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Convergence of the Phase-Field Equations to the Mullins-Sekerka Problem with Kinetic Undercooling

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Dedicated to Mort Gurtin on the occasion of his sixtieth birthday

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Abstract

I prove that the solutions of the phase-field equations, on a subsequence, converge to a weak solution of the Mullins-Sekerka problem with kinetic undercooling. The method is based on energy estimates, a monotonicity formula, and the equipartition of the energy at each time. I also show that for almost all t, the limiting interface is $(d-1)$-rectifiable with a square-integrable mean-curvature vector.

1. Introduction

Phase-field equations for solidification were introduced by CAGINALP [7, 8], COLLINS & LEVINE [15], FIX [19] and LANGER [24] to treat phenomena not covered by the classical Stefan problem. These equations, for the temperature (deviation) θ and the phase field φ, consist of a heat equation

$$c\theta_t + l\varphi_t = k\Delta\theta \tag{1.1}$$

and a Ginzburg-Landau equation

$$\beta\varphi_t = \lambda\Delta\varphi - \nu W'(\varphi) + l\theta \tag{1.2}$$

where c, l, k, β, λ and ν are positive constants and W is a double-well potential whose wells, of equal depth, correspond to the solid and liquid phases.

Recently thermodynamically consistent models have been developed in FRIED & GURTIN [20], PENROSE & FIFE [27], WANG et al. [33] and in references therein; in particular, [20], [27], and [33] allow the latent heat l to depend on the order parameter φ.

The main goal here is to rigorously study the global-time asymptotics of (1.1) and (1.2) in the limit $\varepsilon \downarrow 0$ for

$$c, k = 1, \quad \beta = \lambda = \varepsilon, \quad \nu = \frac{1}{\varepsilon}, \quad l = l(\varphi), \tag{1.3}$$

and for specificity with

$$W(\varphi) = \tfrac{1}{2}(1 - \varphi^2)^2, \quad l = 1 - \varphi^2, \tag{1.4}$$

a choice that is essentially the same as Model II in [33]. My analysis can be modified to analyze any smooth function l vanishing at the minimizers of W; i.e., any l of the form

$$l(\varphi) = (1 - \varphi^2)H(\varphi),$$

where $H \geqq 0$ is an arbitrary smooth function. A specific choice of H corresponds to Model I in [33], while the particular choice $l = W$ would simplify some of the analysis (cf. Remark 4.1 below). Observe that, granted (1.4), the nonlinearity $-\nu W'(\varphi) + l\theta$ in (1.2) vanishes at ± 1 for any value of θ.

The formal analyses of [7, 9, 15, 19] at least indicate that solutions of the Ginzburg-Landau equation (1.2) form a sharp interface whose normal velocity depends linearly on the mean curvature and the temperature of the interface. To describe this result precisely, let $(\theta^\varepsilon, \varphi^\varepsilon)$ be the solution of the phase-field equations with parameters consistent with (1.3) and assume that $(\theta^\varepsilon, \varphi^\varepsilon)$ converges to (θ, φ). Since the two minima of W are ± 1, it is easy to prove that $|\varphi| = 1$ almost everywhere. Let $\Gamma(t)$ be the interface separating the two regions

$$\Omega(t) = \{x : \varphi(t, x) = -1\}$$

and $\{\varphi = 1\}$. Then, formally, (θ, Ω) is a solution of the heat equation

$$\theta_t - \Delta\theta = -(h(\varphi))_t = \tfrac{4}{3}(\chi_{\Omega(t)})_t, \quad h(\varphi) = \varphi - \tfrac{1}{3}\varphi^3 \tag{1.5}$$

everywhere, coupled with the geometric equation

$$\vec{V} = H - \theta n \tag{1.6}$$

at the interface $\Gamma(t)$, where χ_Ω is the indicator of the set Ω, and where $\vec{V}$, n and H are, respectively, the normal velocity vector, the outward unit vector, and the mean-curvature vector of the interface $\Gamma(t)$. A derivation of these sharp interface equations from thermodynamics as well as an exhaustive list of earlier references are given in GURTIN's book [21, Chapter 3]. In 1964, MULLINS & SEKERKA [26] studied the linear stability of a related system of equations obtained by replacing (1.6) by the Gibbs-Thompson condition: $\theta = -K$. They showed that planar interfaces are unstable under some perturbations, thus explaining the dendritic growth observed in solidification. I refer to equations (1.5), (1.6) as the *Mullins-Sekerka problem* with kinetic undercooling.

My chief result is that, in the limit, θ and Ω constitute a weak solution of the Mullins-Sekerka problem with kinetic undercooling. This result is global in time; I do not assume the existence of a solution of (1.5), (1.6). Therefore I also provide an existence result for this limit problem, extending a previous result of CHEN & REITICH [12] for local-time existence. To the best of my knowledge, the only other global results are due to LUCKHAUS [25] and ALMGREN & WANG [3]. They proved the global existence of weak solutions for the heat equation (1.5) coupled with the Gibbs-Thompson condition: $\theta = -K$.

There are two essential difficulties in the analysis of (1.5), (1.6): a solution (θ, Ω) of (1.5), (1.6) can start out smooth and yet, in finite time, the boundary of Ω may develop geometric singularities, and θ may blow up pointwise (see the example in the Appendix). These difficulties also complicate the analysis of convergence. Since θ is unbounded, θ^ε does not converge to θ uniformly. For that reason I cannot use results of [4] concerning the convergence of (1.2) with a given continuous temperature field. The asymptotics of the Cahn-Allen equation, which is (1.2) with $l = 0$, is studied in [18] via sub- and supersolutions constructed from the weak solutions of the mean-curvature flow; unfortunately the approach of [18] is not directly applicable to the phase-field equations, as they do not have a maximum principle and there is no a priori weak theory for the limit equations.

I overcome these difficulties by utilizing the energy estimates in §2.2, and a monotonicity result in §5. The latter is an extension of the monotonicity formula of CHEN & STRUWE [13], which originates from STRUWE's formula for parabolic flow of harmonic maps [32], and a later result of ILMANEN [23] for the Cahn-Allen equation, which originates form HUISKEN's formula for smooth mean-curvature flows [22]. My main observation is that the geometric equation (1.6) is not simply a perturbation of the mean-curvature flow, and therefore the monotonicity should involve the mathematical energy

$$\int_{\mathcal{R}^d} \frac{\varepsilon}{2} |\nabla \varphi^\varepsilon|^2 + \frac{1}{\varepsilon} W(\varphi^\varepsilon) + \frac{1}{2} (\theta^\varepsilon)^2 \, dx$$

related to the system (1.5) and (1.6). The main technical difficulty is then to show that the discrepancy measure

$$\xi^\varepsilon(t; A) = \int_A \frac{\varepsilon}{2} |\nabla \varphi^\varepsilon|^2 - \frac{1}{\varepsilon} W(\varphi^\varepsilon) \, dx \tag{1.7}$$

has non-positive limiting value. For the Cahn-Allen equation, $\xi^\varepsilon \leqq 0$ follows easily from the maximum principle. For the phase-field equations, however, it follows from a series of estimates obtained in §4. In later sections, following ILMANEN [23], I prove that the weak* limit of ξ^ε is indeed equal to zero.

I close this introduction with a brief survey of related results. Equations (1.1), (1.2) with $c = \beta = 0$, $l = 1$ form the Cahn-Hilliard equation. Recently the convergence of the Cahn-Hilliard equation to the Hele-Shaw problem was proved by ALIKAKOS, BATES & CHEN [2] using a spectral estimate of CHEN [11]. In contrast to this paper, they assume the existence of a smooth solution to the limiting problem. Briefly, their method is to construct approximate solutions for the "ε problem" that are close to the smooth solution of the limit problem. They then use the spectral estimates to bound the error terms. Also, STOTH [30] studied the asymptotic limit of the phase-field equations with radial symmetry. Independently, a radially symmetric problem in an annular domain with one interface was studied in [10]. Asymptotics of the Cahn-Allen equation, obtained by setting l to zero in (1.2), have been studied extensively. An exhaustive list of references related to the Cahn-Allen equation can be found in my paper [29].

This paper is organized as follows. In the next section I outline the background and state the main results. In §3, several elementary estimates are obtained. A gradient estimate is proved in §4; this estimate implies that ξ^ε is non-positive in the limit. In §5, I derive a monotonicity result which I use in § 6 to prove a clearing-out lemma. I then establish the equipartition of energy in §7. In that section, I also show that the Hausdorff dimension of the interface is $d-1$. I complete the proof in Section 8. In the appendix, for a simple radially symmetric example studied jointly with ILMANEN, I prove the pointwise blowup of the temperature.

2. Preliminaries

The following notation is used throughout the paper. $C_c^\infty(A \to B)$ denotes the set of all compactly supported, smooth functions on A, with values in B. $\mathscr{D}'(A)$ denotes the set of all distributions defined on A. For a measure space (A, μ) and for $p \in [1, \infty]$, $L^p(A; d\mu)$ denotes the set of all functions that are p-integrable with respect to the measure μ. When μ is the Lebesgue measure, we use the notation $L^p(A)$. For $T < \infty$ and $p \in [1, \infty]$, $\|\cdot\|_{p,T}$ denotes the norm in $L^p((0, T) \times \mathscr{R}^d)$. For $R > 0$ and $x \in \mathscr{R}^d$,

$$B_R = \{y \in \mathscr{R}^d : |y| \leqq R\}, \quad B_R(x) = \{y \in \mathscr{R}^d : |y - x| \leqq R\}.$$

For two $d \times d$ matrices M and N,

$$M : N = \sum_{i,j=1}^{d} M_{ij} N_{ij}.$$

S^{d-1} denotes the set of all unit vectors in $\mathscr{R}^d$. For $p \in \mathscr{R}^d$, $p \otimes p$ denotes the $d \times d$ matrix with entries $p_i p_j$.

For a Radon measure μ on $\mathscr{R}^d$ and a continuous bounded function ψ,

$$\mu(\psi) := \int_{\mathscr{R}^d} \psi(x) d\mu(x).$$

$\mathscr{H}^k$ denotes the k-dimensional Hausdorff measure (cf. [28]). Finally

$$Q = (0, \infty) \times \mathscr{R}^d$$

and for $(\tau, \xi) \in Q$, $G(\tau, \xi)$ is the heat kernel:

$$G(\tau, \xi) = (4\pi\tau)^{-d/2} \exp\left(\frac{|\xi|^2}{4\tau}\right).$$

2.1. *Equations*

For a scalar u, set

$$W(u) = \tfrac{1}{2}(u^2 - 1)^2,$$

$$h(u) = u - \tfrac{1}{3}u^3, \quad g(u) = h'(u) = (1 - u^2) = \sqrt{2W(u)}.$$

The heat equation (1.1) and the order-parameter equation (1.2) — with these functions and with parameters as in (1.3), (1.4) — take the form,

$$\varphi_t^\varepsilon - \Delta\varphi^\varepsilon + \frac{1}{\varepsilon^2} W'(\varphi^\varepsilon) - \frac{1}{\varepsilon} g(\varphi^\varepsilon)\theta^\varepsilon = 0 \quad \text{in } (0, \infty) \times \mathscr{R}^d, \tag{OPE}$$

$$\theta_t^\varepsilon - \Delta\theta^\varepsilon + g(\varphi^\varepsilon)\varphi_t^\varepsilon = 0 \quad \text{in } (0, \infty) \times \mathscr{R}^d. \tag{HE}$$

For $\varepsilon > 0$, let $(\varphi^\varepsilon, \theta^\varepsilon)$ be the unique, smooth, bounded solution of the phase-field equations satisfying the initial data

$$\varphi^\varepsilon(x, 0) = \varphi_0^\varepsilon(x), \quad \theta^\varepsilon(x, 0) = \theta_0^\varepsilon(x), \quad x \in \mathscr{R}^d. \tag{IC}$$

We assume that

$$|\varphi_0^\varepsilon(x)| \leqq 1 \quad \forall x \in \mathscr{R}^d. \tag{A1}$$

Then since $W'(\pm 1) = g(\pm 1) = 0$, by the maximum principle,

$$|\varphi^\varepsilon(t, x)| < 1 \quad \forall (t, x) \in (0, \infty) \times \mathscr{R}^d.$$

For a real number τ, $q(\tau) = \tanh(\tau)$ satisfies

$$q'' = W'(q), \quad q' = \sqrt{2W(q)} = g(q),$$

and q is the standing wave associated with the reaction diffusion equation with nonlinearity W'. Since $|\varphi^\varepsilon| < 1$, we may define z^ε by

$$\varphi^\varepsilon(t, x) = q\left(\frac{z^\varepsilon(t, x)}{\varepsilon}\right) \Leftrightarrow z^\varepsilon = \varepsilon q^{-1}(\varphi^\varepsilon).$$

Then z^ε satisfies

$$z_t^\varepsilon - \Delta z^\varepsilon - \theta^\varepsilon + \frac{2\varphi^\varepsilon}{\varepsilon}(|\nabla z^\varepsilon|^2 - 1) = 0 \tag{ZE}$$

(observe that $g(\varphi^\varepsilon) = g(q(z^\varepsilon/\varepsilon)) = q'(z^\varepsilon/\varepsilon)$ and $q'' = 2\varphi^\varepsilon q'$).

2.2. *Energy*

For a Borel subset $A \subset \mathscr{R}^d$, define

$$\mu^\varepsilon(t; A) = \int_A \tfrac{\varepsilon}{2}|\nabla\varphi^\varepsilon|^2 + \tfrac{1}{\varepsilon} W(\varphi^\varepsilon)\, dx,$$

$$\hat{\mu}^\varepsilon(t; A) = \mu^\varepsilon(t; A) + \int_A \tfrac{1}{2}(\theta^\varepsilon)^2\, dx.$$

In terms of z^ε,

$$\mu^\varepsilon(t; dx) = \frac{1}{2\varepsilon}\left(q'\left(\frac{z^\varepsilon(t,x)}{\varepsilon}\right)\right)^2 [|\nabla z^\varepsilon(t,x)|^2 + 1]\,dx,$$

and the discrepancy measure ξ^ε (cf. (1.7)) is given by

$$\xi^\varepsilon(t; dx) = \frac{1}{2\varepsilon}\left(q'\left(\frac{z^\varepsilon(t,x)}{\varepsilon}\right)\right)^2 [|\nabla z^\varepsilon(t,x)|^2 - 1]\,dx. \tag{2.1}$$

By differentiation and integration by parts we obtain

$$\frac{d}{dt}\hat{\mu}^\varepsilon(t; \mathscr{R}^d) = -\int_{\mathscr{R}^d} \varepsilon(\varphi_t^\varepsilon)^2 + |\nabla\theta^\varepsilon|^2\,dx.$$

We assume that the initial data satisfy

$$\hat{\mu}^\varepsilon(0; \mathscr{R}^d) \leqq C_1^*, \quad \varepsilon > 0. \tag{A2}$$

Then

$$\hat{\mu}^\varepsilon(t; \mathscr{R}^d) + \int_0^t \int_{\mathscr{R}^d} \varepsilon(\varphi_t^\varepsilon)^2 + |\nabla\theta^\varepsilon|^2\,dx\,dt \leqq C_1^*, \quad \varepsilon, t \geqq 0. \tag{2.2}$$

(Assumption (A2) can be relaxed as in [29]). We now localize this estimate. Let ψ be any *positive*, smooth, compactly supported function. Then

$$\begin{aligned}\frac{d}{dt}\hat{\mu}^\varepsilon(t;\cdot)(\psi) &= -\int_{\mathscr{R}^d} \varepsilon\psi\left[\left(\varphi_t^\varepsilon + \frac{\nabla\varphi^\varepsilon\cdot\nabla\psi}{2\psi}\right)^2 + \left|\nabla\theta^\varepsilon + \frac{\theta^\varepsilon\nabla\psi}{2\psi}\right|^2\right]\\ &\quad + \int_{\mathscr{R}^d} \frac{\varepsilon}{4\psi}(\nabla\varphi^\varepsilon\cdot\nabla\psi)^2 + \frac{(\theta^\varepsilon)^2}{4\psi}|\nabla\psi|^2\\ &\leqq \left[\sup_x \frac{|\nabla\psi(x)|^2}{2\psi(x)}\right]\int_{\{\psi>0\}} \frac{\varepsilon}{2}|\nabla\varphi^\varepsilon|^2 + \frac{1}{2}(\theta^\varepsilon)^2\\ &\leqq \|D^2\psi\|_\infty\,\hat{\mu}^\varepsilon(t; \{\psi > 0\}).\end{aligned}$$

Here we have used the fact that, for any positive C^2 function,

$$\frac{|\nabla\psi(x)|^2}{2\psi(x)} \leqq \|D^2\psi\|_\infty.$$

Hence there is a constant $C(\psi)$, independent of ε, such that the map

$$t \mapsto \int \psi(x)\hat{\mu}^\varepsilon(t; dx) - C(\psi)t \tag{2.3}$$

is non-increasing.

We close this subsection by obtaining a similar local energy identity for the classical solutions of the limit equation. Suppose that $(\theta(t,x), \Omega(t))$ is a classical solution of the Mullins-Sekerka problem (1.5), (1.6). Let $\vec{V}$, H, and n be, respectively, the normal velocity vector, the mean-curvature vector and the outward normal of the interface $\Gamma(t) = \partial\Omega(t)$. Let $\mu(t;\cdot)$ be equal to $\frac{4}{3}$ times the surface measure of $\partial\Omega(t)$. Then (1.5) is equivalent to

$$(\theta_t - \Delta\theta)\, dx = \vec{V}\cdot n\mu(t; dx).$$

For a smooth, compactly supported $\psi(x)$, we have

$$\frac{d}{dt}\hat{\mu}(t;\cdot)(\psi) = \int [-\vec{V}\cdot H\psi + \vec{V}\cdot \nabla\psi]\mu(t; dx) + \int \theta\theta_t\psi\, dx$$

$$= \int [\vec{V}\cdot(-H + \theta n)\psi + \vec{V}\cdot\nabla\psi]\mu(t; dx) + \int \tfrac{1}{2}\theta^2\Delta\psi - |\nabla\psi|^2\psi\, dx.$$

Since $\vec{V} = H - \theta n$ by (1.6), it follows that

$$\frac{d}{dt}\hat{\mu}(t;\cdot)(\psi) = \int [-|\vec{V}|^2\psi + \vec{V}\cdot\nabla\psi]\mu(t; dx) + \int \tfrac{1}{2}\theta^2\Delta\psi - |\nabla\theta|^2\psi\, dx. \quad (2.4)$$

Observe that this identity is very similar to that used by BRAKKE to develop a weak theory for mean-curvature flows (cf. [5], [23, §1]).

2.3. *Subsequence*

The energy estimate (2.2) yields

$$\sup_{\varepsilon, t>0} \|\theta^\varepsilon(t,\cdot)\|_2 < \infty.$$

Hence there are a subsequence, denoted by ε, and an L^2 function θ such that

$$\theta^\varepsilon \rightharpoonup \theta \quad \text{in weak } L^2((0,T)\times\mathscr{R}^d),$$

for every $T > 0$. We will show that this convergence is, in fact, in the strong topology (see Proposition 3.4 below). Moreover, by the arguments of BRONSARD & KOHN [6], this sequence can be chosen so that

$$h(\varphi^\varepsilon) \to h(\varphi) = \tfrac{2}{3}\varphi \quad \textit{in } L^1_{\text{loc}}((0,\infty)\times\mathscr{R}^d), \quad \varphi^\varepsilon \to \varphi \quad \text{a.e.},$$

where φ is a function of bounded variation, and $|\varphi(t,x)| = 1$, for almost every (t,x). Since $(h')^2 = 2W$, (2.2) implies that, for $0 \leqq s < t$ and $\varepsilon > 0$,

$$\|h(\varphi^\varepsilon(t,\cdot)) - h(\varphi^\varepsilon(s,\cdot))\|_1 \leqq \int_s^t \int |h'(\varphi^\varepsilon(r,x))\varphi_t^\varepsilon(r,x)|\, dx\, dr$$

$$\leqq \left(\int_s^t \int \tfrac{\varepsilon}{2}(\varphi_t^\varepsilon(r,x))^2\, dx\, dr\right)^{1/2} \left(\int_s^t \mu^\varepsilon(r;\mathscr{R}^d)\, dr\right)^{1/2}$$

$$\leqq C_1^*\sqrt{t-s}. \quad (2.5)$$

Hence

$$h(\varphi^\varepsilon(t;\cdot)) \to h(\varphi(t;\cdot)) \quad \text{in } L^1_{\text{loc}}(\mathscr{R}^d), \quad (2.6)$$

uniformly in the variable t.

The energy estimate implies that, for each $t \geqq 0$, the sequence $\{\hat{\mu}^\varepsilon(t,\cdot)\}_{\varepsilon>0}$ is precompact in the weak* topology of Radon measures. By a diagonalization argument we construct a sequence, denoted by ε again, such that as $\varepsilon \downarrow 0$, $\hat{\mu}^\varepsilon(t,\cdot)$ is weak* convergent for all rational $t \geqq 0$. Then, by the monotonicity estimate (2.3), we construct a further sequence so that $\{\hat{\mu}^\varepsilon(t,\cdot)\}_{\varepsilon>0}$ is convergent for all $t \geqq 0$. Therefore there are a subsequence, denoted by ε, and a family of Radon measures $\hat{\mu}(t,\cdot)$ that satisfy

$$\lim_{\varepsilon\downarrow 0} \hat{\mu}^\varepsilon(t,\cdot) \rightharpoonup \hat{\mu}^\varepsilon(t,\cdot) \quad \forall t \geqq 0$$

in the weak* topology of Radon measures. See [23, §5.4] for further details of this argument. Now for a Borel subset $B \subset [0,\infty)\times\mathscr{R}^d$, define

$$\hat{\mu}(B) = \iint_B \hat{\mu}(t; dx)\,dt, \quad \mu(B) = \hat{\mu}(B) - \tfrac{1}{2}\iint_B \theta^2\,dx\,dt.$$

Then the strong convergence of θ^ε to θ (cf. Proposition 3.4 below) implies that $\mu \geqq 0$ and

$$\mu^\varepsilon(t,\cdot) \rightharpoonup \mu(t,\cdot) \quad \forall t \geqq 0.$$

Since the interface condition (1.6) involves not only the mean-curvature vector, which is independent of orientation, but also the normal vector, we introduce yet another measure, m^ε, that keeps track of the normal direction. For $(t,x,n) \in [0,\infty)\times\mathscr{R}^d\times S^{d-1}$, define

$$v^\varepsilon(t,x) = \begin{cases} \dfrac{\nabla\varphi^\varepsilon(t,x)}{|\nabla\varphi^\varepsilon(t,x)|} & \text{if } |\nabla\varphi^\varepsilon(t,x)| \neq 0, \\ v_0 & \text{if } \nabla\varphi^\varepsilon(t,x) = 0, \end{cases}$$

$$dm^\varepsilon(t,x,n) = dt\,\mu^\varepsilon(t;dx)\,\delta_{\{v^\varepsilon(t,x)\}}(dn),$$

where $v_0 \in S^{d-1}$ is arbitrary and $\delta_{\{v^\varepsilon\}}$ is the Dirac measure located at v^ε. Observe that m^ε is independent of the choice of v_0.

