Arch. Rational Mech. Anal. 131 (1995) 101–119. © Springer-Verlag 1995

On the Asymptotic Structure of Plane Steady Flow of a Viscous Fluid in Exterior Domains

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Communicated by R. KOHN

1. Introduction

One of the most challenging questions in the mathematical theory of the Navier-Stokes equation, which in many respects is still open, is the problem of steady plane flow around an obstacle. Such a problem consists in determining the velocity $\boldsymbol{v} = (v_1, v_2)$ of the particles of the fluid and the associated pressure field p satisfying the system of equations

$$\left.\begin{array}{l}\Delta \boldsymbol{v} = \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \nabla p + \boldsymbol{f}\\ \nabla\cdot\boldsymbol{v} = 0\end{array}\right\}\text{ in }\Omega \tag{1.1}$$

where Ω (the region of flow) is a two-dimensional domain lying in the complement of a compact region (the obstacle) and $\boldsymbol{f}$ is a prescribed vector field (the negative of the body force). To (1.1) one must add the condition on $\boldsymbol{v}$ at the boundary $\partial\Omega$:

$$\boldsymbol{v}(x) = \boldsymbol{v}_*(x), \quad x \in \partial\Omega, \tag{1.2}$$

and at large distances

$$\lim_{|x|\to\infty} \boldsymbol{v}(x) = \boldsymbol{v}_\infty \tag{1.3}$$

where $\boldsymbol{v}_*$ is a prescribed function and $\boldsymbol{v}_\infty$ is a given constant vector. For simplicity, we set the coefficient of kinematical viscosity equal to one.

The investigation of the existence of solutions to problem (1.1)–(1.3) traces back to the work of LERAY (1933) who proved that, for sufficiently smooth data, there exists at least one smooth solution to (1.1), (1.2). Concerning the condition at infinity (1.3), LERAY was only able to show that these solutions have a finite Dirichlet integral, namely,

$$\int_\Omega \nabla\boldsymbol{v} : \nabla\boldsymbol{v} \leqq M, \tag{1.4}$$

where M depends only on the data. However, it is clear that (1.4) alone is not able to control the convergence of $\boldsymbol{v}(x)$ to a constant vector $\boldsymbol{v}_0$ as $|x|$ tends to infinity and, in fact, it is easy to find examples of solenoidal vector functions that satisfy (1.4) and that grow at large distances. The asymptotic behaviour of solutions to (1.1), (1.2) that satisfy (1.4) was first studied by GILBARG & WEINBERGER (1974, 1978). They show, in particular, that the velocity field $\boldsymbol{v}$ of all these solutions corresponding to $\boldsymbol{f}$ of bounded support either converges in a well-defined sense to some vector $\boldsymbol{v}_0$ or is such that the L^2-norm of $\boldsymbol{v}$ over the unit circle approaches infinity at large distances. Moreover, they show that the solution constructed by the method of LERAY (1933) is bounded and therefore converges at infinity. More recently AMICK (1988) has shown that *any* (sufficiently smooth) $\boldsymbol{v}$ satisfying (1.1), (1.2) with $\boldsymbol{f} \equiv \boldsymbol{v}_* \equiv 0$ and satisfying (1.4) is necessarily bounded. The techniques used by these authors are essentially based on maximum principles and topological arguments. However, the problem of the coincidence of $\boldsymbol{v}_0$ and $\boldsymbol{v}_\infty$ remains open.

Even if these solutions also satisfy (1.3), do they exhibit the basic features expected from the physical point of view? For instance, we expect that they satisfy the energy equation and that, for $\boldsymbol{v}_\infty \neq 0$, the associated flow presents an infinite wake in the direction of $\boldsymbol{v}_\infty$. These properties are related to the *asymptotic structure* of solutions at large distances. In this regard, one may argue that $\boldsymbol{v}$ can be represented asymptotically by an expansion in "reasonable" functions of $|x|$ with coefficients independent of $|x|$. However, if $\boldsymbol{v}_\infty = 0$, then not every such solution can be represented in this way, because one can exhibit examples of solutions to (1.1)–(1.3) with $\boldsymbol{v}_\infty = 0$ that obey (1.4) and decay more slowly than any prescribed negative power of $|x|$ (cf. HAMEL (1916); *cf.* also GALDI (1994b, Chapter X, Section 2)). Motivated by these considerations and in the wake of the work of R. FINN for the three-dimensional case (*cf.*, *e.g.*, FINN (1965)), SMITH (1965) introduced the class of *physically reasonable* solutions (*PR* solutions) which satisfy

$$\boldsymbol{v}(x) - \boldsymbol{v}_\infty = O(|x|^{-1/4-\varepsilon}) \tag{1.5}$$

for all large $|x|$ and some $\varepsilon > 0$. SMITH showed that every (smooth) *PR* solution to (1.1) corresponding to $\boldsymbol{f} = 0$[1] and to $\boldsymbol{v}_\infty \neq 0$ possesses the desired regularity at large distances since it behaves there like the Oseen fundamental tensor $\boldsymbol{E}$ (*cf.* Section 2). This latter property means that there exists a (constant) vector $\boldsymbol{m}$ such that

$$\boldsymbol{v}(x) - \boldsymbol{v}_\infty = \boldsymbol{m} \cdot \boldsymbol{E}(x) + \mathcal{R}(x) \tag{1.6}$$

where $\mathcal{R}(x) = O(|x|^{-1}\log^2|x|)$. Observe that, due to the properties of $\boldsymbol{E}$ (*cf.* Section 2), (1.6) shows a behaviour better than that originally assumed in (1.5) and that, in particular, $\boldsymbol{v}(x) - \boldsymbol{v}_\infty = O(|x|^{-1/2})$.

Existence in the class *PR* was proved by FINN & SMITH (1967) and more recently, by a different approach and in a more general context, by GALDI (1993).[2]

The problem is then to investigate under which condition a solution to (1.1)–(1.3) with $\boldsymbol{v}_\infty \neq 0$ and satisfying (1.4) is in the class *PR*. In this connection, it is

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[1] This assumption is made for the sake of simplicity. The result continues to hold provided $\boldsymbol{f}$ decays sufficiently fast at large distances.

