u}\|_0^2 + 2\frac{\nu}{\alpha}\|\boldsymbol{u}\|_0^2 \leqq \frac{\nu}{\alpha}(\|\boldsymbol{v}\|_1^2 + \|\boldsymbol{u}\|_0^2) + 2C\|\boldsymbol{u}\|_0^3$$

and thus the differential inequality

$$\frac{d}{dt}\|\boldsymbol{u}\|_0^2 + \left(\frac{\nu}{\alpha} - 2C\|\boldsymbol{u}\|_0\right)\|\boldsymbol{u}\|_0^2 \leqq \frac{\nu}{\alpha}\|\boldsymbol{v}\|_1^2, \tag{2.23}$$

which we can write in the form

$$y'(t) + \left(\frac{\nu}{\alpha} - 2C\sqrt{y(t)}\right)y(t) \leqq F(t). \tag{2.24}$$

Since the right-hand side of (2.24) is controlled by the initial data in the sense that

$$\int_0^T F(s)\,ds \leqq \sigma \quad \text{(from the estimate (2.22))}$$

for some $\sigma > 0$, then according to Lemma 2.5, there exist $\sigma_1, \sigma_2 > 0$ such that

$$\|\boldsymbol{u}_0\|_0 < \sigma_1 \quad \text{implies that} \quad \|\boldsymbol{u}(t)\|_0 < \sigma_2 \quad \forall t \in [0,T].$$

Using this last estimate together with (2.22) in inequality (2.23) yields

$$\|u\|_{0,T}^2 + \int_0^T \|u(s)\|_0^2\,ds \leqq \sigma_3 \|u_0\|_0^2 \tag{2.25}$$

for some $\sigma_3 > 0$. Now, let us write (2.11) for $m = 1$ as

$$\frac{1}{2}\frac{d}{dt}\|u\|_1^2 + \frac{\nu}{\alpha}\|u\|_1^2 = \frac{\nu}{\alpha}(\mathbf{curl}\, v, u)_1 + (u\cdot\nabla v, u)_1 - (v\cdot\nabla u, u)_1.$$

Using the estimates (2.12) for $m = 1$, we get

$$\frac{1}{2}\frac{d}{dt}\|u\|_1^2 + \frac{\nu}{\alpha}\|u\|_1^2 \leqq \frac{\nu}{2\alpha}(\|v\|_2^2 + \|u\|_1^2) + \bar{C}\|v\|_4\|u\|_1^2$$

and consequently

$$\frac{d}{dt}\|u\|_1^2 + \left(\frac{\nu}{\alpha} - 2\bar{C}\|v\|_4\right)\|u\|_1^2 \leqq \frac{\nu}{\alpha}\|v\|_2^2. \tag{2.26}$$

Now, by (2.25) and Lemma 2.2, we find that

$$\int_0^T \|v(t)\|_2^2\,dt \leqq \sigma_3\|u_0\|_0^2.$$

Therefore, by Lemma 2.5, (2.25), (2.26), and by a reasoning similar to that which we used before, it readily follows that there exist constants $\sigma_4, \sigma_5 > 0$ such that, if $\|u_0\|_1 < \sigma_4$, then

$$\|u\|_{1,T}^2 + k\int_0^T \|u(s)\|_1^2\,ds \leqq \sigma_5\|u_0\|_1^2$$

with $k > 0$. Thus we conclude that the estimate $(2.19)_1$ is valid for $m = 1$. We now prove the general case $m \geqq 2$, by induction. Thus, assuming that $(2.19)_1$ holds for m, let us show that it also holds for $m + 1$. By using the estimates (2.12) in (2.11) written for $m + 1$ we deduce that

$$\frac{d}{dt}\|u\|_{m+1}^2 + \left(\frac{\nu}{\alpha} - 2C\|u\|_{m+1}\right)\|u\|_{m+1}^2 \leqq \frac{\nu}{\alpha}\|v\|_{m+2}^2.$$

In view of Lemma 2.5 and of the inductive assumption, it follows, for $\|u_0\|_{m+1}$ sufficiently small, that

$$\|u\|_{m+1,T}^2 + \int_0^T \|u(s)\|_{m+1}^2\,ds \leqq \sigma_6\|u_0\|_{m+1}^2.$$

Finally, we observe that the estimate $(2.19)_2$ follows easily from (2.15), in view of $(2.19)_1$. □

Remark. Concerning the two-dimensional case we must observe that the main results of the preceding lemmas remain essentially the same and are even stronger.

