FLOW THROUGH A POROUS ANNULUS

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Summary

The problem of laminar flow through a porous annulus with constant velocity of suction at the walls and with swirl is reduced to the solution of four non-linear differential equations. The significance of each of these equations is discussed. By taking the swirl to be zero series solutions are obtained for (i) small suction or blowing (ii) when the total flow into the channel through the walls is small. Finally the asymptotic behaviour of the flow for large suction or blowing is discussed.

§ 1. *Introduction.* During the past few years a number of solutions for steady laminar flow in porous channels have appeared in the literature. The first solution for laminar flow in a uniformly porous channel was given by Berman[1]); he showed that a solution for the flow between porous parallel plates with constant and equal suction at both walls could be obtained by assuming that the velocity component normal to the walls was independent of the distance along the channel. The behaviour of the resultant flow depends on a parameter R, called the suction Reynolds number. Several series solutions of this problem can be obtained depending on whether R is large or small, positive or negative; a full discussion of these solutions and references to other work is given in Terrill[2,3]). The important characteristic of the solutions is that they appear to be well-behaved for all values of R.

If, in a two-dimensional channel, the velocities of suction or blowing at the wall are not equal a number of possible asymptotic solutions arise depending on the relative magnitude of the wall velocities. Proudman[4]) has given a complete discussion of the nature of the asymptotic solution for this flow.

A solution for the flow in a porous pipe in which there is constant suction or blowing at the wall has been given by Yuan and Finklestein[5]). They obtained series solutions for large negative R corresponding to large injection and for small R. The solution for small positive R corresponding to small suction at the wall converges slowly and therefore White[6]) made a numerical investigation of the solution for the case of suction at both walls. White showed that for positive R there are either dual solutions or no solutions, which indicated that the flow through a porous tube would break down for small suction Reynolds numbers. To try to obtain solutions for all positive R Prager[7]) introduced a non-zero swirl velocity and succeeded in obtaining numerical solutions in the range of suction Reynolds numbers for which White could not obtain a numerical solution. Although no analytic investigation has yet been made it seems probable that to obtain a solution that does not break down for flow in a porous pipe when there is suction at the walls, swirl will have to be included.

An interesting approximate solution for flow in a porous wedge or cone has been given by G. I. Taylor[8]). Although Taylor's assumptions*) on the boundary condition normal to the wall are different from those taken in the above papers his solutions exhibit many of the same features. For instance, for large blowing at both walls his solution indicates a discontinuity at the centre of the wedge corresponding to a viscous layer.

As far as the author is aware the only solution for flow in a porous annulus has been given by Berman[9]). This solution is the simple case where the amount of fluid entering through the outer wall is equal to the amount of fluid leaving through the inner wall. The flow in an annulus is particularly interesting in that, as the radius ratio (the ratio of the inner radius to the outer radius of the annulus) tends to one in the correct way the solution should reduce to the well-behaved solution in a two-dimensional channel whereas for small values of radius ratio and an impermeable inner wall the flow should behave more like the flow in a porous pipe (see § 3).

§ 2. *Equations of motion.* The flow to be studied is steady incom-

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*) G. I. Taylor takes the wedge to be made of a porous material, the resistance of which is proportional to the square of the velocity through it.

pressible laminar flow through an annulus. Choose a cylindrical polar co-ordinate system (r, θ, z) where the axis Oz lies along the centre of the annulus. Let v_r, v_θ, and v_z be the velocity components in the directions of r, θ and z increasing respectively. Then if the resultant flow is assumed to be independent of θ the equation of continuity is

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{\partial v_z}{\partial z} = 0 \tag{1}$$

and the equations of momentum are

$$v_r\frac{\partial v_r}{\partial r} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 v_r - \frac{v_r}{r^2}\right), \tag{2}$$

$$v_r\frac{\partial v_\theta}{\partial r} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} = \nu\left(\nabla^2 v_\theta - \frac{v_\theta}{r^2}\right), \tag{3}$$

$$v_r\frac{\partial v_z}{\partial r} + v_z\frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 v_z, \tag{4}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

and where p is the pressure, ρ the density and ν is the kinematic viscosity. Let the inner and outer walls of the annulus be given by $r = a$ and $r = b$ respectively. The boundary conditions are the no-slip conditions at the walls and the constant velocity of injection at the walls so that

$$\begin{array}{lll} \text{at } r = a & v_r = -V_1, \; v_\theta = 0, \; v_z = 0 & \\ \text{and at } r = b & v_r = V_2, \; v_\theta = 0, \; v_z = 0 & (b > a) \end{array} \tag{5}$$

The above boundary conditions imply that fluid is being extracted with velocity V_1 at $r = a$ and extracted with velocity V_2 at $r = b$.