[2] It is interesting to note that the results known for the uniqueness of *PR* solutions are not enough to ensure that (for small data) the solutions of FINN & SMITH coincide with those of GALDI.

interesting to recall that in the three-dimensional case, due to a result of BABENKO (1973) (*cf.* also GALDI (1992)), every (smooth) solution to (1.1)–(1.3) corresponding to f of bounded support,[3] $\boldsymbol{v}_\infty \neq 0$, and satisfying (1.4) is in the class *PR*.[4] For the case at hand, a recent result of AMICK (1991) ensures that every *symmetric* solution to (1.1)–(1.3) corresponding to $\boldsymbol{f} \equiv \boldsymbol{v}_* \equiv 0$, $\boldsymbol{v}_\infty \neq 0$, and satisfying (1.4) is also in the class *PR*. We recall that if $\partial\Omega$ is symmetric about the x_1-axis and $\boldsymbol{v}_\infty = (a, 0)$ $(a \neq 0)$, then a solution $\{\boldsymbol{v}(x_1, x_2) = (v_1(x_1, x_2), v_2(x_1, x_2)), p(x_1, x_2)\}$ is said to be symmetric if v_1 and p are even in x_2 while v_2 is odd in x_2. The method used by AMICK is again based on a clever use of maximum principles, and, while it could possibly be extended to non-symmetric flow, it is not obvious how to generalize it to include the case of non-zero body forces and boundary data. This last circumstance should not be overlooked, in that the pathological solutions of HAMEL (in the case $\boldsymbol{v}_\infty = 0$), which we mentioned before, are just generated by a non-zero total flux of the velocity field at the boundary.

The main objective of this paper is to show the following result. Let $\{\boldsymbol{v} = (v_1, v_2), p\}$ be a (smooth) solution to (1.1) corresponding to $\boldsymbol{f}$ of bounded support[5] and satisfying (1.3) with $\boldsymbol{v}_\infty = (1, 0)$[6] and (1.4). If for some $s \in (1, \infty)$ and $\rho > 0$

$$\int_{|x| > \rho} |v_2(x)|^s \, dx < \infty, \tag{1.7}$$

then $(\boldsymbol{v}, p)$ is a *PR* solution. It is important to emphasize that our result requires the vanishing of neither the boundary data nor the total flux of the velocity field through the boundary. Our method of proof is different from those of SMITH and AMICK, and relies essentially on the coupling of certain L^q-estimates for the Oseen problem in the plane (*cf.* GALDI (1991, 1994a) and Section 2) with an estimate of the type of Saint-Venant for the Dirichlet integral of the velocity field over a suitable neighbourhood of infinity (*cf.* Section 4). However, the key tool is the use of these L^q-estimates to show that every solution to (1.1) satisfying the assumptions stated previously in fact enjoys the same summability properties as the Oseen fundamental tensor. This is achieved by means of a suitable perturbation technique based on a simple "cut-off" argument (*cf.* Section 3). For this argument to hold, it is of the utmost importance that certain components of the Oseen fundamental tensor present no "wake region", in the sense that their uniform asymptotic behaviour is basically isotropic and optimal. We remark that the solutions constructed by FINN & SMITH (1967) and GALDI (1993) satisfy condition (1.7). In this respect, we note that hypothesis (1.7) is needed to ensure uniqueness for solutions to a problem which is a suitable perturbation to the Oseen problem (*cf.* (3.1) and

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[3] See footnote 1.

[4] Of course, in the three-dimensional case, the velocity field of a *PR* solution has a behaviour different from that stated in (1.5), and the estimates on the "remnant" $\mathscr{R}$ in (1.6) are likewise different, *cf.* FINN (1959, 1965).

[5] See footnote 1.

[6] Clearly, this condition causes no loss of generality.

Remark 3.1). If we were able to show such uniqueness under the sole assumption that $\mathbf{v}$ satisfies (1.3) and (1.4), our result would coincide exactly with that proved by BABENKO for the three-dimensional case.

2. Preliminaries

Let us first introduce some notations. $\mathbb{N}$ is the set of all positive integers. $\mathbb{R}$ is the real line and $\mathbb{R}^2$ is the two-dimensional Euclidean space. The disc of radius R centered at the origin is denoted by B_R. By Ω we always denote a domain (open connected set) in $\mathbb{R}^2$. By $\bar{\Omega}$ we mean the closure of Ω and by $\partial\Omega$ its boundary. We also set $\Omega^c = \mathbb{R} - \Omega$. For $\mathscr{B} \subset \mathbb{R}^2$ we indicate by $\delta(\mathscr{B})$ its diameter. If Ω is a domain which is the complement of a (not necessarily connected) compact set Ω^c with a non-empty interior, *i.e.*, Ω is an *exterior* domain, then taking the origin of coordinates into the interior of Ω^c, we put

$$\Omega_R = \{x \in \Omega : |x| < R\},$$

$$\Omega^R = \{x \in \Omega : |x| > R\},$$

$$\Omega_{R_1 R_2} = \{x \in \Omega : R_2 > |x| > R_1\}$$

for $R > \delta(\Omega^c)$ and $R_2 > R_1 > \delta(\Omega^c)$. We indicate by $C_0^\infty(\Omega)$ the class of functions in Ω which are infinitely differentiable and of compact support in Ω.[7] For $k = 1, 2$ we set $D_k = \partial/\partial x_k$. Likewise, for $\alpha = (\alpha_1, \alpha_2)$, $\alpha_i \geqq 0$, we let

$$D^\alpha = \frac{\partial^{\alpha_1 + \alpha_2}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}, \quad |\alpha| = \alpha_1 + \alpha_2.$$

By $W^{m,q}(\Omega)$, $m \in \mathbb{N} \cup \{0\}$, $q \in [1, \infty]$, we indicate the Sobolev space of order (m, q) endowed with the norm

$$\|u\|_{m,q,\Omega} = \left(\sum_{|\alpha|=0}^{m} \int_\Omega |D^\alpha u|^q \right)^{1/q},$$

where the subscript Ω will be omitted if no confusion arises.[8] We have $W^{0,q}(\Omega) = L^q(\Omega)$ and set $\|u\|_{0,q,\Omega} \equiv \|u\|_{q,\Omega}$.

For $m \in \mathbb{N} \cup \{0\}$ and $q \in (1, \infty)$, we define the *homogeneous* Sobolev space (*cf.* SIMADER & SOHR (1994, Chapter I), GALDI (1994a, Chapter I))

$$D^{m,q} = D^{m,q}(\Omega) = \{u \in L^1_{\mathrm{loc}}(\Omega) : D^l u \in L^q(\Omega), |l| = m\}.$$

One can prove that if $u \in D^{m,q}(\Omega)$, then

$$D^l u \in L^q(\Omega'),\ 0 \leqq |l| \leqq m \quad \text{for all compact } \Omega' \text{ with } \bar{\Omega}' \subset \Omega.$$

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[7] As a rule, if Y denotes a space of scalar functions, we use the same symbol to denote the space of vector functions with components in Y.

[8] Unless their use clarifies the context, we also omit the infinitesimal volume and surface elements in the integrals.

In $D^{m,q}(\Omega)$ we introduce the seminorm

$$|u|_{m,q,\Omega} \equiv \left(\sum_{|\alpha|=m} \int_\Omega |D^\alpha u|^q \right)^{1/q},$$

where, as before, the subscript Ω will be omitted if no confusion arises. It is simple to show that $\{D^{m,q}, |\cdot|_{m,q}\}$ is a complete normed space, provided that we identify two functions $u_1, u_2 \in D^{m,q}(\Omega)$ whenever $|u_1 - u_2|_{m,q} = 0$; that is, when u_1 and u_2 differ by a polynomial of degree $m-1$ at most.