In fact, since the conditions determining the vector potential are more stringent than those defining the two-dimensional stream function, Lemmas 2.1 and 2.2 are still valid. On the other hand, in $\boldsymbol{R}^2$ we rewrite problem (2.16) in the form

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} + \frac{v}{\alpha}(u - \operatorname{curl} \boldsymbol{v}) = -\boldsymbol{v}\cdot\nabla u \\ &\operatorname{curl}(\boldsymbol{v} - \alpha\Delta\boldsymbol{v}) = u \\ &\nabla\cdot\boldsymbol{v} = 0 \end{aligned}\right\} \text{ in } \Omega\times(0,T), \tag{2.27}$$

$$\boldsymbol{v}|_{\partial\Omega} = \boldsymbol{0}, \quad t\in[0,T]$$

$$u(x,0) = \operatorname{curl}(\boldsymbol{v}_0 - \alpha\Delta\boldsymbol{v}_0) \equiv u_0,$$

where u is now a scalar function. This system is obtained from problem (1.1), (1.2) by taking the scalar curl on both sides of the main equation and using the two-dimensional result

$$\operatorname{curl}(\operatorname{curl}(\boldsymbol{v} - \alpha\Delta\boldsymbol{v})\times\boldsymbol{v}) = \boldsymbol{v}\cdot\nabla\operatorname{curl}(\boldsymbol{v} - \alpha\Delta\boldsymbol{v}).$$

Following the proofs of the preceding lemmas we observe that the results remain valid in the two-dimensional case, in spite of the absence of the term $\boldsymbol{u}\cdot\nabla\boldsymbol{v}$ on the right-hand side of the system $(2.27)_1$.

3. Global existence of classical solutions

In this section we prove the existence of a unique global solution for problem (1.1), (1.2) (equivalent to problem (1.4)–(1.7)) when the initial data are small enough in the following way. First, by the Schauder fixed-point theorem, we prove the existence of a local solution in $I \equiv (t_0, t_0 + T)$ for all $t_0 \geqq 0$ where T depends on an upper bound for $\|\boldsymbol{u}(t_0)\|_m$, but is otherwise independent of t_0. Then using the global *a priori* estimates of Lemma 2.6 we can show the existence of a solution to (1.4)–(1.7) for all $t > 0$. We begin by recalling the following well-known result.

Theorem 3.1 (Schauder Fixed-Point Theorem). *A compact mapping Φ of a closed bounded convex set G in a Banach space Y into itself has a fixed point.*

Take the Banach space

$$Y = C(I; \boldsymbol{V}_{m-1}), \quad m \geqq 1,$$

and for $D > 0$ define

$$G = \{\boldsymbol{\varphi}\in Y : \boldsymbol{\varphi}\in L^\infty(I; \boldsymbol{H}^m(\Omega)),\ \|\boldsymbol{\varphi}\|_{m,T} \leqq D,\ \boldsymbol{\varphi}(x,t_0) = \boldsymbol{u}(x,t_0)\in\boldsymbol{V}_m\}.$$

Consider now the map

$$\Phi: \boldsymbol{\varphi} \mapsto \boldsymbol{u}$$

defined in G as the composition of the operator $\varphi \mapsto v$ defined by

$$\begin{aligned} \mathbf{curl}(v - \alpha\,\Delta v) &= \varphi \quad \text{in } \Omega_T, \\ \nabla \cdot v &= 0 \quad \text{in } \Omega_T, \\ v|_{\partial\Omega} &= 0 \end{aligned} \tag{3.1}$$

(where the time t is a parameter), with the operator $v \mapsto u$ defined by

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{v}{\alpha}(u - \mathbf{curl}\, v) &= u \cdot \nabla v - v \cdot \nabla u \quad \text{in } \Omega_T, \\ u(\cdot, t_0) &= u(t_0) \qquad\qquad\quad \text{in } \Omega. \end{aligned} \tag{3.2}$$

Notice that proving the existence of a solution to our original problem (1.4)–(1.7) is equivalent to showing that the map

$$\Phi\colon G \subset Y \to Y$$

admits a fixed point.

First we prove

Lemma 3.1. *For all t_0 and D the map Φ transforms the closed bounded convex set G into a relatively compact subset of Y. Moreover, for D sufficiently small, Φ is continuous in the topology of Y.*[4]

Proof. The closedness of set G is obvious. In fact, every sequence φ_n $(n = 1, \ldots)$ in G converging to φ in Y has a subsequence φ_{n_k} which converges weakly to a certain ψ in $L^\infty(I; \boldsymbol{H}^m(\Omega))$ such that $\|\psi\|_{m,T} \leqq \liminf \|\varphi_{n_k}\|_{m,T}$. Thus $\|\varphi\|_{m,T} = \|\psi\|_{m,T} \leqq D$.

