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Impulsive motion of a non-Newtonian fluid between two oscillating parallel plates

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Abstract. An exact solution for the flow of an incompressible viscoelastic fluid between two infinitely extended parallel plates, due to the harmonic oscillations of the upper plate and the impulsively started harmonic oscillations of the lower plate from rest, in the respective planes of the plates, has been obtained. The momentum transfer towards the central region and the skin friction of the lower plate are found to be greater for the viscoelastic fluid than that for viscous fluid. The effect of out-of-phase oscillations of the plates with different amplitudes on the flow characteristics has also been investigated.

1. Introduction

The flow of fluid between two parallel plates has been a subject of considerable interest and importance to theoretical as well as experimental investigators because of its occurrence in rheometric experiments to determine the constitutive properties of the fluid, in lubrication engineering wherein the rheological characteristics of effective lubricants with higher load carrying capacity are obtained and in transport phenomena encountered in chemical engineering. Since the gap width between the two plates is usually small compared to the lengths, the edge effects are assumed to be negligible and the plates are considered to be infinitely long. The flow of fluid may be induced by the pressure gradient along the length of the plates or the motions of the plates or both. The flow of fluid between the plates may also be caused by the presence of source of fluid on the plates. The governing equations for the flow often yield exact solutions and thus providing basic motion for the stability analysis of such flows between the plates which is of vital interest in the study of the onset of turbulent motion of fluid. The investigation of heat and mass transfer in the flow between two parallel plates with constant or time-dependent wall conditions is significantly important in view of the applications in chemical engineering processes and porous bearings.

The momentum, heat and mass transfer in the flow of Newtonian as well as non-Newtonian fluids between two long plates, has been studied extensively by several authors [1–13]. In these studies, the flow is considered to be driven by uniform motion of the plates or by constant or time-dependent pressure gradient. However, the literature on the impulsively set second order fluid motions between two parallel plates is scarce. The unsteady motion of a second order fluid between parallel plates when the upper plate is moving with uniform velocity and lower plate is performing simple harmonic oscillations in its own plane impulsively has been considered by Sacheti and Bhatt [4]. The classical Stokes problem of viscous flow near an impulsively oscillating plate is well known [1]. The flow confined between harmonically oscillating plate and another plate performing impulsively started harmonic oscillations has not been analysed so far. The objective of the present paper is to examine analytically the unsteady motion of an incompressible second order fluid between two infinite parallel plates due to the harmonic oscillations of the upper plate and the impulsively set harmonic oscillations of the lower plate and to compare the flow characteristics with the Newtonian case. The results obtained are of special interest to rheometric experimenters and lubrication technologists dealing with the influence of non-Newtonian character of the fluid on the start-up motion of the fluid between two closely placed parallel plates oscillating in their own planes.

In the following the basic equations governing the flow of an incompressible second order fluid are given in Section 2. Sections 3 and 4 contain the mathematical formulation of the problem and its solution respectively. Section 5 deals with the coefficient of skin friction. The large time solution of the problem is dealt in Section 6. The discussion on the results obtained are presented in Section 7.

2. Basic equations

The constitutive equation of an incompressible second order fluid based on the postulate of gradually fading memory is given by Coleman and Noll [15] as

$$\tilde{T} = p'\tilde{I} + \mu\tilde{A}_1 + \alpha_1\tilde{A}_2 + \alpha_2\tilde{A}_1^2, \tag{1}$$

where $\tilde{T}$ is the stress tensor, p' the pressure, μ, α_1, α_2 are material constants with $\alpha_1 < 0$, and $\tilde{A}_1$ and $\tilde{A}_2$ are Rivlin–Ericksen tensors defined as

$$\tilde{A}_1 = (\text{grad } V) + (\text{grad } V)^T, \tag{2a}$$

$$\tilde{A}_2 = \dot{\tilde{A}}_1 + \tilde{A}_1 \cdot \text{grad } V + (\text{grad } V)^T \cdot \tilde{A}_1. \tag{2b}$$

The equation of continuity is

$$\operatorname{div} V = 0, \tag{3}$$

and the linear momentum equation is

$$\operatorname{div} \tilde{T} + \varrho f = \varrho \dot{V}. \tag{4}$$

Here V is the velocity, ϱ the density and f the body force per unit mass.

3. Formulation of the problem

We consider the fully developed oscillatory flow of a second order fluid, represented by Coleman and Noll's constitutive equation [15], between two infinitely long parallel plates, with a gap width of y_0, due to the simple harmonic oscillations $V_0 \cos(nt + \phi)$ of the upper plate. The x-axis is chosen along the lower plate and the y-axis perpendicular to it. Impulsively, the lower plate is set to perform the simple harmonic oscillations $V \cos nt$ at some instant $t = 0$. The development of flow for $t > 0$ is analysed subsequently. Due to the parallel nature of the flow, the flow variables depend on y and t only. Thus

$$u = u(y, t), \quad v = 0, \quad p' = p'(y, t), \tag{5}$$

and the momentum equations (4) become

$$\varrho \frac{\partial u}{\partial t} = -\frac{\partial p^*}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} + \alpha_1 \frac{\partial^3 u}{\partial t\, \partial y}, \tag{6}$$

$$0 = -\frac{\partial p^*}{\partial y}, \tag{7}$$

in the absence of body forces, where p^* is the modified pressure given by

$$p^* = p' - (2\alpha_1 + \alpha_2)\left(\frac{\partial u}{\partial y}\right)^2. \tag{8}$$

From the equations (5) and (8), it follows that

$$\frac{\partial p^*}{\partial x} = 0, \tag{9}$$

and the equation (6) becomes

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} - K \frac{\partial^3 u}{\partial t\, \partial y^2}, \tag{10}$$

where $\nu = \mu/\varrho$ and $K = -\alpha_1/\varrho$.

The solution of the equation (10) is subject to the following conditions

$$u(y_0, t) = V_0 \cos(nt + \phi), \tag{11a}$$

$$u(0, t) = \begin{cases} 0, & \text{for } t \leqslant 0, \\ V \cos nt, & \text{for } t > 0. \end{cases} \tag{11b}$$

4. Solution of the problem

We seek the solution of the equation (10) in the form

$$u = u_0(y, t) + u_1(y, t), \tag{12}$$

where u_0 is the solution of the equation

$$\frac{\partial u_0}{\partial t} = \nu \frac{\partial^2 u_0}{\partial y^2} - K \frac{\partial^3 u_0}{\partial t\, \partial y^2}, \tag{13}$$

subject to the conditions

$$u_0(y_0, t) = V_0\, e^{i(nt+\phi)}, \tag{14a}$$

$$u_0(0, t) = 0, \tag{14b}$$

and u_1 is the solution of the equation

$$\frac{\partial u_1}{\partial t} = \nu \frac{\partial^2 u_1}{\partial y^2} - K \frac{\partial^3 u_1}{\partial t\, \partial y^2}, \tag{15}$$

subject to the conditions

$$t \leqslant 0: \quad u_1(y, t) = 0, \tag{16a}$$

$$t > 0: \quad u_1(0, t) = V e^{int}, \quad u_1(y_0, t) = 0, \tag{16b}$$

and only the real part of the complex quantities have physical meaning.