We now recall some known properties concerning the Oseen system in $\mathbb{R}^2$

$$\Delta \boldsymbol{u} - \frac{\partial \boldsymbol{u}}{\partial x_1} = \boldsymbol{F} + \nabla \pi, \quad \nabla \cdot \boldsymbol{u} = g, \tag{2.1}$$

where $\boldsymbol{F}$ and g are given functions. Specifically, we have the following result (*cf.* Galdi, 1994a, Chapter VII, Section 4).

Lemma 2.1. *Given*

$$\boldsymbol{F} \in L^q(\mathbb{R}^2), \quad g \in W^{1,q}(\mathbb{R}^2), \quad 1 < q < \infty,$$

there exists a pair of functions $\boldsymbol{u}$, π *with*

$$\boldsymbol{u} \in W^{2,q}(B_R), \quad \pi \in W^{1,q}(B_R) \quad \textit{for every } R > 0,$$

satisfying a.e. the Oseen system (2.1). *Moreover,*

$$\boldsymbol{u} \in D^{2,q}(\mathbb{R}^2), \quad u_2 \in D^{1,q}(\mathbb{R}^2), \quad \frac{\partial \boldsymbol{u}}{\partial x_1} \in L^q(\mathbb{R}^2), \quad \pi \in D^{1,q}(\mathbb{R}^2),$$

$$|u_2|_{1,q} + \left\| \frac{\partial \boldsymbol{u}}{\partial x_1} \right\|_q + |\boldsymbol{u}|_{2,q} + |\pi|_{1,q} \leqq c(\|\boldsymbol{F}\|_q + \|g\|_{1,q}).$$

In addition, the following properties hold. If $1 < q < 3$, *then*

$$\boldsymbol{u} \in D^{1,3q/(3-q)}(\mathbb{R}^2),$$

$$|\boldsymbol{u}|_{1,3q/(3-q)} \leqq c(\|\boldsymbol{F}\|_q + \|g\|_{1,q}).$$

If $1 < q < 2$, *then*

$$u_2 \in L^{2q/(2-q)}(\mathbb{R}^2), \quad \pi \in L^{2q/(2-q)}(\mathbb{R}^2),$$

$$\|u_2\|_{2q/(2-q)} + \|\pi\|_{2q/(2-q)} \leqq c(\|\boldsymbol{F}\|_q + \|g\|_{1,q}).$$

If $1 < q < \frac{3}{2}$ *then*

$$\boldsymbol{u} \in L^{3q/(3-2q)}(\mathbb{R}^2),$$

$$\|\boldsymbol{u}\|_{3q/(3-2q)} \leqq c(\|\boldsymbol{F}\|_q + \|g\|_{1,q}),$$

where the constant c depends only on q. Finally, if $(\boldsymbol{w}, \tau)$ *is another solution to* (2.1) *with*

$$\frac{\partial \boldsymbol{w}}{\partial x_1} \in L^q(\mathbb{R}^2), \quad D^2\boldsymbol{w} \in L^q(\mathbb{R}^2), \quad 1 < q < \infty,$$

then

$$\frac{\partial \boldsymbol{w}}{\partial x_1} = \frac{\partial \boldsymbol{u}}{\partial x_1}, \quad D^2\boldsymbol{w} = D^2\boldsymbol{u}, \quad \pi = \tau + \text{const.} \quad \textit{a.e. in } \mathbb{R}^2.$$

We end this section by recalling some properties of the Oseen fundamental tensor. For more detailed information and for the proof of the stated results, see GALDI (1994a, Chapter VII, Section 3). Following OSEEN (1927, Section 4), for $(x, y) \in \mathbb{R}^2$ we denote by $\boldsymbol{E}$ and e a tensor field and a vector field such that $(i, j = 1, 2)$

$$\begin{aligned} E_{ij}(x-y) &= \left(\delta_{ij}\Delta - \frac{\partial^2}{\partial y_i \partial y_j}\right)\Phi(x-y), \\ e_j(x-y) &= -\frac{\partial}{\partial y_j}\left(\Delta - \frac{\partial}{\partial y_1}\right)\Phi(x-y) \end{aligned} \tag{2.2}$$

where

$$\Phi(x-y) = -\frac{1}{2\pi}\int_{x_1}^{y_1}\{\log\sqrt{(\tau - x_1)^2 + (x_2 - y_2)^2} + K_0\left(\tfrac{1}{2}\sqrt{(\tau - x_1)^2 + (x_2 - y_2)^2}\right)e^{-(\tau - x_1)}\}\,d\tau \tag{2.3}$$

with $K_0(z)$ the modified Bessel function of the second kind of order zero. Moreover,

$$e_j(x-y) = \frac{1}{2\pi}\frac{x_j - y_j}{|x-y|^2}, \quad i, j = 1, 2.$$

By a direct (and tedious) calculation, one can show that

$$\begin{aligned} \left(\Delta + \frac{\partial}{\partial y_1}\right)E_{ij}(x-y) - \frac{\partial}{\partial y_i}e_j(x-y) &= 0, \\ \frac{\partial}{\partial y_l}E_{lj}(x-y) &= 0 \end{aligned}$$

for all $x \neq y$.[9] Denoting by $\mathscr{A}$ the exterior of any circle centered at the origin, from (2.2) and (2.3) one can deduce the uniform bound[10]

$$|\boldsymbol{E}(x)| \le c|x|^{-1/2} \quad \text{for all } x \in \mathscr{A}. \tag{2.4}$$

[9] We adopt the Einstein summation convention over repeated indices.
[10] Notice that $\boldsymbol{E}$ depends on x, y only through $x - y$.

In addition, setting

$$E_1 = (E_{11}, E_{12}), \quad E_2 = (E_{12}, E_{22}),$$

we obtain the summability properties

$$\begin{aligned} &E_1 \in L^q(\mathscr{A}) \quad \text{for all } q > 3, \\ &E_2 \in L^q(\mathscr{A}) \quad \text{for all } q > 2, \\ &\frac{\partial E_i}{\partial x_1} \in L^q(\mathscr{A}) \quad \text{for all } q > 1, i = 1, 2, \\ &\frac{\partial E_i}{\partial x_2} \in L^q(\mathscr{A}) \quad \text{for all } q > \tfrac{3}{2}, i = 1, 2. \end{aligned} \tag{2.5}$$

3. Summability properties at large distances

The aim of this section is to give conditions on the velocity field $\boldsymbol{v}$ of a solution to the problem (1.1) tending to a nonzero vector at large distances, ensuring that its components v_1, v_2 satisfy the same summability properties (2.5) as the vectors E_1, E_2, respectively. As we show in the next section, these conditions allow us to conclude that the solution is physically reasonable in the sense of FINN, that is, it behaves pointwise at large distances like the Oseen fundamental tensor.