Since S^{d-1} is compact, there is a further sequence, denoted by ε, such that dm^ε is weak* convergent. By a slicing argument (cf. [16, Theorem 10, page 14]) we conclude that there exist probability measures $N(t,x,\cdot)$ on S^{d-1} such that as ε tends to zero,

$$dm^\varepsilon \rightharpoonup dm = dt\,\mu(t;dx)\,N(t,x;dn).$$

Finally define $\bar{m}^\varepsilon$ by

$$d\bar{m}^\varepsilon = -z_t^\varepsilon\,dm^\varepsilon.$$

In §8 we show that there are a subsequence, denoted by ε, and

$$v \in L^2((0,T)\times\mathscr{R}^d\times S^{d-1}; dm) \quad \forall T > 0$$

such that

$$\bar{m}^\varepsilon \rightharpoonup \bar{m}, \quad d\bar{m} = v(t,x,n)\,dm.$$

2.4. Initial data and assumptions

In addition to (A1), (A2) we assume that

$$\| \nabla z_0^\varepsilon \|_\infty \leqq 1, \tag{A3}$$

$$\sup_{0<\varepsilon\leqq 1} \varepsilon \| D^2 z_0^\varepsilon \|_\infty < \infty, \tag{A4}$$

$$\sup_{0<\varepsilon\leqq 1} \sup_{\substack{x\in\mathscr{R}^d \\ R>0}} \frac{\mu^\varepsilon(0; B_R(x))}{R^{d-1}} < \infty, \tag{A5}$$

where $B_R(x)$ is the sphere centered at x with radius R. We also assume that

$$\sup_{0<\varepsilon\leqq 1} \{ \| \theta_0^\varepsilon \|_1 + \| \theta_0^\varepsilon \|_\infty + \sqrt{\varepsilon} \| \nabla \theta_0^\varepsilon \|_\infty \} < \infty, \tag{A6}$$

$$\sup_{0<\varepsilon\leqq 1} \{ \varepsilon^3 \| D^3 \varphi_0^\varepsilon \|_\infty + \varepsilon^2 \| D^2 \theta_0^\varepsilon \|_\infty \} < \infty. \tag{A7}$$

Since

$$\| \theta_0^\varepsilon \|_p^p \leqq \| \theta_0^\varepsilon \|_1 \| \theta_0^\varepsilon \|_\infty^{p-1} \quad \text{for } 1 \leqq p < \infty,$$

observe that (A6) implies that

$$\sup_{0<\varepsilon\leqq 1} \| \theta_0^\varepsilon \|_p = K(p) < \infty. \tag{2.7}$$

Finally we assume that there is $\theta_0 \in L^2(\mathscr{R}^d)$ such that

$$\theta_0^\varepsilon \to \theta_0 \quad \text{in } L^2\text{-strong}. \tag{A8}$$

While (A1)–(A8) may seem restrictive, in fact, they are merely technical assumptions which are consistent with approximations to any smooth initial data. Indeed, if $\theta_0^\varepsilon = \theta_0$ is a smooth, compactly supported function, then θ_0 satisfies (A6), (A7) and (A8) trivially. Suppose that Γ_0 is a bounded, smooth hypersurface in $\mathscr{R}^d$. Let $d(x)$ be the signed distance of x to Γ_0 and let $\hat{d}$ be an appropriate modification of d outside of a tubular neighborhood of Γ_0 such that all derivaties of $\hat{d}$ up to order three are bounded and such that $2|\hat{d}| \geqq |d|$. Then $z_0^\varepsilon = \hat{d}$ satisfies (A1)–(A7).

Finally we note that the term $\sqrt{\varepsilon}$ appearing in (A6) is not essential. Indeed if (A6) holds with ε^ν for some $\nu \geqq \frac{1}{2}$, then we can prove the same results with minor changes.

2.5. Varifolds, rectifiable measures, etc.

In this subsection, we recall several definitions and results from geometric measure theory ([28], [23, §1]).

Following [23, §1.7], we call a Radon measure μ on $\mathscr{R}^d$ *k-rectifiable*, if there are a $\mathscr{H}^k$-measurable, locally k-rectifiable set $X \subset \mathscr{R}^d$ and

$$f \in L^1_{\text{loc}}(X; d\mathscr{H}^k \lfloor X) \quad (\mathscr{H}^k \lfloor X(A) = \mathscr{H}^k(A \cap X))$$

such that

$$\mu(A) = \int_{A\cap X} f(x) d\mathscr{H}^k(x) \quad \text{for any Borel set } A.$$

When μ is k-rectifiable, for μ-almost every x, the *measure-theoretic tangent plane*

$$T_x\mu = \lim_{\lambda\downarrow 0} \mu_{x,\lambda}, \quad (\mu_{x,\lambda}(A) = \lambda^{-k}\mu(x + \lambda A))$$

exists and is a positive multiple of $\mathscr{H}^k$ restricted to a k-plane. With an abuse of notation, we use $T_x\mu$ to denote this k-plane.

A general *k-varifold* is a Radon measure on $\mathscr{R}^d \times G_k(\mathscr{R}^d)$, where $G_k(\mathscr{R}^d)$ is the Grassman manifold of unoriented k-planes in $\mathscr{R}^d$. The mass measure $\|V\|$ is defined by

$$\|V\|(A) = V(A \times G_k(\mathscr{R}^d)).$$

For every k-rectifiable Randon measure μ, there is a corresponding (rectifiable) k-varifold V_μ defined by

$$dV_\mu(x, S) = d\mu(x)\, d\delta_{\{T_x\mu\}}(S),$$

where $\delta_{\{T_x\mu\}}$ is the Dirac measure located at $T_x\mu$. Note that $\|V_\mu\| = \mu$. We say that a k-rectifiable Radon measure μ has a generalized *mean-curvature vector*

$$H \in L^1_{\text{loc}}(\mathscr{R}^d \to \mathscr{R}^d; d\mu)$$

if for any smooth, compactly supported vector field $Y(x)$,

$$\int \operatorname{tr}(\nabla Y(x)P(x))\, d\mu = -\int Y(x)\cdot H(x)\, d\mu, \tag{2.8}$$

where $P(x)$ is the projection on the tangent plane $T_x\mu$. In the terminology of geometric measure theory, the left-hand side of (2.8) is the first variation δV_μ of the varifold V_μ [1].

2.6. Main results

First we recall the convergence results stated in §2.3.

Theorem 2.1 (Convergence). *There are a sequence, denoted by ε, functions $\theta \in L^2_{\text{loc}}((0, \infty)\times\mathscr{R}^d)$, $v \in L^2_{\text{loc}}((0, \infty)\times\mathscr{R}^d\times S^{d-1}; dm)$, non-negative Radon measures $\{\mu(t;\cdot), \hat{\mu}(t;\cdot)\}_{t\geq 0}$ and probability measures $\{N(t, x;\cdot)\}_{(t,x)\in(0,\infty)\times\mathscr{R}^d}$ such that, as ε tends to zero,*

$$\theta^\varepsilon(t,\cdot) \to \theta(t,\cdot) \quad \textit{strongly in } L^2_{\text{loc}}(\mathscr{R}^d) \quad \forall t \geqq 0,$$

$$h(\varphi^\varepsilon) \to h(\varphi) \quad \textit{strongly in } L^1_{\text{loc}}((0,\infty)\times\mathscr{R}^d).$$

Moreover, $|\varphi| = 1$, $\varphi^\varepsilon(t, x) \to \varphi(t, x)$ for almost every $(t, x) \in (0, \infty)\times\mathscr{R}^d$, and

$$h(\varphi^\varepsilon(t,\cdot)) \to h(\varphi(t,\cdot)) \quad \textit{strongly in } L^1_{\text{loc}}(\mathscr{R}^d),$$

uniformly for $t \geqq 0$. *In addition, the following convergence results in the weak* topology of Radon measures, are valid*:

$$\mu^\varepsilon(t,\cdot) \rightharpoonup \mu(t,\cdot) \quad \forall t \geqq 0,$$

$$\hat{\mu}^\varepsilon(t,\cdot) \rightharpoonup \hat{\mu}(t,\cdot) \quad \forall t \geqq 0 \quad (\hat{\mu}(t; dx) = \mu(t; dx) + \tfrac{1}{2}\theta^2\, dx),$$

$$m^\varepsilon \rightharpoonup m, \quad dm(t, x, n) = \mu(t; dx)\, dt\, N(t, x; dn),$$

$$\bar{m}^\varepsilon \rightharpoonup \bar{m}, \quad d\bar{m}^\varepsilon = -\, z_t^\varepsilon dm^\varepsilon, \quad d\bar{m} = v(t, x, n)\, dm.$$

For every $T > 0$, *the functions* θ *and* v *satisfy*

$$\sup_{t \geqq 0} \|\theta(t,\cdot)\|_2 + \|\nabla\theta\|_{2,T} < \infty, \tag{2.9}$$

$$\|v\|_{L^2((0,T)\times\mathscr{R}^d\times S^{d-1};\, dm)} < \infty.$$

The strong convergence of θ^ε is proved in Proposition 3.4, and the convergence of $\bar{m}^\varepsilon$ and the integrability of v are proved in §8. The remaining assertions were established in §2.3.

Set $d\mu(t, x) = \mu(t; dx)\, dt$. Let Γ be the support of μ and Γ_t be the t cross section of Γ. In the terminology of §2.5, we have the following regularity result.

Theorem 2.2 (Regularity). *For almost every* $t \geqq 0$, $\mu(t,\cdot)$ *is* $(d-1)$*-rectifiable and has a generalized mean-curvature vector* $H(t, x)$. *Moreover for every* $T > 0$,

$$|H| \in L^2((0, T)\times\mathscr{R}^d; d\mu),$$

$$\sup_{t \leqq T} \mathscr{H}^{d-1}(\Gamma_t) < \infty,$$

$$\theta \in L^1_{\mathrm{loc}}((0, \infty)\times\mathscr{R}^d; d\mu),$$

and the support of the probability measure $N(t, x; \cdot)$ *is orthogonal to* $T_x\mu(t;\cdot)$ *for* μ*-almost every* (t, x). *In particular*,

$$\iiint \nabla Y(t, x) : (I - n \otimes n)\, dm = -\iint Y(t, x)\cdot H(t, x)\, d\mu$$

for all $Y \in C_c^\infty((0, \infty)\times\mathscr{R}^d \to \mathscr{R}^d)$.

The estimate of Hausdorff measure is proved in Proposition 7.2. The existence and the square-integrability of H, the integrability of θ with respect to μ, and the orthogonality of N are all proved in §8. The final assertion of the theorem follows from the orthogonality of N and the defining property of H.

The next result states that the limit of $(\theta^\varepsilon, \varphi^\varepsilon)$ weakly satisfies (1.5), (1.6). However the lack of regularity of the limit functions necessitates the use of measures μ and m.

Let θ, φ, μ, $\hat{\mu}$, N, and v be as in Theorem 2.1. Recall that $Q = (0, \infty)\times\mathscr{R}^d$.

Theorem 2.3 (Limit Equations). *For any* $\psi \in C_c^\infty(Q \to \mathscr{R})$ *and* $Y \in C_c^\infty(Q \to \mathscr{R}^d)$,

$$\int\int -(\psi_t + \Delta\psi)\theta \, dx\, dt = \int\int\int v\psi \, dm \tag{2.10}$$

$$= \int\int \psi_t h(\varphi)\, dx\, dt, \tag{2.11}$$

$$\int\int\int Y\cdot n(\theta + v)\, dm = -\int\int\int \nabla Y : (I - n\otimes n)\, dm$$

$$= \int\int Y\cdot H\, d\mu, \tag{2.12}$$

$$\int\int\int Y\cdot n\, dm = \int\int Y\cdot \nabla(h(\varphi))\, dx\, dt.$$

For any $0 \leqq s \leqq t$ *and* $\phi \in C_c^\infty(Q \to [0, \infty))$, *the following Brakke-type inequality holds*:

$$\hat{\mu}(\phi)(t) - \hat{\mu}(\phi)(s) \leqq \int_s^t \int\int (-v^2\phi - vn\cdot \nabla\phi)\, dm$$

$$+ \int_s^t \int (\tfrac{1}{2}\theta^2 \Delta\phi - |\nabla\theta|^2\phi)\, dx\, dt. \tag{2.13}$$

This theorem is proved in §8.

The system (2.10)–(2.13) constitutes a weak formulation of the Mullins-Sekerka equations (1.5) and (1.6). Indeed, set

$$V(t, x) = \int v(t, x, n) N(t, x; dn), \quad \vec{V}(t, x) = \int nv(t, x, n) N(t, x; dn),$$

$$\Omega(t) = \{x : \varphi(t, x) = -1\}.$$

Then (2.11) yields

$$\theta_t - \Delta\theta + (h(\varphi))_t = 0 \quad \text{in } \mathscr{D}'(Q).$$

Since $(h(\varphi))_t = -(\frac{4}{3})(\chi_\Omega)_t$, by (2.10) and (2.11), V is formally equal to $\frac{4}{3}$ times the normal velocity of the interface $\partial\Omega$. Suppose that $N(t, x; \cdot)$ is a Dirac measure located at $n(t, x) \in S^{d-1}$. Then the orthogonality of N to the tangent plane implies that $n(t, x)$ is orthogonal to $\partial\Omega$, and (2.12) is equivalent to

$$\vec{V} = Vn = H - \theta n \quad \text{on } \partial\Omega.$$

The equation (2.12) is therefore a weak formulation of (1.6) provided that V is the normal velocity of $\mu(t; \cdot)$. Indeed, the inequality (2.13) provides a weak formulation of this statement; compare (2.13) to (2.4).

Remarks on regularity and uniqueness

1. Suppose that $\Gamma(t)$ is smooth. Then does the weak formulation proved in Theorem 2.3 imply that $\Gamma(t)$ and θ satisfy the Mullins-Sekerka problem classically? Alternatively, suppose that there is a smooth solution of the Mullins-Sekerka problem in $(0, T)\times\mathscr{R}^d$. Then does this classical solution agree with the limit functions constructed in this paper?

2. An attendant modification of [23, §9] together with the results of this paper imply that for any compactly supported smooth function $\phi(x)$ and for $t > 0$, if

$$d := \limsup_{s \to t} \frac{1}{s-t} \int \phi(x)(\hat{\mu}(s; dx) - \hat{\mu}(t; dx)) > -\infty,$$

then $\mu(t, \cdot)$ restricted to $\{\phi > 0\}$ is $(d-1)$-rectifiable with a generalized mean-curvature vector $H(t, \cdot)$. Moreover,

$$\begin{aligned} d \leqq & -\int\int \phi(x) |H(t,x) + \theta(t,x) n|^2 \mu(t; dx) N(t, x; dn) \\ & -\int \phi(x) |\nabla\theta(t,x)|^2 \, dx + \tfrac{1}{2} \int \Delta\phi(x) \theta^2(t,x) \, dx \\ & -\int\int D\phi(x) \cdot [H(t,x) + n\theta(t,x)] \mu(t; dx) N(t, x; dn). \end{aligned}$$

It seems that this inequality together with some of the formulae proved in Theorem 2.3 provide a Brakke-type weak-formulation of the Mullins-Sekerka problem. Further analysis of these equations may yield a generalization of a partial-regularity result of BRAKKE [5].

3. Simple examples indicate that $N(t, x; dn)$ may not be a Dirac measure at some points (t, x). This corresponds to interface "piling-up" at such points. An interesting question is to estimate the dimension of these points at which $N(t, x; dn)$ is not a Dirac measure. Since the heat equation (1.5) does not have any external forcing term, we expect this set to be of lower dimension.

4. A related question is whether the equation

$$[v(t, x, n) + \theta(t, x)] n = H(t, x) \tag{2.14}$$

holds for dm almost every (t, x, n). The equation (2.12) implies (2.14) only after integration with respect to $N(t, x; dn)$. Radially symmetric examples indicate that (2.14) may be true.

Suppose that (2.14) holds. Then, formally, if $\theta(t, x) \neq 0$ and $N(t, x, \cdot)$ is not a Dirac measure, then $v(t, x, n)$ is different for each n in the support of $N(t, x; \cdot)$. Therefore formally $N(s, x, \cdot)$ would become a Dirac measure for $s > t$ and s near t.

3. Elementary Estimates

In this section we obtain several elementary estimates by using the heat kernel

$$G(\tau, \xi) = (4\pi\tau)^{-d/2} \exp\left(-\frac{|\xi|^2}{4\tau}\right), \quad (\tau, \xi) \in (0, \infty) \times \mathscr{R}^d.$$

Since $h' = g$, the heat equation (HE) and integration by parts yield

$$\theta^\varepsilon(t, x) = A^\varepsilon(t, x) + B^\varepsilon(t, x), \tag{3.1}$$

where

$$A^\varepsilon(t,x) = (G(t,\cdot) * [\theta_0^\varepsilon + H^\varepsilon(0,\cdot) - H^\varepsilon(t,\cdot)])(x),$$

$$B^\varepsilon(t,x) = \int_0^t (G_\tau(\tau,\cdot) * (H^\varepsilon(t,\cdot) - H^\varepsilon(t-\tau,\cdot)))(x)\, d\tau,$$

$$H^\varepsilon(t,x) = h(\varphi^\varepsilon(t,x)),$$

and $*$ denotes convolution in the x-variable.

All the constants in this and later sections depend on T but we generally suppress this dependence in our notation.

Constants independent of ε are denoted by K; this constant may change from one line to the next.

Lemma 3.1. *There is a constant K such that*

$$\varepsilon \| \nabla\varphi^\varepsilon \|_{\infty,T} \leqq K, \tag{3.2}$$

$$\| \theta^\varepsilon \|_{\infty,T} \leqq K[1 + |\ln \varepsilon|]. \tag{3.3}$$

Proof.

1. Fix $T > 0$ and set

$$m^\varepsilon(T) = \varepsilon \| \nabla\varphi^\varepsilon \|_{\infty,T}, \quad n^\varepsilon(T) = \| \theta^\varepsilon \|_{\infty,T},$$

$$f^\varepsilon = -\frac{1}{\varepsilon^2} W'(\varphi^\varepsilon) + \frac{1}{\varepsilon} g(\varphi^\varepsilon)\theta^\varepsilon.$$

Then

$$\| f^\varepsilon \|_{\infty,T} \leqq \frac{1}{\varepsilon^2}[2 + \varepsilon n^\varepsilon(T)]$$

and the order-parameter equation (OPE) may be rewritten as

$$\varphi_t^\varepsilon - \Delta\varphi^\varepsilon = f^\varepsilon.$$

2. Fix $(t,x) \in [0,T] \times \mathscr{R}^d$. Then for any $\sigma \in (0,t]$, (OPE) yields

$$\nabla\varphi^\varepsilon(t,x) = (\nabla G(\sigma,\cdot) * \varphi^\varepsilon(t-\sigma,\cdot))(x) + \int_0^\sigma (\nabla G(\tau,\cdot) * f^\varepsilon(t-\tau,\cdot))(x)\, d\tau.$$

Observe that for any $\tau > 0$,

$$\sqrt{\tau}\, \| \nabla G(\tau,\cdot) \|_1 = \frac{(\pi)^{-d/2}}{2} \int_{\mathscr{R}^d} |y| e^{-|y|^2}\, dy = K.$$

Therefore

$$\left| \int_0^\sigma [\nabla G(\tau,\cdot) * f^\varepsilon(t-\tau,\cdot)](x)\, dt \right| \leqq \frac{2K(2 + \varepsilon n^\varepsilon(T))}{\varepsilon^2} \sqrt{\sigma},$$

$$|(\nabla G(\sigma,\cdot) * \varphi^\varepsilon(t-\sigma,\cdot))(x)| \leqq \frac{K}{\sqrt{\sigma}}.$$

Also, if $\sigma = t$, then

$$|(\nabla G(\sigma,\cdot) * \varphi^\varepsilon(t-\sigma,\cdot))(x)| = |(G(t,\cdot) * \nabla\varphi_0^\varepsilon(x)| \leqq \|\nabla\varphi_0^\varepsilon\|_\infty.$$

3. Now use the foregoing inequalities with $\sigma = \varepsilon^2 \wedge t$; the result is

$$|\nabla\varphi^\varepsilon(t,x)| \leqq \begin{cases} \dfrac{K}{\varepsilon}[5 + 2\varepsilon n^\varepsilon(T)] & \text{if } t \geqq \varepsilon^2, \\ \dfrac{K}{\varepsilon}[\varepsilon\|\nabla\varphi_0^\varepsilon\|_\infty + 2(2 + \varepsilon n^\varepsilon(T))] & \text{if } t \leqq \varepsilon^2. \end{cases}$$

By (A3), $\varepsilon\|\nabla\varphi_0^\varepsilon\|_\infty \leqq K$; we therefore conclude that

$$m^\varepsilon(T) \leqq K[1 + \varepsilon n^\varepsilon(T)]. \tag{3.4}$$

4. Let A^ε, B^ε be as in (3.1). Then

$$|A^\varepsilon(t,x)| \leqq \|\theta_0^\varepsilon\|_\infty + 2.$$

For $\sigma \in (0, t]$,

$$\left|\int_\sigma^t [G_\tau(\tau,\cdot) * (H^\varepsilon(t,\cdot) - H^\varepsilon(t-\tau,\cdot))](x)\,d\tau\right| \leqq 2\int_\sigma^t \|G_\tau(\tau,\cdot)\|_1\,d\tau.$$

Observe that

$$\tau\|G_\tau(\tau,\cdot)\|_1 \leqq \frac{(\pi)^{-d/2}}{2}\int_{\mathscr{R}^d}\left(\frac{d}{2} + |y|^2\right)e^{-|y|^2}\,dy \leqq K.$$

Hence

$$\left|\int_\sigma^t [G_\tau(\tau,\cdot) * (H^\varepsilon(t,\cdot) - H^\varepsilon(t-\tau,\cdot))](x)\,d\tau\right| \leqq K\ln\left(\frac{t}{\sigma}\right).$$

Since $\Delta G = G_\tau$ and $\nabla H = g\nabla\varphi^\varepsilon$, integrating by parts we obtain

$$\begin{aligned} &\left|\int_0^\sigma [G_\tau(\tau,\cdot) * (H^\varepsilon(t,\cdot) - H^\varepsilon(t-\tau,\cdot))](x)\,d\tau\right| \\ &\quad = \left|\int_0^\sigma\int_{\mathscr{R}^d} \nabla G(\tau, x-y)\cdot[\nabla H(t,y) - \nabla H(t-\tau,y)]\,dy\,d\tau\right| \\ &\quad \leqq \int_0^\sigma \|\nabla G(\tau,\cdot)\|_1(\|\nabla\varphi^\varepsilon(t-\tau,\cdot)\|_\infty + \|\nabla\varphi^\varepsilon(t,\cdot)\|_\infty)\,d\tau \\ &\quad \leqq \frac{K}{\varepsilon}m^\varepsilon(T)\sqrt{\sigma}. \end{aligned}$$

5. Estimates obtained in Step 4 and (3.1) yield

$$|\theta^\varepsilon(t,x)| \leqq \|\theta_0^\varepsilon\|_\infty + 2 + K\ln\left(\frac{t}{\sigma}\right) + \frac{K\sqrt{\sigma}}{\varepsilon}m^\varepsilon(T).$$

Choose $\sigma = \varepsilon^2 \wedge t$; then

$$n^\varepsilon(T) \leqq \|\theta_0^\varepsilon\|_\infty + 2 + K \ln T + K\,|\ln \varepsilon| + Km^\varepsilon(T), \tag{3.5}$$

and hence (3.4) and (A6) yield

$$n^\varepsilon(T) \leqq K_0(1 + |\ln \varepsilon| + \varepsilon n^\varepsilon(T)).$$

Hence (3.2) and (3.3) hold for all $\varepsilon > 0$ satisfying

$$2K_0\varepsilon \leqq 1 \;\Leftrightarrow\; \varepsilon \leqq \varepsilon_0 = (\tfrac{1}{2}K_0).$$

For $1 \geqq \varepsilon \geqq \varepsilon_0$, (3.2) and (3.3) can be proved easily. □

Remark 3.1. The family θ^ε is not necessarily uniformly bounded in ε. Indeed consider the Mullins-Sekerka problem with radial symmetry and one interface. If the radius R_0 of the initial interface is sufficiently small and the initial temperature θ_0 is sufficiently large, then the radius $R(t)$ of the interface becomes zero in a finite time T. Since the phase-field equations with radial symmetry are known to approximate the Mullins-Sekerka problem [30], this example, which is discussed in the Appendix, shows that θ^ε is not uniformly bounded in ε.