Separating the variables in the form

$$u_0(y, t) = f(y)\, e^{i(nt+\phi)}, \tag{17}$$

The solution of the equation (13) subject to the conditions (14) is obtained as

$$u_0 = V_0 \frac{\sinh a\bar{y}}{\sinh a}\, e^{i(nt+\phi)}, \tag{18}$$

where

$$a = \left(\frac{in}{\nu - inK}\right)^{1/2} y_0, \qquad \bar{y} = \frac{y}{y_0}. \tag{19}$$

Defining the Laplace transform of $u_1(y, t)$ as

$$u_1(y, p) = \int_0^\infty e^{-pt}\, u_1(y, t)\, dt, \quad p > 0, \tag{20}$$

the equation (15) is transformed to

$$\frac{\partial^2 \bar{u}_1}{\partial y^2} - \frac{p}{\nu - pK}\bar{u}_1 = 0, \tag{21}$$

and the condition (16b) is transformed to

$$\bar{u}_1(0, p) = \frac{V}{p - in}, \quad \bar{u}_1(y_0, p) = 0. \tag{22}$$

The equation (21) subject to the conditions (22) yields the solution as

$$\bar{u}_1 = \frac{V \sinh\left[\left(\dfrac{p}{\nu - Kp}\right)^{1/2}(y_0 - y)\right]}{(p - in)\sinh\left[\left(\dfrac{p}{\nu - Kp}\right)^{1/2} y_0\right]}. \tag{23}$$

The inverse Laplace transform of equation (23), after employing the calculus of residues [16], is obtained in the form

$$u_1 = \frac{V \sinh[a(1 - \bar{y})]}{\sinh a}\, e^{int} + \sum_{k=1}^{\infty} \frac{V(-1)^k 4\pi k \alpha^2 \sin[k\pi(1 - \bar{y})]}{[k^2\pi^2 + i(2\alpha^2 + k^2\pi^2\beta)](2\alpha^2 + k^2\pi^2\beta)} \exp\left(-\frac{k^2\pi^2 nt}{2\alpha^2 + k^2\pi^2\beta}\right), \tag{24}$$

where

$$\alpha = \left(\frac{n}{2\nu}\right)^{1/2} y_0, \quad \beta = -\frac{Kn}{\nu}. \tag{25}$$

Using equations (18) and (24) in (12), we obtain

$$\bar{u} = \frac{\sinh a\bar{y}}{\sinh a} e^{i(nt+\phi)} + \frac{\bar{V}\sinh[a(1-\bar{y})]}{\sinh a} e^{int}$$

$$+ \sum_{k=1}^{\infty} \frac{\bar{V}(-1)^k 4k\pi\alpha^2 \sin[k\pi(1-\bar{y})]}{[k^2\pi^2 + i(2\alpha^2 + k^2\pi^2\beta)](2\alpha^2 + k^2\pi^2\beta)} \exp\left(\frac{-k^2\pi^2 nt}{2\alpha^2 + k^2\pi^2\beta}\right), \tag{26}$$

where

$$\bar{u} = \frac{u}{V_0}, \quad \bar{V} = \frac{V}{V_0}. \tag{27}$$

The equation (26) can be written in the form

$$\bar{u} = |\bar{u}|\, e^{i(nt+\Phi)}, \tag{28}$$

where $|\bar{u}|$ is the amplitude of the velocity field oscillations and Φ is the phase difference of the oscillations of the fluid layers with respect to the lower plate oscillations (refer to Appendix).

5. Coefficient of friction

The coefficient of friction c_f is given by

$$c_f = \frac{\partial \bar{u}}{\partial \bar{y}} - \frac{K}{\nu}\frac{\partial^2 \bar{u}}{\partial t\, \partial \bar{y}}. \tag{29}$$

The second term in the expression (29) represents the effect of viscoelasticity of the fluid on c_f. The coefficient of drag experienced by the impulsively started lower plate is given by

$$c_{fo} = (c_f)_{\bar{y}=0} = \frac{a(1 + \mathrm{i}\beta)}{\sinh a} \mathrm{e}^{\mathrm{i}(nt+\phi)} - \frac{\bar{V}a(1 + \mathrm{i}\beta)\cosh a}{\sinh a} \mathrm{e}^{\mathrm{i}nt}$$

$$- \bar{V} \sum_{k=1}^{\infty} \frac{8k^2\pi^2\alpha^4}{[k^2\pi^2 + \mathrm{i}(2\alpha^2 + k^2\pi^2\beta)](2\alpha^2 + k^2\pi^2\beta)^2}$$

$$\times \exp\left(-\frac{k^2\pi^2 nt}{2\alpha^2 + k^2\pi^2\beta}\right). \tag{30}$$

The equation (30) can be written in the form

$$c_{fo} = |c_{fo}|\,\mathrm{e}^{\mathrm{i}(nt+\phi_c)}, \tag{31}$$

where $|c_{fo}|$ is the amplitude of the skin friction oscillations and ϕ_c is the phase angle (refer to Appendix).

6. Large time solution

The solution given in equation (26) is useful for small values of time only, since $2\alpha^2 + k^2\pi^2\beta$ will be negative beyond some value of k which leads to very slow convergence of the summation during numerical computation for large values of time.

However, the asymptotic expansion of the transform function in equation (23) in ascending powers of small p retaining the simple pole at $p = \mathrm{i}n$ and the application of inversion theorem leads to the large time solution in the closed form as

$$\bar{u} = \frac{\sinh a\bar{y}}{\sinh a} \mathrm{e}^{\mathrm{i}(nt+\phi)} + \frac{\bar{V}\sinh[a(1 - \bar{y})]}{\sinh a} \mathrm{e}^{\mathrm{i}nt}, \tag{32}$$

and the coefficient of skin friction at the lower plate is given by

$$c_{fo} = \frac{a(1 + \mathrm{i}\beta)}{\sinh a} \mathrm{e}^{\mathrm{i}(nt+\phi)} - \frac{\bar{V}a(1 + \mathrm{i}\beta)\cosh a}{\sinh a} \mathrm{e}^{\mathrm{i}nt}. \tag{33}$$

7. Results and discussion

The numerical computation of the velocity distribution and coefficient of friction at the lower plate has been done for $\alpha = 4$ and $\beta = -1/3, 0$. The Newtonian case corresponds to $\beta = 0$. Figures 1–6 show the amplitude of velocity distribution at different times $T = nt$. It can be observed that the momentum transfer towards the central line $y = 0.5$ is greater for the second order fluid than that for Newtonian fluid. When the plates are oscillating with a phase difference $\phi = \pi/2$, the momentum transfer from the lower plate $y = 0$ towards the central line is greater than that for $\phi = 0$ and $\phi = \pi$. As $\bar{V}$ increases, the momentum transfer towards the central line also increases. The high shear region formed in the flow domain by the out-of-phase oscillations of the plate is narrower for second order fluid as compared to the Newtonian case. For instance, in Fig. 6, we observe that the high shear region in which $|\bar{u}| \leqslant 0.2$, at time $T = \pi$, is of width 0.277 for Newtonian fluid and it is found to be of width 0.216 for second-order fluid. That is, the high shear region becomes narrower by 22% approximately. As ϕ increases, the high shear region becomes thinner. These effects can be attributed to the release of the strain energy stored in the shearing fluid layers due to the elasticity of the fluid. Figures 7–12 show the phase angle of velocity distribution at different times T. The phase difference is found to increase in case of second order fluids in comparison with the Newtonian case. When the two plates are performing out-of-phase oscillations, the fluid layer oscillating in phase with the lower plate is at a higher level in case of non-Newtonian fluid than that in Newtonian case. As $\bar{V}$ increases, this layer moves up and is at the lowest level when $\phi = \pi$. The large time ($T \to \infty$) behaviour of the velocity distribution is shown in Figs. 13–16.