We begin by showing some existence and uniqueness results for a suitable linearization of (1.1). Specifically, let us consider the problem

$$\Delta \boldsymbol{u} - \frac{\partial \boldsymbol{u}}{\partial x_1} = a \frac{\partial \boldsymbol{u}}{\partial x_1} + \boldsymbol{A} u_2 + \nabla \pi + \boldsymbol{G}, \quad \nabla \cdot \boldsymbol{u} = g, \tag{3.1}$$

where $a, \boldsymbol{A}$, $\boldsymbol{G}$ and g are prescribed functions.

Lemma 3.1. *Let*

$$\boldsymbol{G} \in L^q(\mathbb{R}^2), \quad g \in W^{1,q}(\mathbb{R}^2), \quad q \in (1, \tfrac{3}{2})$$

$$\boldsymbol{A} \in L^2(\mathbb{R}^2), \quad a \in L^\infty(\mathbb{R}^2).$$

Moreover, let $(\boldsymbol{u}, \pi)$ *be any solution to* (3.1) *such that*

$$u_2 \in L^{2\bar{q}/(2-\bar{q})}(\mathbb{R}^2), \quad \textit{and} \quad D^2 \boldsymbol{u}, \frac{\partial \boldsymbol{u}}{\partial x_1} \in L^{\bar{q}}(\mathbb{R}^2), \quad \textit{for some } \bar{q} \in (1, 2)$$

$$\lim_{|x| \to \infty} u_1(x) = 0.$$

Then, there exists a positive constant $c = c(q, \bar{q})$ (*cf.* (3.4) *and* (3.12)) *such that if*

$$\|a\|_\infty + \|\boldsymbol{A}\|_2 < c,$$

then

$$u \in D^{2,q}(\mathbb{R}^2) \cap D^{1,3q/(3-q)}(\mathbb{R}^2) \cap L^{3q/(3-2q)}(\mathbb{R}^2),$$

$$u_2 \in D^{1,q}(\mathbb{R}^2) \cap L^{2q/(2-q)}(\mathbb{R}^2),$$

$$(\pi - \pi_0) \in D^{1,q}(\mathbb{R}^2) \cap L^{2q/(2-q)}(\mathbb{R}^2)$$

for some $\pi_0 \in \mathbb{R}$.

Proof. Let X_q, $1 < q < \frac{3}{2}$, denote the Banach space of solenoidal functions $w \in L^1_{\text{loc}}(\mathbb{R}^2)$ such that the norm

$$\|w\|_{X_q} := \|w_2\|_{2q/(2-q)} + \|\nabla w_2\|_q + \left\| \frac{\partial w}{\partial x_1} \right\|_q + \|\nabla w\|_{3q/(3-q)} + \|D^2 w\|_q + \|w\|_{3q/(3-2q)}$$

is finite. Denote by $B_q^{(\delta)}$ the ball in X_q of radius $\delta(>0)$ and consider the map

$$\mathscr{L}: w' \in B_q^{(\delta)} \to w \in X_q,$$

where w satisfies the problem

$$\Delta w - \frac{\partial w}{\partial x_1} = a \frac{\partial w'}{\partial x_1} + A w_2' + \nabla \tau + G, \quad \nabla \cdot w = g. \tag{3.2}$$

For all $q \in (1, 2)$, we have

$$\left\| a \frac{\partial w'}{\partial x_1} + A w_2' \right\|_q \leqq \|a\|_\infty \left\| \frac{\partial w'}{\partial x_1} \right\|_q + \|A\|_2 \|w_2'\|_{2q/(2-q)} \tag{3.3}$$

by the Hölder inequality. Thus, by the hypotheses made on G and g, the map $\mathscr{L}$ is well defined for all $q \in (1, \frac{3}{2})$. Furthermore, by Lemma 2.1 and (3.3), we find that

$$\|w\|_{X_q} \leqq c_1 [(\|a\|_\infty + \|A\|_2) \|w'\|_{X_q} + \|G\|_q + \|g\|_{1,q}]$$

for all $w' \in B_q^{(\delta)}$ and for some $c_1 = c_1(q)$. Thus, assuming (for instance) that

$$\|a\|_\infty + \|A\|_2 < \frac{1}{2c_1}, \tag{3.4}$$

and choosing $\delta \geqq 2c_1(\|G\|_q + \|g\|_{1,q})$, we find that $\mathscr{L}$ transforms $B_q^{(\delta)}$ into itself. Moreover, from (3.2) with $G \equiv g \equiv 0$, and by (3.3), (3.4) we obtain

$$\|w\|_{X_q} \leqq \tfrac{1}{2} \|w'\|_{X_q},$$

and so the existence of a solution w, τ to (3.1) with $w \in X_q$ follows from the contraction mapping theorem. Moreover, $\tau \in D^{1,q}(\mathbb{R}^2)$ and therefore by the Sobolev theorem, $\tau - \tau_0 \in L^{2q/(2-q)}(\mathbb{R}^2)$, for some $\tau_0 \in \mathbb{R}$. We now show that $u = w$, $\tau = \pi + \text{const}$ a.e. in $\mathbb{R}^2$. To this end, we let

$$\bar{w} = w - u, \quad s = \tau - \pi,$$

so that

$$\Delta\bar{w} - \frac{\partial \bar{w}}{\partial x_1} = a\frac{\partial \bar{w}}{\partial x_1} + A\bar{w}_2 + \nabla s, \quad \nabla\cdot\bar{w} = 0. \tag{3.5}$$

It is easy to show that

$$\bar{w}_2 \in L^{2\bar{q}/(2-\bar{q})}(\mathbb{R}^2), \quad \frac{\partial \bar{w}}{\partial x_1} \in L^{\bar{q}}(\mathbb{R}^2).$$

To this end, it is enough to prove that

$$w_2 \in L^{2\bar{q}/(2-\bar{q})}(\mathbb{R}^2), \quad \frac{\partial w}{\partial x_1} \in L^{\bar{q}}(\mathbb{R}^2). \tag{3.6}$$

Assume that $q < \bar{q}$ (the other case $q > \bar{q}$ can be treated likewise by interchanging the role of q and $\bar{q}$). Since $w \in D^{2,q}(\mathbb{R}^2)$, $q < 2$, by the Sobolev theorem we have

$$w_2 \in D^{1,2q/(2-q)}(\mathbb{R}^2). \tag{3.7}$$

Since $w_2 \in L^{2q/(2-q)}(\mathbb{R}^2)$, from (3.7), and from known embedding results in exterior domains (*cf.*, *e.g.*, Remark 7.2 in Chapter II of Galdi (1994a)) we obtain

$$w_2 \in L^{\infty}(\mathbb{R}^2),$$

which proves the first relation in (3.6). Furthermore, by the properties

$$w \in D^{2,q}(\mathbb{R}^2), \quad \frac{\partial w}{\partial x_1} \in L^{q}(\mathbb{R}^2),$$

and a well-known interpolation inequality of Nirenberg (1959, Theorem at p. 125), we find that

$$\frac{\partial w}{\partial x_1} \in L^{s}(\mathbb{R}^2) \quad \text{for all } s \in [q, 2],$$

and also the second relation in (3.6) follows. From (3.4) with w' replaced by $\bar{w}$ we thus conclude that

$$F \equiv a\frac{\partial \bar{w}}{\partial x_1} + A\bar{w}_2 \in L^{\bar{q}}(\mathbb{R}^2).$$