To try to obtain a solution of equations (1)–(4) subject to boundary conditions (5) let us assume that the radial velocity compont is a function of r alone; then we may write v_r in the form

$$v_r = bV\frac{f(r)}{r}, \tag{6}$$

where $f(r)$ is a non-dimensional function. The outer radius of the annulus b has been taken to be a typical dimension of the system and V to be a typical velocity. The precise form of V will be discussed later but it will be based on the velocities of suction at the wall V_1 and V_2 and not the mainstream velocity in the z-direction. The reason for this choice is that it is the cross-velocity which produces the change from steady flow in the z-direction and is the significant velocity in the production of viscous layers in the flow.

The equation of continuity (1) yields

$$v_z = -bV \frac{f'(r)}{r} z + Vh(r), \tag{7}$$

where $h(r)$ is an arbitrary non-dimensional function of r. If the above expressions for v_r and v_z are substituted into the equation of motion (4) it immediately follows that the kinematic pressure is given by

$$-\frac{p}{\rho} = P(r)\, z^2 + Q(r)\, z + R(r), \tag{8}$$

where $P(r)$, $Q(r)$ and $R(r)$ are arbitrary functions of r to be determined. For (2) to be consistent with (6), (7) and (8) the swirl velocity must be of the form

$$rv_\theta = Vg(r)\, z + Vbk(r), \tag{9}$$

where $g(r)$ and $k(r)$ are arbitrary non-dimensional functions of r. If (2) is differentiated with respect to z we obtain

$$-\frac{1}{\rho} \frac{\partial^2 p}{\partial r \partial z} = -\frac{2V^2 g}{r^3} (gz + bk). \tag{10}$$

In particular it may be noted that if there is no swirl the righthand of (10) will be identically zero. If we use the relationship (10) to eliminate the pressure p from equation (4) equating the $C(z)$ coefficients of z to zero yields

$$r^2\{f'f'' - ff'''\} + r\{3ff'' - f'^2\} - 3ff' = \\ = -2rg^2/b^2 + \nu/Vb\{-r^3f^{\text{iv}} + 2r^2f''' - 3rf'' + 3f'\}. \tag{11}$$

The above equation for $f(r)$ can be considerably simplified by introducing the non-dimensional independent variable η defined by

$$\eta = r^2/b^2. \tag{12}$$

If we write the functions occurring in the velocity components as

$$\begin{aligned} f(r) &= F(\eta) \qquad g(r) = G(\eta) \\ h(r) &= H(\eta) \qquad k(r) = K(\eta) \end{aligned} \tag{13}$$

then equation (11) reduces to

$$(\eta F^{iv} + 2F''') + R(F'F'' - FF''') + \tfrac{1}{4}RG^2/\eta^2 = 0, \tag{14}$$

where the prime denotes differentiation with respect to η and where $R = Vb/2\nu$ is a Reynolds number. The boundary conditions which $F(\eta)$ must satisfy are

$$\begin{aligned} F(1) &= V_2/V, \quad F(\eta_0) = -V_1\eta_0^{\frac{1}{2}}/V, \\ F'(1) &= 0, \qquad F'(\eta_0) = 0, \end{aligned} \tag{15}$$

where

$$\eta_0 = a^2/b^2. \tag{16}$$

The above equation for $F(\eta)$ is the most important equation in the problem being the only equation with non-zero boundary conditions.

In a similar way it can be shown that $H(\eta)$ satisfies the non-linear differential equation

$$\eta H''' + 2H'' + R(HF'' - H''F) + \frac{RKG}{2\eta^2} = 0 \tag{17}$$

and the boundary conditions

$$H(\eta_0) = H(1) = 0 \ . \tag{18}$$

The absence of a third boundary condition for this third order differential equation is not of great significance. If $R = 0$ then the solution of (17) would be the usual solution for laminar incompressible flow through an impermeable annulus and the third boundary condition would be given by the total fluid crossing a section. The third boundary condition on equation (17) may be regarded in the same way. (It should be noted that this is different to the solution of (14) which, having four boundary conditions, actually gives the quantity of fluid crossing a section).

The equations for $G(\eta)$ and $K(\eta)$ can be obtained by substituting for v_θ from (9) and (13) into (3) and equating the coefficients of z and z^0 respectively. Hence $G(\eta)$ satisfies

$$\eta G'' = R(FG' - F'G) \tag{19}$$

and the boundary conditions

$$G(\eta_0) = G(1) = 0, \tag{20}$$

and $K(\eta)$ satisfies

$$\eta K'' = R(FK' + \tfrac{1}{2}HG) \tag{21}$$

and the boundary conditions

$$K(\eta_0) = K(1) = 0. \tag{22}$$

With the assumptions that the resultant flow is independent of θ and that v_r is a function of r alone, the problem has now reduced to the solution of four non-linear differential equations for $F(\eta)$, $G(\eta)$, $H(\eta)$ and $K(\eta)$. The most important differential equation is the equation for $F(\eta)$ since to obtain a solution representing flow through a porous annulus it is necessary for $F \neq 0$ whereas the solutions for $G(\eta)$, $H(\eta)$ and $K(\eta)$ may be taken to be identically zero.