The small time behaviour of the coefficient of skin friction at the lower plate $y = 0$ is shown in Figs. 17 and 18. As soon as the lower plate starts oscillating impulsively, there is a surge in the values of coefficient of skin friction experienced by the lower plate. The skin friction is found to be

Table 1. Large time values of coefficient of skin friction $c_{fo} = |c_{fo}| \cos(nt + \phi_c)$ at the lower plate $y = 0$

$\bar{V}$	ϕ	Newtonian ($\alpha = 4, \beta = 0$)		Non-Newtonian ($\alpha = 4, \beta = -1/3$)	
		Amplitude $\lvert c_{fo}\rvert$	Phase angle ϕ_c	Amplitude $\lvert c_{fo}\rvert$	Phase angle ϕ_c
0.5	0	2.967 79	−2.409 71	3.040 59	−2.666 79
0.5	$\pi/2$	2.988	−2.311 5	3.346 68	−2.485 23
0.5	π	2.697 22	−2.298 7	2.820 25	−2.358 65
1.0	0	5.793 92	−2.383 92	5.920 06	−2.594 48
1.0	$\pi/2$	5.814 65	−2.333 55	6.241 57	−2.500 69
1.0	π	5.523 04	−2.328 47	5.697 74	−2.439 66

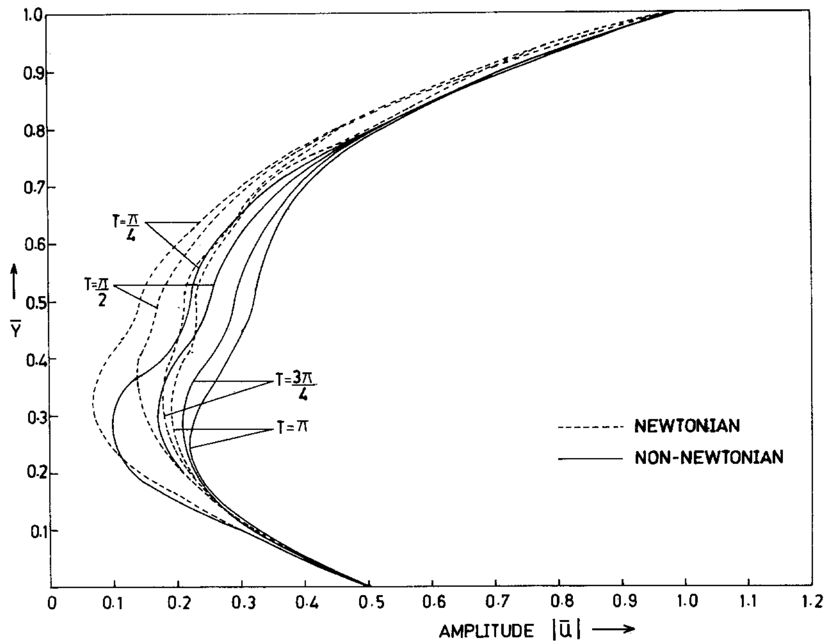

Fig. 1. Amplitude of velocity distribution at different times T for $\bar{V} = 0.5$ and $\phi = 0$.

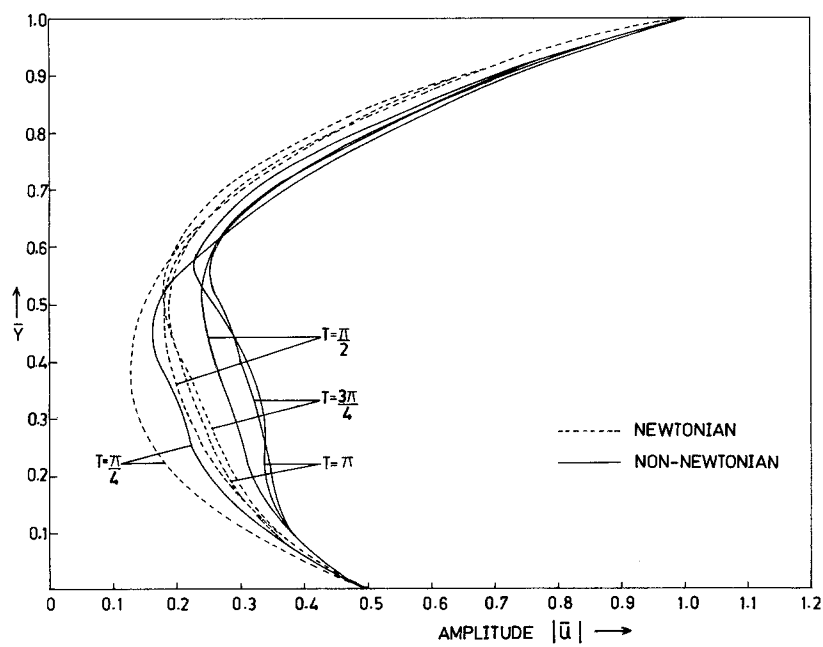

Fig. 2. Amplitude of velocity distribution at different times T for $\bar{V} = 0.5$ and $\phi = \pi/2$.

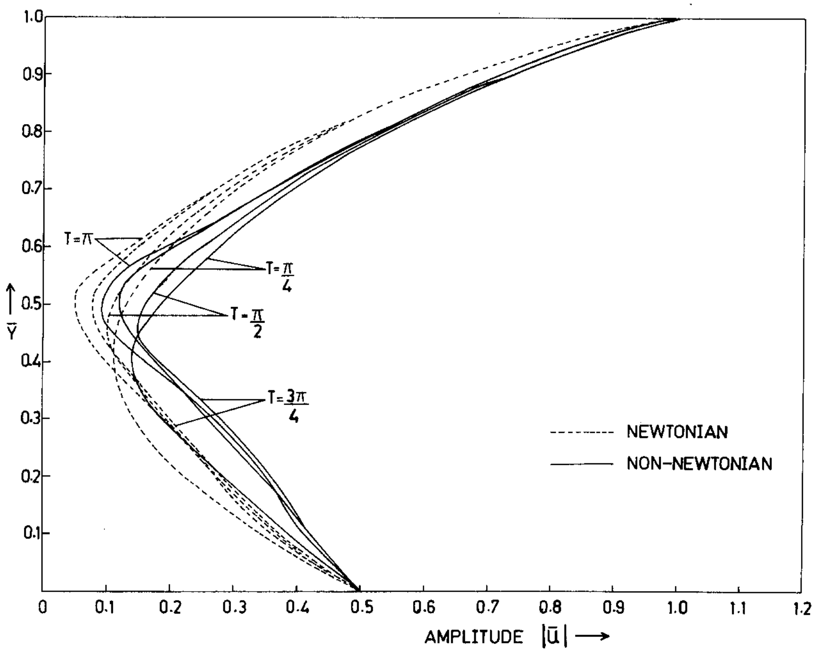

Fig. 3. Amplitude of velocity distribution at different times T for $\bar{V} = 0.5$ and $\phi = \pi$.