Next we use the techniques developed in this section to obtain uniform bounds for $\varepsilon^2|D^2\varphi^\varepsilon|$ and $\varepsilon|\nabla\theta^\varepsilon|$.

Lemma 3.2.

$$\sup_{0<\varepsilon\leqq 1} \{\varepsilon^2[\|D^2\varphi^\varepsilon\|_{\infty,T} + \|\varphi_t^\varepsilon\|_{\infty,T}] + \varepsilon\|\nabla\theta^\varepsilon\|_{\infty,T}\} < \infty. \tag{3.6}$$

Proof.
1. Differentiate the (OPE) to obtain

$$\varphi_{x_j t}^\varepsilon - \Delta\varphi_{x_j}^\varepsilon = F_j^\varepsilon,$$

$$F_j^\varepsilon = -\frac{1}{\varepsilon^2} W''(\varphi^\varepsilon)\varphi_{x_j}^\varepsilon + \frac{1}{\varepsilon} g'(\varphi^\varepsilon)\varphi_{x_j}^\varepsilon\theta^\varepsilon + \frac{1}{\varepsilon} g(\varphi^\varepsilon)\theta_{x_j}^\varepsilon.$$

Using (3.2) and (3.3) we conclude that

$$\|F_j^\varepsilon\|_{\infty,T} \leqq K\left[\frac{1}{\varepsilon^3} + \frac{1}{\varepsilon}\|\nabla\theta^\varepsilon\|_{\infty,T}\right]$$

for some constant K. Set

$$\bar{m}^\varepsilon(T) = \varepsilon^2\|D^2\varphi^\varepsilon\|_{\infty,T}, \quad \bar{n}^\varepsilon(T) = \varepsilon\|\nabla\theta^\varepsilon\|_{\infty,T}.$$

Then we use (A4) and (3.2) as in Step 2 of the proof of Lemma 3.1; the result is

$$\bar{m}^\varepsilon(T) \leqq K[1 + \varepsilon\bar{n}^\varepsilon(T)]; \tag{3.7}$$

hence (3.3), (3.7), and (OPE) yield

$$\varepsilon^2\|\varphi_t^\varepsilon\|_{\infty,T} \leqq K[1 + \varepsilon\bar{n}^\varepsilon(T)]. \tag{3.8}$$

2. Let A^ε and B^ε be as in (3.1). Then,

$$\nabla A^\varepsilon(t,x) = (G(t,\cdot) * \nabla(\theta_0^\varepsilon + H^\varepsilon(0,\cdot) - H^\varepsilon(t,\cdot)))(x),$$

$$\nabla B^\varepsilon(t,x) = \int_0^t (\nabla G_\tau(\tau,\cdot) * [H^\varepsilon(t,\cdot) - H^\varepsilon(t-\tau,\cdot)])(x)\,d\tau.$$

Fix $(t,x) \in [0,T] \times \mathscr{R}^d$. In view of (A6) and (3.2),

$$|\nabla A^\varepsilon(t,x)| \leqq \|\nabla\theta_0^\varepsilon\|_\infty + 2\|\nabla\varphi^\varepsilon\|_{\infty,T} \leqq \frac{K}{\varepsilon}.$$

Also for $\sigma \in (0, t \wedge 1]$,

$$\left|\int_\sigma^t [G_\tau(\tau,\cdot) * (H^\varepsilon(t,\cdot) - H^\varepsilon(t-\tau,\cdot))](x)\,d\tau\right|$$

$$\leqq 2\int_\sigma^t \|\nabla G_\tau(\tau,\cdot)\|_1\,d\tau \leqq \int_\sigma^t K\tau^{-3/2}\,d\tau \leqq K\left(\frac{1}{\sqrt{\sigma}} - \frac{1}{\sqrt{t}}\right).$$

3. Integrating by parts in the τ-variable, we obtain

$$\left|\int_0^\sigma [G_\tau(\tau,\cdot) * (H^\varepsilon(t,\cdot) - H^\varepsilon(t-\tau,\cdot)])(x)\,d\tau\right|$$

$$\leqq |\nabla G(\sigma,\cdot) * (H^\varepsilon(t,\cdot) - H^\varepsilon(t-\sigma,\cdot)])(x)| + \left|\int_0^\sigma (\nabla G(\tau,\cdot) * (H_t^\varepsilon(t-\tau,\cdot))(x)\,d\tau\right|$$

$$\leqq \frac{K}{\sqrt{\sigma}}\|H^\varepsilon(t,\cdot) - H^\varepsilon(t-\sigma,\cdot)\|_\infty + \int_0^\sigma \|\nabla G(\tau,\cdot)\|_1 \|H_t^\varepsilon\|_{\infty,T}\,d\tau.$$

$$\leqq K\sqrt{\sigma}\,\|H_t^\varepsilon\|_{\infty,T} \leqq K\sqrt{\sigma}\,\|\varphi_t^\varepsilon\|_{\infty,T}.$$

4. Combine Steps 2 and 3, and choose $\sigma = \varepsilon^2 \wedge t$; then

$$|\nabla\theta^\varepsilon(t,x)| \leqq \tfrac{K}{\varepsilon}(1 + \varepsilon^2\|\varphi_t^\varepsilon\|_{\infty,T}).$$

As in the last step of the previous lemma, this estimate together with (3.7) and (3.8) imply (3.6) for sufficiently small $\varepsilon \leqq \varepsilon_0$. But for $\varepsilon \geqq \varepsilon_0$, (3.6) holds trivially. □

Assumption (A7) and the arguments of Lemma 3.2 yield

$$\sup_{0<\varepsilon\leqq 1} \{\varepsilon^3\|D^3\varphi^\varepsilon\|_{\infty,T} + \varepsilon^2\|D^2\theta^\varepsilon\|_{\infty,T} + \varepsilon^2\|\theta_t^\varepsilon\|_{\infty,T}\} < \infty. \tag{3.9}$$

Lemma 3.3. *For* $1 \leqq p < \infty$ *and* $T \geqq 0$,

$$\sup_{0<\varepsilon\leqq 1,\, t\leqq T} \|\theta^\varepsilon(t,\cdot)\|_{L^p(\mathscr{R}^d)} < \infty. \tag{3.10}$$

Proof.
1. Recall that, by (2.5),

$$\|H^\varepsilon(t_1,\cdot) - H^\varepsilon(t_0,\cdot)\|_1 \leqq C_1^*\sqrt{t_1 - t_0}.$$

Since $|H^\varepsilon| \leqq 1$,

$$\| H^\varepsilon(t_1, \cdot) - H^\varepsilon(t_0, \cdot) \|_p^p \leqq 2^p C_1^* \sqrt{t_1 - t_0}.$$

2. Let A^ε and B^ε be as in (3.1). Then, by (2.7) and the previous step,

$$\| A^\varepsilon \|_p \leqq \| \theta_0^\varepsilon \|_p + \| H^\varepsilon(t, \cdot) - H^\varepsilon(0, \cdot) \|_p \leqq K(1 + t^{1/2p}).$$

3. By Step 1,

$$\| B^\varepsilon \|_p \leqq \int_0^t \| G_\tau(\tau, \cdot) * (H^\varepsilon(t, \cdot) - H^\varepsilon(t - \tau, \cdot)) \|_p \, d\tau$$

$$\leqq \int_0^t \| G_\tau(\tau, \cdot) \|_1 \| H^\varepsilon(t, \cdot) - H^\varepsilon(t - \tau, \cdot) \|_p \, d\tau$$

$$\leqq K \int_0^t \tau^{-1 + 1/2p} \, d\tau. \quad \square$$

We close this section by proving the strong convergence of the sequence θ^ε.

Proposition 3.4. *For every $t \geqq 0$, $\theta^\varepsilon(t, \cdot)$ converges to $\theta(t, \cdot)$ strongly in $L^2_{\mathrm{loc}}(\mathscr{R}^d)$. In particular,*

$$\mu^\varepsilon(t; \cdot) \rightharpoonup \mu(t; \cdot) \quad \forall t \geqq 0,$$

in the weak topology of Radon measures.*

Proof.
1. Let $\bar\theta^\varepsilon = \theta^\varepsilon - \hat\theta^\varepsilon$ with $\hat\theta^\varepsilon$ the unique solution of

$$\hat\theta_t^\varepsilon - \Delta \hat\theta^\varepsilon = 0 \quad \text{in } (0, \infty) \times \mathscr{R}^d$$

with initial data $\hat\theta^\varepsilon(0, x) = \theta_0^\varepsilon(x)$. Then (A8) implies that $\hat\theta^\varepsilon(t, \cdot)$ converges to

$$\hat\theta(t, \cdot) = G(t, \cdot) * \theta_0$$

strongly in $L^2(\mathscr{R}^d)$.

2. By integration by parts, $\bar\theta^\varepsilon = \bar\theta^{\varepsilon,1} + \bar\theta^{\varepsilon,2}$, where

$$\bar\theta^{\varepsilon,1}(t, \cdot) = G(t, \cdot) * [H^\varepsilon(0, \cdot) - H^\varepsilon(t, \cdot)],$$

$$\bar\theta^{\varepsilon,2}(t, \cdot) = \int_0^t G_\tau(\tau, \cdot) * [H^\varepsilon(t, \cdot) - H^\varepsilon(t - \tau, \cdot)] \, d\tau,$$

and $H(t, x) = h(\varphi(t, x))$. Clearly (2.6) implies that for every $t \geqq 0$, $\bar\theta^{\varepsilon,1}(t, \cdot)$ converges to

$$\bar\theta^1(t, \cdot) = G(t, \cdot) * [H(0, \cdot) - H(t, \cdot)]$$

strongly in $L^2_{\mathrm{loc}}(\mathscr{R}^d)$.

3. For $t, \sigma > 0$, set $\delta = \min\{\sigma, t\}$. Then

$$\int_{\delta}^{t} G_\tau(\tau,\cdot) * [H^\varepsilon(t,\cdot) - H^\varepsilon(t-\tau,\cdot)]\, d\tau$$

converges to

$$\int_{\delta}^{t} G_\tau(\tau,\cdot) * [H(t,\cdot) - H(t-\tau,\cdot)]\, d\tau$$

strongly in $L^2_{\text{loc}}(\mathscr{R}^d)$. And by (2.5),

$$\begin{aligned}\left\| \int_0^\delta (G_\tau(\tau,\cdot) * [H^\varepsilon(t,\cdot) - H^\varepsilon(t-\tau,\cdot)]\, d\tau \right\|_2 &\leqq \int_0^\delta \| G_\tau(\tau,\cdot)\|_1 \, \| H^\varepsilon(t,\cdot) - H^\varepsilon(t-\tau,\cdot)\|_2 \, d\tau, \\ &\leqq K \int_0^\delta \tfrac{1}{\tau} \| H^\varepsilon(t,\cdot) - H^\varepsilon(t-\tau,\cdot)\|_1^{1/2}\, d\tau, \\ &\leqq K(\delta)^{1/4} \leqq K\sigma^{1/4}.\end{aligned}$$

A similar argument shows that

$$\left\| \int_0^\delta G_\tau(\tau,\cdot) * [H(t,\cdot) - H(t-\tau,\cdot)]\, d\tau \right\|_2 \leqq K\sigma^{1/4}.$$

Therefore, for all $\sigma, t, R > 0$,

$$\limsup_{\varepsilon\to 0} \| \theta^\varepsilon(t,\cdot) - \theta(t,\cdot)\|_{L^2(B_R)} \leqq K\sigma^{1/4},$$

where

$$\begin{aligned}\theta(t,x) = \hat{\theta}(t,x) &+ G(t,\cdot) * [H(0,\cdot) - H(t,\cdot)])(x) \\ &+ \int_0^t G_\tau(\tau,\cdot) * [H(t,\cdot) - H(t-\tau,\cdot)]\, d\tau. \quad \square\end{aligned}$$

An elementary argument, very similar to the proof of Proposition 3.4, shows that the map

$$t \mapsto \|\theta^\varepsilon(t,\cdot)\|_2$$

is uniformly Hölder continuous in $\varepsilon \in (0, 1]$, with exponent $\frac{1}{4}$. This fact will not be used in our analysis.

4. A Gradient Estimate

The main result of this section is

Theorem 4.1. *For $T > 0$, there exists a constant $K^* = K^*(T)$ satisfying*

$$|\nabla z^\varepsilon(t,x)|^2 \leqq 1 + \sqrt{\varepsilon} K^*(1 + |z^\varepsilon(t,x)|), \tag{4.1}$$

for all $(t,x) \in [0,T] \times \mathscr{R}^d$, and $0 < \varepsilon \leqq 1$.

This estimate is an essential ingredient of the monotonicity result that is proved in §5. In particular, (4.1) implies that the weak* limit of the discrepancy measure ξ^ε introduced earlier (cf. (1.7), (2.1)) is nonpositive. Later we show that this limit is zero (see Proposition 7.3.).

The proof of this estimate, which is tangential to the main thrust of this paper, will be completed in several steps. Before I start the lengthy analysis, I briefly explain the main idea. Set

$$w^\varepsilon = |\nabla z^\varepsilon|^2.$$

In view of the equation (ZE),

$$w_t^\varepsilon + \mathscr{L}_t^\varepsilon w^\varepsilon + R^\varepsilon(t, x, w^\varepsilon) - 2\nabla\theta^\varepsilon \cdot \nabla z^\varepsilon \leqq 0, \tag{4.2}$$

where for $\psi \in C^2(\mathscr{R}^d)$,

$$\mathscr{L}_t^\varepsilon \psi(x) = -\Delta\psi(x) + \frac{4\varphi^\varepsilon(t, x)}{\varepsilon} \nabla z^\varepsilon(t, x) \cdot \nabla\psi(x),$$

$$R^\varepsilon(t, x, r) = \frac{4}{\varepsilon^2} q'\left(\frac{z^\varepsilon(t, x)}{\varepsilon}\right) r(r-1), \quad r \geqq 0.$$

In [29, §8], I obtained pointwise estimates for a differential inequality obtained by setting the last term involving $\nabla\theta^\varepsilon$ in (4.2) to zero. Here we start by using the technique developed in [29]. Using (3.6), we first obtain the crude estimate that

$$|2\nabla\theta^\varepsilon \cdot \nabla z^\varepsilon| \leqq 2\|\nabla\theta^\varepsilon\|_{\infty, T} w^\varepsilon \leqq \frac{K}{\varepsilon} w^\varepsilon$$

for $w^\varepsilon \geqq 1$. Then the proof of Proposition 8.1 in [29] yields that w^ε is uniformly bounded in ε.

Our next step is to obtain a uniform bound for $\varepsilon|z_t^\varepsilon|$ (see (4.8)). Using these estimates, we shall obtain a bound for $|\nabla\theta^\varepsilon|$, which is slightly better than (3.6). Finally, we shall use this new estimate of $|\nabla\theta^\varepsilon|$ in (4.2) together with an argument similar to the ones used in [29] to obtain (4.1).

Remark 4.1. In the phase-field equations if, in contrast to the choice we actually made, we take

$$g = 2W = (q')^2,$$

then w satisfies

$$w_t^\varepsilon + \mathscr{L}_t^\varepsilon w^\varepsilon + R^\varepsilon(t, x, w^\varepsilon) - 2q'\nabla\theta^\varepsilon \cdot \nabla z^\varepsilon - \frac{2}{\varepsilon} q''\theta^\varepsilon w^\varepsilon \leqq 0,$$

and the proof of the estimate (4.1) simplifies greatly. Indeed an attendant modification of the proof of Proposition 4.1 below yields this estimate.

As in §3, we fix $T > 0$ and denote by K all constants depending only on T.

Proposition 4.2. *There is a $K = K(T)$ satisfying*

$$|\nabla z^\varepsilon(t,x)|^2 \leqq K(1 + |z^\varepsilon(t,x)|), \quad (t,x) \in [0,T] \times \mathscr{R}^d. \tag{4.3}$$

Proof.

1. Fix $T > 0$ and set

$$K_0 = 2 \sup_{0<\varepsilon\leqq 1} \varepsilon \|\nabla\theta^\varepsilon\|_{\infty,T},$$

$$\hat{\mathscr{L}}\psi = \mathscr{L}_t^\varepsilon\psi - 2|\nabla\theta^\varepsilon(t,x)|\psi.$$

Then

$$w_t^\varepsilon + \hat{\mathscr{L}}w^\varepsilon + R^\varepsilon(t,x,w^\varepsilon) \leqq 0. \tag{4.4}$$

In the next several steps we construct a "supersolution" to (4.4).

2. Let $z_0 > 0$ be the point that satisfies

$$q'\left(\frac{z_0}{\varepsilon}\right) = \varepsilon^{1/4} \Rightarrow z_0 = \varepsilon(q')^{-1}(\varepsilon^{1/4}).$$

Then z_0 behaves like $\varepsilon|\ln\varepsilon|$ as ε tends to zero. Indeed,

$$\lim_{\varepsilon\to 0} \frac{z_0}{\varepsilon|\ln\varepsilon|} = \frac{1}{8}.$$

Now define

$$h_\varepsilon(r) = \begin{cases} \frac{1}{2}C_\varepsilon r^2 + 1, & |r| \leqq z_0, \\ (K_0+1)[|r| - z_0] + h_\varepsilon(z_0), & |r| > z_0, \end{cases}$$

where

$$C_\varepsilon = \frac{K_0 + 1}{z_0},$$

so that h_ε is continuously differentiable with Lipschitz derivatives. Finally we set

$$W = 1 + h_\varepsilon(z^\varepsilon).$$

3. By (ZE) and a direct calculation, we obtain

$$\begin{aligned} I &= W_t + \hat{\mathscr{L}}W + R^\varepsilon(t,x,W) \\ &\geqq h_\varepsilon'(z^\varepsilon)[z_t^\varepsilon - \Delta z^\varepsilon + \tfrac{4}{\varepsilon}\varphi^\varepsilon w^\varepsilon] - h_\varepsilon''(z^\varepsilon)w^\varepsilon - \frac{K_0}{\varepsilon}W + \frac{4}{\varepsilon^2}q'\left(\frac{z^\varepsilon}{\varepsilon}\right)h_\varepsilon(z^\varepsilon)W \\ &\geqq \frac{2}{\varepsilon}\varphi^\varepsilon h_\varepsilon'(z^\varepsilon)(w^\varepsilon + 1) + \frac{4}{\varepsilon^2}q'\left(\frac{z^\varepsilon}{\varepsilon}\right)h_\varepsilon(z^\varepsilon)W - \frac{K_0}{\varepsilon}W - h_\varepsilon''(z^\varepsilon)w^\varepsilon + h_\varepsilon'(z^\varepsilon)\theta^\varepsilon. \end{aligned}$$

Observe that $h_\varepsilon \geqq 1$, $|h_\varepsilon'| \leqq K_0 + 1$ and

$$\|h_\varepsilon''\|_\infty = C_\varepsilon, \quad \lim_{\varepsilon\to 0} \varepsilon C_\varepsilon = 0. \tag{4.5}$$

Hence

$$I \geqq \frac{2}{\varepsilon} \varphi^\varepsilon h_\varepsilon'(w^\varepsilon + 1) + \frac{4}{\varepsilon^2} q' W - \frac{K_0}{\varepsilon} W - C_\varepsilon w^\varepsilon - (K_0 + 1) \| \theta^\varepsilon \|_{\infty, T}.$$

4. Suppose that

$$|z^\varepsilon(t, x)| \leqq z_0.$$

(The opposite case will be discussed in the next step). Then

$$q'\left(\frac{z^\varepsilon}{\varepsilon}\right) \geqq q'\left(\frac{z_0}{\varepsilon}\right) = \varepsilon^{1/4}.$$

Since $\varphi^\varepsilon h_\varepsilon' \geqq 0$, $W \geqq 1$, by (3.3), we obtain

$$\begin{aligned} I &\geqq \frac{4}{\varepsilon^2} q' W - \frac{K_0}{\varepsilon} W - C_\varepsilon w^\varepsilon - (K_0 + 1) \| \theta^\varepsilon \|_{\infty, T} \\ &\geqq \left(\frac{4}{\varepsilon^{7/4}} - \frac{K_0}{\varepsilon} \right) W - C_\varepsilon w^\varepsilon - (K_0 + 1) K (1 + |\ln \varepsilon|) \\ &\geqq C_\varepsilon (W - w^\varepsilon) \end{aligned}$$

for sufficiently small $\varepsilon > 0$.

5. Suppose that $|z^\varepsilon(t, x)| \geqq z_0$. Then

$$h_\varepsilon'(|z^\varepsilon(t, x)|) = K_0 + 1,$$

$$|\varphi^\varepsilon(x, t)| = q\left(\frac{|z^\varepsilon(t, x)|}{\varepsilon}\right) \geqq q\left(\frac{z_0}{\varepsilon}\right) \geqq \frac{1}{2}$$

for sufficiently small $\varepsilon > 0$. Therefore

$$\varphi^\varepsilon(t, x) h_\varepsilon'(z^\varepsilon(t, x)) = |\varphi^\varepsilon(t, x)| h_\varepsilon'(|z^\varepsilon(t, x)|) \geqq \tfrac{1}{2}[K_0 + 1].$$

Since $q' \geqq 0$,

$$\begin{aligned} I &\geqq \frac{2\varphi^\varepsilon}{\varepsilon} h_\varepsilon'(z^\varepsilon)(w^\varepsilon + 1) - \frac{K_0}{\varepsilon} W - C_\varepsilon w^\varepsilon - (K_0 + 1) \| \theta^\varepsilon \|_{\infty, T} \\ &\geqq \left(\frac{K_0 + 1}{\varepsilon} - C_\varepsilon \right) w^\varepsilon - \frac{K_0}{\varepsilon} W + (K_0 + 1) \left[\frac{1}{\varepsilon} - \| \theta^\varepsilon \|_{\infty, T} \right], \end{aligned}$$

and (3.3) and (4.5) imply that $I \geqq 0$ on $\{w^\varepsilon \geqq W\}$.

6. In Steps 3, 4 and 5, we proved that for every $T > 0$ there is an $\varepsilon_0 = \varepsilon_0(T) > 0$ that satisfies

$$W_t + \hat{\mathscr{L}} W + R^\varepsilon(t, x, W) \geqq C_\varepsilon (W - w^\varepsilon)$$

on $(0, T)\times\mathscr{R}^d\cap\{w^\varepsilon \geqq W\}$ for all $0<\varepsilon\leqq\varepsilon_0(T)$. Also in Step 1, we showed that

$$w_t^\varepsilon + \hat{\mathscr{L}}w^\varepsilon + R^\varepsilon(t, x, w^\varepsilon) \leqq 0 \quad \text{in } (0, T)\times\mathscr{R}^d.$$

Since $W \geqq 1 \geqq w^\varepsilon(0, x)$, by the maximum principle we conclude that $W \geqq w^\varepsilon$ on $(0, T)\times\mathscr{R}^d$. See the proof of Proposition 4.2 in [29, §8] for an application of the maximum principle in a very similar situation.

