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Wave fronts in elastic media with temperature dependent properties

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Abstract. This paper deals with the propagation of shock and acceleration wave fronts in elastic media with temperature dependent properties. The partial differential equations governing the evolution of such waves are derived and solved using the method of Charpit. Solutions for wave front propagation in a thermoelastic layer with exponentially temperature dependent properties are obtained.

1. Introduction

Some elastic materials, for instance metals, change their mechanical properties under temperature loadings, and a representation by means of a classical Hookean model does not adequately describe their behaviour [8]. To determine the interaction between the fields of deformation and temperature it can be applied the theory of thermoelasticity for isotropic materials with temperature dependent properties. The list of references connected with the problems of thermoelastic bodies with temperature variable elastic properties is given in [9].

Consider a disturbance suddenly applied over a surface S in an isotropic homoegenous elastic medium, which is initially undeformed and at rest. Then it is well known that the subsequent positions of the wavefront S_t which separate the disturbed from the quiescent regions of the medium, will constitute a system of parallel surfaces and will propagate with constant speed, cf. [2, 11]. Hence for a given initial wave shape the problem of wave evolution is solved. In non-homogeneous, but isotropic media, the situation is complicated by the fact that the wave speed is no longer position independent, cf. [3, 5].

In this paper our aim is to investigate the effect of material inhomogeneity caused by temperature dependent properties of the elastic bodies on the propagation of wave fronts. We confine attention to the class of singular waves of the order $k \geqslant 1$, which include both shock and acceleration waves.

Within the framework of the theory of thermal stresses [7], we restrict our consideration on finding the shape of the wavefronts arising from given initial disturbances in elastic bodies exhibiting certain specific choices of the temperature-dependent properties.

2. The fundamental equations

Restrict considerations to the certain dynamic problems within the framework of the linear thermoelasticity for materials with temperature dependent properties. Let λ, μ denote the Lamé coefficients, α_t, ω be the coefficients of linear thermal expansion and heat conduction, respectively. We assume that

$$\lambda = \tilde{\lambda}(\theta), \quad \mu = \tilde{\mu}(\theta), \quad \gamma = \tilde{\gamma}(\theta), \quad \omega = \tilde{\omega}(\theta), \tag{2.1}$$

where $\tilde{\lambda}(\cdot)$, $\tilde{\mu}(\cdot)$, $\tilde{\gamma}(\cdot)$, $\tilde{\omega}(\cdot)$ are given functions, $\theta = \theta(x_i, t)$ denotes the temperature increment measured from the reference state, (x_i), $i = 1, 2, 3$ and t comprise rectangular Cartesian coordinates and time, respectively, and $\tilde{\gamma} = (3\lambda + 2\mu)\alpha_t$.

The equations of motion and heat conduction in the case under consideration take the following form, cf. [7, 9]:

$$\begin{aligned} &\mu u_{i,kk} + [(\lambda + 2\mu) u_{k,k}]_{,i} - \mu u_{k,ki} - 2\mu_{,i} u_{k,k} \\ &\quad + (u_{i,k} + u_{k,i})\mu_{,k} + F_i - (\tilde{\gamma}\theta)_{,i} = \varrho \ddot{u}_i \\ &(\omega\theta_{,k})_{,k} - c\varrho\dot{\theta} = -\hat{W}, \end{aligned} \tag{2.2}$$

where u_i are components of the displacement vector, F_i are components of the body forces, ϱ is the mass density, c is the quantity of heat required for unit increase of temperature of unit mass and $\hat{W}$ denotes the quantity of heat generated in unit volume and unit time. The comma and dot denote partial differentiation with respect to the space variables x_i and the time t, respectively. Subscripts i, k run over 1, 2, 3 and summation convention holds.

Solving the nonlinear partial differential equation $(2.2)_4$ with adequate initial and boundary conditions we obtain the temperature distribution in the body. Knowing the temperature field and the dependence of the coefficients λ, μ, γ on the temperature (given by (2.1)) and hence also on position, the equations $(2.2)_{1-3}$ constitute a system of linear partial differential equations with variable coefficients. Let Ω, $\Omega \subset R^3$ be the region which is occupied by the thermoelastic body under consideration and $I \equiv (t_0, t_1)$ denote the time interval. It has to be emphasized that Eqs. (2.2) hold for every $(x, t) \in \Omega \times I$.