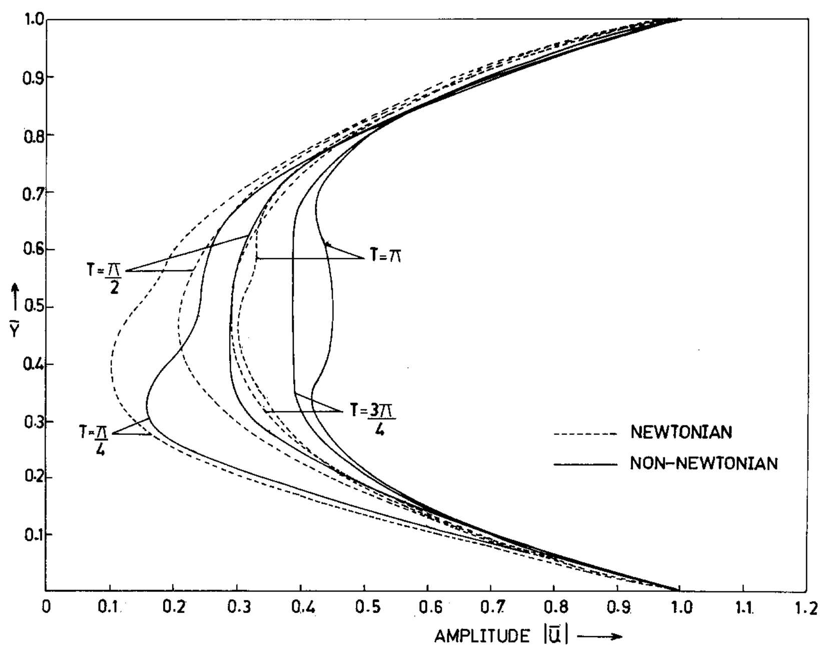

Fig. 4. Amplitude of velocity distribution at different times T for $\bar{V} = 1.0$ and $\phi = 0$.

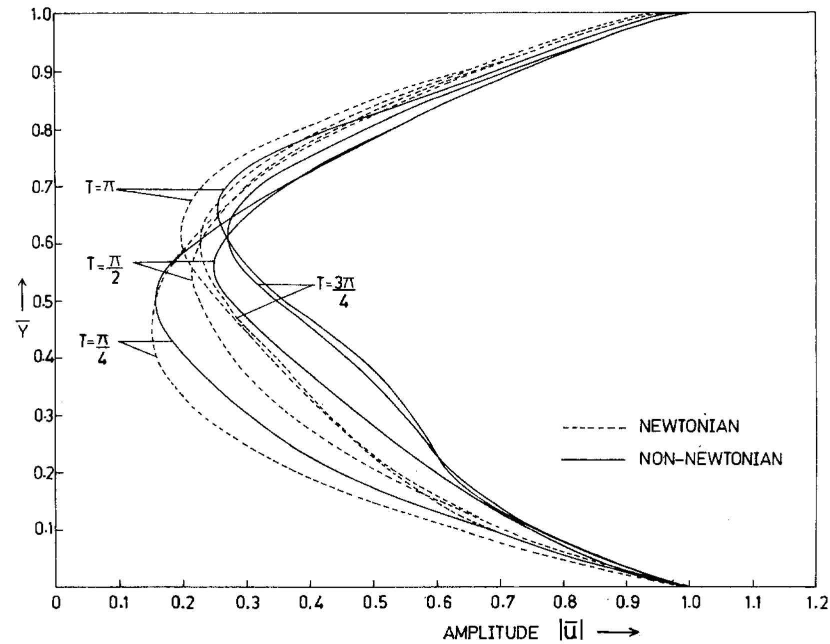

Fig. 5. Amplitude of velocity distribution at different times T for $\bar{V} = 1.0$ and $\phi = \pi/2$.

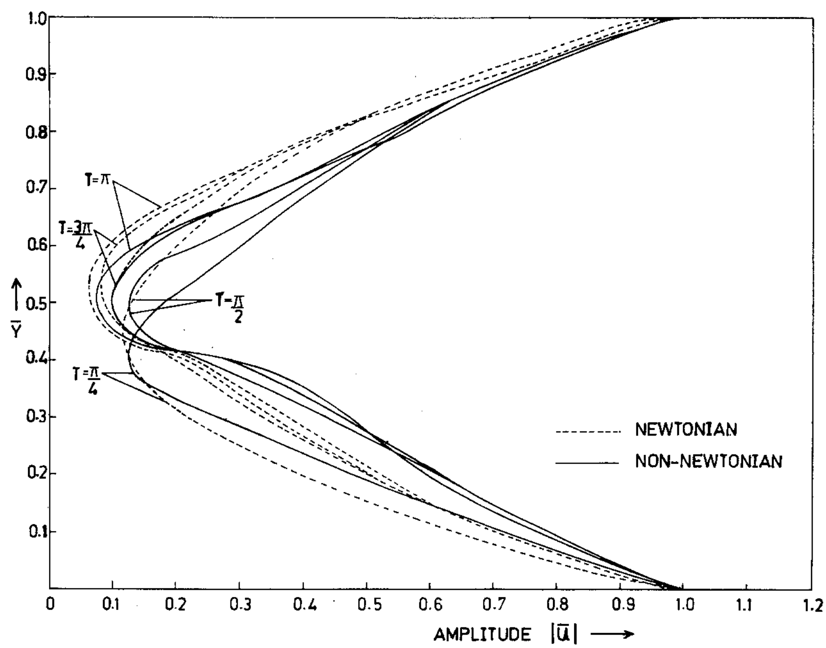

Fig. 6. Amplitude of velocity distribution at different times T for $\bar{V} = 1.0$ and $\phi = \pi$.