7. Since

$$h_\varepsilon(z) \leqq (K_0 + 1)|z| + 1,$$

we conclude that

$$w^\varepsilon \leqq W = 1 + h_\varepsilon(z^\varepsilon) \leqq 1 + (K_0 + 1)(|z^\varepsilon| + 1). \quad \square$$

Our next step is a crude estimate of $|z_t^\varepsilon|$. We obtain a better estimate in Lemma 4.4.

Lemma 4.3. *For* $0<\varepsilon\leqq 1$,

$$|z_t^\varepsilon(t, x)| \leqq \frac{K}{\varepsilon^2}, \quad (t, x)\in[0, T]\times\mathscr{R}^d. \tag{4.6}$$

Proof.

1. For $\alpha > 0$, set

$$\Omega = \left\{(t, x)\in[0, T)\times\mathscr{R}^d : \left|\frac{z^\varepsilon(t, x)}{\varepsilon}\right| > \alpha\right\}.$$

By (3.6),

$$|z_t^\varepsilon| = \varepsilon\left(q'\left(\frac{z^\varepsilon}{\varepsilon}\right)\right)^{-1}|\varphi_t^\varepsilon| \leqq \frac{K}{\varepsilon}\left(q'\left(\frac{z^\varepsilon}{\varepsilon}\right)\right)^{-1}.$$

Hence, on the complement of Ω,

$$|z_t^\varepsilon| \leqq \frac{K}{\varepsilon q'(\alpha)}.$$

2. Set $v = z_t^\varepsilon$. Differentiate (ZE) to obtain

$$v_t - \Delta v + \frac{4\varphi^\varepsilon}{\varepsilon}\nabla z^\varepsilon\cdot\nabla v + \frac{2}{\varepsilon^2}q'\left(\frac{z^\varepsilon}{\varepsilon}\right)(|\nabla z^\varepsilon|^2 - 1)v = \theta_t^\varepsilon. \tag{4.7}$$

3. For $K_1 > 0$, let

$$V = \frac{K_1}{\varepsilon}\left(1 + \frac{|z^\varepsilon|}{\varepsilon}\right).$$

We show that for appropriately chosen K_1 and α, the function V is a supersolution of (4.7) in Ω. Indeed in Ω,

$$\begin{aligned} I &= V_t - \Delta V + \frac{4\varphi^\varepsilon}{\varepsilon} \nabla z^\varepsilon \cdot \nabla V + \frac{2}{\varepsilon^2} q'\left(\frac{z^\varepsilon}{\varepsilon}\right)(|\nabla z^\varepsilon|^2 - 1) V \\ &\geqq \frac{K_1 z^\varepsilon}{\varepsilon^2 |z^\varepsilon|}\left[z_t^\varepsilon - \Delta z^\varepsilon + \frac{4\varphi^\varepsilon}{\varepsilon}|\nabla z^\varepsilon|^2\right] - \frac{2K_1}{\varepsilon^3} q'\left(\frac{z^\varepsilon}{\varepsilon}\right)\left[\frac{|z^\varepsilon|}{\varepsilon} + 1\right] \\ &\geqq \frac{K_1}{\varepsilon^3}\left\{2|\varphi^\varepsilon|(1 + |\nabla z^\varepsilon|^2) - \varepsilon \|\theta^\varepsilon\|_{\infty, T} - 2 \sup_{r \geqq \alpha} q'(r)(r+1)\right\} \\ &\geqq \frac{K_1}{\varepsilon^3}\left\{2q(\alpha) - \varepsilon \|\theta^\varepsilon\|_{\infty, T} - 2 \sup_{r \geqq \alpha} q'(r)(r+1)\right\}. \end{aligned}$$

Since $q'(r)$ is exponentially small for large values of r, (3.3) and (3.9) imply that there are constants K_1 and α such that

$$I \geqq \|\theta_t^\varepsilon\|_{\infty, T} \qquad \text{in } \Omega$$

for all sufficiently small ε. By redefining K_1, if necessary, we may assume that

$$\inf_{\partial\Omega} V = \frac{K_1}{\varepsilon}(1 + \alpha) \geqq \frac{K}{\varepsilon q'(\alpha)} = \sup_{\Omega^c} |z_t^\varepsilon|.$$

4. We proved that there is an $\varepsilon_0 > 0$ such that for, $0 < \varepsilon \leqq \varepsilon_0$, V is a supersolution of (4.7) in Ω. Moreover $V \geqq v$ on $\partial\Omega$. Therefore, by the maximum principle,

$$V \geqq v \quad \text{in } \Omega,\ \varepsilon \leqq \varepsilon_0.$$

Hence, by Step 1,

$$v = z_t^\varepsilon \leqq \frac{K}{\varepsilon}\left(1 + \frac{|z^\varepsilon|}{\varepsilon}\right)$$

for all $0 < \varepsilon \leqq \varepsilon_0$. For $\varepsilon_0 \leqq \varepsilon < 1$ this last estimate is easier to prove. These arguments also yield the same bound for $-z_t^\varepsilon$.

5. Set

$$\hat{\Omega} = \{|z^\varepsilon| \geqq 1\}, \quad \hat{V} = \frac{K}{\varepsilon^2}(1 + t).$$

Then on $[0, T] \times \mathscr{R}^d \cap \hat{\Omega}$,

$$\begin{aligned} &\hat{V}_t - \Delta \hat{V} + \frac{4\varphi^\varepsilon}{\varepsilon} \nabla z^\varepsilon \cdot \nabla \hat{V} + \frac{2}{\varepsilon^2} q'\left(\frac{z^\varepsilon}{\varepsilon}\right)(|\nabla z^\varepsilon|^2 - 1)\hat{V} \\ &\geqq \frac{K}{\varepsilon^2} - \frac{K(1+T)}{\varepsilon^4} q'\left(\frac{1}{\varepsilon}\right) \geqq \|\theta_t^\varepsilon\|_{\infty, T} \end{aligned}$$

for sufficiently small ε. Also, by Step 4, $\hat{V} \geqq |v|$ on $\partial\hat{\Omega}$. Hence (4.6) follows from the maximum principle. □

Next we improve (4.6).

Lemma 4.4.

$$|z_t^\varepsilon(t,x)| + |D^2 z^\varepsilon(t,x)| \leqq \frac{K}{\varepsilon}(1 + |z^\varepsilon(t,x)|), \quad (t,x) \in [0,T] \times \mathscr{R}^d. \tag{4.8}$$

Proof. Fix $T > 0$. All the constants in this proof depend in T. Set

$$k^\varepsilon = \sup \left\{ \frac{\varepsilon |z^\varepsilon_{x_i x_j}(t,x)|}{1 + |z^\varepsilon(t,x)|} : (t,x) \in [0,T] \times \mathscr{R}^d,\, i,j = 1,\ldots,d \right\}.$$

1. In view of (4.3), for any $t \leqq T$ and $x, y \in \mathscr{R}^d$,

$$(|z^\varepsilon(t,y)| + 1) \leqq e^{K|x-y|}(|z^\varepsilon(t,x)| + 1).$$

Also (4.6) implies that there is a K^* such that for all $\tau \in [0,t]$,

$$\begin{aligned}(|z^\varepsilon(t-\tau,y) + 1) &\leqq \left(1 + \frac{K^*\tau}{\varepsilon^2}\right)(|z^\varepsilon(t,y)| + 1) \\ &\leqq \left(1 + \frac{K^*\tau}{\varepsilon^2}\right) e^{K^*|x-y|}(|z^\varepsilon(t,x)| + 1). \end{aligned}\tag{4.9}$$

2. Fix $(t_0, x_0) \in [0,T] \times \mathscr{R}^d$. For any $h \in (0, t_0 \wedge 1]$, (ZE) yields

$$z^\varepsilon_{x_i x_j}(t_0, x_0) = a + b + c,$$

where

$$\begin{aligned} a &= (G_{x_i}(h,\cdot) * z^\varepsilon_{x_j}(t_0 - h, \cdot))(x_0), \\ b &= \int_0^h (G_{x_i}(\tau,\cdot) * \theta^\varepsilon_{x_j}(t_0 - \tau, \cdot))(x_0)\, d\tau, \\ c &= \int_0^h (G_{x_i}(\tau,\cdot) * F^\varepsilon_{x_j}(t_0 - \tau, \cdot))(x_0)\, d\tau, \\ F^\varepsilon &= \frac{2\varphi^\varepsilon}{\varepsilon}(1 - |\nabla z^\varepsilon|^2). \end{aligned}$$

3. If $h = t_0$, by (A4), we obtain

$$|a| \leqq \|G(h,\cdot)\|_1 \|D^2 z_0^\varepsilon\|_\infty \leqq \frac{K}{\varepsilon}.$$

When $h < t_0$, (4.3) and (4.9) imply that

$$\begin{aligned} |a| &\leqq \int_{\mathscr{R}^d} K |\nabla G(h, x_0 - y)| [1 + |z^\varepsilon(t_0 - h, y)|]^{1/2}\, dy \\ &\leqq K\left(1 + \frac{K^*h}{\varepsilon^2}\right)^{1/2} [1 + |z^\varepsilon(t_0, x_0)|]^{1/2} \int_{\mathscr{R}^d} e^{\frac{1}{2}K^*|w|} |\nabla G(h,w)|\, dw \\ &\leqq \frac{K}{\sqrt{h}}\left(1 + \frac{K^*h}{\varepsilon^2}\right)^{1/2} (1 + |z^\varepsilon(t_0, x_0)|)^{1/2}. \end{aligned}$$

4. By (3.6),

$$|b| \leqq \frac{K}{\varepsilon}\sqrt{h}.$$

5. Differentiate F^ε to obtain

$$\nabla F^\varepsilon = \frac{2}{\varepsilon^2} q'\left(\frac{z^\varepsilon}{\varepsilon}\right) \nabla z^\varepsilon(1 - |\nabla z^\varepsilon|^2) - \frac{4\varphi^\varepsilon}{\varepsilon} D^2 z^\varepsilon \nabla z^\varepsilon.$$

The definition of k^ε, (4.3) and (4.9) yield

$$\begin{aligned} |F^\varepsilon_{x_j}(t_0 - \tau, y)| &\leqq \frac{K}{\varepsilon^2}\left[\sup_r q'(r)(1 + \varepsilon r)^{3/2} + (1 + |z^\varepsilon(t_0 - \tau, y)|)^{3/2} k^\varepsilon\right] \\ &\leqq \frac{K}{\varepsilon^2}\left[1 + k^\varepsilon\left(1 + \frac{K^*\tau}{\varepsilon^2}\right)^{3/2}(1 + |z^\varepsilon(t_0, x_0)|)^{3/2} \exp\left(\frac{3}{2}K^*|x_0 - y|\right)\right]. \end{aligned}$$

Therefore

$$|c| \leqq C^* \frac{\sqrt{h}}{\varepsilon^2}\left[1 + k^\varepsilon\left(1 + \frac{K^* h}{\varepsilon^2}\right)^{3/2}(1 + |z^\varepsilon(t_0, x_0)|)^{3/2}\right]$$

for some C^*, and without loss of generality we may assume that $C^* \geqq 1$.
6. Choose

$$h = \min\{t_0, \varepsilon^2[4(1 + K^*)^3(1 + |z^\varepsilon(t_0, x_0)|)(C^*)^2]^{-1}\}.$$

Since $h \leqq \varepsilon^2$, $(1 + K^* h/\varepsilon^2) \leqq (1 + K^*)$ and therefore

$$|c| \leqq \frac{1}{2\varepsilon}(1 + |z^\varepsilon(t_0, x_0)|)k^\varepsilon + \frac{C^*}{\varepsilon},$$

and, by Step 3,

$$|a| \leqq \frac{K}{\varepsilon}(1 + K^*)^2 C^*(1 + |z^\varepsilon(t_0, x_0)|).$$

Therefore

$$|z^\varepsilon_{x_i x_j}(t_0, x_0)| \leqq \frac{1}{\varepsilon}(1 + |z^\varepsilon(t_0, x_0)|)[K + \tfrac{1}{2} k^\varepsilon].$$

The inequality (4.8) follows from this estimate and (ZE). □

We continue by improving the estimate for $|\nabla\theta^\varepsilon|$.

Lemma 4.5. *For every* $(t, x) \in [0, T] \times \mathscr{R}^d$,

$$|\nabla\theta^\varepsilon(t, x)| \leqq \frac{K}{\sqrt{\varepsilon[\varepsilon + (|z^\varepsilon(t, x)| \wedge 1)]}} \tag{4.10}$$

for some constant $K = K(T)$.

Proof. Fix $(t_0, x_0) \in [0, T] \times \mathscr{R}^d$ and set

$$p^\varepsilon = \frac{|z^\varepsilon(t_0, x_0)| \wedge 1}{\varepsilon}.$$

If $p^\varepsilon \leqq 1$, (4.10) at (t_0, x_0) follows from (3.6); so we may assume that $p^\varepsilon \geqq 1$.

1. For $\varepsilon, \lambda > 0$, set

$$O_{\varepsilon,\lambda} = [t_0 - \varepsilon^2\lambda p^\varepsilon, t_0 + \varepsilon^2\lambda p^\varepsilon] \times B^\varepsilon, \quad B^\varepsilon = \{|x - x_0| \leqq \varepsilon\lambda p^\varepsilon\}.$$

We assert that there exists $\lambda = \lambda(T) > 0$ satisfying

$$|z^\varepsilon(t, x)| \geqq \frac{\varepsilon}{2} p^\varepsilon \quad \forall (t, x) \in O_{\varepsilon,\lambda}.$$

Use (4.3) and (4.8) to construct a constant $K = K(T)$ such that

$$|z^\varepsilon(s, y)| + 1 \leqq (|z^\varepsilon(t, x)| + 1) \exp K\left(\frac{|t - s|}{\varepsilon} + |x - y|\right)$$

for all $s, t \leqq T$. Now suppose that

$$z^\varepsilon(t_0 + \varepsilon^2\tau, x_0 + \varepsilon y) = \tfrac{1}{2}\varepsilon p^\varepsilon \quad \text{for some } (\tau, y) \in \mathscr{R}^{d+1}.$$

We use the previous estimate with $(s, y) = (t_0, x_0)$ and $(t, x) = (t_0 + \varepsilon^2\tau, x_0 + \varepsilon y)$ to obtain

$$1 + \varepsilon p^\varepsilon \leqq 1 + |z^\varepsilon(t_0, x_0)| \leqq \left(1 + \frac{\varepsilon}{2} p^\varepsilon\right) e^{\varepsilon K(|\tau| + |y|)}.$$

Since $\varepsilon p^\varepsilon \leqq 1$,

$$\varepsilon K(|\tau| + |y|) \geqq \ln\left(\frac{1 + \varepsilon p^\varepsilon}{1 + \frac{\varepsilon}{2} p^\varepsilon}\right) \geqq \ln\left(1 + \frac{\varepsilon}{4} p^\varepsilon\right) \geqq \frac{\varepsilon}{8} p^\varepsilon.$$

Hence, for $\lambda = 1/8K$,

$$|z^\varepsilon(t, x)| \geqq \tfrac{1}{2}\varepsilon p^\varepsilon \quad \forall (t, x) \in O_{\varepsilon,\lambda}.$$

2. Set

$$\sigma = \min\{t_0, \varepsilon^2\lambda p^\varepsilon\}.$$

By integrating by parts and by (3.1), we obtain

$$\nabla\theta^\varepsilon(t_0, x_0) = a + b + c$$

where

$$a = (\nabla G(t_0, \cdot) * [\theta_0^\varepsilon + H^\varepsilon(0, \cdot) - H^\varepsilon(t_0, \cdot)])(x_0),$$

$$b = \int_\sigma^{t_0} (\nabla G_\tau(\tau, \cdot) * [H^\varepsilon(t_0, \cdot) - H^\varepsilon(t_0 - \tau, \cdot)])(x_0)\, d\tau,$$

$$c = \int_0^\sigma (\nabla G_\tau(\tau, \cdot) * [H^\varepsilon(t_0, \cdot) - H^\varepsilon(t_0 - \tau, \cdot)])(x_0)\, d\tau.$$

3. Since $\varepsilon p^\varepsilon \leqq 1$, (A6) yields

$$|(\nabla G(t_0,\cdot) * \theta_0^\varepsilon)(x_0)| = |(G(t_0,\cdot) * \nabla\theta_0^\varepsilon)(x_0)| \leqq \frac{K}{\sqrt{\varepsilon}} \leqq \frac{K}{\varepsilon\sqrt{p^\varepsilon}}.$$

Also, when $t_0 \geqq \varepsilon^2\lambda p^\varepsilon$,

$$|(\nabla G(t_0,\cdot) * [H^\varepsilon(0,\cdot) - H^\varepsilon(t_0,\cdot)])(x_0)| \leqq \frac{K}{\sqrt{t_0}} \leqq \frac{K}{\varepsilon\sqrt{\lambda p^\varepsilon}}.$$

However, if $t_0 \leqq \varepsilon^2\lambda p^\varepsilon$, then

$$\{0\} \times B^\varepsilon \subset O_{\varepsilon,\lambda}$$

so that by (3.2) and Step 1,

$$|\nabla H^\varepsilon(0,y)| = |g(\varphi^\varepsilon(0,y))\nabla\varphi^\varepsilon(0,y)| \leqq \frac{K}{\varepsilon} q'\left(\frac{z^\varepsilon(0,y)}{\varepsilon}\right)$$
$$\leqq \frac{K}{\varepsilon} q'\left(\frac{p^\varepsilon}{2}\right) \quad \forall y \in B^\varepsilon.$$

Hence,

$$\begin{aligned}|(\nabla G(t_0,\cdot) * H^\varepsilon(0,\cdot))(x_0)| &= |(G(t_0,\cdot) * \nabla H^\varepsilon(0,\cdot))(x_0)| \\ &\leqq \int_{B^\varepsilon} G(t_0, x_0 - y) q'\left(\frac{p^\varepsilon}{2}\right)\frac{K}{\varepsilon}\,dy + \int_{\mathscr{R}^d - B^\varepsilon} G(t_0, x_0 - y)\frac{K}{\varepsilon}\,dy \\ &\leqq \frac{K}{\varepsilon}\left[q'\left(\frac{p^\varepsilon}{2}\right) + \int G(1,w)\chi_{\{\sqrt{4t_0}|w| \geqq \varepsilon\lambda p^\varepsilon\}}\,dw\right] \\ &\leqq \frac{K}{\varepsilon}\left[q'\left(\frac{p^\varepsilon}{2}\right) + G\left(1, \frac{\varepsilon\lambda p^\varepsilon}{\sqrt{4t_0}}\right)\right].\end{aligned}$$

Since $t_0 \leqq \varepsilon^2\lambda p^\varepsilon$ and $p^\varepsilon \geqq 1$,

$$|(\nabla G(t_0,\cdot) * H^\varepsilon(0,\cdot))(x_0)| \leqq \frac{K}{\varepsilon\sqrt{p^\varepsilon}}.$$

Indeed, we can estimate this quantity by a function decaying faster than the square root, but this sharper estimate does not improve the final estimate.

Next, we estimate $|(\nabla G(t_0,\cdot) * H^\varepsilon(t_0,\cdot))(x_0)|$ exactly the same way to obtain

$$|a| \leqq \frac{K}{\varepsilon\sqrt{p^\varepsilon}}.$$

4. Since $\|\nabla G_\tau(\tau,\cdot)\|_1 \leqq K\tau^{-3/2}$,

$$|b| \leqq K\left(\frac{1}{\sqrt{\sigma}} - \frac{1}{\sqrt{t_0}}\right) \leqq \frac{K}{\varepsilon\sqrt{p^\varepsilon}}.$$

5. By integration by parts in the variable t,

$$|c| \leqq \left| \int_0^\sigma (\nabla G(\tau,\cdot) * g(\varphi^\varepsilon(t_0-\tau,\cdot))\varphi_t^\varepsilon(t_0-\tau,\cdot))(x_0)\,d\tau \right| + |(\nabla G(\sigma,\cdot) * [H^\varepsilon(t_0,\cdot) - H^\varepsilon(t_0-\sigma,\cdot)])(x_0))|.$$

Since $\sigma \leqq \varepsilon^2 \lambda p^\varepsilon$,

$$[t_0-\sigma, t_0] \times B^\varepsilon \subset O_{\varepsilon,\lambda}.$$

Therefore, for any $y \in B^\varepsilon$, $\tau \in [0,\sigma]$, (3.6) yields

$$|g^\varepsilon(\varphi^\varepsilon(t_0-\tau,y))\varphi_t^\varepsilon(t_0-\tau,y)| \leqq |\varphi_t^\varepsilon| q'\left(\frac{p^\varepsilon}{2}\right) \leqq \frac{K}{\varepsilon^2} q'\left(\frac{p^\varepsilon}{2}\right).$$

As in Step 3,

$$\left| \int_0^\sigma (\nabla G(\tau,\cdot) * g(\varphi^\varepsilon(t_0-\tau,\cdot))\varphi_t^\varepsilon(t_0-\tau,\cdot))(x_0)\,d\tau \right|$$

$$\leqq \int_0^\sigma \int_{B^\varepsilon} |\nabla G(\tau, x_0-y)| \frac{K}{\varepsilon^2} q'\left(\frac{p^\varepsilon}{2}\right) d\tau + \int_0^\sigma \int_{\mathscr{R}^d - B^\varepsilon} |\nabla G(\tau, x_0-y)| \frac{K}{\varepsilon^2}\,d\tau$$

$$\leqq \frac{K\sqrt{\sigma}}{\varepsilon^2}\left[q'\left(\frac{p^\varepsilon}{2}\right) + \left| \nabla G\left(1, \frac{\varepsilon\lambda p^\varepsilon}{\sqrt{4\sigma}}\right)\right| \right] \leqq \frac{K}{\varepsilon\sqrt{p^\varepsilon}}.$$

Also, if $\sigma = \varepsilon^2\lambda p^\varepsilon$,

$$|(\nabla G(\sigma,\cdot) * [H^\varepsilon(t_0,\cdot) - H^\varepsilon(t_0-\sigma,\cdot)])(x_0)| \leqq K \|\nabla G(\sigma,\cdot)\|_1 \leqq \frac{K}{\sqrt{\sigma}} \leqq \frac{K}{\varepsilon\sqrt{p^\varepsilon}}.$$

If $\sigma = t_0$, then by step 3,

$$|(\nabla G(\sigma,\cdot) * [H^\varepsilon(t_0,\cdot) - H^\varepsilon(t_0-\sigma,\cdot)])(x_0)|$$

$$= |(G(t_0,\cdot) * [\nabla H^\varepsilon(t_0,\cdot) - \nabla H^\varepsilon(0,\cdot)])(x_0)| \leqq \frac{K}{\varepsilon\sqrt{p^\varepsilon}}.$$

6. Finally, we combine Steps 3, 4 and 5 to conclude that

$$|\nabla\theta^\varepsilon(t_0,x_0)| \leqq \frac{K}{\varepsilon\sqrt{p^\varepsilon}}. \quad \square$$

We are now in a position to prove Theorem 4.1.