We can also observe that the equations for $F(\eta)$ and $G(\eta)$ do not involve $H(\eta)$ and $K(\eta)$. Thus in looking for a solution with or without swirl the major question is to solve (14) and (19) and in doing this we may take $H(\eta) \equiv K(\eta) \equiv 0$. The equations for $H(\eta)$ and $K(\eta)$ are subsidiary equations in which, having found a solution to the problem, we look to see if there are any eigensolutions. It will make the discussion of the solutions of the problem easier if no eigensolutions exist since the question of which solution the real flow tends to far downstream does not then arise. In Terrill[11]) the possibility of eigensolutions for laminar flow through a porous channel with equally porous walls was examined and it was found that none existed in that particular case. For the present it is convenient to assume that $H(\eta)$ *) and $K(\eta)$ are identically zero.

We now turn our attention to $G(\eta)$. It has already been pointed out that for the flow through a porous pipe White[6]) could not obtain a solution of his equation satisfying the boundary conditions for a certain range of suction Reynold's number but Prager[7]) was able to obtain some numerical solutions in that range by including the swirl term. The flow through an annulus will behave like the flow through a pipe when the ratio of the inner radius a to the

*) In fact in § 3 we will take $H(\eta) \propto F'(\eta)$ but the question of eigensolutions will be ignored.

outer radius b of the annulus is small, that is, when η_0 is small. Thus for η_0 small we expect that to obtain a solution it may be necessary to include a swirl term. Also it seems possible that the swirl term will introduce further solutions in the range of Reynolds number in which there is already a solution. Before we attempt to answer these questions we should try to find some solutions for $F(\eta)$ and so in the following sections a discussion of certain solutions of (14) is undertaken. A particular case of interest is when $F(\eta) \equiv 0$ everywhere which gives $G(\eta) = K(\eta) = 0$ so that we are not able to find a solutions for flow through an impermeable annulus with swirl.

§ 3. *Series solution for small suction or blowing.* In the following sections we will assume $G(\eta) = K(\eta) \equiv 0$. If $G = 0$ a particular solution of (17) with boundary conditions (18) is $H(\eta) \propto F'(\eta)$ and this is the only non-zero $H(\eta)$ which will be considered. Thus the solutions considered in this and subsequent sections is equivalent to assuming that the velocity components take the form

$$v_r = \frac{VF(\eta)}{\sqrt{\eta}}, \quad v_\theta = 0, \quad v_z = \left[U(0) - 2V\frac{z}{b}\right]F'(\eta), \tag{23}$$

where $U(0)$ is an arbitrary constant. (If the suction commences at $z = 0$ then $U(0)\, F'(\eta)$ can be interpreted as the velocity profile at $z = 0$ corresponding to flow through an annulus with impermeable walls.)

Before we obtain some solutions for $F(\eta)$ the choice of the typical velocity V for the Reynolds number will have to be considered. In previous papers on flow through porous channels and pipes it has been the convention to take the Reynolds number positive when fluid is extracted from the channel and negative when fluid is injected and we will follow this system. The most obvious choice of a typical velocity would be given by $bV = bV_2 + aV_1$ so that the Reynolds number would then be based on the total fluid entering the annulus through the walls. However if V_2 and V_1 are opposite in sign then bV would not be typical of the cross-flow; for instance, there could be a boundary layer on one wall although bV was small. This is the main reason for rejecting the above method and selecting the typical velocity in such a way that the magnitude of one of the boundary conditions is unity and the magnitude of the other is less than one. The problem is divided into two cases depending on whether $|bV_2| \geqslant |aV_1|$ or $|bV_2| \leqslant |aV_1|$ as follows

(i) $|bV_2| \geqslant |aV_1|$. Choose $V = V_2$ so that $R_2 = V_2 b/2\nu$; the equation for $F(\eta)$ becomes

$$\eta F^{iv} + 2F''' + R_2(F'F'' - FF''') = 0 \tag{24}$$

and the boundary conditions (15) take the form

$$\begin{aligned} F(\eta_0) &= -\frac{V_1}{V_2}\eta_0^{\frac{1}{2}} = -\alpha; \quad & F(1) &= 1, \\ F'(\eta_0) &= 0; & F'(1) &= 0, \end{aligned} \tag{25}$$

where $|\alpha| \leqslant 1$. It should be noted that at the outer wall suction and blowing correspond to $R_2 > 0$ and to $R_2 < 0$ respectively.

(ii) $|aV_1| \geqslant |bV_2|$. Take $V = V_1\eta_0^{\frac{1}{2}}$ so that $R_1 = V_1 a/2\nu$; the equation for $F(\eta)$ is

$$\eta F^{iv} + 2F''' + R_1(F'F'' - FF''') = 0 \tag{26}$$

with boundary conditions

$$\begin{aligned} F(\eta_0) &= -1; \quad & F(1) &= V_2\eta_0^{-\frac{1}{2}}/V_1 = \beta, \\ F'(\eta_0) &= 0; & F'(1) &= 0, \end{aligned} \tag{27}$$

where $|\beta| \leqslant 1$. It may be noted that suction and blowing at the inner wall correspond to $R_1 > 0$ and to $R_1 < 0$ respectively.