Therefore, in view of Lemma 2.1, the problem

$$\Delta z - \frac{\partial z}{\partial x_1} = F + \nabla\sigma, \quad \nabla\cdot z = 0$$

admits at least one solution w^*, s^* such that

$$\begin{aligned} \|D^2 w^*\|_{\bar{q}} + \|w_2^*\|_{2\bar{q}/(2-\bar{q})} + \left\|\frac{\partial w^*}{\partial x_1}\right\| &\leqq c_2\|F\|_{\bar{q}} \\ &\leqq c_2(\|a\|_\infty + \|A\|_2)\left(\left\|\frac{\partial \bar{w}}{\partial x_1}\right\|_{\bar{q}} + \|\bar{w}_2\|_{2\bar{q}/(2-\bar{q})}\right) \end{aligned} \tag{3.8}$$

with $c_2 = c_2(\bar{q})$. We now show that

$$D^2(\boldsymbol{w}^* - \bar{\boldsymbol{w}}) \equiv \frac{\partial(\boldsymbol{w}^* - \bar{\boldsymbol{w}})}{\partial x_1} \equiv 0, \quad w_2^* \equiv \bar{w}_2. \tag{3.9}$$

Actually, with $\boldsymbol{v} = \boldsymbol{w}^* - \bar{\boldsymbol{w}}$, $p = s^* - s$, it follows that

$$\Delta \boldsymbol{v} - \frac{\partial \boldsymbol{v}}{\partial x_1} = \nabla p, \quad \nabla \cdot \boldsymbol{v} = 0.$$

We now use a local representation for $\boldsymbol{v}$ in terms of the Oseen-Fujita truncated fundamental tensor (*cf.* GALDI (1994a, p. 400)):

$$\begin{aligned} D^\alpha v_j(x) &= - \int_{B_R(x)} \mathscr{H}_{ij}(x-y) D^\alpha v_i(y)dy \\ &\equiv - \int_{B_R(x)} \mathscr{H}_{ij}^{(R)}(x-y) D^\alpha (w_i^*(y) - w_i(y) - u_i(y))dy. \end{aligned} \tag{3.10}$$

Here, for any fixed x, $\mathscr{H}_{ij}^{(R)}(x-y)$ is an infinitely differentiable function with compact support in $B_R(x)$ satisfying the estimate

$$|\mathscr{H}_{ij}^{(R)}(x-y)| \leqq CR^{-3/2} \tag{3.11}$$

for all large R with C independent of R. Recalling the summability properties of $\boldsymbol{w}^*$, $\boldsymbol{w}$ and $\boldsymbol{u}$ and using (3.11) and the Hölder inequality on the right-hand side of (3.10) for various values of α, we can easily show the validity of (3.9). For instance, with $|\alpha| = 2$ we find that

$$|D^2 v_j(x)| \leqq C_1 [R^{-3/2} R^{2(1-1/\bar{q})}(|\boldsymbol{v}|_{2,\bar{q}} + |\boldsymbol{u}|_{2,\bar{q}}) + R^{-3/2} R^{2(1-1/q)} |\boldsymbol{w}|_{2,q}],$$

and so, noticing that $-\frac{3}{2} + 2(1 - 1/s) < 0$ for all $s < 4$, we prove the first relation in (3.9) by letting $R \to \infty$ in this last inequality. The other relations in (3.9) follow in a similar manner. From (3.9) and (3.8) we then obtain

$$\|D^2 \bar{\boldsymbol{w}}\|_{\bar{q}} + \|\bar{w}_2\|_{2\bar{q}/(2-\bar{q})} + \left\| \frac{\partial \bar{\boldsymbol{w}}}{\partial x_1} \right\|_{\bar{q}} \leqq c_2(\|a\|_\infty + \|A\|_2) \left(\left\| \frac{\partial \bar{\boldsymbol{w}}}{\partial x_1} \right\|_{\bar{q}} + \|\bar{w}_2\|_{2\bar{q}/(2-\bar{q})} \right).$$

Thus, if

$$\|a\|_\infty + \|A\|_2 < \frac{1}{2c_2}, \tag{3.12}$$

we conclude that

$$D^2(\boldsymbol{w} - \boldsymbol{u}) \equiv \frac{\partial(\boldsymbol{w} - \boldsymbol{u})}{\partial x_1} \equiv 0, \quad w_2 \equiv u_2,$$

and the lemma follows from the properties of $\boldsymbol{w}$ and the fact that u_1 tends to zero as $|x|$ tends to infinity.

Remark 3.1. We do not know if the conclusion of Lemma 3.1 continues to hold under the alternative hypotheses on $\boldsymbol{u}$:

$$D^2\boldsymbol{u}, \frac{\partial \boldsymbol{u}}{\partial x_1} \in L^2(\mathbb{R}^2), \quad \lim_{|x|\to\infty} \boldsymbol{u}(x) = 0.$$

If it did, then assumption (3.14) in the main Theorem 3.1 could be weakened to require only that $v_2(x)$ tend to zero uniformly as $|x| \to \infty$. Nevertheless, we can treat the case $\bar{q} = 2$ if we suppose that $\boldsymbol{A} \equiv 0$, as shown in

Lemma 3.2. *Let* $(\boldsymbol{u}, \pi)$ *be a solution to* (3.1) *with* $\boldsymbol{A} \equiv 0$, *such that*

$$\boldsymbol{u} \in D^{2,2}(\mathbb{R}^2), \quad \frac{\partial \boldsymbol{u}}{\partial x_1} \in L^2(\mathbb{R}^2).$$

Suppose, further, that

$$\boldsymbol{G} \in L^r(\mathbb{R}^2), \quad g \in W^{1,r}(\mathbb{R}^2) \quad \text{for some } r \in (1, 2).$$

Then, there exists a positive constant $c = c(r)$ *such that if*

$$\|a\|_\infty < c,$$

then

$$D^2\boldsymbol{u}, \frac{\partial \boldsymbol{u}}{\partial x_1} \in L^r(\mathbb{R}^2).$$