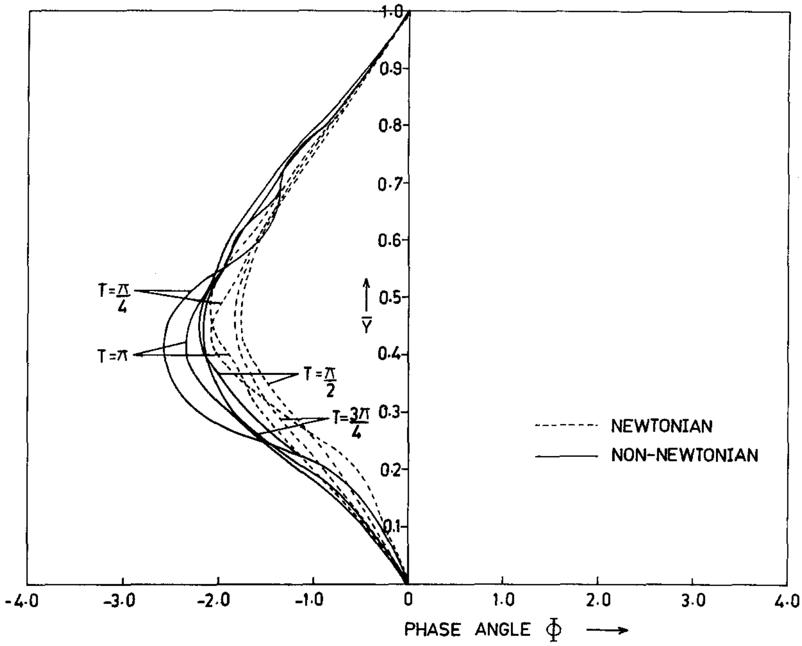

Fig. 7. Phase angle of the velocity distribution at different times T for $\bar{V} = 0.5$ and $\phi = 0$.

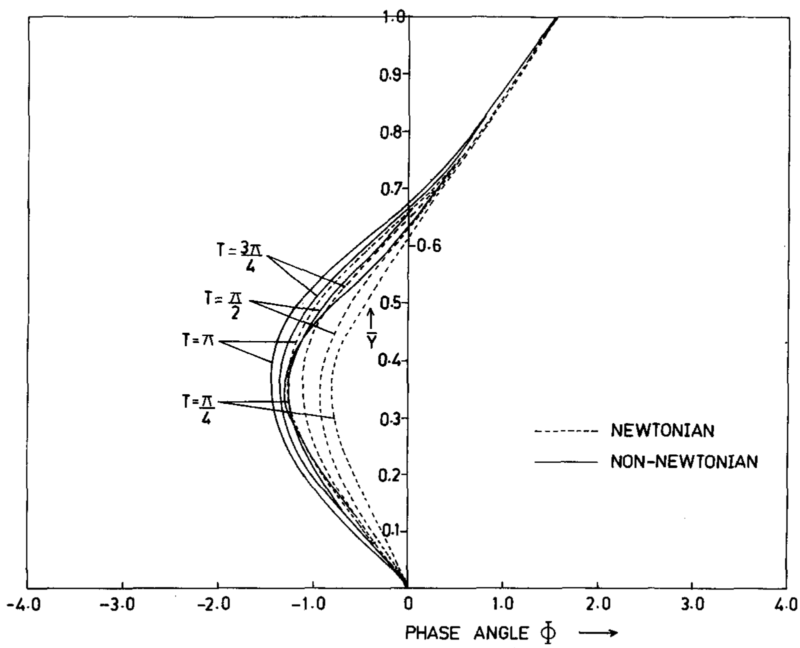

Fig. 8. Phase angle of the velocity distribution at different times T for $\bar{V} = 0.5$ and $\phi = \pi/2$.

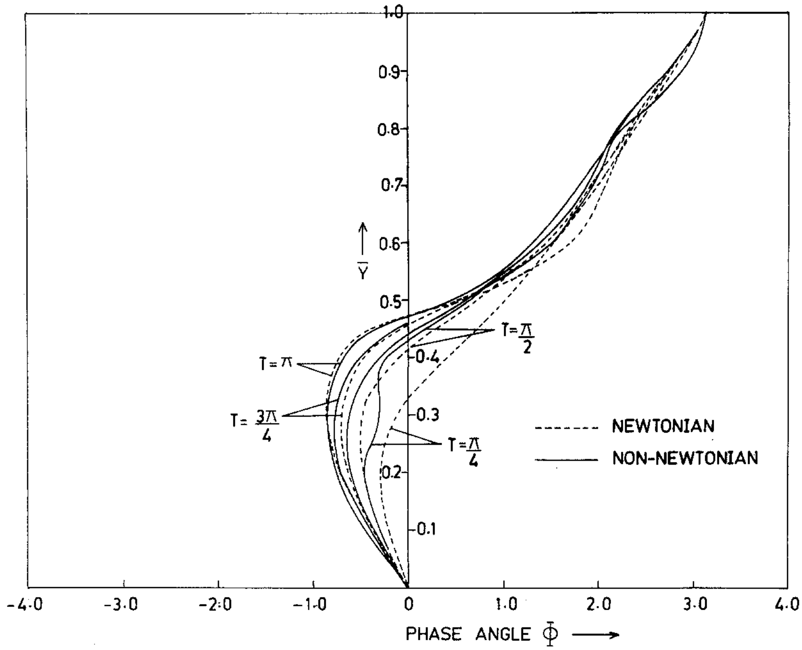

Fig. 9. Phase angle of the velocity distribution at different times T for $\bar{V} = 0.5$ and $\phi = \pi$.

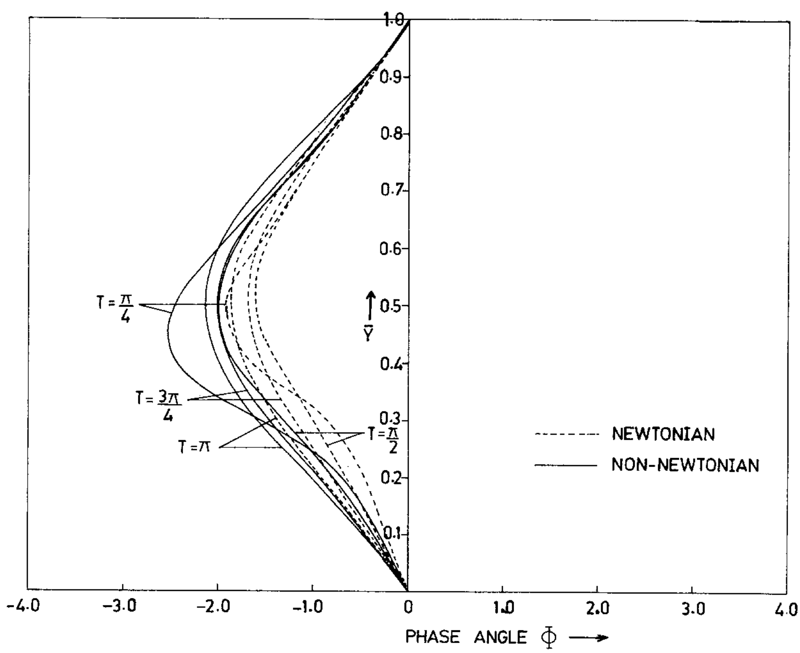

Fig. 10. Phase angle of the velocity distribution at different times T for $\bar{V} = 1.0$ and $\phi = 0$.