To prove the compactness property of Φ, let us denote by v_n a sequence of solutions to problem (3.1), corresponding to the data $\varphi_n \in G$, such that v_n is uniformly bounded on I for the $\boldsymbol{H}^{m+3}(\Omega)$-norm (Lemma 2.2). Let $u_n \in G$ be the associated sequence of solutions to problem (3.2). Since

$$\begin{aligned} u_n \quad &\text{is bounded in } L^\infty(I; V_m), \\ \frac{du_n}{dt} \quad &\text{is bounded in } L^\infty(I; V_{m-1}), \end{aligned}$$

we have in particular that u_n is bounded in $W^{1,2}(I; V_{m-1})$, and by classical compactness arguments we conclude that $u_n \to u$ in Y.

To treat the continuity of Φ we still denote by $u, u_n \in Y (n = 1, \ldots)$ the corresponding images of $\varphi, \varphi_n \in G (n = 1, \ldots)$ under the map Φ. Let us subtract the

[4] The proof of continuity could be given without imposing restriction on D. However, this would be inessential for our purposes.

equations $(1.9)_1$ written for $\boldsymbol{u}$ and $\boldsymbol{u}_n$, multiply the result by $\boldsymbol{u}_n - \boldsymbol{u}$ and calculate the $\boldsymbol{H}^{m-1}(\Omega)$-inner product. Setting $y(t) = \|\boldsymbol{u}_n - \boldsymbol{u}\|_{m-1}$, after some easy calculations we get

$$y'(t) + \lambda y(t) \leqq C(1 + \|\boldsymbol{u}\|_{m-1,T} + \|\boldsymbol{u}\|_{m,T})\,\|\varphi_n - \varphi\|_Y,$$

where λ can be chosen positive for D sufficiently small. Then noticing that $\|\boldsymbol{u}\|_m$ is bounded (Lemma 2.4) we conclude that $\|\boldsymbol{u}_n - \boldsymbol{u}\|_Y \leqq C_1 \|\varphi_n - \varphi\|_Y$, and thus the continuity of Φ is proved. □

We are now ready to prove

Lemma 3.2 (Local Existence). *Given arbitrary $t_0 \geqq 0$ and $\boldsymbol{u}(x,t_0) \in \boldsymbol{V}_m$, $m \geqq 1$, with $\|\boldsymbol{u}(t_0)\|_m < D$, D sufficiently small, there exists $T > 0$ such that problem* (1.4)–(1.7) *has a unique solution in $[t_0, t_0 + T]$ with*

$$\begin{aligned} &(\boldsymbol{u}, \boldsymbol{v}) \in G \times (C(I; \boldsymbol{V}_{m+2}) \cap L^\infty(I; \boldsymbol{H}^{m+3}(\Omega))), \\ &\frac{d\boldsymbol{v}}{dt} \in L^\infty(I; \boldsymbol{H}^{m+2}(\Omega)). \end{aligned} \tag{3.3}$$

In particular, if $\|\boldsymbol{u}(t_0)\|_m < \frac{1}{2}D$, then T can be chosen as

$$T = \frac{K_1}{D} \ln\left(\frac{1 + K_2}{\frac{1}{4}D + K_2}\right) > 0 \tag{3.4}$$

where the positive constants K_1 and K_2 depend only on Ω, m, ν and α.

Proof. We apply the Schauder fixed-point theorem. In view of Lemma 3.1, for existence and the proof of $(3.3)_1$ we only need to prove that Φ maps G into itself. Let us take $\varphi \in G(\|\varphi\|_{m,T} \leqq D)$ and fix $t_0 \geqq 0$. From (2.13), proceeding as in the proof of Lemma 2.4 and using Gronwall's lemma, we get

$$\|\boldsymbol{u}\|_m^2 \leqq \|\boldsymbol{u}(t_0)\|_m^2 e^{(C_1 + C_2)Mt} + \frac{\nu M}{\alpha(C_1 + C_2)}(e^{(C_1 + C_2)Mt} - 1),$$

where $M = CD$ (with C the constant in Lemma 2.2). Thus, if $\|\boldsymbol{u}(x,t_0)\|_m < D$, it follows that there exists $T > 0$ such that $\Phi(G) \subset G$ and that, in particular, for $\|\boldsymbol{u}(x,t_0)\|_m < \frac{1}{2}D$ we may take T as in (3.4). Condition $(3.3)_2$ can be proved as in (2.15). Finally, the uniqueness of the solution follows from [5]. □

Using the global *a priori* estimates of Lemma 2.6, we can extend the local existence of the previous lemma to deduce our main result:

Theorem 3.2 (Global Existence). *Let $\boldsymbol{u}_0 \in \boldsymbol{V}_m$, $m \geqq 1$. There exists $\varepsilon_0 = \varepsilon_0(\Omega, m, \nu, \alpha) > 0$ such that if*

$$\|\boldsymbol{u}_0\|_m \leqq \varepsilon_0,$$

then problem (1.4)–(1.7) *has a unique solution for all* $t \in [0, \infty)$ *with*

$$\begin{aligned} &v \in C(0, T; V_{m+2}) \cap L^\infty(0, T; H^{m+3}(\Omega)) \\ &\frac{dv}{dt} \in L^\infty(0, T; H^{m+2}(\Omega)), \end{aligned} \tag{3.5}$$

for all $T > 0$. *Moreover, for* $m \geqq 2$,

$$\frac{d^2 v}{dt^2} \in L^\infty(0, T; H^{m+1}(\Omega)). \tag{3.6}$$

Thus in particular, for $m = 4$, v *is a classical solution, i.e.*,

$$v \in C^1(0, T; C^3(\Omega)).$$

Proof. We choose $\|u_0\|_m \leqq \min\{D, \delta\} = \varepsilon$. By Lemma 3.2 we have existence on $[0, T]$. Moreover, by Lemma 2.6 (inequality $(2.19)_1$) we have[5]

$$\|u(T)\|_m \leqq C \|u_0\|_m$$

for C independent of $\|u_0\|_m$ and T. Choosing $\|u_0\|_m \leqq \varepsilon/C = \varepsilon_0$ we get existence on $[T, 2T]$ and again by Lemma 2.6 we get

$$\|u(2T)\|_m \leqq C\|u_0\|_m \leqq \varepsilon_0.$$

Repeating this procedure we obtain the solution on $[0, +\infty)$ which satisfies properties (3.5). It remains to show (3.6). Differentiating (1.5) with respect to time t, we get

$$\frac{\partial^2 u}{\partial t^2} + \frac{\nu}{\alpha}\left(\frac{\partial u}{\partial t} - \mathbf{curl}\frac{\partial v}{\partial t}\right) = \frac{\partial}{\partial t}(u \cdot \nabla v) - \frac{\partial}{\partial t}(v \cdot \nabla u).$$

From Lemmas 2.6 and 2.2 we know that

$$\begin{aligned} &u \in L^\infty(0, T; H^m(\Omega)), \\ &\frac{\partial u}{\partial t} \in L^\infty(0, T; H^{m-1}(\Omega)), \\ &v \in L^\infty(0, T; H^{m+3}(\Omega)), \\ &\frac{\partial v}{\partial t} \in L^\infty(0, T; H^{m+2}(\Omega)), \end{aligned}$$

so that we have the estimates (*cf.* Lemma 2.3)

$$\left\|\frac{\partial}{\partial t}(u \cdot \nabla v)\right\|_{m-2,T} \leqq C_1\left(\left\|\frac{\partial u}{\partial t}\right\|_{m-2,T} \|v\|_{m+1,T} + \|u\|_{m-2,T}\left\|\frac{\partial v}{\partial t}\right\|_{m+1,T}\right) < +\infty,$$

$$\left\|\frac{\partial}{\partial t}(v \cdot \nabla u)\right\|_{m-2,T} \leqq C_2\left(\left\|\frac{\partial v}{\partial t}\right\|_{m,T} \|u\|_{m-1,T} + \|v\|_{m,T}\left\|\frac{\partial u}{\partial t}\right\|_{m-1,T}\right) < +\infty.$$

[5] It is clear that $\|u(T)\|_m$ is finite, since $u \in C(0, T; V_{m-1}) \cap L^\infty(0, T; H^m(\Omega))$.

Hence we find that

$$\frac{\partial^2 \boldsymbol{u}}{\partial t^2} \in L^\infty(0, T; \boldsymbol{H}^{m-2}(\Omega)),$$

and consequently from Lemma 2.2 we conclude (3.5). Now choosing $m = 4$ we have

$$\frac{\partial^2 \boldsymbol{u}}{\partial t^2} \in L^\infty(0, T; \boldsymbol{H}^2(\Omega)) \quad \text{which implies that} \quad \frac{\partial^2 \boldsymbol{v}}{\partial t^2} \in L^\infty(0, T; \boldsymbol{H}^5(\Omega)).$$

By a Sobolev embedding theorem we get

$$\boldsymbol{v} \in W^{2,\infty}(0, T; \boldsymbol{X}_5) \hookrightarrow C^1(0, T; \boldsymbol{C}^3(\Omega)). \quad \square$$

Remark. The results of Theorem 3.2 can also be obtained by using the Leray-Schauder fixed-point theory, as shown in a previous version of this paper [11]. However, the proof is more complicated than the one presented here.

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