The solution of cases (i) and (ii) can thus be obtained by solving the equation

$$\eta F''' + F'' + R(F'^2 - FF'') = c \tag{28}$$

where c is a constant, subject to the boundary conditions

$$\begin{aligned} F(\eta_0) &= -\alpha; \quad & F(1) &= \beta, \\ F'(\eta_0) &= 0; & F'(1) &= 0. \end{aligned} \tag{29}$$

Then case (i) is given by $R = R_2$, $\beta = 1$ and case (ii) is given by $R = R_1$, $\alpha = 1$.

It is fairly obvious that if we let $\eta_0 \to 1$ in the correct way then (28) and (29) reduce to the flow through porous parallel plates. (See § 3 (b)). On the other hand the suggestion that as $\eta_0 \to 0$ in the correct way the flow behaves like the flow through a porous pipe seems to conflict with the boundary condition $F'(\eta_0) = 0$. However suppose we take $\alpha = 0$, $\beta = 1$, $\eta_0 = 0$ and exclude the boundary condition $F'(\eta_0) = 0$ in conditions (29). Then the solution of (28) satisfying these revised conditions is the flow through a porous pipe. Since such a solution yields $F'(0)$ non-zero it seems that there is a line discontinuity at $\eta = 0$. Further investigation of (28) and (29) near $\eta = \eta_0$ shows that

$$F'(\eta) \simeq A\left[\frac{\log \eta}{\log \eta_0} - 1\right],$$

where A is a constant and η and η_0 are small. Thus there is a narrow layer in which there is a sharp transition in $F'(\eta)$ corresponding to an inner solution and as $\eta_0 \to 0$ the thickness of this layer tends to zero. The flow through a porous pipe corresponds to an outer solution. The previous reasoning is confirmed by the behaviour of the series solution for small R (§ 3 (b)) and for large negative R (§ 5.1) as $\eta_0 \to 0$.

For small suction Reynolds numbers a solution of the form

$$F(\eta) = \sum_{r=0}^{\infty} F_r(\eta)\, R^r, \quad c = \sum_{r=0}^{\infty} c_r R^r, \tag{30}$$

where $F_r(\eta)$, c_r are independent of R, is sought. The equation for $F_0(\eta)$ is

$$\eta F_0''' + F_0'' = c_0 \tag{31}$$

with boundary conditions

$$\begin{aligned} F_0(\eta_0) &= -\alpha; \quad F_0'(\eta_0) = 0, \\ F_0(1) &= \beta; \quad F_0'(1) = 0. \end{aligned} \tag{32}$$

The solution of (31) subject to boundary conditions (32) is

$$F_0(\eta) = \beta + c_0\left\{\frac{(1-\eta)^2}{2} + \frac{(1-\eta_0)}{\log \eta_0}(\eta \log \eta + 1 - \eta)\right\}, \tag{33}$$

where c_0 is given by

$$\left[\frac{(1-\eta_0)^2}{\log \eta_0} + \frac{1-\eta_0^2}{2}\right] c_0 = -(\alpha + \beta). \tag{34}$$

As $V \to 0$ and, therefore, $R \to 0$ we expect the solution to reduce to the flow through an impermeable annulus. When $V \to 0$ equation

(23) gives

$$v_r = v_\theta = 0; \quad v_z = U(0)\, F_0'(\eta). \tag{35}$$

By differentiating (33) and substituting for η and η_0, it can be easily shown that

$$v_z = K\left[a^2 - r^2 + \frac{(b^2 - a^2)}{\log(b/a)} \log(r/a)\right], \tag{36}$$

where K satisfies

$$K\left[\frac{(b^2 - a^2)^2}{\log(a/b)} + b^4 - a^4\right] = 2(\beta + \alpha)\, U(0). \tag{37}$$

The above solution represents the fully developed flow through an impermeable annulus. If $\bar{U}$ is the average velocity of the fluid at $z = 0$, that is, the amount of fluid entering the channel per unit time is $\pi(b^2 - a^2)\, \bar{U}$, then it can easily be shown that

$$\bar{U}(b^2 - a^2) = U(0)\, b^2(\beta + \alpha).$$

The equation for $F_1(\eta)$ is

$$\eta F_1''' + F_1'' = F_0 F_0'' - F_0'^2 + c_1 =$$

$$= c_1 + \beta\left(c_0 + \frac{d}{\eta}\right) - \frac{c_0^2(\eta - 1)^2}{2} +$$

$$+ d^2\left\{\log\eta - \log^2\eta + \frac{1}{\eta} - 1\right\} + c_0 d\left\{(2 - \eta)\log\eta - \frac{(\eta^2 - 1)}{2\eta}\right\}, \tag{38}$$