Proof of Theorem 4.1. This proof is very similar to the proof of Proposition 4.2.
1. Let z_0 be as in Proposition 4.3, i.e.,

$$q'\left(\frac{z_0}{\varepsilon}\right) = \varepsilon^{1/4}.$$

Set

$$K_\varepsilon = (z_0)^{-3/2}.$$

Since $q'(r)$ decays exponentially,

$$\lim_{\varepsilon\to 0} \frac{z_0}{\varepsilon|\ln\varepsilon|} = \frac{1}{8}, \quad \lim_{\varepsilon\to 0} (\varepsilon|\ln\varepsilon|)^{3/2} K_\varepsilon < \infty. \tag{4.11}$$

2. For a real number r, set

$$f_\varepsilon(r) = \begin{cases} \frac{1}{2}K_\varepsilon r^2 + 1, & |r| \leqq z_0, \\ 2[\sqrt{r} - \sqrt{z_0}] + f_\varepsilon(z_0), & |r| \in [z_0, 1], \\ |r| - 1 + f_\varepsilon(1), & |r| \geqq 1. \end{cases}$$

Observe that f_ε is continuously differentiable with Lipschitz continuous derivatives.
3. For $K^* > 1$ define

$$W = 1 + \sqrt{\varepsilon} K^* f_\varepsilon(z^\varepsilon).$$

In the next three steps, we show that for K^* large enough, W is a "supersolution" of (4.4).

Let $\hat{\mathscr{L}}$, R^ε and w^ε be as in Proposition 4.2. Then

$$\begin{aligned} I &= W_t + \hat{\mathscr{L}} W + R^\varepsilon(t, x, W) \\ &= \sqrt{\varepsilon} K^* f'_\varepsilon \left[z^\varepsilon_t - \Delta z^\varepsilon + \frac{4\varphi^\varepsilon}{\varepsilon} w^\varepsilon \right] - \sqrt{\varepsilon} K^* f''_\varepsilon w^\varepsilon \\ &\quad + \frac{4}{\varepsilon^2} q'\left(\frac{z^\varepsilon}{\varepsilon}\right) \sqrt{\varepsilon} K^* f_\varepsilon W - 2|\nabla\theta^\varepsilon(t, x)| W \\ &\geqq \sqrt{\varepsilon} K^* \left\{ f'_\varepsilon \left[\frac{2\varphi^\varepsilon}{\varepsilon}(w^\varepsilon + 1) + \theta^\varepsilon \right] - f''_\varepsilon w^\varepsilon + \frac{4}{\varepsilon^2} q'\left(\frac{z^\varepsilon}{\varepsilon}\right) W \right\} - 2|\nabla\theta^\varepsilon| W. \end{aligned}$$

4. We split the estimate of I into the three cases:

$$\text{(a) } |z^\varepsilon| \leqq z_0, \quad \text{(b) } |z^\varepsilon| \in [z_0, 1], \quad \text{(c) } |z^\varepsilon| \geqq 1,$$

and start with case a. Since $z_0 = \varepsilon(q')^{-1}(\varepsilon^{1/4})$,

$$q'\left(\frac{z^\varepsilon}{\varepsilon}\right) \geqq q'\left(\frac{z_0}{\varepsilon}\right) = \varepsilon^{1/4}.$$

By (4.11),

$$\sqrt{\varepsilon}|f'_\varepsilon(z^\varepsilon)| = \sqrt{\varepsilon} K_\varepsilon |z^\varepsilon| \leqq \sqrt{\varepsilon} z_0 K_\varepsilon \leqq \frac{K}{\sqrt{|\ln\varepsilon|}}$$

for some constant K. Since $f'_\varepsilon \varphi^\varepsilon \geqq 0$,

$$I \geqq K^* \left[-\frac{K}{\sqrt{|\ln\varepsilon|}} \|\theta^\varepsilon\|_{\infty, T} - \sqrt{\varepsilon} K_\varepsilon w^\varepsilon + 4\varepsilon^{-5/4} W \right] - 2\|\nabla\theta^\varepsilon\|_{\infty, T} W.$$

Using (3.3), (3.6), (4.11), and the fact that $W \geqq 1$, we construct $\varepsilon_0 = \varepsilon_0(T) > 0$ such that

$$I \geqq \sqrt{\varepsilon} K^* K_\varepsilon (W - w^\varepsilon), \quad \varepsilon \leqq \varepsilon_0,\, t \leqq T$$

for any $K^* \geqq 1$.

5. Suppose that $|z^\varepsilon| \geqq 1$. Then for sufficiently small ε,

$$f'_\varepsilon \varphi^\varepsilon = |\varphi^\varepsilon| \geqq \tfrac{1}{2}.$$

Moreover, $f''_\varepsilon(z^\varepsilon) = 0$ and, by (4.10),

$$|\nabla \theta^\varepsilon(t, x)| \leqq \frac{\hat{K}}{\sqrt{\varepsilon}}.$$

Since $q' \geqq 0$,

$$I \geqq \frac{K^*}{\sqrt{\varepsilon}}(w^\varepsilon + 1) - \sqrt{\varepsilon} K^* \|\theta^\varepsilon\|_{\infty, T} - \frac{2\hat{K}}{\sqrt{\varepsilon}} W.$$

So if $K^* \geqq 2\hat{K}$, (3.3) implies that $I \geqq 0$ on $\{w^\varepsilon \geqq W\}$ for all sufficiently small ε.

6. Finally we consider the case $|z^\varepsilon| \in [z_0, 1]$. Then, for sufficiently small $\varepsilon > 0$,

$$f'_\varepsilon(z^\varepsilon)\varphi^\varepsilon = f'_\varepsilon(|z^\varepsilon|)|\varphi^\varepsilon| \geqq \frac{1}{2} f'_\varepsilon(|z^\varepsilon|) = \frac{1}{2\sqrt{|z^\varepsilon|}}.$$

Moreover, by the construction of f_ε,

$$\sqrt{\varepsilon} f'_\varepsilon(|z^\varepsilon|) \leqq 1, \quad f''_\varepsilon(z^\varepsilon) \leqq 0 \Rightarrow -f''_\varepsilon w^\varepsilon \geqq 0.$$

Since $\varepsilon \leqq z_0 \leqq |z^\varepsilon| \leqq 1$, by (4.10),

$$|\nabla \theta^\varepsilon(t, x)| \leqq \frac{\hat{K}}{\sqrt{\varepsilon |z^\varepsilon|}},$$

$$I \geqq \frac{K^*}{\sqrt{\varepsilon^* z^\varepsilon|}}(w^\varepsilon + 1) - K^* \|\theta^\varepsilon\|_{\infty, T} - \frac{2\hat{K}}{\sqrt{\varepsilon |z^\varepsilon|}} W.$$

Hence, there exists a constant $\varepsilon_0 > 0$ such that, on $\{w^\varepsilon \geqq W\}$, $I \geqq 0$ for all $K^* \geqq 2\hat{K}$ and $\varepsilon \in (0, \varepsilon_0]$.

7. Steps 4, 5 and 6 yield

$$I \geqq \sqrt{\varepsilon} K^* K_\varepsilon (W - w^\varepsilon) \quad \text{on } \{w^\varepsilon \geqq W\}$$

for $K^* \geqq 2\hat{K}$ and $\varepsilon \in (0, \varepsilon_0]$. By (A3), $W(0, x) \geqq 1 \geqq |\nabla z_0^\varepsilon|^2$ and therefore the maximum principle implies that $W \geqq w^\varepsilon$ for $\varepsilon \leqq \varepsilon_0$ (see [29, §8] for the details of an application of the maximum principle in a similar situation). Since by construction

$$f_\varepsilon(r) \leqq |r| - f_\varepsilon(1) \leqq |r| + 4,$$

this proves (4.1) for all $\varepsilon \leqq \varepsilon_0$. For $\varepsilon \geqq \varepsilon_0$, (4.1) follows from (4.3). □

The following lemma will be useful in the next section.

Lemma 4.6. *Suppose that there is a bounded, open set $O \subset (0, \infty) \times \mathscr{R}^d$ for which*

$$\beta = \liminf_{\varepsilon \to 0} \inf_{(s,y) \in \bar{O}} |\varphi^\varepsilon(s, y)| > 0.$$

Then for every $(s, y) \in O$,

$$\liminf_{(s', y') \to (s, y) \varepsilon \to 0} |z^\varepsilon(s', y')| \geqq \inf\{|\hat{y} - y| : (s, \hat{y}) \notin O\}.$$

Proof.

1. Since $\bar{O}$ is compact, there is an $\varepsilon_0 > 0$ satisfying

$$|\varphi^\varepsilon(s, y)| \geqq \tfrac{1}{2}\beta \quad \forall (s, y) \in \bar{O}, \ \varepsilon \leqq \varepsilon_0.$$

Since φ^ε is continuous and $h(\varphi^\varepsilon)$ is convergent in L^1_{loc}, either

$$\varphi^\varepsilon(s, y) \geqq \tfrac{1}{2}\beta \quad \forall (s, y) \in \bar{O}, \ \varepsilon \leqq \varepsilon_0 \tag{4.12}$$

or

$$\varphi^\varepsilon(s, y)| \leqq -\tfrac{1}{2}\beta \quad \forall (s, y) \in \bar{O}, \ \varepsilon \leqq \varepsilon_0. \tag{4.13}$$

2. Multiply (ZE) (of Section 2.1) by ε to get

$$2\varphi^\varepsilon(|\nabla z^\varepsilon|^2 - 1) = \varepsilon(-z_t^\varepsilon + \Delta z^\varepsilon) + \varepsilon\theta^\varepsilon. \tag{4.14}$$

In view of (3.3),

$$\lim_{\varepsilon \to 0} \varepsilon \|\theta^\varepsilon\|_{\infty, T} = 0.$$

Set

$$z^*(t, x) = \limsup_{\varepsilon \to 0, (s,y) \to (t,x)} z^\varepsilon(s, y), \quad z_*(t, x) = \liminf_{\varepsilon \to 0, (s,y) \to (t,x)} z^\varepsilon(s, y).$$

Then, as ε approaches zero, (4.14) yields

$$|Dz_*| - 1 \geqq 0 \quad \text{in } O \text{ if (4.12) holds,} \tag{4.15}$$

$$|Dz^*| - 1 \leqq 0 \quad \text{in } O \text{ if (4.13) holds.}$$

These inequalities are to be understood in the viscosity sense [14]; the details of this argument are given in [29, Lemma 4.1].

For $(s, y) \in \bar{O}$ set

$$d(s, y) = \inf\{|y' - y| : (s, y') \notin O\} \text{ if (4.12) holds,}$$

$$d(s, y) = -\inf\{|y' - y| : (s, y') \in O\} \text{ if (4.13) holds.}$$

Then $d(s, y)$ satisfies (4.15) in the viscosity sense and the comparison theorems for the eikonal equation [14] imply that

$$z_*(s, y) \geqq d(s, y) \quad \text{if (4.12) holds,}$$

$$z^*(s, y) \leqq d(s, y) \quad \text{if (4.13) holds.} \quad \square$$

5. Monotonicity Formula

In this section we obtain an extension of the monotonicity formula of CHEN & STRUWE [13], which originates from STRUWE's formula for parabolic flow of harmonic maps [32], and a later result of ILMANEN [23] for the Cahn-Allen equation, which originates from HUISKEN's formula for smooth mean-curvature flows [22].

For $x, x_0 \in \mathscr{R}^d$, $0 \leqq t < t_0$, let

$$\rho(t, x; t_0, x_0) = (4\pi(t_0 - t))^{1/2} G(t_0 - \tau, x_0 - x)$$
$$= (4\pi(t_0 - t))^{-(d-1)/2} \exp\left(-\frac{|x - x_0|^2}{4(t_0 - t)}\right).$$

Then

$$\nabla_x \rho = -\frac{(x - x_0)}{2(t_0 - t)} \rho,$$

$$\rho_t = \left[\frac{d-1}{2(t_0 - t)} - \frac{|x - x_0|^2}{4(t_0 - t)^2}\right] \rho,$$

$$D_x^2 \rho = \left[-\frac{I}{2(t_0 - t)} + \frac{(x - x_0) \otimes (x - x_0)}{4(t_0 - t)^2}\right] \rho,$$

where I is the identity matrix and $\otimes$ is the tensor product. For $t \geqq 0$ and any Borel set $A \subset \mathscr{R}^d$, let $\mu^\varepsilon(t; A)$, $\hat{\mu}^\varepsilon(t; A)$ and ξ^ε be as in §2.2 and (1.7), and set

$$\alpha^\varepsilon(t; t_0, x_0) = \int_{\mathscr{R}^d} \rho(t, x; t_0, x_0) \hat{\mu}^\varepsilon(t; dx).$$

Theorem 5.1. *There is a constant C_d, depending only on the dimension, such that*

$$\frac{d}{dt} \alpha^\varepsilon(t; t_0, x_0) \leqq \frac{1}{2(t_0 - t)} \int_{\mathscr{R}^d} \rho(t, x; t_0, x_0) \xi^\varepsilon(t; dx) + \frac{C_d}{\sqrt{t_0 - t}}. \tag{5.1}$$

Proof. Fix (t_0, x_0). We suppress the dependence on (t_0, x_0) in our notation.

1. By (OPE) and (HE),

$$\frac{d}{dt} \alpha^\varepsilon(t) = \int \rho_t \cdot \hat{\mu}^\varepsilon + \int \rho\left(\varepsilon \nabla\varphi^\varepsilon \cdot \nabla\varphi_t^\varepsilon + \frac{1}{\varepsilon} W'(\varphi^\varepsilon)\varphi_t^\varepsilon + \theta^\varepsilon \theta_t^\varepsilon\right)$$

$$= \int \rho_t \hat{\mu}^\varepsilon - \int \varepsilon \nabla\rho \cdot \nabla\varphi^\varepsilon \varphi_t^\varepsilon$$
$$+ \int \left[\varepsilon \varphi_t^\varepsilon \left(-\Delta\varphi^\varepsilon + \frac{1}{\varepsilon^2} W'(\varphi^\varepsilon)\right) + \theta^\varepsilon(\Delta\theta^\varepsilon - g(\varphi^\varepsilon)\varphi_t^\varepsilon)\right] \rho$$

$$= \int \rho_t \hat{\mu}^\varepsilon - \int \varepsilon \nabla\rho \cdot \nabla\varphi^\varepsilon \varphi_t^\varepsilon - \int \varepsilon(\varphi_t^\varepsilon)^2 \rho + \int \theta^\varepsilon \Delta\theta^\varepsilon \rho =$$

$$= \int \rho_t \hat{\mu}^\varepsilon + \varepsilon \int \left[\nabla\rho \cdot \nabla\varphi^\varepsilon \varphi_t^\varepsilon + \frac{(\nabla\rho \cdot \nabla\varphi^\varepsilon)^2}{\rho} \right]$$

$$- \varepsilon \int \left(\varphi_t^\varepsilon + \frac{\nabla\rho \cdot \nabla\varphi^\varepsilon}{\rho} \right)^2 \rho - \int |\nabla\theta^\varepsilon|^2 \rho + \int \Delta\rho \frac{1}{2} (\theta^\varepsilon)^2 .$$

Since

$$\rho_t + \Delta\rho = -\frac{1}{2(t_0 - t)} \rho, \quad \hat{\mu}^\varepsilon = \mu^\varepsilon + \frac{1}{2} (\theta^\varepsilon)^2,$$

it follows that

$$\frac{d}{dt} \alpha^\varepsilon(t) = - \varepsilon \int \left(\varphi_t^\varepsilon + \frac{\nabla\rho \cdot \nabla\varphi^\varepsilon}{\rho} \right)^2 \rho$$

$$+ \int \left[\rho_t \mu^\varepsilon + \nabla\rho \cdot \nabla\varphi^\varepsilon \left(\varepsilon\Delta\varphi^\varepsilon - \frac{1}{\varepsilon} W'(\varphi^\varepsilon) \right) + \varepsilon \frac{(\nabla\rho \cdot \nabla\varphi^\varepsilon)^2}{\rho} \right]$$

$$- \int \rho \left[|\nabla\theta^\varepsilon|^2 + \frac{1}{4(t_0 - t)} (\theta^\varepsilon)^2 \right] + \int \nabla\rho \cdot \nabla\varphi^\varepsilon g(\varphi^\varepsilon) \theta^\varepsilon .$$

2. Let $v = v^\varepsilon$ be as in §2.3, i.e.,

$$v = \frac{\nabla\varphi^\varepsilon}{|\nabla\varphi^\varepsilon|}$$

for $|\nabla\varphi^\varepsilon| \neq 0$. Set

$$\hat{T} = \varepsilon \left(v \otimes v - \frac{1}{2} I \right) |\nabla\varphi^\varepsilon|^2 - \frac{1}{\varepsilon} W(\varphi^\varepsilon) I.$$

Then

$$\hat{T} = \varepsilon \nabla\varphi^\varepsilon \otimes \nabla\varphi^\varepsilon - \left(\frac{\varepsilon}{2} |\nabla\varphi^\varepsilon|^2 + \frac{1}{\varepsilon} W(\varphi^\varepsilon) \right) I,$$

$$\sum_{i=1}^{d} \frac{\partial}{\partial x_i} \hat{T}_{ij} = \varphi_{x_j}^\varepsilon \left(\varepsilon\Delta\varphi^\varepsilon - \frac{1}{\varepsilon} W'(\varphi^\varepsilon) \right).$$

Since $\xi^\varepsilon + \mu^\varepsilon = \varepsilon |\nabla\varphi^\varepsilon|^2 \, dx$,

$$\hat{T} \, dx = (v \otimes v) \xi^\varepsilon - (I - v \otimes v) \mu^\varepsilon .$$

Let k be the second term appearing in the expression at the end of Step 1:

$$k = \int \left[\rho_t \mu^\varepsilon + \nabla\rho \cdot \nabla\varphi^\varepsilon \left(\varepsilon\Delta\varphi^\varepsilon - \frac{1}{\varepsilon} W'(\varphi^\varepsilon) \right) + \varepsilon \frac{(\nabla\rho \cdot \nabla\varphi^\varepsilon)^2}{\rho} \right].$$

Integration by parts and the identity $\xi^\varepsilon + \mu^\varepsilon = \varepsilon |\nabla\varphi^\varepsilon|^2 \, dx$ yield

$$k = \int \rho_t \mu^\varepsilon - D^2\rho : \hat{T} \, dx + \frac{(\nabla\rho \cdot v)^2}{\rho} (\xi^\varepsilon + \mu^\varepsilon)$$

$$= \int \left[\rho_t + D^2\rho : (I - v \otimes v) + \frac{(\nabla\rho \cdot v)^2}{\rho} \right] \mu^\varepsilon + \int \left[\frac{(\nabla\rho \cdot v)^2}{\rho} - D^2\rho : v \otimes v \right] \xi^\varepsilon,$$

where $M:N = \text{trace } MN$, for symmetric matrices M, N. Explicit formulae for the derivatives of ρ imply that, for any unit vector v,

$$\rho_t + D^2\rho:(I - v \otimes v) + \frac{(\nabla\rho \cdot v)^2}{\rho} = 0,$$

$$\frac{(\nabla\rho \cdot v)^2}{\rho} - D^2\rho : v \otimes v = \frac{\rho}{2(t_0 - t)}.$$

Hence

$$k = \int \frac{\rho}{2(t_0 - t)} \xi^\varepsilon(t; dx).$$

3. Recall that $H^\varepsilon = h(\varphi^\varepsilon)$ and $\nabla H^\varepsilon = \nabla\varphi^\varepsilon g(\varphi^\varepsilon)$. By an integration by parts,

$$\int \nabla\rho \cdot \nabla\varphi^\varepsilon g(\varphi^\varepsilon)\theta^\varepsilon = -\int [\Delta\rho H^\varepsilon \theta^\varepsilon + H^\varepsilon \nabla\rho \cdot \nabla\theta^\varepsilon].$$

In view of Steps 1 and 2,

$$\frac{d}{dt}\alpha^\varepsilon(t) \leqq \frac{1}{2(t_0 - t)} \int \rho\xi^\varepsilon(t; dx) + I + J,$$

where

$$I = -\int [\rho|\nabla\theta^\varepsilon|^2 + H^\varepsilon \nabla\rho \cdot \nabla\theta^\varepsilon],$$

$$J = -\int \left[\frac{\rho}{4(t_0 - t)}(\theta^\varepsilon)^2 + \Delta\rho H^\varepsilon \theta^\varepsilon\right].$$

4. Since $|\varphi^\varepsilon| < 1$, $|H^\varepsilon| \leqq \frac{2}{3} < 1$. Hence

$$I = -\int \rho \left|\nabla\theta^\varepsilon + H^\varepsilon \frac{\nabla\rho}{2\rho}\right|^2 + \frac{1}{4}\int \frac{|\nabla\rho|^2}{\rho}|H^\varepsilon|^2$$

$$\leqq \frac{1}{4}\int \frac{|x - x_0|^2}{4(t_0 - t)^2} \rho(t, x; t_0, x_0)\ dx$$

$$= \frac{1}{2\sqrt{t_0 - t}} \int_{\mathscr{R}^d} (\pi)^{-(d-1)/2} |y|^2 e^{-|y|^2} dy.$$

5. To estimate J, we observe that

$$|\Delta\rho H^\varepsilon \theta^\varepsilon| \leqq \left[\frac{d-1}{2(t_0 - t)} + \frac{|x - x_0|^2}{4(t_0 - t)^2}\right] |H^\varepsilon||\theta^\varepsilon|\rho$$

$$= \frac{\rho}{2(t_0 - t)}\left[d - 1 + \frac{|x - x_0|^2}{2(t_0 - t)}\right] |H^\varepsilon||\theta^\varepsilon|.$$

Since

$$2|H^\varepsilon||\theta^\varepsilon| \leqq \left[d - 1 + \frac{|x_0 - x|^2}{2(t_0 - t)}\right]^{-1} (\theta^\varepsilon)^2 + \left[d - 1 + \frac{|x_0 - x|^2}{2(t_0 - t)}\right] |H^\varepsilon|^2,$$

we have

$$|\Delta\rho H^\varepsilon\theta^\varepsilon| \leqq \frac{(\theta^\varepsilon)^2\rho}{4(t_0-t)} + C(t,x)\rho,$$

where

$$C(t,x) = \frac{1}{4(t_0-t)}\left[d-1+\frac{|x_0-x|^2}{2(t_0-t)}\right]^2.$$

Hence

$$J \leqq \int C(t,x)\rho dx$$
$$= \frac{1}{2\sqrt{t_0-t}} \int_{\mathscr{R}^d} (\pi)^{-(d-1)/2}(d-1+2|y|^2)e^{-|y|^2}\,dy.$$

6. Combining the previous steps, we obtain (5.1) with

$$C_d = \tfrac{1}{2}(\pi)^{(1-d)/2} \int_{\mathscr{R}^d} [|y|^2 + (d-1+2|y|^2)^2]\, e^{-|y|^2}\,dy. \quad \square$$

Remark 5.1. Suppose that in the heat equation (1.1), $\lambda = c\varepsilon$ with some constant $c > 0$. Then (HE) in §2.1 takes the form

$$\theta_t^\varepsilon - c\Delta\theta^\varepsilon + g(\varphi^\varepsilon)\varphi_t^\varepsilon = 0.$$

This change does not affect the results of the preceeding sections. However, for $c \neq 1$, the monotonicity formula (5.1) has to be modified: For any $\beta > 0$, there is a constant $C_{d,\beta}$ such that

$$\frac{d}{dt}\alpha^\varepsilon \leqq \frac{1}{2(t_0-t)}\int \rho\, d\xi^\varepsilon + \frac{C_{d,\beta}}{(t_0-t)^{1/2+\beta}}.$$

Since the new error term $E(t) = (t_0-t)^{-1/2-\beta}$ is integrable over $(0, t_0)$, the main results of this paper and their proof remain unchanged.