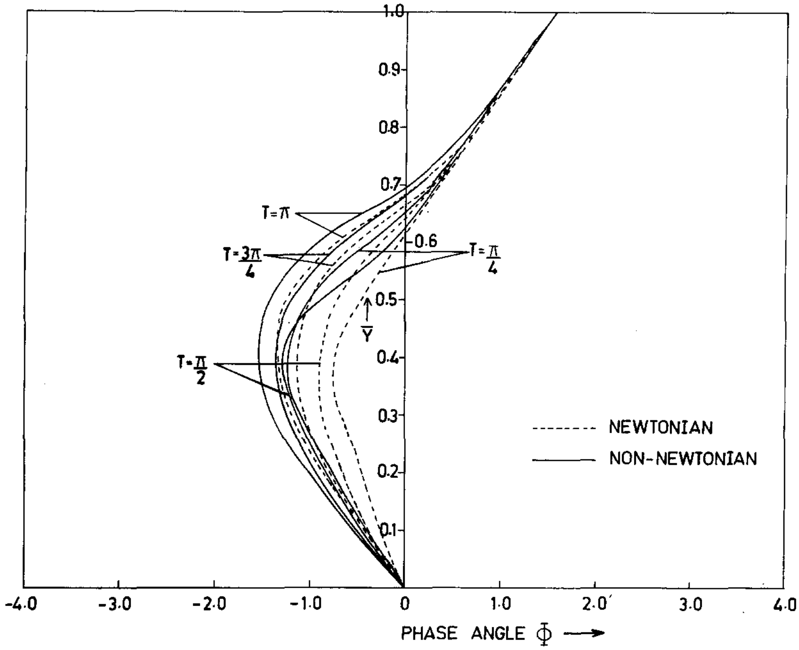

Fig. 11. Phase angle of the velocity distribution at different times T for $\bar{V} = 1.0$ and $\phi = \pi/2$.

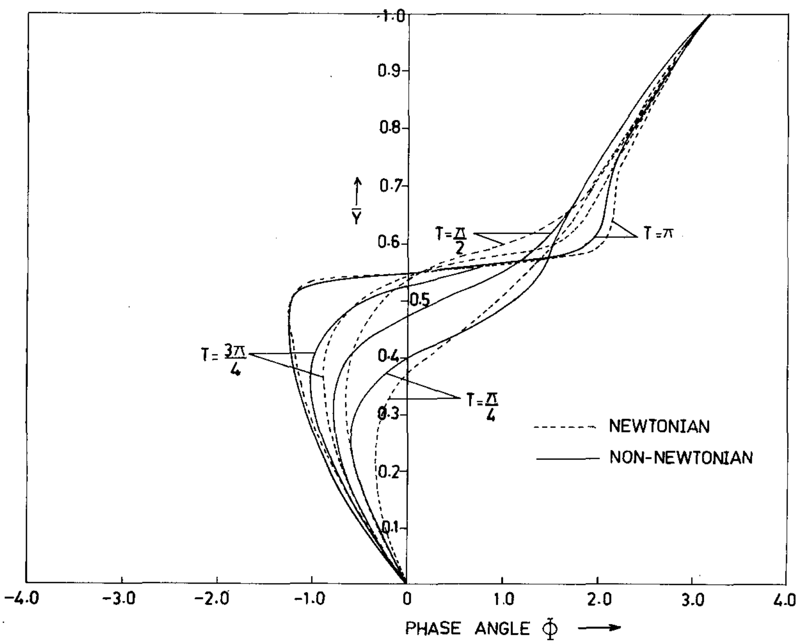

Fig. 12. Phase angle of the velocity distribution at different times T for $\bar{V} = 1.0$ and $\phi = \pi$.

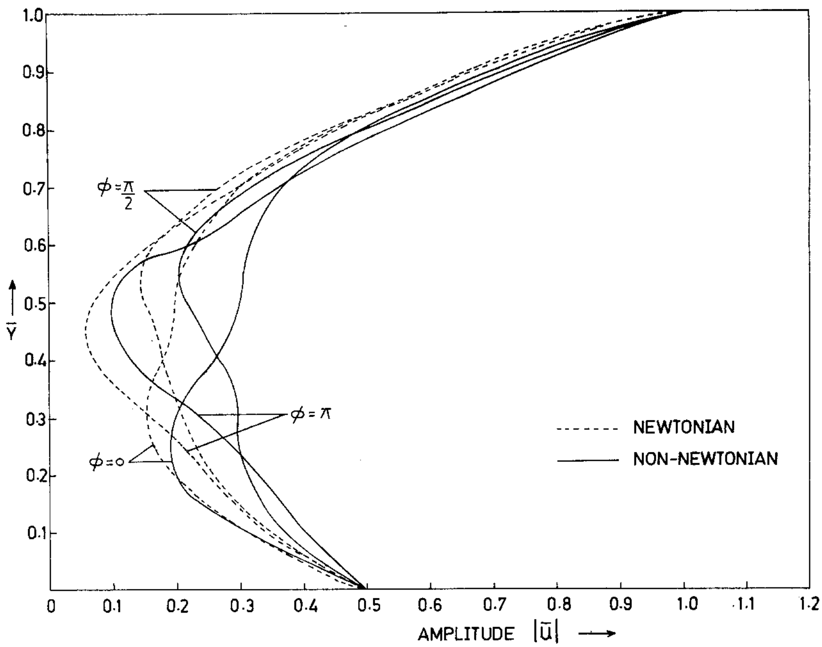

Fig. 13. Amplitude of the velocity distribution at large times for $\bar{V} = 0.5$ and $\phi = 0, \pi/2, \pi$.

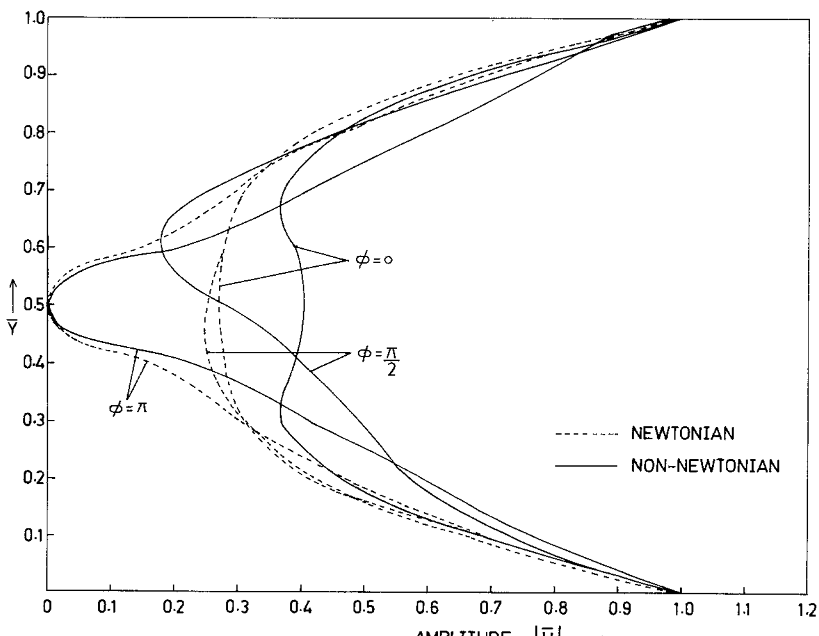

Fig. 14. Amplitude of the velocity distribution at large times for $\bar{V} = 1.0$ and $\phi = 0, \pi/2, \pi$.

