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Chromatic Number of Hasse Diagrams, Eyebrows and Dimension

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Abstract. We construct posets of dimension 2 with highly chromatic Hasse diagrams. This solves a previous problem by Nešetřil and Trotter.

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0. Introduction

In a response to a problem by I. Rival, the following theorem was independently proved by Bollobas [1] and Nešetřil and Rödl [4]:

0.1. THEOREM. *For every n there exists a poset P whose Hasse diagram H(P) has chromatic number* $\geqslant n$. ■

The examples constructed in [1], [4] are complex and have a large dimension. The following question (due to W. Trotter and the second author) arises. Let **N** denote the natural numbers and let $\chi(G)$ be the chromatic number of a graph G.

0.2 PROBLEM. Given a $k \in \mathbf{N}$, is there an $n(k) \in \mathbf{N}$ such that for any poset P with $\chi(H(P)) \geqslant n(k)$ we have $\dim(P) \geqslant k$?

In this note we solve Problem 0.2 negatively for the case $k \geqslant 3$. On the way, we define a new characteristic of an (unoriented) graph $G = (V, E)$. Let $\leqslant$ be a linear ordering on V. We say that $y \in V$ *is between* $x, z \in V$ if either $x < y < z$ or $z < y < x$. An *eyebrow* of $\leqslant$ in G is a triple $(x, y, z) \in V^3$ such that $\{x, z\} \in E$ and

y is between x and z. We define a number

$$\text{eye}(G)$$

as the minimal $k \in \mathbf{N}$ such that there are linear orderings $\leqslant_1, \leqslant_2, \ldots, \leqslant_k$ with no common eyebrow in G. Our motivation to study eyebrows is the following

0.3 PROPOSITION. *Let P be a poset and let $\leqslant_1, \leqslant_2, \ldots, \leqslant_k$ be a collection of linear orderings on P such that $(\leqslant_P) \subseteq (\leqslant_1 \cap \leqslant_2, \ldots, \cup \leqslant_k) =: (\leqslant)$. Then we have*

$$H(P) \subseteq H(\leqslant)$$

if and only if $\leqslant_1, \leqslant_2, \ldots, \leqslant_k$ have no common eyebrow in $H(P)$.

Proof. Look at pairs $x < z$ with $\{x, z\} \in H(P)$. Note that $\{x, z\} \notin H(\leqslant)$ if and only if there is a y with $x < y < z$. ∎

0.4 COROLLARY. *We have* eye $(H(P)) \leqslant \dim(P)$.

Proof. Apply Proposition 0.3 to the case $(\leqslant_P) = (\leqslant)$. ∎

In Section 1 below we study the number eye(G) for general graphs. Section 2 is devoted to the posets.

1. Eyebrows in Graphs

1.1. PROPOSITION. *Let G be a graph and let H be a homomorphic image of G. Then we have*

$$\text{eye}(G) \leqslant \text{eye}(H) + 1.$$

Proof. Let $G = (V, E)$, $H = (V', E')$ and let $f\colon V \to V'$ be a homomorphism onto. Now choose a collection $\leqslant_1', \leqslant_2', \ldots, \leqslant_k'$ of linear orderings on V' with no common eyebrow in H. On G, we first fix a linear ordering $\leqslant$ and then define $\leqslant_1, \leqslant_2, \ldots, \leqslant_k$ by

$$x \leqslant_i y \quad \text{if} \quad \text{either } f(x) <_i' f(y) \text{ or } f(x) = f(y) \text{ and } x \leqslant y. \qquad \blacksquare$$

Unfortunately, $\leqslant_1, \leqslant_2, \ldots, \leqslant_k$ still have common eyebrows. Each of them is of the form $x <_i y <_i z$ where $f(x) \neq f(z)$ but $f(x) = f(y)$ or $f(y) = f(z)$. The situation may be remedied by adding one new ordering $\leqslant_0$ where, say,

$$x \leqslant_0 y \quad \text{if} \quad \text{either } f(x) <_1' f(y) \text{ or } f(x) = f(y) \text{ and } x \geqslant y. \qquad \blacksquare$$

We shall now study the complete graph K_n on the set $\{1, 2, \ldots, n\}$.

1.2. PROPOSITION. *We have*

$$\text{eye}(K_n) = \lceil \log \log n \rceil + 1.$$

(The logarithm is with base 2.)