where

$$d = \frac{c_0(1 - \eta_0)}{\log\eta_0}$$

and the boundary conditions are

$$F_1(\eta_0) = F_1'(\eta_0) = F_1(1) = F_1'(1) = 0. \tag{39}$$

The solution of (38) subject to boundary conditions (39) is

$$F_1(\eta) = \beta\frac{d}{2}(\eta\log^2\eta) - \frac{c_0^2}{72}(\eta - 1)^2(\eta^2 - 4\eta) +$$

$$+ d^2\{\tfrac{1}{2}(\eta - \eta^2)\log^2\eta + 3\eta^2\log\eta - 3(\eta - 1)\} +$$

$$+ c_0 d \left\{ \tfrac{1}{4}\eta \log^2 \eta - \left(\frac{\eta^3}{12} - \eta^2 \right) \log \eta + \right.$$
$$\left. + \frac{(\eta - 1)(5\eta^2 - 71)}{72} \right\} + \tfrac{1}{2}k(\eta - 1)^2 + \mu\{ \log \eta + 1 - \eta\}, \quad (40)$$

where the constants μ and k are given by

$$2\mu + (1 - \eta_0) \left\{ \beta c_0 + \frac{c_0^2}{2} + \tfrac{1}{18} c_0^2 \frac{(2\eta_0^2 + 11\eta_0 + 20)}{\log \eta_0} + \right.$$
$$\left. + \frac{3(1 - \eta)_0^2}{\log^2 \eta_0} c_0^2 = \frac{5}{4} \frac{(1 - \eta_0)^4}{\log \eta_0} \cdot \right.$$
$$\cdot c_0^2/[2(1 - \eta_0) + (1 + \eta_0) \log \eta_0],$$

$$k = 5(1 - \eta_0)^3 c_0^2/8[2(1 - \eta_0) + (1 + \eta_0) \log \eta_0] +$$
$$+ \beta c_0 + \tfrac{1}{4} c_0^2 (3\eta_0^2 + 2) + \frac{(1 - \eta_0)(243\eta_0 - 107) c_0^2}{72 \log \eta_0} -$$
$$- \frac{3(1 - \eta_0)^2}{\log^2 \eta_0} c_0^2. \quad (41)$$

Clearly to obtain any further terms of the series expansion for small R would be extremely complicated. To make further analytic progress it seems necessary to make some assumptions about the ratio of the inner and outer walls of the annulus. Presumably this would involve either letting η_0 be small and obtaining a solution resembling flow through a porous pipe or letting η_0 be nearly one and obtaining a solution similar to flow between porous parallel plates. We will only consider the limiting solutions and show that they agree with known results.

a. *Flow in a circular pipe.* To obtain fully developed laminar flow through a circular pipe we let $\alpha \to 0$, $\beta \to 1$ and $\eta_0 \to 0$. Equation (34) gives $c_0 \to -2$ and the solutions for $F_0(\eta)$ and $F_1(\eta)$, from (33) and (40) respectively, reduce to

$$F_0(\eta) + RF_1(\eta) = (2\eta - \eta^2) + \frac{R_2}{18} (\eta - 1)^2 (4\eta - \eta^2) \quad (42)$$

where the Reynold's number $R_2 = bV_2/2\nu$ and where $\eta = r^2/b^2$. The above expression was first obtained by Yuan and Finkelstein[5]) for the flow in a circular pipe with constant suction or blowing at the walls.

b. *Flow between parallel plates.* The solution for the flow between porous parallel plates of different permeability which has been given by Terrill and Shrestha[10]) may be obtained by taking $r = a + \varepsilon b\lambda$ and letting $\varepsilon \to 0$. To derive a solution with the notation of Terrill and Shrestha choose $a = b(1 - \varepsilon)$. Hence $\eta = [1 - \varepsilon(1 - \lambda)]^2$ and the points $\eta = \eta_0$ and $\eta = 1$ become $\lambda = 0$ and $\lambda = 1$ respectively.

We will only consider case (i) in which $R = R_2$, $\beta = 1$. Then if we substitute for η_0 in (34) and let $\varepsilon \to 0$ we find that

$$c_0 = -3\alpha_2/2\varepsilon^3,$$

where $\alpha_2 = 1 + \alpha$. By substituting for η and η_0 in (33) and letting $\varepsilon \to 0$, $F_0(\eta)$ is found to be

$$F_0(\eta) = 1 - \alpha_2(1 - \lambda)^2\,(2\lambda + 1), \tag{43}$$

which is identical with the solution given by Terrill and Shrestha[10]) [equation (22)]. Before the solution for $F_1(\eta)$ is obtained the suction Reynolds number appropriate to the flow must be discussed. For the flow between parallel plates the channel width $h = b - a = \varepsilon b$, and, therefore, the Reynolds number $R_2^* = hV_2/\nu$ for the flow through parallel plates is related to Reynolds number R_2 for the annulus by

$$R_2^* = 2\varepsilon R_2.$$

By substituting for η_0 and η in equations (40) and (41) and letting $\varepsilon \to 0$ it can be shown that $F_1(\eta)$ is given by

$$F_1(\eta) = 2\varepsilon\alpha_2\lambda^2(1 - \lambda)^2 \cdot \\ \cdot\left\{-\frac{1}{2} - \frac{\alpha_2}{70}\lambda^2(1 - \lambda)^2\,(4\lambda^3 - 6\lambda^2 + 5\lambda - 19)\right\}, \tag{44}$$

which is the expression obtained by Terrill and Shrestha[10]) (equation (25)). Thus the first two terms of the series for $|V_2| \gg |V_1|$ give

$$F(\eta) = F_0 + R_2F_1(\eta).$$

The series solution for $|V_1| \gg |V_2|$ can be obtained by a similar method.