Consider now the problem of singular wave front propagation in the thermoelastic bodies with temperature dependent properties. We assume that the singular moving surface S_t of the order $k \geqslant 1$ (i.e., the surface of discontinuity in the k-th order of derivative of the displacement vector $\boldsymbol{u}$) in space–time is given in the form:

$$S_t\colon \Phi(t, x_1, x_2, x_3) \;=\; 0, \tag{2.3}$$

where $\Phi(\cdot)$ is a function of class $C^2(\Omega \times I)$.

A straightforward calculation, using the geometrical and kinematical compatibility relations, cf. [2, 11], and Eqs. (2.2) reveals that this surface must satisfy the nonlinear partial differential equations:

$$\begin{aligned} &[\Phi_{,1}(t, x_1, x_2, x_3)]^2 + [\Phi_{,2}(t, x_1, x_2, x_3)]^2 + [\Phi_{,3}(t, x_1, x_2, x_3)]^2 \\ &\quad = c_\gamma^{-2}(\theta(t, x_1, x_2, x_3))[\dot{\Phi}(t, x_1, x_2, x_3)]^2 \\ &\gamma = 1, 2, \end{aligned} \tag{2.4}$$

where

$$c_1^{-2} \equiv \varrho(\lambda + 2\mu)^{-1}, \quad c_2^{-2} \equiv \varrho\mu^{-1}. \tag{2.5}$$

Equation (2.4) governs the front of the longitudinal wave of the k-th order, $k \geqslant 1$, for $\gamma = 1$ and the front of the transversal wave for $\gamma = 2$. We observe that wave speeds c_γ, $\gamma = 1, 2$, given by Eqs. (2.5) are dependent on the temperature (according with Eqs. (2.1)). The temperature $\theta(\cdot)$ must be determined from the heat conduction equation $(2.3)_2$ and adequate initial and boundary conditions.

The nonlinear partial differential equations (2.4) may be solved by application of the Monge and Cauchy method, cf. [10]. Since the basic equations (2.4) are of the same form for both cases of the longitudinal and transversal wave fronts we shall drop the subscript γ from this point on. The Monge and Cauchy method reduces the solution of Eq. (2.4) to the solution of the system of ordinary differential equations:

$$\begin{aligned} \frac{\mathrm{d}x_1}{2p} &= \frac{\mathrm{d}x_2}{2q} = \frac{\mathrm{d}x_3}{2w} = \frac{\mathrm{d}t}{2rf(x_1, x_2, x_3, t)} \\ &= \frac{\mathrm{d}\Phi}{2(p^2 + q^2 + w^2 - r^2 f(x_1, x_2, x_3, t))} = \frac{\mathrm{d}p}{r^2 f_{,1}(x_1, x_2, x_3, t)} \\ &= \frac{\mathrm{d}q}{r^2 f_{,2}(x_1, x_2, x_3, t)} = \frac{\mathrm{d}w}{r^2 f_{,3}(x_1, x_2, x_3, t)} = \frac{\mathrm{d}r}{r^2 \dot{f}(x_1, x_2, x_3, t)}, \end{aligned} \tag{2.6}$$

where

$$f \equiv c_1^{-2} \quad \text{or} \quad f \equiv c_2^{-2} \tag{2.7}$$

and

$$p \equiv \Phi_{,1}, \quad q \equiv \Phi_{,2}, \quad w \equiv \Phi_{,3}, \quad r \equiv \dot{\Phi}, \quad f_{,i} = \frac{\partial f}{\partial \theta}\theta_{,i}, \quad \dot{f} = \frac{\partial f}{\partial \theta}\dot{\theta}. \tag{2.8}$$

The system of equations (2.6) can be solved for special cases of material properties by using the Charpit method, cf. [6]. The procedure will be applied in Section 3.