Proof. Reasoning exactly as in the proof of Lemma 3.1, we can show the existence of a solution $\boldsymbol{w}$, τ to problem (3.1) with $\boldsymbol{A} \equiv 0$ satisfying

$$\boldsymbol{w} \in D^{2,r}(\mathbb{R}^2), \quad \tau \in D^{1,r}(\mathbb{R}^2), \quad \frac{\partial \boldsymbol{w}}{\partial x_1} \in D^{1,r}(\mathbb{R}^2).$$

Letting $\bar{\boldsymbol{w}} = \boldsymbol{w} - \boldsymbol{u}$, $s = \tau - \pi$, we find that

$$\Delta \bar{\boldsymbol{w}} - \frac{\partial \bar{\boldsymbol{w}}}{\partial x_1} = a \frac{\partial \bar{\boldsymbol{w}}}{\partial x_1} - \nabla s, \quad \nabla \cdot \bar{\boldsymbol{w}} = 0.$$

Again as in the proof of Lemma 3.1, we may use the Nirenberg interpolation inequality to show that

$$a \frac{\partial \bar{\boldsymbol{w}}}{\partial x_1} \in L^2(\mathbb{R}^2),$$

and so, by Lemma 2.1, and by means of the same procedure used in Lemma 3.1, we obtain

$$\|D^2 \bar{\boldsymbol{w}}\|_2 + \left\| \frac{\partial \bar{\boldsymbol{w}}}{\partial x_1} \right\|_2 \leq c\|a\|_\infty \left\| \frac{\partial \bar{\boldsymbol{w}}}{\partial x_1} \right\|_2.$$

Therefore, if $\|a\|_\infty$ is sufficiently small, we find that

$$D^2\bar{w} \equiv \frac{\partial \bar{w}}{\partial x_1} \equiv 0,$$

and the lemma is proved.

With these lemmas in hand, we are now able to establish the main result of this section.

Theorem 3.1. *Let $(\boldsymbol{v}, p)$ be a solution to the Navier-Stokes system* (1.1) *in* Ω *with* $f \in L^q(\Omega^\rho)$ *for all* $q \in (1, 2]$ *and for some* $\rho > 4\delta(\Omega^c)$ *such that*

$$\boldsymbol{v} \in D^{1,2}(\Omega^\rho), \quad \lim_{|x| \to \infty} v_1(x) = 1. \tag{3.13}$$

Assume, further, that

$$v_2 \in L^s(\Omega^\rho), \quad \textit{for some } s \in (1, \infty). \tag{3.14}$$

Then

$$\begin{aligned}
(v_1 - 1) &\in L^{t_1}(\Omega^\rho) \quad \textit{for all } t_1 > 3,\\
v_2 &\in L^{t_2}(\Omega^\rho) \quad \textit{for all } t_2 > 2,\\
\frac{\partial v_1}{\partial x_2} &\in L^{t_3}(\Omega^\rho) \quad \textit{for all } t_3 > \tfrac{3}{2},\\
\frac{\partial v_1}{\partial x_1}, \nabla v_2 &\in L^{t_4}(\Omega^\rho) \quad \textit{for all } t_4 > 1,\\
(p - p_0) &\in L^{t_5}(\Omega^\rho) \quad \textit{for all } t_5 > 2,
\end{aligned}$$

where p_0 is a constant.

Proof. From the assumption on $\boldsymbol{v}$ and Lemma X.3.2 of GALDI (1994b) it follows that

$$\nabla \boldsymbol{v} \in W^{1,2}(\Omega^\rho). \tag{3.15}$$

This condition, in turn, combined with (3.14) and Remark 7.2 in Chapter II of GALDI (1994a), yields

$$\lim_{|x| \to \infty} v_2(x) = 0,$$

and so, setting $\boldsymbol{e}_1 = (1, 0)$, we find that

$$\lim_{|x| \to \infty} \boldsymbol{v}(x) = \boldsymbol{e}_1. \tag{3.16}$$

From the hypothesis, (3.15), (3.16) and (1.1) we also have

$$\nabla p \in L^2(\Omega^\rho). \tag{3.17}$$

For $R \geqq \rho$, let ψ_R be a smooth cut-off function defined by

$$\psi_R(x) = \begin{cases} 0 & \text{if } |x| < R/2, \\ 1 & \text{if } |x| \geqq R. \end{cases}$$

Setting

$$\boldsymbol{u} = \psi_R(\boldsymbol{v} - \boldsymbol{e}_1) \equiv \psi_R \bar{\boldsymbol{v}}, \quad \pi = \psi_R p,$$

from (1.1) we deduce that $(\boldsymbol{u}, \pi)$ satisfies the system

$$\begin{aligned} &\Delta \boldsymbol{u} - \frac{\partial \boldsymbol{u}}{\partial x_1} = (\psi_{R/2} \bar{v}_1) \frac{\partial \boldsymbol{u}}{\partial x_1} + \left(\psi_{R/2} \frac{\partial \boldsymbol{v}}{\partial x_2} \right) u_2 + \nabla \pi + \boldsymbol{G}_1, \\ &\nabla \cdot \boldsymbol{u} = g, \end{aligned} \tag{3.18}$$

where

$$\begin{aligned} \boldsymbol{G}_1 &= \psi_R \boldsymbol{f} + 2 \nabla \psi_R \cdot \nabla \boldsymbol{v} + \Delta \psi_R \bar{\boldsymbol{v}} - \frac{\partial \psi_R}{\partial x_1} \bar{\boldsymbol{v}} - \bar{v}_1 \boldsymbol{v} \frac{\partial \psi_R}{\partial x_1} - p \nabla \psi_R, \\ g &= \bar{\boldsymbol{v}} \cdot \nabla \psi_R. \end{aligned}$$

Clearly,

$$\boldsymbol{G}_1 \in L^q(\mathbb{R}^2), \quad g \in W^{1,q}(\mathbb{R}^2) \quad \text{for all } q \in (1, 2].$$

Moreover, we observe that in view of the assumption, if we take R sufficiently large, then the quantities

$$\| \psi_{R/2} \bar{v}_1 \|_\infty, \quad \left\| \psi_{R/2} \frac{\partial \boldsymbol{v}}{\partial x_2} \right\|_2$$

can be made less than any prescribed constant. Setting

$$\bar{q} = \frac{2s}{2 + s} \; (< 2),$$

by the Hölder inequality and assumption we find that

$$u_2 \frac{\partial \boldsymbol{v}}{\partial x_2} \in L^{\bar{q}}(\mathbb{R}^2).$$

Thus by Lemma 3.2 with $a = \psi_{R/2} \bar{v}_1$, $\boldsymbol{G} = \boldsymbol{G}_1 + u_2 \partial \boldsymbol{v} / \partial x_2$, $r = \bar{q}$, by (3.15), (3.18), and by the properties of ψ_R, we deduce that

$$D^2 \boldsymbol{v}, \; \frac{\partial \boldsymbol{v}}{\partial x_1} \in L^{\bar{q}}(\Omega^R).$$