§ 4. *A solution when $\alpha + \beta$ is nearly zero.* When the quantity of fluid entering through the outer wall is nearly equal to the quantity of fluid being extracted through the inner wall a solution to (28) can be readily found. For this distribution of wall velocities it is unimportant whether we choose R_1 or R_2 to be the suction Reynolds number. Let R_2 be the typical Reynolds number and take $\alpha = -1 + \varepsilon$ where ε is a small quantity which may be negative or positive; the solution to both cases is given by solving

$$\eta F''' + F'' + R_2(F'^2 - FF'') = c \tag{45}$$

subject to boundary conditions

$$\begin{aligned} F(\eta_0) &= 1 - \varepsilon; \quad & F(1) &= 1, \\ F'(\eta_0) &= 0; & F'(1) &= 0. \end{aligned} \tag{46}$$

We look for a solution of the form

$$F(\eta) = 1 + \sum_{r=1}^{\infty} \varepsilon^r F_r(\eta); \quad c = \sum_{r=0}^{\infty} c_r \varepsilon^r, \tag{47}$$

where $F_r(\eta)$ and c_r are independent of ε. The equation for $F_1(\eta)$ is

$$\eta F_1''' - (R_2 - 1)\, F_1'' = c_1 \tag{48}$$

and the required solution satisfying the boundary conditions is

$$F_1(\eta) = K\left\{-\left(\frac{1-\eta}{2}\right)^2 - \left(\frac{1-\eta_0}{1-\eta_0^{R_2}}\right)\left(\eta - 1 - \left[\frac{\eta^{R_2+1}-1}{R_2+1}\right]\right)\right\}, \tag{49}$$

where the constant K is given by

$$\frac{K(1-\eta_0)}{2(R_2+1)}\left\{(R_2+1) - (R_2-1)\,\eta_0 - \frac{2R_2(1-\eta_0)}{1-\eta_0^{R_2}}\right\} = 1. \tag{50}$$

The special cases $R_2 = -1, 0, 1$ can be obtained by finding the limits of (49) and (50) as R_2 approaches the appropriate value. For instance, the limit as R_2 tends to one gives

$$F_1(\eta) = K_1\left\{\frac{\eta_0 \log \eta_0}{1-\eta_0}\,(\eta-1)^2 + \eta^2 \log \eta - \tfrac{1}{2}(\eta^2 - 1)\right\},$$

where

$$K_1\{\tfrac{1}{2}(\eta_0^2 - 1) - \eta_0 \log \eta_0\} = 1.$$

The solutions for the successive terms of the series $F_2(\eta)$, $F_3(\eta)$ ... can be obtained in a similar way.

A particularly interesting solution is when the amount of fluid entering through the outer wall is exactly equal to the amount of fluid leaving through the inner wall, that is, when $\varepsilon \to 0$. From (23) the velocity components are

$$V_r = \frac{V_2}{\sqrt{\eta}} = \frac{V_2 b}{r}; \quad V_\theta = 0, \qquad (51)$$
$$V_z = [U(0) - 2V_z/b]\, F_1'(\eta),$$

where

$$F_1'(\eta) = K\left\{(1-\eta) - \frac{(1-\eta_0)}{1-\eta_0^{R_2}}\,(1-\eta^{R_2})\right\} \qquad (52)$$

and K is given by (50). The above solution was first given by Berman [9]).

In certain cases the flow given by (51) will have inflexion points in the axial profile and these indicate that the larger is R, the sooner will the flow become unstable. To obtain points of inflexion η we put the second derivative with respect to r of the axial velocity equal to zero so that

$$\eta^{R_2-1} = \frac{1-\eta_0^{R_2}}{(1-\eta_0)\,R_2(2R_2-1)} \qquad \eta_0 \leqslant \eta \leqslant 1.$$

Therefore, for a point of inflexion to lie in the range $[\eta_0, 1]$ it is necessary for

$$\eta_0^{R_2-1} \leqslant \frac{1-\eta_0^{R_2}}{(1-\eta_0)(R_2)(2R_2-1)} \leqslant 1.$$

Thus, it may be immediately concluded that for $R_2 < 1$ there are no inflexion points in the flow. For $R_2 > 1$ the existence of points of inflexion in the flow depends on the values of η_0 and R_2; Berman gives a graph showing the values of (η_0, R_2) for which they occur. In particular he shows that the smaller the value of η_0, the larger the range of values of R_2 for which inflexion points occur. Hence, we may expect that the closer the annulus approaches to being parallel plates the more likely the flow will be well-behaved.