3. Wave fronts in a thermoelastic layer

Consider now the thermoelastic layer which occupies the region $\Omega = \{(x_1, x_2, x_3) \in R^3 : 0 \leqslant x_1 \leqslant h, x_2 \in R, x_3 \in R\}$. We assume that there are any heat sources in the body under consideration and that the coefficient of heat condition is described by

$$\omega = \omega_0 e^{\alpha\theta}, \tag{3.1}$$

where ω_0, α are given constants.

Let the boundary surfaces of the body be kept in constant but not the same temperatures. We take the boundary conditions of the heat conduction problem in the form

$$\theta(x_1 = 0, x_2, x_3, t) = 0, \quad \theta(x_1 = h, x_2, x_3, t) = \theta_1,$$
$$(x_2, x_3) \in R^2, \quad t \in R, \tag{3.2}$$

where θ_1 is a given constant temperature.

We confine attention to the steady temperature problem assuming that

$$\dot{\theta}(x_1, x_2, x_3, t) = 0, \quad (x_1, x_2, x_3, t) \in \Omega \times R_+. \tag{3.3}$$

The equation of heat conduction $(2.2)_4$ and the assumptions given by Eqs. (3.1) and (3.3) lead to the following equation

$$\nabla(e^{\alpha\theta}\nabla\theta(x)) = 0. \tag{3.4}$$

Let U be a new dependent variable defined according to Kirchhoff transformation, cf. [10], as

$$U(x) \equiv \int_0^{\theta(x)} e^{\alpha\theta'} d\theta' = \alpha^{-1}(e^{\alpha\theta(x)} - 1). \tag{3.5}$$

Then by substitution of Eq. (3.5) into Eq. (3.4) we obtain the Laplace equation for unknown function $U(\cdot)$

$$\nabla^2 U(x) = 0. \tag{3.6}$$

The boundary conditions (3.2) and Eq. (3.5) lead to the boundary conditions for function $U(\cdot)$

$$U(x_1 = 0, x_2, x_3) = 0, \quad U(x_1 = h, x_2, x_3) = \alpha^{-1}(e^{\alpha\theta_1} - 1),$$
$$(x_2, x_3) \in R^2. \tag{3.7}$$

The solution of the problem given by Eqs. (3.6) and (3.7) takes the form

$$U(x_1, x_2, x_3) = Ax_1, \tag{3.8}$$

where

$$A = (\alpha h)^{-1}(e^{\alpha\theta_1} - 1).$$

From Eqs. (3.8) and (3.5) it follows that

$$\theta(x_1, x_2, x_3) = \alpha^{-1} \ln (A\alpha x_1 + 1), \tag{3.9}$$

for $A\alpha x_1 + 1 > 0$, i.e., for every $x_1 \in \langle 0, h\rangle$. Assume that the density of mass ϱ is a constant and the Lamé coefficients λ, μ given by Eq. (2.1) are chosen as

$$\tilde{\lambda}(\theta) = \lambda_0 e^{\beta\theta}, \quad \tilde{\mu}(\theta) = \mu_0 e^{\beta\theta}, \quad \varrho = \text{const.}, \tag{3.10}$$

where λ_0, μ_0, β are given constants, $\lambda_0, \mu_0 > 0$. This choice corresponds to the Poisson ratio ν and the Young modulus E defined by

$$\nu = \frac{\lambda_0}{2(\lambda_0 + \mu_0)} = \text{const.}, \quad E = \frac{\mu_0(3\lambda_0 + 2\mu_0)}{\lambda_0 + \mu_0} e^{\beta\theta}. \tag{3.11}$$

The identical material properties given by Eq. (3.11) were considered in numerous papers, cf. [9].