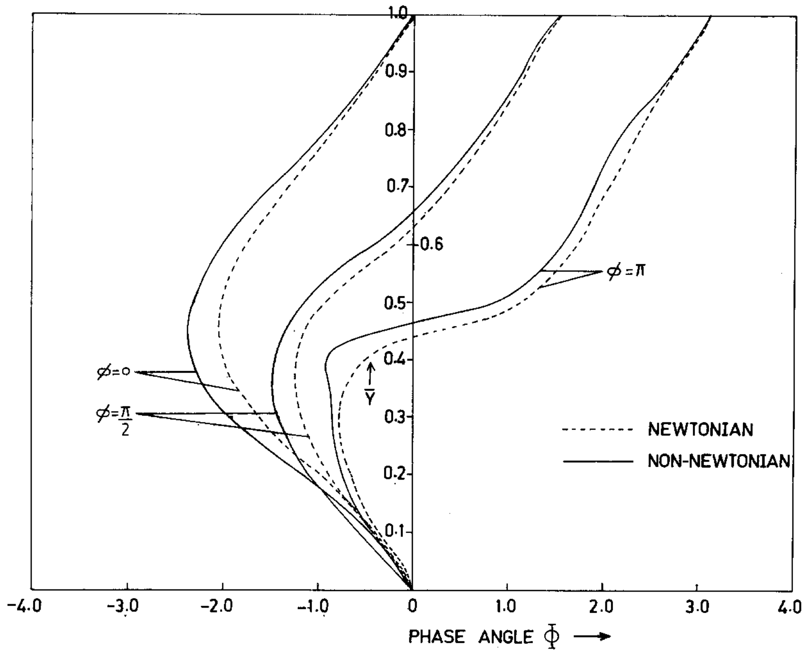

Fig. 15. Phase angle of the velocity distribution at large times for $\bar{V} = 0.5$ and $\phi = 0, \pi/2, \pi$.

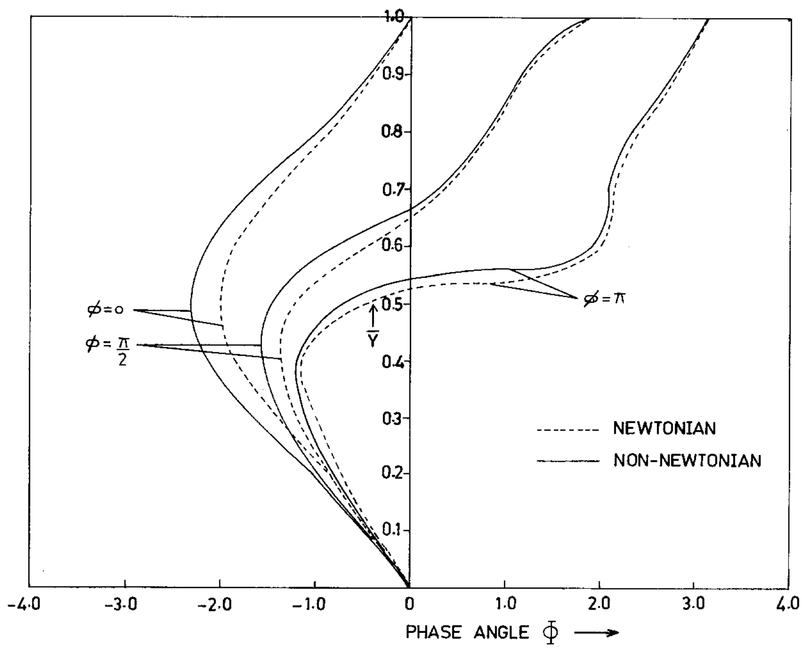

Fig. 16. Phase angle of the velocity distribution at large times for $\bar{V} = 1.0$ and $\phi = 0, \pi/2, \pi$.

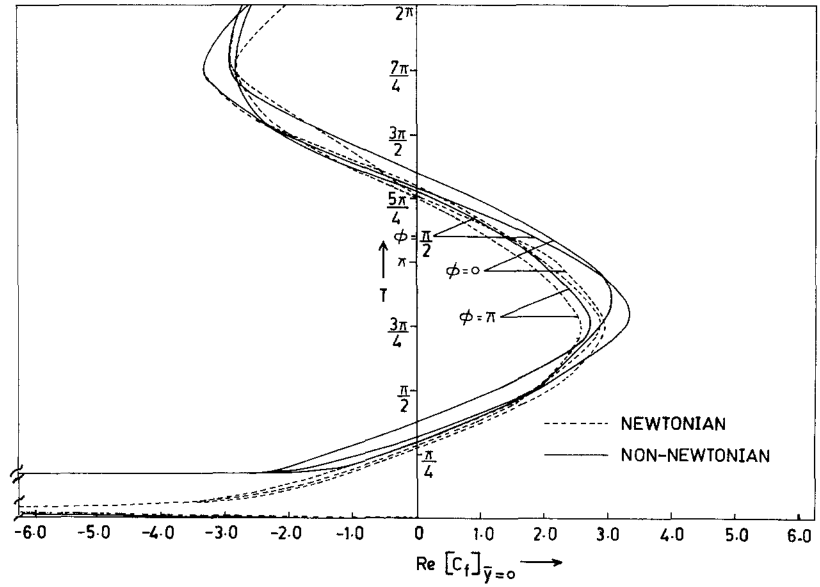

Fig. 17. Coefficient of skin friction versus T for $\bar{V} = 0.5$ and $\phi = 0, \pi/2, \pi$.

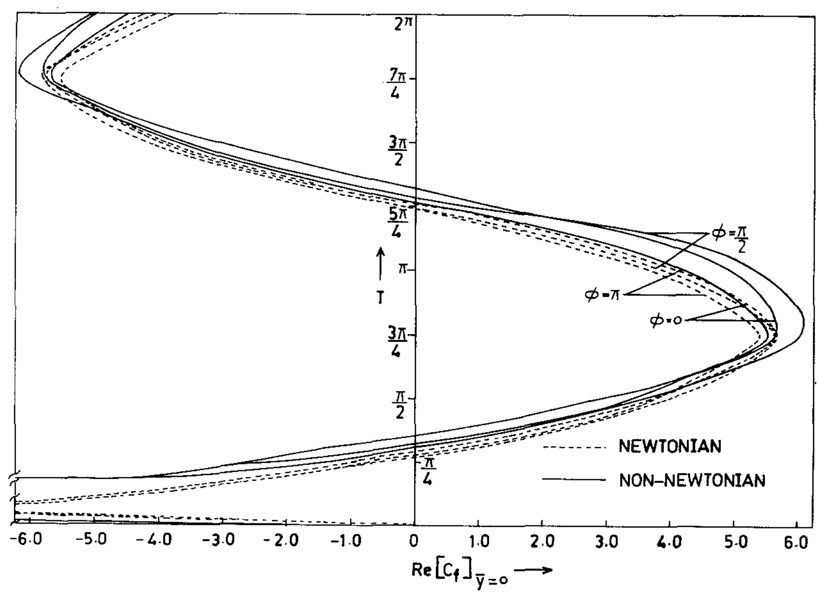

Fig. 18. Coefficient of skin friction versus T for $\bar{V} = 1.0$ and $\phi = 0, \pi/2, \pi$.

maximum when the phase difference of the oscillations of the plates is $\pi/2$. The skin friction is greater in case of the second order fluid as compared to the Newtonian case. As $\bar{V}$ increases, the skin friction is also found to increase. After large time, the skin friction oscillations reach a steady pattern whose values are shown in Table 1 for different values of the parameters $\bar{V}$ and ϕ.