The proof of the modified monotonicity result is very similar to that of (5.1). Indeed, in Step 1 we now have an additional error term

$$L := \frac{1-c}{2}\int \rho_t(\theta_\varepsilon)^2\,dx.$$

For any $p > 1$, Hölder's inequality and (3.10) yield

$$|L| \leqq \frac{|1-c|}{c}\,\|\rho_t\|_q\,\|(\theta_\varepsilon(t,\cdot))^2\|_p \leqq K_{p,c}(t_0-t)^{-1/2-d/2p}.$$

For a given $\beta > 0$, choose $p = d/2\beta$. $\square$

The monotonicity formula and the gradient estimate (4.1) yield

Corollary 5.2. *For any $T > 0$, there exists a constant $K = K(T)$ such that, for any $x_0 \in \mathscr{R}^d$ and $0 \leqq t \leqq r < t_0 \leqq T$,*

$$\alpha^\varepsilon(r; t_0, x_0) \leqq \alpha^\varepsilon(t; t_0, x_0)\left(\frac{t_0-t}{t_0-r}\right)^{K\sqrt{\varepsilon}} + K\int_t^r \left(\frac{t_0-\tau}{t_0-r}\right)^{K\sqrt{\varepsilon}} \frac{d\tau}{\sqrt{t_0-\tau}}. \tag{5.2}$$

Moreover, as ε tends to zero,

$$\alpha(r; t_0, x_0) \leqq \alpha(t; t_0; x_0) + C_d[\sqrt{t_0 - t} - \sqrt{t_0 - r}]. \tag{5.3}$$

Proof. Since

$$\xi^\varepsilon(t; dx) = \frac{1}{2\varepsilon}\left(q'\left(\frac{z^\varepsilon}{\varepsilon}\right)\right)^2 (|\nabla z^\varepsilon|^2 - 1),$$

(4.1) and (5.1) yield

$$\frac{d}{dt}\alpha^\varepsilon(t) \leqq \frac{1}{2(t_0 - t)}\int \rho \frac{1}{2\varepsilon}\left(q'\left(\frac{z^\varepsilon}{\varepsilon}\right)\right)^2 K^* \sqrt{\varepsilon}(1 + |z^\varepsilon|) + \frac{C_d}{\sqrt{t_0 - t}}.$$

Observe that

$$\frac{1}{2\varepsilon}\left(q'\left(\frac{z^\varepsilon}{\varepsilon}\right)\right)^2 dx \leqq \frac{1}{2\varepsilon}\left(q'\left(\frac{z^\varepsilon}{\varepsilon}\right)\right)^2 [1 + |\nabla z^\varepsilon|^2]\, dx = \mu^\varepsilon(t; dx),$$

$$\left(q'\left(\frac{z^\varepsilon}{\varepsilon}\right)\right)^2 |z^\varepsilon|^2 \leqq \varepsilon^2 \left(\sup_{r \geqq 0} q'(r) r\right)^2 \leqq 4\varepsilon^2.$$

Hence

$$\frac{1}{2\varepsilon}\left(q'\left(\frac{z^\varepsilon}{\varepsilon}\right)\right)^2 (1 + |z^\varepsilon|)\, dx \leqq \frac{1}{2\varepsilon}\left(q'\left(\frac{z^\varepsilon}{\varepsilon}\right)\right)^2 \left(\frac{3}{2} + \frac{1}{2}|z^\varepsilon|^2\right) dx \leqq \frac{3}{2}\mu^\varepsilon(t; dx) + \varepsilon\, dx$$

and consequently

$$\begin{aligned}
\frac{d}{dt}\alpha^\varepsilon(t) &\leqq \frac{K\sqrt{\varepsilon}}{(t_0 - t)}\int \rho(\mu^\varepsilon + \varepsilon\, dx) + \frac{C_d}{\sqrt{t_0 - t}} \\
&\leqq \frac{K\sqrt{\varepsilon}}{(t_0 - t)}\alpha^\varepsilon(t) + \frac{K\varepsilon\sqrt{\varepsilon}}{t_0 - t}\int \rho\, dx + \frac{C_d}{\sqrt{t_0 - t}} \\
&\leqq \frac{K\sqrt{\varepsilon}}{(t_0 - t)}\alpha^\varepsilon(t) + \frac{K}{\sqrt{t_0 - t}}.
\end{aligned}$$

Now an application of Gronwall's inequality yields (5.2). □

6. Clearing-Out

In this section we follow the proof of [29, Theorem 5.1] to prove an extension of the clearing-out lemma established in [23, 29].

Theorem 6.1. *For every $T > 0$, there are positive constants η, $t^* > 0$, depending on T, such that if*

$$\int \rho(t, x; t_0, x_0)\mu(t; dx) \leqq \eta \tag{6.1}$$

for some t, t_0, x_0 *satisfying*

$$(t_0 - t^*) \leqq t < t_0 \leqq T, \tag{6.2}$$

then there exists a neighborhood O *of* (t_0, x_0) *such that*

$$\lim_{(s', y') \to (s, y)\, \varepsilon \to 0} |z^\varepsilon(s', y')| > 0, \quad \forall (s, y) \in O. \tag{6.3}$$

In particular,

$$(t_0, x_0) \notin \overline{\bigcup_{t \geqq 0} \{t\} \times \operatorname{spt} \mu(t, \cdot)}. \tag{6.4}$$

Proof. Fix t, t_0, x_0. Suppose that (6.1), (6.2) hold with some η, t^* that will be chosen later.

1. Hölder's inequality yields

$$\begin{aligned}\int \rho(t, x; t_0, x_0)(\theta^\varepsilon(t, x))^2\, dx &\leqq \| \rho(t, \cdot; t_0, x_0) \|_{p'} \| (\theta^\varepsilon(t, \cdot))^2 \|_p \\ &= (4\pi(t_0 - t))^{1/2} \| G(t_0 - t, \cdot) \|_{p'} \| \theta^\varepsilon(t, \cdot))^2 \|_p\end{aligned}$$

for any $1 \leqq p \leqq \infty$, where p' is the conjugate of p: $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. Since

$$\| G(\tau, \cdot) \|_{p'} \leqq K(p) \tau^{-d/2p},$$

we have

$$\int \rho(t, x; t_0, x_0)(\theta^\varepsilon(t, x))^2\, dx \leqq \hat{K}(p)(t_0 - t)^{1/2(1 - d/p)} \| (\theta^\varepsilon(t, \cdot))^2 \|_p.$$

Choose $p = d + 1$ and use (3.10) to obtain

$$\int \rho(t, x; t_0, x_0)(\theta^\varepsilon(t, x))^2\, dx \leqq K^*(t_0 - t)^\gamma, \quad 0 < \varepsilon \leqq 1,$$

for some constants K^* and $\gamma > 0$.

2. The continuity of ρ, the convergence of μ^ε to μ and (2.2) imply that there are a constant $\varepsilon_0 > 0$ and a neighborhood U of (t_0, x_0) such that for all $\varepsilon \leqq \varepsilon_0$ and $(s, y) \in U$,

$$t + \varepsilon^2 < \frac{t + t_0}{2} < s, \tag{6.5}$$

$$\int \rho(t, x; s, y) \mu^\varepsilon(t; dx) \leqq 2\eta.$$

Step 1 yields

$$\begin{aligned}\alpha^\varepsilon(t; s, y) &= \int \rho(t, x; s, y)[\mu^\varepsilon(t; dx) + \tfrac{1}{2}(\theta^\varepsilon(t, x))^2\, dx] \\ &\leqq 2\eta + \tfrac{1}{2} K^*(s - t)^\gamma, \quad (s, y) \in U,\ \varepsilon \leqq \varepsilon_0.\end{aligned}$$

Note that ε_0 may depend on η, t and U.

3. Since $s - \varepsilon^2 > t$, we may use (5.2) with $(t_0, x_0) = (s, y)$ and $r = s - \varepsilon^2$ to obtain

$$\alpha^\varepsilon(s - \varepsilon^2; s, y) \leqq \left(\frac{s-t}{\varepsilon^2}\right)^{K\sqrt{\varepsilon}} \alpha^\varepsilon(t; s, y) + K \int_t^{s-\varepsilon^2} \left(\frac{s-\tau}{\varepsilon^2}\right)^{K\sqrt{\varepsilon}} \frac{d\tau}{\sqrt{s-\tau}}$$

$$\leqq \left(\frac{s-t}{\varepsilon^2}\right)^{K\sqrt{\varepsilon}} \left[\alpha^\varepsilon(t; s, y) + K \int_t^{s-\varepsilon^2} \frac{d\tau}{\sqrt{s-\tau}} d\tau\right]$$

$$\leqq \left(\frac{s-t}{\varepsilon^2}\right)^{K\sqrt{\varepsilon}} [\alpha^\varepsilon(t; s, y) + 2K\sqrt{s-t}].$$

As ε approaches to zero, $\varepsilon^{-2K\sqrt{\varepsilon}}$ converges to 1 and therefore, by (6.5), there is a constant $0 < \hat{\varepsilon}_0 \leqq \varepsilon_0$ that satisfies

$$\left(\frac{s-t}{\varepsilon^2}\right)^{K\sqrt{\varepsilon}} \leqq 2, \quad \varepsilon \leqq \hat{\varepsilon}_0, \ (s, y) \in U.$$

Then, by Step 2,

$$\alpha^\varepsilon(s - \varepsilon^2; s, y) \leqq 4\eta + K^*(s-t)^\gamma + 4K\sqrt{s-t}$$

for all $(s, y) \in U$ and $\varepsilon \leqq \hat{\varepsilon}_0$. Set

$$\hat{U} = U \cap (t_0 - t^*, t_0 + t^*) \times \mathscr{R}^d,$$

so that, for any $(s, y) \in \hat{U}$ and t, t_0 satisfying (6.2), $(s - t) \leqq 2t^*$, and consequently for an appropriately chosen $t^* = t^*(\eta)$,

$$K^*(s-t)^\gamma + 4K\sqrt{s-} \leqq K^*(2t^*)^\gamma + 4K\sqrt{2t^*} \leqq \eta.$$

Therefore

$$\alpha(s - \varepsilon^2, s, y) \leqq 5\eta, \quad (s, y) \in \hat{U}, \ \varepsilon \leqq \hat{\varepsilon}_0.$$

Recall that this estimate is obtained under the assumption that (6.1) holds with t, t_0 satisfying (6.2) with $t^* = t^*(\eta)$ and that we have not chosen η yet.

4. Let $B_\varepsilon(y)$ be the sphere centered at y with radius ε. For any $x \in B_\varepsilon(y)$,

$$\rho(s - \varepsilon^2, x; s, y) = (4\pi\varepsilon^2)^{-(d-1)/2} \exp\left(-\frac{|x-y|^2}{4\varepsilon^2}\right)$$

$$\geqq [(4\pi)^{-(d-1)/2} e^{-1/4}] \varepsilon^{-(d-1)} = (K_* \varepsilon^{d-1})^{-1}$$

for some constant K_*. Therefore

$$\hat{\mu}^\varepsilon(s - \varepsilon^2; B_\varepsilon(y)) \leqq \left[\min_{x \in B_\varepsilon(y)} \rho(s - \varepsilon^2, x; s, y)\right]^{-1} \alpha^\varepsilon(s - \varepsilon^2; s, y)$$

$$\leqq 5K_* \eta \varepsilon^{d-1} \quad \forall (s, y) \in \hat{U}, \ \varepsilon \leqq \hat{\varepsilon}_0. \tag{6.6}$$

5. Define

$$\beta = \liminf_{\varepsilon \to 0} \inf_{(s,y) \in \hat{U}} |\varphi^\varepsilon(s, y)|.$$

In this step we show that for a carefully chosen η, we have $\beta \geqq \frac{7}{8}$.

Suppose that $\beta < \frac{7}{8}$. Then there are $\varepsilon_n \to 0$ and $(s_n, y_n) \in \hat{U}$ satisfying

$$|\varphi^{\varepsilon_n}(s_n - \varepsilon_n^2, y_n)| < \tfrac{7}{8} \;\Rightarrow\; |z^{\varepsilon_n}(s_n - \varepsilon_n^2, y_n)| < \varepsilon_n q^{-1}(\tfrac{7}{8}).$$

Using (4.1) we construct K_0 and n_0, independent of η, such that

$$|z^{\varepsilon_n}(s_n - \varepsilon_n^2, x)| < \varepsilon_n[q^{-1}(\tfrac{7}{8}) + K_0] \quad \forall x \in B_{\varepsilon_n}(y_n), \quad n \geqq n_0,$$

and therefore

$$W(\varphi^{\varepsilon_n}(s_n - \varepsilon_n^2, x)) > W(q(q^{-1}(\tfrac{7}{8}) + K_0)) \quad \forall x \in B_{\varepsilon_n}(y_n), \quad n \geqq n_0.$$

Hence, for $n \geqq n_0$,

$$\begin{aligned}\mu^{\varepsilon_n}(s_n - \varepsilon_n^2; B_{\varepsilon_n}(y_n)) &\geqq \int_{B_{\varepsilon_n}(y_n)} \frac{1}{\varepsilon_n} W(\varphi^{\varepsilon_n}(\varepsilon_n - \varepsilon_n^2, x))\,dx \\ &> w_d W(q(q^{-1}(\tfrac{7}{8}) + K_0))(\varepsilon_n)^{d-1},\end{aligned}$$

where w_d is the volume of the d-dimensional unit sphere. Now choose

$$\eta = \frac{w_d}{5K_*} W(q(q^{-1}(\tfrac{7}{8}) + K_0)), \tag{6.7}$$

where K_* is the constant appearing in (6.6). With this choice of η, (6.7) contradicts (6.6). Hence $\beta \geqq \frac{7}{8}$.

In the foregoing discussion we have established the following: If (6.1) and (6.2) hold for some t, t_0 with η as in (6.7) and t^* as in Step 3, then there exists a neighborhood $\hat{U}$ of (t_0, x_0) such that

$$\beta = \liminf_{\varepsilon \to 0} \inf_{(s,y) \in \hat{U}} |\varphi^\varepsilon(s, y)| \geqq \tfrac{7}{8}.$$

Now, by Lemma 4.6, (6.3) holds on any open set O satisfying $\bar{O} \subset \hat{U}$ and

$$\liminf_{\varepsilon \to 0} \inf_{(s,y) \in \bar{O}} |z^\varepsilon(s, y)| > 0.$$

Then (4.1) yields

$$\begin{aligned}\mu^\varepsilon(\bar{O}) &= \int\int_{\bar{O}} \tfrac{1}{2\varepsilon}(q'(\tfrac{z^\varepsilon}{\varepsilon}))^2(|\nabla z^\varepsilon|^2 + 1)\,dx\,dt \\ &\leqq \int\int_{\bar{O}} \tfrac{1}{2\varepsilon}(q'(\tfrac{z^\varepsilon}{\varepsilon}))^2(2 + K\sqrt{\varepsilon}(1 + |z^\varepsilon|))\,dx\,dt,\end{aligned}$$

and therefore $\mu^\varepsilon(\bar{O})$ converges to zero as ε tends to zero, proving (6.4). □

7. Dimension of Γ and Equipartition of Energy

Let $\Lambda(t) \subset \mathscr{R}^d$ be the support of $\mu(t;\cdot)$ and $\Gamma \subset \bar{Q}$ be the support of $d\mu = \mu(t; dx)\,dt$. Then

$$\Gamma \subset \overline{\bigcup_{t \geqq 0} \{t\} \times \Lambda(t)}.$$

Suppose that $(t_0, x_0) \notin \Gamma$. Then there is a neighborhood U of (t_0, x_0) such that $U \cap \Gamma = \emptyset$. Therefore

$$\lim_{t \uparrow t_0} \int \rho(t, x; t_0, x_0)\mu(t; dx) = 0$$

and, by Theorem 6.1, (t_0, x_0) satisfy (6.4). Hence

$$\Gamma = \overline{\bigcup_{t \geqq 0} \{t\} \times \Lambda(t)}.$$

Let Γ_t be the t-section of Γ. In this section we first estimate the Hausdorff dimension of Γ_t (cf. [17]). Then we show that the discrepancy measure ξ^ε, defined in (1.7), converges to zero, hence proving the equipartition of energy. Our arguments closely follow Sections 6, 7 and 8 in [23].

The next theorem follows from [34, Theorem 5.12.4.].

Theorem 7.1. *Let μ be a positive Borel measure satisfying*

$$M(\mu) = \sup_{x \in \mathscr{R}^d,\, R > 0} \frac{\mu(B_R(x))}{R^{d-1}} < \infty.$$

Then there is a constant K_d, depending only on the dimension d but not on μ, such that

$$\left| \int \varphi(x)\mu(dx) \right| \leqq K_d M(\mu) \| \nabla\varphi \|_1 \quad \forall \varphi \in C_c^\infty(\mathscr{R}^d).$$

We continue with an estimate of the dimension of the interface.

Proposition 7.2. *For every $T > 0$ there is $K(T) > 0$ such that*

$$\mu^\varepsilon(r; B_R(x)) \leqq K(T) R^{d-1}, \tag{7.1}$$

$$\mathscr{H}^{d-1}(\Gamma_r) \leqq K(T), \tag{7.2}$$

for all $0 < \varepsilon \leqq 1$, $R > 0$ and $0 \leqq r \leqq T$.

Proof.

1. Theorem 7.1, (A5) and (A6) imply that

$$\begin{aligned} \alpha^\varepsilon(0; t_0, x_0) &= \int \rho(0, x; t_0, x_0)\left[\mu^\varepsilon(0; dx) + \tfrac{1}{2}(\theta_0^\varepsilon(x))^2\, dx\right] \\ &\leqq K\left[\| \nabla_x \rho(0, \cdot; t_0, x_0) \|_1 + \| \rho(0, \cdot; t_0, x_0) \|_1\right] \\ &\leqq K(\sqrt{t_0} + 1) \end{aligned}$$

for some constant K, independent of ε.

2. If $R \geqq (C_1^*)^{1/(d-1)} = R_0$, then the energy estimate (2.2) yields

$$\mu^\varepsilon(r; B_R(x)) \leqq \hat{\mu}^\varepsilon(r; R^d) \leqq C_1^* \leqq R^{d-1}.$$

Hence (7.1) holds for all $r \geqq 0$ and $R \geqq R_0$ with constant $K(T) = 1$.
3. Fix $0 \leqq r \leqq T$, $\varepsilon \leqq R \leqq R_0$ and $x_0 \in \mathscr{R}^d$. Then for $t_0 > r$,

$$\begin{aligned} \mu^\varepsilon(r; B_R(x_0)) &\leqq \hat{\mu}^\varepsilon(r; B_R(x_0)) \\ &\leqq \left[\inf_{x \in B_R(x_0)} \rho(r, x; t_0, x_0)\right]^{-1} \alpha^\varepsilon(r; t_0, x_0) \qquad (7.3) \\ &= (4\pi(t_0 - r))^{(d-1)/2} \exp\left(\frac{R^2}{4(t_0 - r)}\right) \alpha^\varepsilon(r; t_0, x_0). \end{aligned}$$

Choose $t_0 = r + R^2$ so that $t_0 \leqq T + R_0^2 = T_*$ and, by Step 1 and (5.2),

$$\begin{aligned} \alpha^\varepsilon(r; t_0, x_0) &\leqq \alpha^\varepsilon(0; t_0, x_0) \left(\frac{t_0}{R^2}\right)^{K\sqrt{\varepsilon}} + K \int_0^r \left(\frac{t_0 - \tau}{R^2}\right)^{K\sqrt{\varepsilon}} \frac{d\tau}{\sqrt{t_0 - \tau}} \\ &\leqq K \left(\frac{t_0}{R^2}\right)^{K\sqrt{\varepsilon}} (\sqrt{t_0} + 1). \end{aligned}$$

Since $R \geqq \varepsilon$, there is a constant $K = K(T)$ satisfying

$$\alpha^\varepsilon(r; r + R^2, x_0) \leqq K, \quad 0 < \varepsilon \leqq 1, \quad r \leqq T.$$

Then (7.3) implies that

$$\mu^\varepsilon(r; B_R(x_0)) \leqq (4\pi)^{(d-1)/2} e^{1/4}\, K R^{d-1}, \quad \varepsilon \leqq R \leqq R_0;$$

hence (7.1) holds for all $R \geqq \varepsilon$.
4. In this step we study the case $0 < R \leqq \varepsilon$. The inequality (3.2) yields

$$\mu^\varepsilon(r; B_R(x_0)) = \int_{B_R(x_0)} \frac{\varepsilon}{2} |\nabla\varphi^\varepsilon|^2 + \frac{1}{\varepsilon} W(\varphi^\varepsilon) \leqq \frac{K}{\varepsilon} |B_R(x_0)| = \frac{K}{\varepsilon} R^d$$

for $0 \leqq r \leqq T$. Since $R \leqq \varepsilon$, $R^d \varepsilon^{-1} \leqq R^{d-1}$, this completes the proof of (7.1) for all R.
5. The inequality (7.2) follows from Theorem 6.1, (7.1) and an application of the Besicovitch covering theorem (see the proof of [23, §6.3]). □

In the remainder of this section we prove that ξ^ε converges to zero. Our proof is a direct modification of Sections 7 and 8 in [23].

Let η be as in Theorem 6.1. Define

$$Z^- = \left\{(t, x) \in \Gamma \cap [0, T] \times \mathscr{R}^d : \sup_{s \downarrow t} \int \rho(t, x; s, y) \mu(s; dy) < \eta\right\}.$$

Then Section 7 in [23] implies that for any $\delta > 0$,

$$\mathscr{H}^{d-2+\delta}(Z_t^-) = 0 \quad \text{for almost every } t \in [0, T]. \qquad (7.4)$$

Let ξ^ε be as in Section 5. For a Borel set $A \subset [0, T] \times \mathscr{R}^d$ define

$$\xi^\varepsilon(A) = \int_A \xi^\varepsilon(t; dx)\, dt.$$

Since $|\xi^\varepsilon| \leqq \mu^\varepsilon$, by passing to a further subsequence we assume that ξ^ε converges to a Borel measure ξ in the weak* topology of Radon measures.

Proposition 7.3. $\xi = 0$.