Suppose that the thermoelastic layer under consideration propagates a singular wave of the order k, $k \geqslant 1$. Making appeal to the equations describing the wave front propagation given by Eqs. (2.3), (2.4) and (2.5), for the case of material properties defined by Eqs. (3.10), (2.1) and the temperature distribution determined by Eq. (3.9), we have

$$\Phi(t, x_1, x_2, x_3) \equiv t - \psi(x_1, x_2, x_3) = 0, \tag{3.12}$$

where unknown function $\psi(\cdot)$ satisfies the following equation:

$$[\psi_{,1}(x_1, x_2, x_3)]^2 + [\psi_{,2}(x_1, x_2, x_3)]^2 + [\psi_{,3}(x_1, x_2, x_3)]^2$$
$$= c_0^{-2}(A\alpha x_1 + 1)^{-\beta/\alpha}. \tag{3.13}$$

Here we have $c_0^{-2} = \varrho(\lambda_0 + 2\mu_0)^{-1}$ or $c_0^{-2} = \varrho\mu_0^{-1}$ in the cases of the longitudinal and transversal waves, respectively. The equation (3.13) constitutes the special case of Eq. (2.4) and it can be solved by the Monge and Cauchy method. Using Eqs. (2.6), (2.7) and (2.8) the equation (3.13) reduces to the system of ordinary differential equations

$$\frac{\mathrm{d}x_1}{2p} = \frac{\mathrm{d}x_2}{2q} = \frac{\mathrm{d}x_3}{2w} = \frac{\mathrm{d}t}{2c_0^{-2}(A\alpha x_1 + 1)^{-\beta/\alpha}}$$
$$= \frac{\mathrm{d}p}{-2c_0^{-2}\beta A(A\alpha x_1 + 1)^{-1-\beta/\alpha}} = \frac{\mathrm{d}q}{0} = \frac{\mathrm{d}w}{0}. \tag{3.14}$$

To solve Eqs. (3.14) we can employ the Charpit method, cf. [6], which was used in our previous papers [3–5]. The complete integral of Eqs. (3.14) can be written in the form

$$t = \psi(x_1, x_2, x_3) = \int \{c_0^{-2}(A\alpha x_1 + 1)^{-\beta/\alpha} - \varepsilon - \delta\}^{1/2}\,\mathrm{d}x_1$$
$$+ \sqrt{\varepsilon}x_2 + \sqrt{\delta}x_3 + \hat{\Phi}(\varepsilon, \delta), \quad \varepsilon, \delta > 0, \tag{3.15}$$

where ε, δ are parameters and $\hat{\Phi}(\cdot)$ is an unknown function. If we adopt the notations:

$$H(x_1, \varepsilon, \delta) \equiv \int \{c_0^{-2}(A\alpha x_1 + 1)^{-\beta/\alpha} - \varepsilon - \delta\}^{1/2}\,\mathrm{d}x_1 \tag{3.16}$$

and

$$G(x_i, t, \varepsilon, \delta, \hat{\Phi}(\varepsilon, \delta)) \equiv t - H(x_1, \varepsilon, \delta) - \sqrt{\varepsilon}x_2 - \sqrt{\delta}x_3 - \hat{\Phi}(\varepsilon, \delta), \tag{3.17}$$

the complete integral (3.15) takes the form

$$G(x_i, t, \varepsilon, \delta, \hat{\Phi}(\varepsilon, \delta)) = 0. \tag{3.18}$$

Consider now the initial data for the singular wave under consideration given over a surface $S_0 \cap \Omega$ in space–time by

$$S_0: t = t(s, z), \quad x_i = x_i(s, z), \quad i = 1, 2, 3, \tag{3.19}$$

where s, z are parameters. We require that the complete integral (3.17) satisfies initial conditions (3.19). This requirement is met along an envelope of (3.18) if we impose the conditions:

$$\begin{aligned} \Gamma(s, z, \varepsilon, \delta, \hat{\Phi}(\varepsilon, \delta)) &= 0, \\ \Gamma_{,s}(s, z, \varepsilon, \delta, \hat{\Phi}(\varepsilon, \delta)) &= 0, \\ \Gamma_{,z}(s, z, \varepsilon, \delta, \hat{\Phi}(\varepsilon, \delta)) &= 0, \end{aligned} \tag{3.20}$$

where Γ is defined by

$$\Gamma(s, z, \varepsilon, \delta, \hat{\Phi}(\varepsilon, \delta)) \equiv G(x_i(s, z), t(s, z), \varepsilon, \delta, \hat{\Phi}(\varepsilon, \delta)). \tag{3.21}$$