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Appendix

Letting $\bar{u} = \bar{u}_r + i\bar{u}_i$ and separating the real and imaginary parts of equation (26), we obtain

$$\bar{u}_r = \frac{1}{\cosh^2 A - \cos^2 B}\{\sinh A\bar{y} \cos B\bar{y} [\sinh A \cos B \cos (nt + \phi)$$

$$+ \cosh A \sin B \sin(nt + \phi)] + \cosh A\bar{y} \sin B\bar{y} [\cosh A \sin B$$

$$\times \cos(nt + \phi) - \sinh A \cos B \sin(nt + \phi)]\} + \frac{\bar{V}}{\cosh^2 A - \cos^2 B}$$

$$\times \{\sinh A \cos B [\sinh A(1 - \bar{y}) \cos B(1 - \bar{y}) \cos nt$$

$$- \cosh A(1 - \bar{y}) \sin B(1 - \bar{y}) \sin nt] + \cosh A \sin B$$

$$\times [\sinh A(1 - \bar{y}) \cos B(1 - \bar{y}) \sin nt$$

$$+ \cosh A(1 - \bar{y}) \sin B(1 - \bar{y}) \cos nt]\}$$

$$+ \bar{V} \sum_{k=1}^{\infty} \frac{(-1)^k 4k^3\pi^3\alpha^3 \sin[k\pi(1 - \bar{y})]}{[k^4\pi^4 + (2\alpha^2 + k^2\pi^2\beta)^2](2\alpha^2 + k^2\pi^2\beta)}$$

$$\times \exp\left(-\frac{k^2\pi^2 nt)}{2\alpha^2 + k^2\pi^2\beta}\right), \tag{A1}$$

and

$$\bar{u}_i = \frac{1}{\cosh^2 A - \cos^2 B} \{\sinh A\bar{y} \cos B\bar{y} [\sinh A \cos B \sin(nt + \phi)$$

$$- \cosh A \sin B \cos(nt + \phi)] + \cosh A\bar{y} \sin B\bar{y}$$

$$\times [\sinh A \cos B \cos(nt + \phi) + \cosh A \sin B \sin(nt + \phi)]\}$$

$$+ \frac{\bar{V}}{\cosh^2 A - \cos^2 B} \{\sinh A \cos B [\sinh A(1 - \bar{y}) \cos B(1 - \bar{y})$$

$$\times \sin nt + \cosh A(1 - \bar{y}) \sin B(1 - \bar{y}) \cos nt] + \cosh A \sin B$$

$$\times [\cosh A(1 - \bar{y}) \sin B(1 - \bar{y}) \sin nt - \sinh A(1 - \bar{y})$$

$$\times \cos B(1 - \bar{y}) \cos nt]\} - \bar{V} \sum_{k=1}^{\infty} \frac{(-1)^k 4\pi k\alpha^2 \sin[k\pi(1 - \bar{y})]}{[k^4\pi^4 + (2\alpha^2 + k^2\pi^2\beta)^2]}$$

$$\times \exp\left(-\frac{k^2\pi^2 nt}{2\alpha^2 + k^2\pi^2\beta}\right), \tag{A2}$$

where,

$$A = \left[\frac{\beta + \sqrt{1 + \beta^2}}{1 + \beta^2}\right]^{1/2} \tag{A3}$$

$$B = \frac{\alpha}{[(1 + \beta^2)(\beta + \sqrt{1 + \beta^2})]^{1/2}}. \tag{A4}$$

Hence, we have in equation (28) the following

$$\bar{u} = (\bar{u}_r^2 + \bar{u}_i^2)^{1/2}, \tag{A5}$$

$$\Phi = \tan^{-1}(\bar{u}_i/\bar{u}_r) - nt. \tag{A6}$$

The real and imaginary parts of the coefficient of friction at the lower plate given in equation (30) is

$$\begin{aligned}
\mathrm{Re}\,[c_{fo}] = {} & \frac{1}{\cosh^2 A - \cos^2 B}\{\sinh A \cos B\,[Q \cos(nt + \phi) \\
& - P \sin(nt + \phi)] + \cosh A \sin B\,[Q \sin(nt + \phi) \\
& + P \cos(nt + \phi)]\} + \frac{\bar{V}}{\cosh^2 A - \cos^2 B} \\
& \times \{\sinh A \cosh A\,[P \sin nt - Q \cos nt] - \sin B \cos B \\
& \times [Q \sin nt + P \cos nt]\} \\
& - \bar{V} \sum_{k=1}^{\infty} \frac{8k^4\pi^4\alpha^4}{[k^4\pi^4 + (2\alpha^2 + k^2\pi^2\beta)^2](2\alpha^2 + \beta k^2\pi^2)^2} \\
& \times \exp\left(-\frac{k^2\pi^2 nt}{2\alpha^2 + k^2\pi^2\beta}\right),
\end{aligned} \tag{A7}$$

and

$$\begin{aligned}
\mathrm{Im}\,[c_{fo}] = {} & \frac{1}{\cosh^2 A - \cos^2 B}\{\sinh A \cos B\,[Q \sin(nt + \phi) \\
& + P \cos(nt + \phi)] + \cosh A \sin B\,[P \sin(nt + \phi)
\end{aligned}$$

$$- Q\cos(nt+\phi)]\} + \frac{\bar{V}}{\cosh^2 A - \cos^2 B}$$

$$\times \{\sin B \cos B\,[Q\cos nt - P\sin nt] - \sinh A \cosh A$$

$$\times [Q\sin nt + P\cos nt]\}$$

$$+ \bar{V}\sum_{k=1}^{\infty} \frac{8k^2\pi^2\alpha^4}{[k^4\pi^4 + (2\alpha^2+\beta k^2\pi^2)^2](2\alpha^2+\beta k^2\pi^2)}$$

$$\times \exp\left(-\frac{ntk^2\pi^2}{2\alpha^2+\beta k^2\pi^2}\right), \tag{A8}$$

where,

$$P = B - \beta A, \tag{A9}$$

$$Q = A - \beta B. \tag{A10}$$

Hence, we have

$$|c_{fo}| = (\mathrm{Re}\,[c_{fo}]^2 + \mathrm{Im}\,[c_{fo}]^2)^{1/2}, \tag{A11}$$

and

$$\phi_c = \tan^{-1}\left(\frac{\mathrm{Im}\,[c_{fo}]}{\mathrm{Re}\,[c_{fo}]}\right) - nt. \tag{A12}$$