§ 5. *The asymptotic behaviour of solutions for large R.* The derivation of a series solution for large Reynolds numbers will be involved since it will require the use of inner expansions in the viscous layers and outer expansions in the remainder of the flow region. For example, a solution for the flow between parallel porous plates for large negative Reynolds numbers has been given by Terrill[2]) by matching inner and outer expansions.

In this section we will only obtain the limiting forms of the solutions. Proudman[4]) discusses a similar problem for flow in a porous two-dimensional channel and, by considering the positions of the viscous layers, obtains asymptotic solutions for large Reynolds numbers. For the flow in an annulus the arguments about the position of the viscous layers are almost identical with those given by Proudman and therefore only the asymptotic solutions will be given.

5.1. *The case of large blowing at both walls.* For large blowing there cannot be a boundary layer on either wall so that limiting inviscid solution must satisfy all four boundary conditions. Then as $R \to -\infty$ equation (28) yields

$$F'^2 - FF'' = K, \tag{53}$$

where K is a constant and the boundary conditions (29) remain as

$$\begin{aligned} F(\eta_0) &= -\alpha; & F(1) &= \beta, \\ F'(\eta_0) &= 0; & F'(1) &= 0. \end{aligned} \tag{54}$$

At the point where $F(\eta)$ vanishes there will be a viscous layer; let this point be $\eta = \eta^*$. Then the required solution of (53) and (54) is

$$\begin{aligned} F(\eta) &= -\alpha \sin \frac{\pi}{2}\left(\frac{\eta - \eta^*}{\eta^* - \eta_0}\right), & \eta < \eta^*, \\ &= \beta \sin \frac{\pi}{2}\left(\frac{\eta - \eta^*}{1 - \eta^*}\right), & \eta > \eta^*, \end{aligned} \tag{55}$$

where η^* is given by

$$\eta^* = \frac{\alpha + \beta\eta_0}{\alpha + \beta}. \tag{56}$$

The solution (55) has a viscous layer at $\eta = \eta^*$ if $F'''(\eta^*)$ is discontinuous, that is, when

$$\alpha(1 - \eta^*)^3 \neq \beta(\eta_0 - \eta^*)^3. \tag{57}$$

To find the structure of the viscous layer at η^* when condition (57) is satisfied we write

$$F(\eta) = A\varepsilon + F_1(\varepsilon),$$

where

$$A = \frac{\alpha\pi}{2(\eta_0 - \eta^*)} = \frac{\beta\pi}{2(1 - \eta^*)} \quad \text{and} \quad \varepsilon = \eta - \eta^*. \tag{58}$$

By following an identical argument to that given by Proudman[4]) (Section 3.6) we find that $F_1(\varepsilon)$ satisfies

$$(\eta^* + \varepsilon)\, F_1'''' + F_1''' + R\{2AF_1'' - A\varepsilon F_1'''\} = 0 \qquad (\eta^* \neq 0)$$

so that for the viscous and inertia terms to be of the same order $\varepsilon = O(R^{-\frac{1}{2}})$ and we find that $F_1'''(\varepsilon)$ is given by

$$F_1'''(\varepsilon) = \alpha_1 \int\limits_0^{\varepsilon R^{\frac{1}{2}}} \exp\left\{\frac{1}{2}\,\frac{A\theta^2}{\eta^*}\right\} d\theta + \beta_1, \tag{59}$$

where α_1 and β_1 are constants of integration. The constants α_1 and β_1 are chosen by making $F'''(\varepsilon)$ continuous at the boundary. Hence the solution (59) gives the behaviour in the viscous layer.

Suppose that $F'''(\eta^*)$ is continuous for the solution (55). It can be shown that the terms in (57) can be equal if

(a) $\eta_0 = \eta^* = 1$

(b) $\alpha = 1,\ \beta = 0, \eta^* = 1$

(c) $\alpha = 0,\ \beta = 1,\ \eta^* = \eta_0$.

By taking the correct limit for $\eta_0 \to 1$, as in § 3, case (a) can be shown to reduce to flow through parallel plates. This flow has been discussed by Proudman[4]) who shows that there is always a viscous layer present; in particular when the velocities of injection at the wall are equal then F''' is continuous but there is still a weak viscous layer at the centre of the channel. A series solution for this case has been given by Terrill[2]).

In case (b) the solution (55) reduces to

$$F(\eta) = \sin\frac{\pi}{2}\left(\frac{1-\eta}{1-\eta_0}\right)$$

where $\eta < 1$. All the boundary conditions except the no-slip condition at $\eta = 1$ are satisfied. Since there is a discontinuity in

$F'(\eta)$ at $\eta = 1$ there must be a shear layer at the outer wall. However if the outer wall is allowed to move in the z direction with a velocity whch makes $F'(1) = -\pi/[2(1 - \eta_0)]$ then there will not be a shear layer at the outer wall.