Solving the system of equations (3.20) for the unknown function $\hat{\Phi}(\varepsilon, \delta)$ we obtain a two parameter family of solutions G which satisfies the initial conditions (3.19) along their envelope. The required solution of wave front problem is obtained by elimination of ε and δ between the equations

$$G = 0, \quad G_{,\varepsilon} + G_{,\hat{\Phi}}\hat{\Phi}_{,\varepsilon} = 0, \quad G_{,\delta} + G_{,\hat{\Phi}}\hat{\Phi}_{,\delta} = 0. \tag{3.22}$$

We consider now special cases of the initial data (3.19).

Case 1

Let the initial data (3.19) be specified as follows

$$S_0: t = 0, \quad x_1 = h, \quad x_2 = s, \quad x_3 = z, \quad s, z \in R. \tag{3.23}$$

Thus, the region of disturbance in the time $t = 0$ is the plane $x_1 = h$.

We consider now the material properties defined by (3.1) and (3.10), where constants α, β satisfy

$$\alpha = -\beta. \tag{3.24}$$

Using the procedure described by Eqs. (3.16)–(3.22) and Eq. (3.24) we obtain for the initial data (3.23) that the solution of the wave front problem takes the form

$$t = \tfrac{2}{3}c_0^{-1}(A\alpha)^{-1}\{(A\alpha x_1 + 1)^{3/2} - (A\alpha h + 1)^{3/2}\}. \tag{3.25}$$

Case 2

We now consider the initial data in the form

$$S_t\colon t = s, \quad x_1 = 0, \quad x_2 = vs, \quad x_3 = z, \quad s, z \in R, \tag{3.26}$$

where $v > 0$, is a constant. Thus, the initial disturbance takes place over the plane $x_1 = 0$ and the wave source is the line $x_2 = vt$ moving with a constant speed v.

We take into account the material properties defined by Eqs. (3.1), (3.10) and (3.24).

From the form of the complete integral (3.18) and (3.17), with the H given by Eq. (3.16), and the initial data (3.26), it follows on solving Eqs. (3.20) that

$$\varepsilon = v^{-2}, \quad \delta = 0, \quad \hat{\Phi}(\varepsilon, \delta) = -\tfrac{2}{3}c_0^2(A\alpha)^{-1}(c_0^{-2} - v^{-2})^{3/2}. \tag{3.27}$$

The solution of the problem under consideration can be written in the form

$$t = \tfrac{2}{3}c_0^2(A\alpha)^{-1}\{[c_0^{-2}(A\alpha x_1 + 1) - v^{-2}]^{3/2} - (c_0^{-2} - v^{-2})^{3/2}\} + v^{-1}x_2. \tag{3.28}$$

In the limit case, $v \to \infty$, we obtain the solution given by Eq. (3.25) for simultaneous disturbance initiation.

Case 3

Now we will consider the thermoelastic layer which propagates a singular surface due to a moving source on its boundary $x_1 = 0$. We assume that the initial data takes the form

$$S_t: t = s, \quad x_1 = 0, \quad x_2 = vs\cos z, \quad x_3 = vs\sin z, \tag{3.29}$$

where $s > 0$, $0 \leqslant z < 2\pi$ and $v > 0$ are parameters. The wave front problem is obviously axisymmetric. The moving source is a circle of radius $r = vt$, which is expanding with constant speed v.