Proof.
1. For $\varepsilon > 0$ and any Borel set $A \subset \bar{Q}$, let

$$v^\varepsilon(A) = \int_A \frac{1}{2\varepsilon}\left(q'\left(\frac{z^\varepsilon}{\varepsilon}\right)\right)^2 (|\nabla z^\varepsilon|^2 - 1)^+\, dx\, dt,$$

$$\lambda^\varepsilon(A) = \int_A \frac{1}{2\varepsilon}\left(q'\left(\frac{z^\varepsilon}{\varepsilon}\right)\right)^2 (|\nabla z^\varepsilon|^2 - 1)^-\, dx\, dt$$

where for any real number b, $(b)^+ = \max\{b, 0\}$, $b^- = \max\{-b, 0\}$. Then

$$\xi^\varepsilon = v^\varepsilon - \lambda^\varepsilon.$$

2. Equation (4.1) and the proof of Corollary 5.2 imply that

$$v^\varepsilon(A) \leqq K\sqrt{\varepsilon}[\mu^\varepsilon(A) + \varepsilon|A|], \quad 0 < \varepsilon \leqq 1 \tag{7.5}$$

for any $A \subset [0, T] \times \mathscr{R}^d$. Hence v^ε converges to zero and λ^ε converges to $-\xi$.
3. Fix $(s, y) \in [0, \infty) \times \mathscr{R}^d$ and $0 < \sigma \leqq s$. Integrate (5.1) on $[0, s - \sigma]$. Using (7.5) and the exponential decay of ρ, we let ε go to zero to obtain

$$\alpha(s - \sigma; s, y) - \alpha(0; s, y) \leqq - \int_0^{s-\sigma} \int_{\mathscr{R}^d} \frac{1}{2(s - t)} \rho(t, x; s, y)\, d\lambda(t, x) + 2C_d(\sqrt{s} - \sqrt{\sigma}).$$

This inequality and Step 1 of Proposition 7.2 yield

$$\int_0^{s-\sigma} \int_{\mathscr{R}^d} \frac{1}{2(s - t)} \rho(t, x; s, y)\, d\lambda(t, x) \leqq K(\sqrt{s} + 1).$$

Fix $T > 0$ and integrate this inequality against $\mu(s; dy)\, ds$ and then use (2.2); the result is

$$\int_0^{T+1} \int_{\mathscr{R}^d} \int_0^{s-\sigma} \int_{\mathscr{R}^d} \frac{1}{2(s - t)} \rho(t, x; s, y)\, d\lambda(t, x) \mu(s; dx)\, ds$$

$$\leqq \int_0^{T+1} \int_{\mathscr{R}^d} K(\sqrt{s} + 1) \mu(s; dy)\, ds \leqq \hat{C}(T)$$

for some constant $\hat{C}(T)$ depending on T.

4. Fubini's theorem and the monotone convergence theorem enable us to send σ to zero to obtain

$$\int_0^{T+1} \int_{\mathscr{R}^d} \int_t^{T+1} \int_{\mathscr{R}^d} \frac{1}{2(s-t)} \rho(t, x; s, y) \mu(s; dy)\, ds\, d\lambda(t, x) \leqq \hat{C}(T).$$

Hence

$$\int_t^{t+1} \frac{1}{2(s-t)} \int_{\mathscr{R}^d} \rho(t, x; s, y) \mu(s; dy)\, ds \leqq C(x, t) < \infty \tag{7.6}$$

for λ almost every $(t, x) \in [0, T] \times \mathscr{R}^d$.

5. Fix (t, x) such that (7.6) holds. For $s \in (t, t+1]$ define

$$\beta = \ln(s-t), \quad h(s) = \int_{\mathscr{R}^d} \rho(t, x; s, y) \mu(s; dy).$$

Then (7.6) implies that

$$\int_{-\infty}^{0} h(t + e^\beta)\, d\beta < \infty. \tag{7.7}$$

We wish to prove that

$$\lim_{s \downarrow t} h(s) = 0.$$

Clearly (7.7) implies that $h(t + e^\beta)$ converges to zero on a subsequence. We now use the monotonicity of h to prove convergence on the whole sequence.

6. Following [23], for $\gamma \in (0, 1]$ we choose a decreasing sequence $\beta_i \to -\infty$ such that

$$|\beta_{i+1} - \beta_i| \leqq \gamma, \quad h(t + e^{\beta_i}) \leqq \gamma.$$

Then, for any $\beta \in [\beta_i, \beta_{i-1})$,

$$\begin{aligned} h(t + e^\beta) &= \int \rho(t, x; t + e^\beta, y) \mu(t + e^\beta; dy) \\ &= \int \rho(t + e^\beta, x; t + 2e^\beta, y) \mu(t + e^\beta; dy) \\ &= \alpha(t + e^\beta; t + 2e^\beta, x). \end{aligned}$$

Use (5.3) to obtain

$$\begin{aligned} h(t + e^\beta) &\leqq \alpha(t + e^{\beta_i}; t + 2e^\beta, x) + C_d[\sqrt{2e^\beta - e^{\beta_i}} - \sqrt{e^\beta}] \\ &\leqq \alpha(t + e^{\beta_i}; t + 2e^\beta, x) + C_d\sqrt{2e^\beta}, \end{aligned} \tag{7.8}$$

and the preceding identity with $\beta = \beta_i$ yields

$$\gamma \geqq h(t + e^{\beta_i}) = \alpha(t + e^{\beta_i}; t + 2e^{\beta_i}, x). \tag{7.9}$$

7. We assert that for any $\delta > 0$ there is a $\gamma(\delta, T) > 0$ satisfying

$$\alpha(t_0; t_0 + R_1, x) \leqq (1 + \delta)\alpha(t_0; t_0 + R_0, x) + \delta \tag{7.10}$$

for all $0 \leqq t_0 \leqq T + 1$, $x \in \mathscr{R}^d$ and $0 \leqq R_0 \leqq R_1 \leqq (\gamma(\delta) + 1)R_0$. This result follows from (7.1) and it is stated in [23, Lemma 3.4(iv)]. We postpone the elementary proof of (7.10) to the next step and complete the proof of the Proposition.

Set

$$t_0 = t + e^{\beta_i}, \quad R_1 = 2e^{\beta} - e^{\beta_i}, \quad R_0 = e^{\beta_i}$$

so that

$$\frac{R_1}{R_0} = \sqrt{2e^{\beta-\beta_i} - 1} \leqq \sqrt{2[e^{\beta-\beta_i} - 1] + 1} \leqq 1 + K\gamma$$

for some constant K. So if $K\gamma \leqq \gamma(\delta)$, then (7.10) holds and, by (7.8) and (7.9),

$$\begin{aligned} h(t + e^{\beta}) &\leqq \alpha(t + e^{\beta_i}; t + 2e^{\beta}, x) + C_d\sqrt{2e^{\beta}} \\ &\leqq (1 + \delta)\alpha(t + e^{\beta_i}; t + 2e^{\beta_i}, x) + \delta + C_d\sqrt{2e^{\beta}} \\ &= (1 + \delta)h(t + e^{\beta_i}) + \delta + C_d\sqrt{2e^{\beta}} \\ &\leqq (1 + \delta)\gamma + \delta + C_d\sqrt{2e^{\beta}} \end{aligned}$$

for all $\delta > 0$ and $0 < \gamma \leqq \gamma_0(\delta)$. Now pass to the limit $i \to \infty$, $\gamma \to 0$ and then $\delta \to 0$, to obtain

$$\lim_{s \downarrow t} h(s) = 0$$

for every (t, x) satisfying (7.6). Recall that (7.6) holds for λ-almost every (t, x). On the other hand, (7.4) and (7.1) imply that

$$\limsup_{s \downarrow t} h(s) \geqq \eta > 0$$

for μ-almost every (t, x). Since $\lambda = -\xi$ is absolutely continuous with respect to μ, we conclude that $\lambda = -\xi = 0$.

8. In this step we establish (7.10). Recall that

$$\alpha(t_0; t_0 + \tau, x_0) = \int \left(\frac{1}{4\pi\tau}\right)^{(d-1)/2} e^{-|x_0 - y|^2/4\tau}\mu(t_0; dy).$$

Without loss of generality we take $x_0 = 0$. Set $\mu(dy) = \mu(t_0; dy)$,

$$f(\tau) = \int \left(\frac{1}{4\pi\tau}\right)^{(d-1)/2} e^{-|y|^2/4\tau}\mu(dy).$$

Then, for any $0 < \alpha < 1$,

$$f\left(\frac{\tau}{1-\alpha}\right) \leqq \int \left(\frac{1}{4\pi\tau}\right)^{(d-1)/2} e^{-|y|^2/4\tau(1-\alpha)}\mu(dy).$$

Furthermore, for $0 < \delta$ and $\alpha \leqq \frac{1}{2}$,

$$I = f\left(\frac{\tau}{1-\alpha}\right) - (1+\delta) f(\tau)$$

$$\leqq \int \left(\frac{1}{4\pi\tau}\right)^{(d-1)/2} e^{-|y|^2/4\tau}\left(e^{\alpha(|y|^2)/4\tau} - (1+\delta)\right)$$

$$\leqq \int\limits_{|y| \geqq \Lambda} \left(\frac{1}{4\pi\tau}\right)^{(d-1)/2} e^{-|y|^2/4\tau} \mu(dy),$$

where

$$\Lambda = \sqrt{\frac{4\tau}{\alpha}\ln(1+\delta)}\,.$$

Since by (7.1), $\mu(\{|y| \leqq R\}) \leqq KR^{d-1}$, and since the integrand is radially symmetric, an integration by parts yields

$$I \leqq \int\limits_{\Lambda}^{\infty} \left(\frac{1}{(4\pi\tau)}\right)^{(d-1)/2} \frac{R}{4\tau} e^{-R^2/4\tau} KR^{d-1}\, dR.$$

By a change of variables,

$$I \leqq K \int\limits_{\Lambda/2\sqrt{2\tau}}^{\infty} |\xi|^d e^{-|\xi|^2} d\xi \leqq K \exp\left(-\frac{\ln(1+\delta)}{2\alpha}\right) \leqq K \exp\left(-\frac{1}{2\alpha}\right) \leqq \delta$$

provided α is sufficiently small. □

8. Passage to the Limit

In this section we complete the proofs of Theorems 2.1, 2.2, and 2.3. We start with the following lemma.

Lemma 8.1. *For any $T > 0$ and $\alpha \geqq 0$ there are constants $K(T, \alpha)$ and $K(T)$ such that, for any Borel set $B \subset \mathscr{R}^d$,*

$$\sup_{0<\varepsilon\leqq 1} \int_0^T \int_B (1 + |z^\varepsilon(t,x)|)^\alpha \mu^\varepsilon(t; dx)\, dt \leqq K(T,\alpha)(1 + |B|), \tag{8.1}$$

$$\int_0^T \int_B (z_t^\varepsilon(t,x))^2 \mu^\varepsilon(t; dx)\, dt \leqq K(T)(1 + \sqrt{\varepsilon}\,|B|). \tag{8.2}$$

Proof. Set

$$\Omega = \{(t,x) \in [0,T] \times B;\, |z^\varepsilon(t,x)| \leqq 1\}.$$

Then the energy estimate (2.2) yields

$$\begin{aligned}\int_0^T\int_B &(1+|z^\varepsilon(t,x)|)^\alpha\mu^\varepsilon(t;dx)\,dt\\ &\leqq 2^\alpha\int_\Omega \mu^\varepsilon(t;dx)\,dt+\int_{\Omega^c}(1+|z^\varepsilon|)^\alpha\mu^\varepsilon(t;dx)\,dt\\ &\leqq (2^\alpha+1)C_1^*T+\int_{\Omega^c}[(1+|z^\varepsilon|)^\alpha-1]\mu^\varepsilon(t;dx)\,dt,\end{aligned}$$

where Ω^c denote the complement of Ω. For any $r \geqq 1$,

$$(1+r)^\alpha - 1 \leqq \alpha r\max\{1,(1+r)^{\alpha-1}\}\leqq \alpha r(1+r)^\alpha.$$

Since $|z^\varepsilon|>1$ on Ω^c, this inequality and (4.3) imply that, on Ω^c,

$$\begin{aligned}[(1+|z^\varepsilon|)^\alpha-1]\mu^\varepsilon(t;dx) &= [(1+|z^\varepsilon|)^\alpha-1]\frac{1}{2\varepsilon}\left(q'\left(\frac{z^\varepsilon}{\varepsilon}\right)\right)^2(|\nabla z^\varepsilon|^2+1)\\ &\leqq K\alpha\frac{|z^\varepsilon|}{\varepsilon}(1+|z^\varepsilon|)^{\alpha+1}\left(q'\left(\frac{z^\varepsilon}{\varepsilon}\right)\right)^2\\ &\leqq K\alpha\sup_{0<\varepsilon\leqq 1}\sup_{r\geqq 1}\frac{r}{\varepsilon}(1+r)^{\alpha+1}\left(q'\left(\frac{r}{\varepsilon}\right)\right)^2\\ &= K\alpha\sup_{0<\varepsilon\leqq 1}\sup_{\bar r>\varepsilon}\bar r(1+\varepsilon\bar r)^{\alpha+1}(q'(\bar r))^2\\ &= K\alpha\sup_{\bar r>0}\bar r(1+\bar r)^{\alpha+1}(q'(\bar r))^2 = C^*(\alpha)<\infty.\end{aligned}$$

This proves (8.1). To prove (8.2), first recall that, by (2.2),

$$\int_0^T\int_{\mathscr{R}^d}\varepsilon(\varphi_t^\varepsilon)^2\,dx\,dt=\int_0^T\int_{\mathscr{R}^d}\frac{1}{\varepsilon}(z_t^\varepsilon)^2(q')^2\,dx\,dt\leqq C_1^*,$$

where q' is evaluated at $(z^\varepsilon/\varepsilon)$. Hence, by (2.2) and (4.1),

$$\begin{aligned}\int_0^T\int_B(z_t^\varepsilon)^2\mu^\varepsilon(t;dx)\,dt &= \int_0^t\int_B\frac{1}{2\varepsilon}(z_t^\varepsilon)^2(q')^2(1+|\nabla z^\varepsilon|^2)\,dx\,dt\\ &\leqq C_1^*+K\sqrt{\varepsilon}\left[\int_0^T\int_B\frac{1}{2\varepsilon}(z_t^\varepsilon)^2(q')^2(1+|z^\varepsilon|)\right]dx\,dt\\ &\leqq C_1^*+K\sqrt{\varepsilon}\left[\int_0^T\int_{\mathscr{R}^d}\frac{\varepsilon}{2}(\varphi_t^\varepsilon)^2\,dx\,dt+\int_0^T\int_B\frac{1}{2\varepsilon}(z_t^\varepsilon)^2(q')^2|z^\varepsilon|\,dx\,dt\right]\end{aligned}$$

By (4.6) and (2.2),

$$\int_0^T\int_B \frac{1}{2\varepsilon}(z_t^\varepsilon)^2(q')^2|z^\varepsilon|\,dx\,dt$$

$$\leqq \int_0^T\int_B \frac{1}{2\varepsilon}(z_t^\varepsilon)^2(q')^2[1+|z^\varepsilon|\chi_{|z^\varepsilon|\geqq 1}]\,dx\,dt$$

$$\leqq \int_0^T\int_{\mathscr{R}^d} \varepsilon(\varphi_t^\varepsilon)^2 + \int_0^T\int_B \frac{K}{\varepsilon^5}(q')^2|z^\varepsilon|\chi_{|z^\varepsilon|\geqq 1}\,dx\,dt$$

$$\leqq C_1^* + \int_0^T\int_B \frac{K}{\varepsilon^5}\sup_{r\geqq 1} rq'\left(\frac{r}{\varepsilon}\right)dx\,dt \leqq K[1+|B|T]. \quad \square$$

Proof of Theorem 2.1. The only assertion left to prove is the convergence of $\bar{m}^\varepsilon$, where

$$d\bar{m}^\varepsilon = -\, z_t^\varepsilon dm^\varepsilon.$$

The L^2 estimate (8.2) implies that

$$\sup_{0<\varepsilon\leqq 1} |\bar{m}^\varepsilon|([0,T]\times B_R\times S^{d-1}) < \infty \quad \text{for } R, T > 0.$$

Hence on a subsequence, denoted by $\varepsilon, \bar{m}^\varepsilon$ converges to a Radon measure $\bar{m}$. Moreover, (8.2) implies that $\bar{m}$ is absolutely continuous with respect to m; let v denote the corresponding Radon-Nikodym derivative, so that, by (8.2),

$$v \in L^2((0,T)\times\mathscr{R}^d\times S^{d-1}; dm). \quad \square$$

Proof of Theorem 2.2. We first prove the existence of the mean-curvature vector H. Following [23, §9.3], let $V^\varepsilon(t;\cdot)$ be the varifold (cf. [28])

$$V^\varepsilon(t; dx\,dS) = \delta_{\{(v^\varepsilon)^\perp\}}(dS)\mu^\varepsilon(t; dx)$$

so that $(V^\varepsilon(t;\cdot))^{(x)}$ is supported at $(v^\varepsilon(t,x))^\perp$ and

$$\| V^\varepsilon(t;\cdot)\| = \mu^\varepsilon(t;\cdot).$$

1. In this step we show that

$$\sup_{0<\varepsilon\leqq 1}\int_0^T\int \varepsilon\left[-\Delta\varphi^\varepsilon + \frac{1}{\varepsilon^2}W'(\varphi^\varepsilon)\right]^2 dx\,dt < \infty. \tag{8.3}$$

Since $g^2 = 2W$, (OPE) implies that

$$\varepsilon\left[-\nabla\varphi^\varepsilon + \frac{1}{\varepsilon^2}W'(\varphi^\varepsilon)\right]^2 = \varepsilon\left[-\varphi_t^\varepsilon + \frac{1}{\varepsilon}g(\varphi^\varepsilon)\theta^\varepsilon\right]^2 \leqq 2\varepsilon(\varphi_t^\varepsilon)^2 + \frac{4}{\varepsilon}W(\varphi^\varepsilon)(\theta^\varepsilon)^2,$$

while Theorem 7.1 and (7.1) yield

$$\tfrac{1}{\varepsilon}\int W(\varphi^\varepsilon)(\theta^\varepsilon)^2\,dx \leqq \int (\theta^\varepsilon)^2\,d\mu^\varepsilon \leqq K\,\|\nabla(\theta^\varepsilon)^2\|_1 \leqq K\int (\theta^\varepsilon)^2 + |\nabla\theta^\varepsilon|^2.$$

Hence

$$\int_0^T \int \varepsilon\left[-\Delta\varphi^\varepsilon + \frac{1}{\varepsilon^2}W'(\varphi^\varepsilon)\right]^2 dx\,dt$$

$$\leqq \int_0^T \int [2\varepsilon(\varphi_t^\varepsilon)^2 + K|\nabla\theta^\varepsilon|^2 + K(\theta^\varepsilon)^2]\,dx\,dt,$$

and (8.3) follows from (2.2).

2. For any smooth vector field $Y(x)$, the definition of V^ε and the definition of the first variation (cf. [28]) imply that

$$\delta V^\varepsilon(t;\cdot)(Y) = \int \nabla Y : S V^\varepsilon(t; dx\,dS) = \int \nabla Y : (I - \nu^\varepsilon \otimes \nu^\varepsilon)\mu^\varepsilon(t; dx).$$

Let $\hat{T}$ be as in Step 2 of Theorem 5.1. Recall that

$$(I - \nu^\varepsilon \otimes \nu^\varepsilon)\mu^\varepsilon = (\nu^\varepsilon \otimes \nu^\varepsilon)\xi^\varepsilon - \hat{T}\,dx,$$

$$\operatorname{div}\hat{T} = -\varepsilon\,\nabla\varphi^\varepsilon\left[-\Delta\varphi^\varepsilon + \frac{1}{\varepsilon^2}W'(\varphi^\varepsilon)\right].$$

As in Theorem 5.1,

$$\delta V^\varepsilon(t;\cdot)(Y) = \int \nabla Y : [(\nu^\varepsilon \otimes \nu^\varepsilon)\,d\xi^\varepsilon - \hat{T}\,dx]$$

$$= \int Y\cdot\operatorname{div}\hat{T}\,dx + \nabla Y : (\nu^\varepsilon \otimes \nu^\varepsilon)\,d\xi^\varepsilon$$

$$= -\int \varepsilon Y\cdot\nabla\varphi^\varepsilon\left[-\Delta\varphi^\varepsilon + \frac{1}{\varepsilon^2}W'(\varphi^\varepsilon)\right]dx + \int \nabla Y : \nu^\varepsilon \otimes \nu^\varepsilon\,d\xi^\varepsilon.$$

Hence

$$|\delta V^\varepsilon(t;\cdot)(Y)| \leqq \left(\int \varepsilon|Y|^2|\nabla\varphi^\varepsilon|^2\,dx\right)^{1/2}\left(\int \varepsilon\left[-\Delta\varphi^\varepsilon + \frac{1}{\varepsilon^2}W(\varphi^\varepsilon)\right]^2\right)^{1/2}$$

$$+ \int |\nabla Y|\,d|\xi^\varepsilon|.$$

3. In view of (8.3), Proposition 7.3 and (2.2), for every $T > 0$ there is a constant $K(T)$ satisfying

$$\limsup_{\varepsilon\downarrow 0} \int_0^T |\delta V^\varepsilon(t;\cdot)(Y)|\,dt \leqq K(T)\|Y\|_\infty$$

for all $Y \in C_c^\infty(\mathscr{R}^d \to \mathscr{R}^d)$. Choose a further subsequence $\varepsilon_n \downarrow 0$ such that the Radon measures $dV^{\varepsilon_n}(t;\cdot)\,dt$ on $\mathscr{R}^d \times G_{d-1}(\mathscr{R}^d) \times [0,\infty)$ are convergent in the weak* topology. By a slicing argument [16, Theorem 10, page 14], we conclude that there are varifolds $\tilde{V}(t;\cdot)$ that satisfy

$$dV^{\varepsilon_n}(t;\cdot)\,dt \rightharpoonup d\tilde{V}(t;\cdot)\,dt.$$

4. By definition,

$$\int_0^T |\delta\tilde{V}(t;\cdot)(Y)|\,dt = \sup_{|h(t)|\leqq 1} \int_0^T \left(\int Y(x) : S\,\tilde{V}(t; dx\,dS) \right) h(t)\,dt,$$

so that Step 3 yields

$$\int_0^T |\delta\tilde{V}(t,\cdot)(Y)|\,dt \leqq K(T)\|Y\|_\infty \quad \forall Y \in C_c^\infty(\mathscr{R}^d \to \mathscr{R}^d).$$

Since $C_c^\infty(\mathscr{R}^d \to \mathscr{R}^d)$ is separable,

$$K(t) := \sup_{|Y|\leqq 1} |\delta V^\varepsilon(t,\cdot)(Y)| < \infty$$

for almost every $t \geqq 0$.

5. Let $t \geqq 0$ be a point with $K(t) < \infty$. Then (7.2) and ALLARD's theorem of rectifiability [1, 5.5(2)] imply that $\|\tilde{V}(t;\cdot)\|$ is $d-1$ rectifiable. Moreover, by the definition of the varifolds $V^\varepsilon(t;\cdot)$,

$$\| V^\varepsilon(t;\cdot) = \mu^\varepsilon(t;\cdot) \;\Rightarrow\; \|\tilde{V}(t,\cdot)\| = \mu(t;\cdot).$$

Since a $(d-1)$-rectifiable varifold is uniquely determined by its mass measure,

$$\tilde{V}(t;\cdot) = V_{\mu(t;\cdot)}.$$

Hence $dV^\varepsilon(t;\cdot)\,dt$ converges on the entire original sequence ε, and more importantly, $\mu(t;\cdot)$ is $(d-1)$-rectifiable.