Proof. We first prove

$$\mathrm{eye}(K_n) \leqslant 1 + \mathrm{eye}(K_{\lceil n^{1/2}\rceil}). \tag{1.2.1}$$

Since obviously $\mathrm{eye}(K_2) = 1$, this implies the '$\leqslant$'-inequality. To prove (1.2.1), it suffices to consider the case of $n = m^2$. Let $\leqslant_1, \leqslant_2, \ldots, \leqslant_r$ be linear orderings on $\{1, \ldots, m\}$ with no common eyebrow in K_m. We consider linear orderings $\leqslant'_1, \leqslant'_2, \ldots, \leqslant'_r$ on $\{1, \ldots, m\}$ given by

$$(i-1)m + k \leqslant'_p (j-1)m + s \quad \text{if} \quad \text{either } i <_p j \text{ or } i = j \;\&\; k \leqslant_p s.$$

Then $\leqslant'_1, \leqslant'_2, \ldots, \leqslant'_r$ have no common eyebrows in K_n with the possible exception of the triples

$$((i-1)m + k, (i-1)m + s, (j-1)m + t) \quad \text{where} \quad i <_p j$$

and

$$((j-1)m + t, (i-1)m + k, (i-1)m + s) \quad \text{where} \quad j <_p i.$$

As in the proof of 1.1, one introduces an additional ordering $\leqslant'_0$ to kill these eyebrows. For example, we may define

$$(i-1)m + k \leqslant'_0 (j-1)m + s \quad \text{if} \quad \text{either } i >_1 j \text{ or } i = j \;\&\; k \leqslant_1 s.$$

(1.2.1) is proved.

To prove that eye $(K_n) \geqslant \lceil \log\log n \rceil + 1$, let $\leqslant_1, \leqslant_2, \ldots, \leqslant_r$ be linear orderings on $\{1, \ldots, n\}$ and let $r < \lceil \log\log n \rceil + 1$. By a classical result of Erdös and Szekeres, two linear orderings on $\{1, \ldots, n\}$ either agree or disagree on a certain subset of cardinality $\lceil n^{1/2} \rceil$. Applying this result repeatedly, we find a set of cardinality $\geqslant 3$ where every two of the orderings $\leqslant_1, \leqslant_2, \ldots, \leqslant_r$ either agree or disagree. This is exactly equivalent to finding a common eyebrow of $\leqslant_1, \leqslant_2, \ldots, \leqslant_r$ in K_n. ■

1.3. COROLLARY. *For an arbitrary graph G, we have*

$$\mathrm{eye}(G) \leqslant \lceil \log\log \chi(G) \rceil + 2. \qquad ■$$

2. Complex Diagrams of Simple Posets

2.1. THEOREM. *For each $k > 0$ there exists a finite set X and linear orderings L1, L2 on X such that*

$$\chi(H(L1 \cap L2)) \geqslant k.$$

Proof. Will be given in 2.5. ■

2.2. DEFINITION. A *preordering* on a set X is reflexive and transitive relation $\leqslant$ on X such that $(\forall x, y \in X)$ $(x \leqslant y$ or $y \leqslant x)$. Given functions $\alpha_1, \ldots, \alpha_k : X \to Z$, we define a preordering $[\alpha_1, \ldots, \alpha_k]$ on X by putting

$$x[\alpha_1, \ldots, \alpha_k]y \text{ if } (\alpha_1(x), \ldots, \alpha_k(x)) <_L (\alpha_1(y), \ldots, \alpha_k(y))$$

where $<_L$ is the lexicographical ordering giving more weight to the coordinates toward the left.

In the sequel, we shall sometimes denote n-tuples $(x_1, \ldots, x_n)$ by juxtaposition $x_1 x_2 \cdots x_n$. The symbol $|X|$ will denote the cardinality of a finite set X.

2.3. CONSTRUCTION. Define a graph $G_n = (V_n, E_n)$ as follows:

$$V_n = (\{0\} \times \{0, \ldots, n-1\} \times \{0, \ldots, n-1\})$$
$$\cup(\{1\} \times \{0, \ldots, n^n\} \times \{0, \ldots, n-1\})$$
$$E_n = \left\{\{0kx, 1ik\} \mid \left\lfloor \frac{i \bmod n^{k+1}}{n^k} \right\rfloor = x\right\}.$$

We shall put $U_n = (\{0\} \times \{0, \ldots, n-1\}) \cup (\{1\} \times \{0, \ldots, n^n\})$ so that $V_n = U_n \times \{0, \ldots, n-1\}$. Let

$$\pi_i : V_n \to \mathbf{N}, \quad i \in \{1, 2, 3\}$$

be the projection to the i-th coordinate. Observe that the components of G_n are stars with centres of the form $0kx$ where $k, x \in \{0, \ldots, n-1\}$. Define

$$\phi : V_n \to V_n$$

by $\phi(v) = 0kx$ where $0kx, v$ are in the same component of G_n.