From this and (3.14) we find that

$$\boldsymbol{u} \in L^{2\bar{q}/(2-\bar{q})}(\mathbb{R}^2), \quad D^2 \boldsymbol{u}, \; \frac{\partial \boldsymbol{u}}{\partial x_1} \in L^{\bar{q}}(\mathbb{R}^2). \tag{3.19}$$

We next apply Lemma 3.1 with $a = \psi_{R/2}\bar{v}_1$, $\boldsymbol{A} = \psi_{R/2}\, \partial \boldsymbol{v}/\partial x_1$ and $\boldsymbol{G} = \boldsymbol{G}_1$. In view of (3.16), (3.18), (3.19), and the properties of ψ_R we find that for any given $q \in (1, \frac{3}{2})$ there exists an R so large that

$$(v_1 - 1) \in L^{3q/(3-2q)}(\Omega^R),$$

$$v_2 \in L^{2q/(2-q)}(\Omega^R),$$

$$\frac{\partial v_1}{\partial x_2} \in L^{3q/(3-q)}(\Omega^R),$$

$$\frac{\partial v_1}{\partial x_1}, \nabla v_2 \in L^q(\Omega^R),$$

$$(p - p_0) \in L^{2q/(2-q)}(\Omega^R).$$

These conditions, along with (3.15)–(3.17), and the Sobolev embedding theorem, allow us to conclude the validity of the summability properties stated in the theorem.

4. Asymptotic structure

In this final section we prove that any solution to (1.1) satisfying the assumptions of Theorem 3.1 and corresponding to a body force of bounded support[11] is physically reasonable in the sense of FINN, that is, behaves at large distances like the Oseen fundamental tensor. We do this by showing that the velocity field $\boldsymbol{v}$ of every such solution satisfies

$$\boldsymbol{v} = \boldsymbol{e}_1 + O(|x|^{-\delta}) \quad \text{for some } \delta > \tfrac{1}{4},$$

with $\boldsymbol{e}_1 = (1, 0)$, since this condition, in turn, implies the desired result; *cf.* SMITH (1965).

We begin by making some estimates of the type of de Saint-Venant on the Dirichlet norm of $\boldsymbol{v} - \boldsymbol{e}_1$. In what follows, we set

$$\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{e}_1.$$

Lemma 4.1. *Let $\boldsymbol{v}$ satisfy the assumptions of Theorem 3.1 with $\boldsymbol{f} = 0$. Then for all $R \geqq \rho > \delta(\Omega^c)$, the following estimate holds:*

$$\int_{\Omega^R} \nabla \boldsymbol{u} : \nabla \boldsymbol{u} \leqq c R^{-1/3+\varepsilon}$$

where ε is an arbitrary positive number and c is independent of R.[12]

[11] See footnote 1.

[12] Of course, c depends on ε in such a way that $c \to \infty$ as $\varepsilon \to 0$.

Proof. Multiplying (1.1) by $\boldsymbol{u}$ and integrating by parts over Ω_{R,R_*}, $\rho \leqq R < R_*$, we find that

$$\int_{\Omega_{R,R*}} \nabla \boldsymbol{u} : \nabla \boldsymbol{u} = F(R) + F(R_*) \tag{4.1}$$

where

$$F(r) = \int_{\partial B_r} \left\{ \boldsymbol{u} \cdot \frac{\partial \boldsymbol{u}}{\partial n} - \frac{1}{2} \boldsymbol{u}^2 \boldsymbol{v} \cdot \boldsymbol{n} - p(\boldsymbol{u} \cdot \boldsymbol{n}) \right\}.$$

Using the summability properties of $\boldsymbol{u}$, p, we can show that there exists at least one sequence $\{R_k\}_{k \in \mathbb{N}}$ with $R_k \to \infty$ as $k \to \infty$ such that $F(R_k)$ goes to zero. Thus, replacing R_* by R_k in (4.1) and letting $k \to \infty$, we find that

$$G(R) = F(R) \tag{4.2}$$

where

$$G(R) \equiv \int_{\Omega^R} \nabla \boldsymbol{u} : \nabla \boldsymbol{u}.$$

Taking into account that, by Theorem 3.1,

$$\boldsymbol{u} \cdot \nabla \boldsymbol{u} \in L^1(\Omega^\rho),$$

we find that

$$g_1(R) \equiv \int_{\partial B_R} \boldsymbol{u} \cdot \frac{\partial \boldsymbol{u}}{\partial n} \in L^1(\rho, \infty). \tag{4.3}$$

Furthermore, recalling that $\boldsymbol{v} \in L^\infty(\Omega^\rho)$, by Young's inequality we obtain

$$R^{-\alpha} g_2(R) \equiv R^{-\alpha} \int_{\partial B_R} |\boldsymbol{u}^2 \boldsymbol{v} \cdot \boldsymbol{n}| \leqq c_1 \left\{ R^{-\alpha q' + 1} + \int_{\partial B_R} \boldsymbol{u}^{2q} \right\}$$

for $\alpha > 0$, and so, by Theorem 3.1,

$$R^{-\alpha} g_2(R) \in L^1(\rho, \infty) \quad \text{for all } \alpha > \tfrac{2}{3}. \tag{4.4}$$

Finally, again by Young's inequality,

$$R^{-\alpha} g_3(R) \equiv R^{-\alpha} \int_{\partial B_R} |p \boldsymbol{u} \cdot \boldsymbol{n}| \leqq c_2 \left\{ R^{-\alpha s' + 1} + \int_{\partial B_R} |p|^s |\boldsymbol{u}|^s \right\}.$$

Since, by Theorem 3.1, $p\boldsymbol{u} \in L^s(\Omega^\rho)$ for all $s > \frac{6}{5}$, we deduce that[13]

$$R^{-\alpha} g_3(R) \in L^1(\rho, \infty) \quad \text{for all } \alpha > \tfrac{1}{3}. \tag{4.5}$$

Observing that

$$R^{-\alpha} G(R) \leqq R^{-\alpha}(g_1(R) + g_2(R) + g_3(R)),$$

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[13] We may take, without loss of generality, the constant p_0 in Theorem 3.1 to be zero.

from (4.3)–(4.5) we find that

$$R^{-\alpha}G(R) \in L^1(\rho, \infty) \quad \text{for all } \alpha > \tfrac{2}{3}$$

and since

$$G'(R) = -\int_{\partial B_R} \nabla \boldsymbol{u} : \nabla \boldsymbol{u} < 0,$$

we conclude from the identity

$$R^{1-\alpha}G(R) = \rho^{1-\alpha}G(\rho) + \int_\rho^R \frac{d}{dr}(r^{1-\alpha}G(r))\,dr$$

that

$$G(R) \leqq cR^{-1+\alpha},$$

which proves the lemma.