We have also proved that

$$|\delta V_{\mu(t;\cdot)}(Y)| \leqq \liminf_{\varepsilon\to 0} \left(\int \varepsilon \left[-\Delta\varphi^\varepsilon + \frac{1}{\varepsilon^2} W'(\varphi^\varepsilon) \right]^2 dx \right)^{1/2} \left(\int |Y|^2 \mu(t; dx) \right)^{1/2}.$$

Hence, for almost every $t \geqq 0$, $\mu(t,\cdot)$ has a generalized mean-curvature vector $H(t,x)$ and

$$\int |H(t,x)|^2 \mu(t; dx) \leqq \liminf_{\varepsilon\to 0} \int \varepsilon \left[-\Delta\varphi^\varepsilon + \frac{1}{\varepsilon^2} W'(\varphi^\varepsilon) \right]^2 dx.$$

Step 1 implies that $H \in L^2((0,T)\times\mathscr{R}^d; d\mu)$ and in Step 2 we have established that, for any $Y \in C_c^\infty(\mathscr{R}^d \to \mathscr{R}^d)$,

$$\int\int Y\cdot H(t,x)\,dt\,\mu(t;dx) = -\int \delta\tilde{V}(t;\cdot)(Y)\,dt = -\lim_{\varepsilon\to 0}\int \delta V^\varepsilon(t;\cdot)(Y)\,dt \quad (8.4)$$

$$= \lim_{\varepsilon\to 0} \int\int \varepsilon Y\cdot\nabla\varphi^\varepsilon \left[-\Delta\varphi^\varepsilon + \frac{1}{\varepsilon^2} W'(\varphi^\varepsilon) \right] dx\,dt.$$

6. In this step we show that $\theta \in L^1_{\text{loc}}(d\mu)$. In view of (2.9), there exists a sequence of smooth functions θ_k satisfying

$$\lim_{k\to\infty} \|\theta_k - \theta\|_{2,T} = 0, \quad \sup_k \|\nabla\theta_k\|_{2,T} < \infty \tag{8.5}$$

for every $T > 0$. Fix $\lambda > 0$ and $T > 0$. Then, by Theorem 7.1, (7.1) and (2.2),

$$\begin{aligned}\int_0^T \int_{\mathscr{R}^d} |\theta_k - \theta_l|\, dt\mu(t; dx) &\leqq \int_0^T \int_{\mathscr{R}^d} \left[\frac{\lambda}{2}|\theta_k - \theta_l|^2 + \frac{1}{2\lambda}\right] dt\, \mu(t; dx)\\ &\leqq K\left[\frac{\lambda}{2}\|\nabla(|\theta_k - \theta_l|^2)\|_{1,T} + \frac{T}{\lambda}\right]\\ &\leqq K\left[\lambda\|\theta_k - \theta_l\|_{2,T}\|\nabla(\theta_k - \theta_l)\|_{2,T} + \frac{T}{\lambda}\right],\end{aligned}$$

which converges to zero as $k, l \to \infty$, since we can take $\lambda \to \infty$. Hence $\theta \in L^1_{\text{loc}}(d\mu)$ and

$$\textstyle\int\int\int \theta(t,x) n\cdot Y(t,x)\, dm(t,x,n) = \lim_{k\to\infty} \int\int\int \theta_k n \cdot Y\, dm.$$

7. For $n \in S^{d-1}$, let $Pn \in G_{d-1}(\mathscr{R}^d)$ be the $(d-1)$-dimensional, unoriented plane orthogonal to n so that $P\colon S^{d-1} \to G_{d-1}(\mathscr{R}^d)$ is a surjective map. Then, by definition,

$$\frac{dV^\varepsilon(t;\cdot)}{\mu^\varepsilon(t; dx)} = \frac{dm^\varepsilon}{\mu^\varepsilon(t; dx)\, dt} \circ P^{-1}.$$

By the weak* convergence of these measures,

$$\delta_{T_x\mu(t;\cdot)} = \frac{dV_{\mu(t;\cdot)}(t;\cdot)}{\mu(t; dx)} = \frac{dm}{\mu(t; dx)\, dt} \circ P^{-1} = N(t,x;\cdot)\circ P^{-1}.$$

Therefore the support of $N(t,x;\cdot)$ is orthogonal to $T_x\mu(t;\cdot)$ for $d\mu$-almost all (t,x). □

Proof of Theorem 2.3.

1. Let $\psi(t,x)$ be a smooth compactly supported function. Then the action of the distribution $\theta_t - \Delta\theta$ on ψ is given by

$$I(\psi) = -\textstyle\int\int (\psi_t + \Delta\psi)\theta\, dx\, dt = \lim_{\varepsilon\to 0} \int\int (\theta^\varepsilon_t - \Delta\theta^\varepsilon)\psi\, dx\, dt,$$

so that, by (HE),

$$\begin{aligned} I(\psi) &= -\lim_{\varepsilon\to 0} \iint g(\varphi^\varepsilon)\varphi^\varepsilon_t \psi\, dx\, dt\\ &= -\lim_{\varepsilon\to 0} \iint \frac{1}{\varepsilon}\left(q'\left(\frac{z^\varepsilon}{\varepsilon}\right)\right)^2 z^\varepsilon_t \psi\, dx\, dt\\ &= \lim_{\varepsilon\to 0} \iint \psi\, d\bar{m}^\varepsilon + \lim_{\varepsilon\to 0}\iint \psi z^\varepsilon_t\, d\xi^\varepsilon\\ &= \iiint v(t,x,n)\psi(t,x)\, dm + \lim_{\varepsilon\to 0}\iint \psi z^\varepsilon_t\, d\xi^\varepsilon. \end{aligned}$$

We assert that the second term in the last expression is equal to zero. Indeed, since $|\xi^\varepsilon| \leqq \mu^\varepsilon$, the Cauchy-Schwarz inequality yields

$$\left|\iint \psi z_t^\varepsilon \, d\xi^\varepsilon\right| \leqq \left(\iint |\psi|^2 d|\xi^\varepsilon|\right)^{1/2} \left(\iint_{\operatorname{spt}\psi} |z_t^\varepsilon|^2 \, d\mu^\varepsilon\right)^{1/2}.$$

By Proposition 7.3 and (8.2), the right-hand side of the previous expression converges to zero as ε tends to zero; hence (2.10) holds. Equation (2.11) follows after an integration by parts in the variable t.

2. Let Y be a compactly supported, smooth vector field. The definitions of $\bar{m}^\varepsilon$ and $v(t, x, n)$ yield

$$\begin{aligned} L(Y) &:= \iiint v(t, x, n) n \cdot Y(t, x) \, dm = \iiint n \cdot Y(t, x) \, d\bar{m} \\ &= -\lim_{\varepsilon \to 0} \iint z_t^\varepsilon v^\varepsilon \cdot Y \mu^\varepsilon(t; dx) \, dt \\ &= -\lim_{\varepsilon \to 0} \iint z_t^\varepsilon v^\varepsilon \cdot Y \frac{1}{\varepsilon}\left(q'\left(\frac{z^\varepsilon}{\varepsilon}\right)\right)^2 dx \, dt - \lim_{\varepsilon \to 0} \iint z_t^\varepsilon v^\varepsilon \cdot Y \, d\xi^\varepsilon \, dt. \end{aligned}$$

As in Step 1, the second term in the above expression is zero. Next we use the identities

$$z_t^\varepsilon q'\left(\frac{z^\varepsilon}{\varepsilon}\right) = \varepsilon \varphi_t^\varepsilon, \quad v^\varepsilon q'\left(\frac{z^\varepsilon}{\varepsilon}\right) = \frac{\varepsilon \nabla \varphi^\varepsilon}{|\nabla z^\varepsilon|}, \quad g(\varphi^\varepsilon) = \sqrt{2W(\varphi^\varepsilon)} = q',$$

together with (OPE) and (8.4) to obtain

$$\begin{aligned} L(Y) &= -\lim_{\varepsilon \to 0} \iint \varepsilon \varphi_t^\varepsilon \nabla \varphi^\varepsilon \cdot Y \frac{1}{|\nabla z^\varepsilon|} \, dx \, dt \\ &= -\lim_{\varepsilon \to 0} \iint q'\left(\frac{z^\varepsilon}{\varepsilon}\right) \nabla \varphi^\varepsilon \cdot Y \frac{1}{|\nabla z^\varepsilon|} \theta^\varepsilon \, dx \, dt \\ &\quad + \lim_{\varepsilon \to 0} \iint \varepsilon Y \cdot \nabla \varphi^\varepsilon \left[-\Delta \varphi^\varepsilon + \frac{1}{\varepsilon^2} W'(\varphi^\varepsilon)\right] \frac{1}{|\nabla z^\varepsilon|} \, dx \, dt \\ &= -\lim_{\varepsilon \to 0} \iint \frac{1}{\varepsilon}\left(q'\left(\frac{z^\varepsilon}{\varepsilon}\right)\right)^2 v^\varepsilon \cdot Y \theta^\varepsilon \, dx \, dt \\ &\quad + \lim_{\varepsilon \to 0} \iint \varepsilon Y \cdot \nabla \varphi^\varepsilon \left[-\Delta \varphi^\varepsilon + \frac{1}{\varepsilon^2} W'(\varphi^\varepsilon)\right] dx \, dt \\ &\quad + \lim_{\varepsilon \to 0} \iint \varepsilon Y \cdot \nabla \varphi^\varepsilon \left[-\Delta \varphi^\varepsilon + \frac{1}{\varepsilon^2} W'(\varphi^\varepsilon)\right] \left(\frac{1 - |\nabla z^\varepsilon|}{|\nabla z^\varepsilon|}\right) dx \, dt \\ &:= \lim_{\varepsilon \to 0} I^\varepsilon + \iint H \cdot Y \, dt \mu(t; dx) + \lim_{\varepsilon \to 0} E^\varepsilon. \end{aligned} \tag{8.6}$$

3. In this step we show that E^ε converges to zero. By (8.3),

$$|E^\varepsilon| \leqq \|Y\|_\infty \left(\iint_{\mathrm{spt}\, Y} \varepsilon \left[-\Delta\varphi^\varepsilon + \frac{1}{\varepsilon^2} W'(\varphi^\varepsilon) \right]^2 \right)^{1/2} \left(\iint_{\mathrm{spt}\, Y} \varepsilon |\nabla\varphi^\varepsilon|^2 \left(\frac{1 - |\nabla z^\varepsilon|}{|\nabla z^\varepsilon|} \right)^2 \right)^{1/2}.$$

$$\leqq K(Y) \left(\iint_{\mathrm{spt}\, Y} \frac{1}{\varepsilon} \left(q'\left(\frac{z^\varepsilon}{\varepsilon} \right) \right)^2 (1 - |\nabla z^\varepsilon|)^2 \right)^{1/2},$$

while (4.1) yields

$$\begin{aligned}(1 - |\nabla z^\varepsilon|)^2 &= (1 - |\nabla z^\varepsilon|)^2 \chi_{\{|\nabla z^\varepsilon| \geqq 1\}} + (1 - |\nabla z^\varepsilon|)^2 \chi_{\{|\nabla z^\varepsilon| < 1\}} \\ &\leqq K\varepsilon(1 + |z^\varepsilon|)^2 + |1 - |\nabla z^\varepsilon|^2|.\end{aligned}$$

Hence

$$E^\varepsilon \leqq K(Y) \left(\iint_{\mathrm{spt}\, Y} K\varepsilon(1 + |z^\varepsilon|)^2 \, d\mu^\varepsilon + d|\xi^\varepsilon| \right)^{1/2},$$

and by Proposition 7.3 and (8.1), the limit of E^ε is zero.

4. Let θ_k be as in (8.5). In Step 6 of the previous proof we have shown that $\theta \in L^1_{\mathrm{loc}}(d\mu)$ and

$$I(\theta) := -\iiint Y \cdot n\theta \, dm = -\lim_{k \to \infty} \iiint \theta_k n \cdot Y \, dm.$$

Then

$$\begin{aligned} I(\theta) - I^\varepsilon &= \iint \frac{1}{\varepsilon} (q')^2 v^\varepsilon \cdot Y(\theta^\varepsilon - \theta_k) \, dx \, dt \\ &+ \iiint \left(\frac{1}{\varepsilon} (q')^2 v^\varepsilon \cdot Y \, dx \, dt - n \cdot Y \, dm^\varepsilon \right) \theta_k \\ &+ \iiint n \cdot Y\theta_k (dm^\varepsilon - dm) + \iiint n \cdot Y(\theta_k - \theta) \, dm. \end{aligned}$$

Since

$$\frac{1}{\varepsilon} (q')^2 v^\varepsilon \cdot Y \, dx \, dt - n \cdot Y \, dm^\varepsilon = v^\varepsilon \cdot Y \, d\xi^\varepsilon,$$

Proposition 7.3 and the convergence of m^ε to m yields

$$\begin{aligned} \limsup_{\varepsilon \to 0} |I^\varepsilon - I(\theta)| &\leqq \limsup_{\varepsilon \to 0} \left| \iint \frac{1}{\varepsilon} (q')^2 v^\varepsilon \cdot Y(\theta^\varepsilon - \theta_k) \, dx \, dt \right| \\ &+ \left| \iiint n \cdot Y(\theta_k - \theta) \, dm \right|. \end{aligned}$$

Recall that θ^ε converges to θ strongly in L^2_{loc} (cf. Proposition 3.4). So as in Step 6 of the previous proof,

$$\limsup_{\varepsilon\to 0}\left|\iint \frac{1}{\varepsilon}(q')^2 v^\varepsilon \cdot Y(\theta^\varepsilon - \theta_k)\,dx\,dt\right|$$

$$\leqq \|Y\|_\infty \limsup_{\varepsilon\to 0} \iint_{\text{spt}\,Y} |\theta^\varepsilon - \theta_k|\,dt\,\mu^\varepsilon(t;dx)$$

$$\leqq K\|Y\|_\infty\left[\lambda\,\|\theta-\theta_k\|_{2,T} + \frac{T}{\lambda}\right],$$

for any $\lambda > 0$. Finally, let k and then λ go to infinity to show that I^ε converges to $I(\theta)$.
5. Combining the previous steps we conclude that

$$\iiint Y\cdot n(v+\theta)\,dm = \iint H\cdot Y\,d\mu$$

for any smooth vector field Y proving (2.12).
6. The computations of §2.3 imply that, for any $\phi \in C_c^\infty(\mathscr{R}^d \to [0,\infty))$,

$$\frac{d}{dt}\int \phi(x)\hat{\mu}^\varepsilon(t;dx) = -\int \phi[\varepsilon(\varphi_t^\varepsilon)^2 + |\nabla\theta^\varepsilon|^2]\,dx + \tfrac{1}{2}\int \Delta\phi(\theta^\varepsilon)^2\,dx$$

$$-\,\varepsilon\int \nabla\phi\cdot\nabla\varphi^\varepsilon\varphi_t^\varepsilon\,dx.$$

By Step 2,

$$\int_s^t\!\!\iint Y\cdot nv\,dm = -\lim \int_s^t\!\!\int \varepsilon\varphi_t^\varepsilon\nabla\varphi^\varepsilon\cdot Y\frac{1}{|\nabla z^\varepsilon|}\,dx\,dr.$$

We now proceed as in Step 3 to obtain

$$\int_s^t\!\!\iint Y\cdot nv\,dm = -\lim\int_s^t\!\!\int \varepsilon\varphi_t^\varepsilon\nabla\varphi^\varepsilon\cdot Y\,dx\,dr.$$

Hence

$$\hat{\mu}(\phi)(t) - \hat{\mu}(\phi)(s) \leqq -\liminf\int_s^t\!\!\int \phi(\varphi_t^\varepsilon)^2\,dx\,dr$$

$$+\int_s^t\!\!\iint vn\cdot\nabla\phi\,dm + \int_s^t\!\!\int (\tfrac{1}{2}\theta^2\Delta\phi - |\nabla\theta|^2\phi)\,dx\,dr.$$

Since ϕ is compactly supported, following the proof of the estimate (8.2) we find that

$$\liminf\int_s^t\!\!\int \phi(\varphi_t^\varepsilon)^2\,dx\,dr = \liminf\int_s^t\!\!\int \phi(z_t^\varepsilon)^2\mu^\varepsilon(r;dx)\,dr.$$

For any $w \in C_c^\infty(Q \times S^{d-1} \to \mathscr{R})$,

$$0 \leqq \liminf \int_s^t \int \int \phi (z_t^\varepsilon + w)^2 \, dm^\varepsilon$$

$$= \liminf \int_s^t \int \phi (z_t^\varepsilon)^2 \mu^\varepsilon(r; dx) \, dr + \int_s^t \int \int \phi (w^2 - 2wv) \, dm;$$

hence

$$\int_s^t \int \int \phi v^2 \, dm = \int_s^t \int \int \phi [(w - v)^2 - (w^2 - 2wv)] \, dm$$

$$\leqq \liminf \int_s^t \int \phi (\varphi_t^\varepsilon)^2 \, dx \, dr + \int_s^t \int \int \phi (w - v)^2 \, dm.$$

Since $v \in L^2(dm)$,

$$\int_s^t \int \int \phi v^2 \, dm \leqq \liminf \int_s^t \int \phi (\varphi_t^\varepsilon)^2 \, dx \, dr,$$

which proves (2.13). □

9. Appendix

In this section we study a simple radially symmetric solution of the Mullins-Sekerka problem. We show that if the initial radius of the interface is sufficiently small, then the temperature is *not* a bounded function.

The Mullins-Sekerka problem (1.5), (1.6) with radial symmetry and one interface, takes the form

$$\theta_t - \Delta \theta = \tfrac{4}{3} (\chi_{\{|x| \leqq R(t)\}})_t$$

while the radius $R(t)$ of the interface is a solution of

$$R'(t) = -\frac{1}{R(t)} - \theta(t, R(t)), \quad t \in (0, T_{\text{ext}}),$$

where the extinction (or melting) time T_{ext} is defined to be the first time at which $R(T_{\text{ext}}) = 0$, if there is such a time; otherwise $T_{\text{ext}} = \infty$. Let $(R(t), \theta(t, r))$ be the solution of this problem with initial data

$$\theta(0, \cdot) \equiv 0, \quad R(0) = R_0.$$

The following result was obtained in collaboration with T. ILMANEN.

Theorem 9.1. *For all sufficiently small R_0, the extinction time $T_{\text{ext}} < \infty$ and*

$$\lim_{\delta \downarrow 0} \frac{\theta(T_{\text{ext}} - \delta, 0)}{|\ln(\delta)|} < 0.$$

Proof.

1. Let G be the heat kernel. Then

$$\theta(t, x) = \tfrac{4}{3}\int_0^t R'(s)\left[\int_{\{|y|=R(s)\}} G(t-s, x-y)\, d\mathscr{H}^{d-1}(y)\right] ds$$

and there is a constant $C(d)$ such that for any $\tau, \rho > 0$ and $x \in \mathscr{R}^d$,

$$\int_{\{|y|=\rho\}} G(\tau, x-y)\, d\mathscr{H}^{d-1}(y) \leqq \frac{3C(d)}{8\sqrt{\tau}}.$$

Set

$$\bar{\theta}(t) := \sup\{|\theta(s, x)| : (s, x) \in [0, t] \times \mathscr{R}^d\},$$

$$\bar{K}(t) := \sup\left\{\frac{1}{R(s)} : s \in [0, t]\right\}$$

so that

$$|\theta(t, x)| \leqq \frac{4}{3}\int_0^t \sup\{|R'(s)| : s \in [0, t]\}\, \frac{3C(d)}{8\sqrt{t-s}}\, ds$$

$$\leqq C(d)\sqrt{t}[\bar{K}(t) + \bar{\theta}(t)].$$

Hence

$$\bar{\theta}(t) \leqq C(d)\sqrt{t}\,[\bar{K}(t) + \bar{\theta}(t)].$$

2. Set

$$t^* = \inf\{t \in [0, T_{\text{ext}}] : R'(t) = 0\},$$

or $t^* = \infty$ if this set is empty. Since $\theta_0 \equiv 0$, it follows that $t^* > 0$ and

$$R'(t) < 0, \quad \bar{K}(t) = \frac{1}{R(t)} \quad \forall t \in [0, t^*].$$

Also, if $t^* < T_{\text{ext}}$, then

$$\bar{\theta}(t^*) \geqq -\theta(t^*, R(t^*)) = \frac{1}{R(t^*)} = \bar{K}(t^*).$$

3. Set

$$t_0 = \min\left\{T_{\text{ext}}, \frac{1}{(3C(d))^2}\right\}$$

so that, for $s \in [0, t_0]$,

$$\bar{\theta}(t) \leqq \frac{1}{3}(\bar{K}(t) + \bar{\theta}(t)) \;\Rightarrow\; \bar{\theta}(t) \leqq \frac{1}{2}\bar{K}(t) = \frac{1}{2R(t)}.$$

Therefore $t_0 < t^*$ and

$$R'(s) \leqq -\frac{1}{2R(s)} \quad \forall t \in [0, t_0]. \tag{9.1}$$

Since $R(t_0) \geqq 0$, the preceeding differential inequality yields

$$R(s) \geqq \sqrt{t_0 - s} \quad \forall s \in [0, t_0].$$

Suppose that

$$R_0 < \frac{1}{3C(d)};$$

then

$$\sqrt{t_0} \leqq R_0 < \frac{1}{3C(d)},$$

and we conclude that $T_{\text{ext}} = t_0 < \infty$. Also, since $\theta \leqq 0$, we have

$$R' \geqq -\frac{1}{R(t)}; \tag{9.2}$$

hence

$$\sqrt{T_{\text{ext}} - s} \leqq R(s) \leqq \sqrt{2(T_{\text{ext}} - s)} \quad \forall s \in [0, T_{\text{ext}}]. \tag{9.3}$$

4. Set

$$\Lambda(\zeta) := \int_{\{|y|=\zeta\}} G(1, y)\, d\mathscr{H}^{d-1}(y).$$

For $\delta > 0$, by (9.1) and (9.3),

$$\begin{aligned}
\theta(T_{\text{ext}} - \delta, 0) &\leqq -\frac{2}{3} \int_0^{T_{\text{ext}}-\delta} \frac{1}{R(s)} \left[\int_{\{|y|=R(s)\}} G(T_{\text{ext}} - \delta - s, y)\, d\mathscr{H}^{d-1}(y) \right] ds \\
&= -\frac{2}{3} \int_0^{T_{\text{ext}}-\delta} \frac{1}{R(s)} \frac{1}{\sqrt{T_{\text{ext}} - \delta - s}} \Lambda\left(\frac{R(s)}{\sqrt{T_{\text{ext}} - \delta - s}} \right) ds \\
&\leqq -\frac{2}{3\sqrt{2}} \int_0^{T_{\text{ext}}-\delta} \frac{1}{\sqrt{\tau}\sqrt{\tau + \delta}} \Lambda\left(\frac{R(T_{\text{ext}} - \delta - \tau)}{\sqrt{\tau}} \right) d\tau.
\end{aligned}$$

For $\tau \geqq \delta$,

$$1 \leqq \sqrt{\frac{\delta + \tau}{\tau}} \leqq \frac{R(T_{\text{ext}} - \delta - \tau)}{\sqrt{\tau}} \leqq \sqrt{\frac{2(\delta + \tau)}{\tau}} \leqq 2\sqrt{2}.$$

Hence, for $\tau \geqq \delta$,

$$\Lambda\left(\frac{R(T_{\text{ext}} - \delta - \tau)}{\sqrt{\tau}} \right) \geqq \lambda_0 > 0,$$

where λ_0 is an appropriate constant. Therefore

$$\theta(T_{\text{ext}} - \delta, 0) \leqq -\frac{2\lambda_0}{3\sqrt{2}} \int_\delta^{T_{\text{ext}}-\delta} \frac{1}{\sqrt{\tau + \delta}\sqrt{\tau}}\, d\tau \leqq -\frac{\lambda_0}{3} \int_\delta^{T_{\text{ext}}-\delta} \frac{1}{\tau}\, d\tau. \quad \square$$

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