2.4. LEMMA. *Let M be an independent set in G_n. Then we have an $xy \in U_n$ such that*

$$M \cap (\{x\} \cap \{y\} \cap \{0, \ldots, n-1\}) = \emptyset. \tag{2.4.1}$$

Proof. Suppose that (2.4.1) is not true for any $xy \in U_n$ with $x = 0$. Then we have for each $y \in \{0, \ldots, n-1\}$ an $a_y \in \{0, \ldots, n-1\}$ such that

$$0ya_y \in M.$$

Let a be the number with the n-adic expansion $a_{n-1}a_{n-2} \cdots a_0$. By definition, we have

$$\{0ya_y, 1ay\} \in E_n.$$

Thus, by the independency of M, $M \cap (\{1\} \times \{a\} \times \{0, \ldots, n-1\}) = \emptyset$. ■

2.5. *Proof of Theorem 1.* By induction on k. Let $R1$, $R2$ be strict linear orderings on $\{0, \ldots, n-1\}$ such that

$$\chi(H(R1 \cap R2)) \geqslant k.$$

Without loss of generality, we have $R1 = (<)$ (the usual ordering of natural numbers). Define a permutation $\alpha: \{0, \ldots, n-1\} \to \{0, \ldots, n-1\}$ by

$$\alpha(i) = |\{j \in \{0, \ldots, n-1\} \mid j(R2)i\}|.$$

Now put

$$X = V_n,$$

$$L1 = [\pi_1, \pi_2, \alpha\pi_3],$$

$$L2 = [\pi_2\phi, \pi_3\phi, \pi_1, -\pi_2].$$

It is easy to check that $L1$, $L2$ are linear orderings. We claim that

$$H(L1 \cap L2) \supseteq E_n \cup \{\{xya, xyb\} \mid \{a, b\} \in H(R1 \cap R2) \ \& \ xy \in U_n\}. \tag{2.5.1}$$

To see this, let first $xy \in U_n$, $\{a, b\} \in H(R1 \cap R2)$, $a < b$. Since trivially

$$xya(L1 \cap L2)xyb,$$

we have to check that there is no $v \in V_n$ with

$$xya \, L1 \, v \, L1 \, xyb \tag{2.5.2}$$

$$xya \, L2 \, v \, L2 \, xyb. \tag{2.5.3}$$

Suppose the contrary. Since $\pi_i(xya) = \pi_i(xyb)$, $i = 1, 2$, it follows from (2.5.2) that $v = xyc$ for some c with

$$\alpha(a) < \alpha(c) < \alpha(b).$$

We conclude that

$$a \, R2 \, c \, R2 \, b. \tag{2.5.4}$$

Now there are two possibilities:

Case 1. $x = 1$. Then, as we easily see, $\pi_2\phi(xyz) = z$ for any z and thus (2.5.3) implies $a < c < b$ or, equivalently,

$$a \, R1 \, c \, R1 \, b. \tag{2.5.5}$$

Case 2. $x = 0$. Then $\phi(xyz) = xyz$ and thus, (2.5.5) follows from (2.5.3) again. In both cases, (2.5.4) together with (2.5.5) contradict the assumption

$$\{a, b\} \in H(R1 \cap R2).$$

We have shown that

$$H(L1 \cap L2) \supseteq \{\{xya, xyb\} \mid \{a, b\} \in H(R1 \cap R2) \ \& \ xy \in U_n\}.$$

To prove (2.5.1), it remains to show that $H(L1 \cap L2) \supseteq E_n$ or, equivalently, that if

$$\{0kx, 1ik\} \in E_n \tag{2.5.6}$$

then there is no v with

$$0kx\ L1\ v\ L1\ 1ik \tag{2.5.7}$$

$$0kx\ L2\ v\ L2\ 1ik \tag{2.5.8}$$

(Observe that we have trivially $0kx(L1 \cap L2)1ik$.) Suppose that the above is false. From (2.5.8) we obtain

$$\pi_2\phi(v) = k = \pi_2\phi(0kx) = \pi_2\phi(1ik),$$

$$\pi_3\phi(v) = x = \pi_2\phi(0kx) = \pi_3\phi(1ik).$$

Since $v \neq 0kx$, we have $\pi_1(v) = 1$ and hence $v = 1jk$ for some $j \in \{0, \ldots, n^n - 1\}$. Now $(v\ L1\ 1ik)$ implies $j < i$, while $(v\ L2\ 1ik)$ implies $i < j$. (2.5.1) is proved.