Solving Eqs. (3.20) for the case of complete integral given by (3.18), (3.17), (3.16) and the initial data (3.29) we obtain

$$z = \tan^{-1}(\delta\varepsilon^{-1}), \quad \delta = v^{-2} - \varepsilon, \quad \hat{\Phi}(\varepsilon, \delta) = -\tfrac{2}{3}c_0^2(A\alpha)^{-1}(c_0^{-2} - v^{-2})^{3/2}. \tag{3.30}$$

Substituting formulas (3.30) into equations (3.17), (3.16) and (3.18) results directly in the complete integral

$$t = \tfrac{2}{3}c_0^2(A\alpha)^{-1}\{[c_0^{-2}(A\alpha x_1 + 1) - v^{-2}]^{3/2} - (c_0^{-2} - v^{-2})^{3/2}\} + \sqrt{\varepsilon}x_2 + \sqrt{v^{-2} - \varepsilon}x_3. \tag{3.31}$$

Now, using $(3.22)_2$ we obtain for this case that

$$\varepsilon = \frac{v^{-2}x_2^2}{x_2^2 + x_3^2}. \tag{3.32}$$

Finally, we substitute (3.32) into Eq. (3.31). We have the required solution, namely

$$t = \tfrac{2}{3}c_0^2(A\alpha)^{-1}\{[c_0^{-2}(A\alpha x_1 + 1) - v^{-2}]^{3/2} - (c_0^{-2} - v^{-2})^{3/2}\} + v^{-1}(x_2^2 + x_3^2)^{1/2}. \tag{3.33}$$

In the limit case $v \to \infty$ we obtain from equation (3.33) that for a simultaneous disturbance initiation

$$t = \tfrac{2}{3}c_0^{-1}(A\alpha)^{-1}[(A\alpha x_1 + 1)^{3/2} - 1]. \tag{3.34}$$

The solution (3.34) coincides with equation (3.25).

4. Final remarks

In this paper within the framework of the theory of thermal stresses, the propagation of shock and acceleration wavefronts are considered in elastic bodies with temperature dependent properties. The obtained equations (2.4) governed the front of the longitudinal and transversal singular waves of the k-th order, $k \geqslant 1$, show that in case of a uniform temperature distribution the wave fronts will propagate as in isotropic homogeneous bodies, i.e., as parallel surfaces to the initial wavefronts. If the temperature field is dependent on position and time, this property of wavefront propagation will be distributed. In this case the shape of the wavefront is dependent on the initial wavefront, the temperature distribution and the temperature-dependent material properties. Hence, the wavefront problem may be treated by using the procedure given in Section 2.

The knowledge of wave front may be applied to solve the dynamic problems of stress distribution in the body under consideration by using the methods of series expansion given in [1, 12].

References

1. J.D. Achenbach: *Wave Propagation in Elastic Solids*. North-Holland, American Elsevier, Amsterdam (1979).
2. P. Chadwick and B. Powdrill: Singular surfaces in linear thermoelasticity, *Int. J. Engng. Sci.* 3 (1965) 561–595.
3. H. Cohen and S.J. Matysiak: Wave fronts in nonhomogeneous elastic media, *Int. J. Engng. Sci.* 21 (1983) 113–122.
4. H. Cohen and S.J. Matysiak: Wave fronts in nonhomogeneous three-dimensional elastodynamics, *S.M. Archives* 9 (1984) 451–471.
5. H. Cohen and S.J. Matysiak: Wave fronts due to moving sources, *Journal de Mecan. Theor. et Appl.* 2 (1983) 1003–1011.
6. A. Dou: *Lectures on Partial Differential Equations of First Order*. University of Notre Dame Press, Indiana (1972).
7. W. Nowacki: *Thermoelasticity*. Pergamon Press and PWN, Warszawa (1962).
8. J.L. Nowinski: Thermoelastic problem for an isotropic sphere with temperature dependent properties, *ZAMP* X (1959) 565–575.
9. J.L. Nowinski: *Theory of Thermoelasticity with Applications*. Sijthoff and Noordhoff Inter. Publishers B.V., Alphen aan den Rijn (1978).
10. M.N. Ozisik: *Heat Conduction*. Wiley-Interscience Publ., New York, Chichester, Brisbane, Toronto (1980).
11. T.Y. Thomas: *Plastic Flow and Fracture in Solids*. Academic Press, New York, London (1961).
12. Z. Wesolowski: Dynamic problems of the nonlinear elasticity (in Polish), PWN (1974).