Our next task is to show that all solutions satisfying the assumptions of Lemma 4.1 are physically reasonable in the sense of FINN. To this end, we recall that every solution of Lemma 4.1 satisfies the asymptotic representation (*cf. e.g.*, GALDI (1994b, p. 211))

$$u_j(x) = \mathscr{M}_i E_{ij}(x) + \int_{\tilde{\Omega}} E_{ij}(x-y)u_l(y)D_l u_i(y)\,dy + s_j(x) \tag{4.6}$$

where

$$\mathscr{M}_i = -\int_{\partial\tilde{\Omega}} [T_{il}(\boldsymbol{u}, p) - \delta_{1l}u_i]n_l,$$

$\tilde{\Omega}$ is the complement of some closed ball which contains Ω^c and the support of $\boldsymbol{f}$, $\boldsymbol{T}$ is the stress tensor associated with $\boldsymbol{u}$, p, and $\boldsymbol{E}$ is the Oseen fundamental tensor. Finally, $\boldsymbol{s}$ satisfies

$$D^\alpha \boldsymbol{s}(x) = O(|x|^{-(2+|\alpha|)/2}).$$

Lemma 4.2. *Let $\boldsymbol{v}$ satisfy the assumptions of Lemma 4.1. Then, for all large $|x|$,*

$$\boldsymbol{u}(x) = O(|x|^{-1/4-\eta}) \quad \textit{for some } \eta > 0.$$

Proof. In view of (4.6) and of the bound (2.4), to show the result it is enough to prove that

$$N_j(x) \equiv \int_{\tilde{\Omega}} E_{ij}(x-y)u_l(y)D_l u_i(y)\,dy = O(|x|^{-1/4-\eta}). \tag{4.7}$$

To this end, setting $|x| = 2R$ (sufficiently large), we split $\boldsymbol{N}$ thus:

$$\begin{aligned} N_j(x) &= \int_{\tilde{\Omega}_R} E_{ij}(x-y)u_l(y)D_l u_i(y)\,dy + \int_{\tilde{\Omega}^R} E_{ij}(x-y)u_l(y)D_l u_i(y)\,dy \\ &\equiv N_j^{(1)} + N_j^{(2)}. \end{aligned} \tag{4.8}$$

Since $|x-y| \geqq R = |x|/2$ for $y \in \tilde{\Omega}_R$, by (2.4) we find that

$$|N_j^{(1)}| \leqq \frac{c}{|x|^{1/2}} \int_{\tilde{\Omega}_R} |\boldsymbol{u} \cdot \nabla \boldsymbol{u}|.$$

Therefore, taking into account that $\boldsymbol{u} \cdot \nabla \boldsymbol{u} \in L^1(\tilde{\Omega})$ by Theorem 3.1 we conclude that

$$|N_j^{(1)}| \leqq \frac{c_1}{|x|^{1/2}}. \tag{4.9}$$

By the Hölder inequality we obtain

$$|N_j^{(2)}| \leqq \left(\int_{\tilde{\Omega}^R} |\boldsymbol{u}|^s\right)^{1/s} \left(\int_{\tilde{\Omega}^R} |\boldsymbol{E}|^q\right)^{1/q} \left(\int_{\tilde{\Omega}^R} |\nabla \boldsymbol{u}|^2\right)^{1/2} \tag{4.10}$$

where

$$\frac{1}{s} + \frac{1}{q} = \frac{1}{2}. \tag{4.11}$$

Choosing, for instance, $s = q = 4$ and using Theorem 3.1, Lemma 4.1 and (2.4) we deduce that

$$|N_j^{(2)}| \leqq \frac{c_2}{|x|^\gamma}$$

for arbitrary $\gamma < \frac{1}{6}$. From this condition, (4.6), (4.8) and (4.9) we thus obtain

$$|\boldsymbol{u}(x)| \leqq \frac{c_3}{|x|^\gamma} \quad \text{for arbitrary } \gamma < \tfrac{1}{6}. \tag{4.12}$$

We now use (4.12) to improve the uniform bound on $\boldsymbol{N}^{(2)}$. To this end, we observe that, by (2.5), we can take the exponent q in (4.11) to be any number greater than 3 which, in turn, by Theorem 3.1, implies that we can choose s arbitrarily in the interval $(3, 6)$. Thus, writing

$$|\boldsymbol{u}|^s = |\boldsymbol{u}|^{6-\varepsilon-\sigma} |\boldsymbol{u}|^\sigma, \tag{4.13}$$

with arbitrary small positive ε, and with σ arbitrarily close to $3-\varepsilon$, from Lemma 4.1, (4.10) and (4.12) we deduce that

$$|N_j^{(2)}| \leqq c_2 \frac{1}{|x|^\gamma} \frac{1}{|x|^{\gamma\sigma/s}} = \frac{c_2}{|x|^{\gamma(1+\sigma/s)}}.$$

Since σ/s can be taken arbitrarily close to $\frac{1}{2}$, this latter estimate yields

$$|N_j^{(2)}| \leqq \frac{c_2}{|x|^{\gamma_1}},$$

for arbitrary $\gamma_1 < \frac{1}{4}$. This condition together with (4.6), (4.8) and (4.9) then gives

$$|\boldsymbol{u}(x)| \leqq \frac{c_3}{|x|^{\gamma_1}}, \quad \text{for arbitrary } \gamma_1 < \tfrac{1}{4}. \tag{4.14}$$

Using this estimate, we can give a further improvement on the bound for $N^{(2)}$ which then leads to (4.7). We again use (4.10), (4.13) and (4.14) to deduce that

$$|N_j^{(2)}| \leqq c_2 \frac{1}{|x|^{\gamma}} \frac{1}{|x|^{\gamma_1 \sigma/s}}.$$

Thus, recalling the properties of γ, σ/s and γ_1 we conclude that

$$|N_j^{(2)}| \leqq c_3 \frac{1}{|x|^{\gamma_2}}$$

for some $\gamma_2 > \frac{1}{4}$, which together with (4.9) and (4.7) implies (4.6), and the lemma is proved.

Lemma 4.2 in conjunction with the result of SMITH (1965) leads at once to the following main theorem.

Theorem 4.1. *Let $\boldsymbol{v}$ be a solution to* (1.1), *with $\boldsymbol{f}$ of bounded support,*[14] *that satisfies the assumptions of Theorem 3.1. Then, there exists a constant vector $\boldsymbol{m}$ such that*

$$\boldsymbol{v}(x) = \boldsymbol{e}_1 + \boldsymbol{m} \cdot \boldsymbol{E}(x) + O(|x|^{-1} \log^2 |x|)$$

for all sufficiently large $|x|$.

Acknowledgments. This work reported here was done in October 1993, while G. P. GALDI was visiting the Department of Mathematics of the University of Paderborn with a Deutsche Forschungsgemeinschaft (DFG) contract. Support is gratefully acknowledged.

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[14] See footnote 1.

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(Accepted October 11, 1994)