Denote the right hand side of 2.5.1 by $\bar{E}_n$. We will show that

$$\chi(V_n, \bar{E}_n) \geqslant k + 1.$$

Let $M_1, \ldots, M_m$ be a partition of V_n into independent sets of $(V_n, \bar{E}_n)$. By Lemma 2.4, we have $M_m \cap (\{d\} \times \{n-1\}) = \emptyset$ for some $d \in U_n$. Thus,

$$\pi_3(M_1 \cap (\{d\} \times \{0, \ldots, n-1\})), \ldots, \pi_3(M_{m-1} \cap (\{d\} \times \{0, \ldots, n-1\})$$

form a partition of $\{0, \ldots, n-1\}$ into independent sets of $H(R1 \cap R2)$. From the induction hypothesis we obtain $k \leqslant m - 1$. ∎

Combining Proposition 1.1 with Theorem 2.1, we have the following result:

2.6. COROLLARY. *For every n, s there exists a graph $G_{n,s}$ with the following properties*:

$$G_{n,s} \text{ has girth } s \tag{2.6.1}$$

$$\chi(G_{n,s}) = n \tag{2.6.2}$$

$$\text{eye}(G_{n,s}) \leqslant 3 \tag{2.6.3}$$

Proof. It is well known that for every graph H and every s there exists a graph G such that $\chi(G) = \chi(H)$, G has girth s and there is a homomorphism $f\colon G \to H$ (see, e.g., [3]). If H has the properties given by Theorem 2.1 then $\text{eye}(G) \leqslant \text{eye}(H) + 1$. ∎

3. Concluding Remarks

3.1. Although one might think that the inequalities 1.1 and 1.2 may be improved by one, in general this is false. For instance, if n is sufficiently large then

$$\text{eye}(K_n^3) = 3$$

(where K_n^3 is the complete tripartite graph). This may be seen as follows: Suppose there were linear orderings $\leqslant_1$, $\leqslant_2$ on the vertices of K_n^3 with no common eyebrow. By Zarankiewicz's theorem, there are vertices v_j^i, $i=0, 1, j=0, 1, 2$ in K_n^3 such that v_j^i belongs to the j-th part and for each $j \neq k$, $s \leqslant 2$, the validity of the formula

$$v_j^\varepsilon \leqslant_p v_k^\delta$$

does not depend on the values of ε, δ. Since $\leqslant_1$ and $\leqslant_2$ have no common eyebrow, we may assume without loss of generality that, say,

$$v_0^{\varepsilon(0)} \leqslant_1 v_1^{\varepsilon(1)},$$

$$v_0^{\varepsilon(0)} \leqslant v_2^{\varepsilon(2)},$$

$$v_0^{\varepsilon(0)} \geqslant_2 v_1^{\varepsilon(1)} \leqslant v_2^{\varepsilon(2)}.$$

Moreover, let $v_1^0 \leqslant_1 v_1^1$. Now if $v_1^0 \leqslant_2 v_1^1$ then

$$(v_1^0, v_1^1, v_2)$$

is a common eyebrow and else

$$(v_0^\varepsilon, v_1^0, v_1^1)$$

is a common eyebrow (ε arbitrary). This proves that

$$\text{eye}(K_n^3) = 3 = \lceil \log \log 3 \rceil + 2.$$

To the contrary,

$$\text{eye}\, G \leqslant 3 = \lceil \log \log m \rceil + 1$$

holds for every graph with $\chi(G) \leqslant 6$. In fact, let $f\colon V(G) \to \{1, 2, 3, 5, 6\}$ be a coloring. Choose an injective function $g\colon V(G) \to \mathbf{N}$. Consider the permutations π_i of $\{1, 2, 3, 4, 6\}$ given by $\pi_1 = (165324)$, $\pi_2 = (264315)$, $\pi_3 = (354216)$. Then the orderings given by $[\pi_1 f, g]$, $[\pi_2 f, g]$, $[\pi_3 f, g]$ have no common eyebrow. This suggests the following

3.2. PROBLEM. For which values of m does $\chi(G) \leqslant m$ imply

$$\text{eye}\, G \leqslant \lceil \log \log m \rceil + 1?$$

(Yes for $m = 5, 6$, No for $m = 2, 3, 4$.)

3.3. At the moment we do not know whether Corollary 2.6 may replace

$$\text{eye}(G_{n,s}) \leqslant 3$$

by $\text{eye}(G_{n,s}) \leqslant 2$.

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