For case (c) the solution (55) becomes

$$F(\eta) = \sin \frac{\pi}{2}\left(\frac{\eta - \eta_0}{1 - \eta_0}\right)$$

where $\eta > \eta_0$ and since the no-slip condition is not satisfied at the inner wall, there must be a shear layer at $\eta = \eta_0$. If the inner wall is allowed to move in such a way that $F'(\eta_0) = \pi/[2(1 - \eta_0)]$ this shear layer will be absent. In particular if we let $\eta_0 \to 0$ and remove the inner wall then the flow through a porous pipe is obtained.

Thus to find the flow through an annulus (with fixed walls) for large injection the behaviour of the viscous layer as well as the outer solution has to be considered except in the particular case where the annulus reduces to the pipe. The solution for the flow through a porous pipe for large injection has been given by Yuan and Finkelstein[5]). In particular it is interesting that for symmetric flow through porous parallel plates a solution for the viscous layer is required whereas this is not necessary for flow in porous pipe.

5.2. *The case of large suction at both walls.* For large suction we expect there to be viscous layers present at each wall and, for R sufficiently large, an inviscid region outside these layers. The solution in the inviscid region has only to satisfy the radial velocity boundary conditions so that

$$F(\eta) = \frac{(\eta - 1)\,\alpha + \beta(\eta - \eta_0)}{1 - \eta_0} \tag{60}$$

is the required solution outside the viscous layers.

In the neighbourhood of the wall $\eta = \eta_0$, by assuming a thin viscous layer, (28) can be replaced by

$$\eta F''' + (R\alpha + 1)\, F'' = c.$$

The required solution of the above equation which satisfies the no-slip condition at $\eta = \eta_0$ and tends asymptotically to the outer solution (60) is

$$F'(\eta) = \frac{(\alpha + \beta)}{1 - \eta_0}\left\{1 - \left(\frac{\eta}{\eta_0}\right)^{-\alpha R}\right\}, \tag{61}$$

provided that αR is sufficiently large. In particular, as $\alpha R \to \infty$,

$$F' \to \frac{(\alpha + \beta)}{1 - \eta_0}\left\{1 - \exp\left[-\alpha R\,\frac{(\eta - \eta_0)}{\eta_0}\right]\right\}$$

which is the usual boundary-layer solution.

Similarly at the outer wall the viscous layer solution is given by

$$F'(\eta) = \frac{(\alpha + \beta)}{1 - \eta_0}\{1 - \eta^{\beta R}\}, \tag{62}$$

provided that βR is sufficiently large. As $\beta R \to \infty$,

$$F' \to \frac{(\alpha + \beta)}{1 - \eta_0}\{1 - \exp[-\beta R(1 - \eta)]\}$$

giving a boundary layer solution.

It should be noted that although a solution of the equation of motion behaves asymptotically in the above way it does not follow that the actual flow will. For example, in the flow through a porous pipe there are dual solutions for large Reynolds numbers so that the actual flow breaks down and does not reach the form given by (60), (61) and (62). On the other hand the solution is valid for the flow through porous parallel plates; it is expected that the solution will be valid for a range of values of η_0 but without a numerical investigation it is difficult to see what the limits of the range of validity are.

5.3. *The case of either* $\alpha < 0$, $\beta > 0$ *or* $\alpha > 0$, $\beta < 0$. The flow pattern is given by large blowing at one surface and large suction at the other surface, so that large radial velocities occur everywhere.

Suppose that $\alpha < 0$, $\beta > 0$ so that a boundary layer occurs at the outer wall. Then we require a solution of

$$F'^2 - FF'' = k, \tag{63}$$

where k is a constant, subject to the boundary conditions

$$F(\eta_0) = -\alpha; \quad F'(\eta_0) = 0; \quad F(1) = \beta. \tag{64}$$

(The no-slip condition at $\eta = 1$ has been omitted.) The solution of (63) satisfying the boundary conditions (64) is

$$F(\eta) = -\alpha \cos\left\{\frac{(\eta-\eta_0)}{(1-\eta_0)} \cos^{-1}\left(-\frac{\beta}{\alpha}\right)\right\}; \quad (\beta+\alpha) < 0$$

$$F(\eta) = -\alpha = 1; \qquad \beta+\alpha = 0 \quad (65)$$

$$F(\eta) = -\alpha \cosh\left\{\frac{(\eta-\eta_0)}{(1-\eta_0)} \cosh^{-1}\left(-\frac{\beta}{\alpha}\right)\right\}; \quad \beta+\alpha > 0.$$

In (65) the smallest positive value of $\cos^{-1}(-\beta/\alpha)$ is taken. At $\eta = 1$ there is a suction boundary layer. The above solution is similar to the solution given by Proudman[4]) for flow through porous parallel plates.

When $\beta + \alpha$ is small the solution (65) can readily be shown to be equal to the solution (47) given in § 4 for large R.

The solution for $\alpha > 0$, $\beta < 0$ can be obtained in a similar way to the above solution.

Received 1st April, 1966.

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