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SLOW STEADY ROTATION OF AN AXIALLY SYMMETRIC BODY IN A MICROPOLAR FLUID

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Abstract

This paper examines the Stokes' flow due to an axially symmetric body rotating about its axis of symmetry in a micropolar fluid which sustains anti-symmetric stress and couple stress. General solutions are obtained to the coupled differential equations governing such a flow and the special case of a sphere is deduced. Then, with the aid of a concentrated couple, a simple formula for the couple experienced by a body is derived in terms of the angular velocity of the flow field.

§ 1. Introduction

In a companion paper [1] the Stokes' flow problem was considered for an axially symmetric body moving with uniform velocity in an infinite incompressible micropolar fluid. In particular, a general expression for the drag was obtained in terms of the stream function by using an argument advanced by Brenner [2] involving an axisymmetric point force. This drag formula was similar to that derived by Payne and Pell [3] in the classical case. The purpose of the present paper is to examine in some detail, the Stokes' flow problem for the case in which an axially symmetric body is uniformly rotating about its axis of symmetry. General solutions are obtained for this type of rotational motion and the special case of flow due to the rotation of a sphere is deduced. We then establish a simple formula for the couple experienced by an axially symmetric body in micropolar fluids in terms of the angular velocity of the flow. This formula is similar to that of Kanwal [4] in the case of classical fluids.

The theory characterizing this particular class of so-called micropolar fluids, was developed by Eringen [5] in 1966 and since then it has been under intensive investigation. Unlike Stokes' couple stress theory [6] which represents the simplest generalization of the classical theory that allows for polar effects, Eringen's theory introduces a kinematically independent rotation vector $\boldsymbol{\nu}$. This theory may serve as a satisfactory model for a description of the flow properties of such rheologically complex fluids as polymeric suspensions, liquid crystals and animal blood for which the classical Navier-Stokes theory is inadequate.

§ 2. General solution

We now consider the Stokes' flow due to the slow steady rotation of an axially symmetric body in an unbounded incompressible micropolar fluid at rest at infinity. Such a flow in a region D exterior to the closed boundary B of the body, is governed by the following basic equations [5]:

The equations of motion

$$t_{ji,j} + f_i = 0, \quad m_{ji,j} + \varepsilon_{ijk} t_{jk} + l_i = 0, \tag{1}$$

and the linear constitutive laws

$$\begin{aligned} t_{ij} &= -p\delta_{ij} + \tfrac{1}{2}(2\mu + \kappa)(v_{i,j} + v_{j,i}) + \kappa\varepsilon_{ijk}(\omega_k - \nu_k), \\ m_{ij} &= \alpha \nu_{l,l}\delta_{ij} + \beta \nu_{i,j} + \gamma \nu_{j,i} \end{aligned} \tag{2}$$

together with the continuity equation

$$\nabla \cdot \boldsymbol{v} = 0, \tag{3}$$

where t_{ij} are components of the stress tensor, m_{ij} are the components of the couple stress, $\boldsymbol{v}$ is the velocity vector, $\boldsymbol{\nu}$ is the micro-rotation vector, $\boldsymbol{f}$ is the body force, $\boldsymbol{l}$ is the body couple, p is the pressure, δ_{ij} is the Kronecker delta, ε_{ijk} is the alternating tensor, $(\alpha, \beta, \gamma, \mu, \kappa)$ are constants characteristic of the particular fluid under consideration and $\omega_i = \frac{1}{2}(\nabla \times \boldsymbol{\omega})_i$.

Equations (1), (2) and (3) reduce to the following system of coupled vector differential equations governing the flow within the region D:

$$-(\mu + \kappa)\nabla \times \nabla \times \boldsymbol{v} + \kappa \nabla \times \boldsymbol{\nu} - \nabla p + \boldsymbol{f} = 0, \tag{4}$$

$$(\alpha + \beta + \gamma)\nabla\nabla\cdot\boldsymbol{\nu} - \gamma\nabla \times \nabla \times \boldsymbol{\nu} + \kappa\nabla \times \boldsymbol{v} - 2\kappa\boldsymbol{\nu} + \boldsymbol{l} = 0, \tag{5}$$

$$\nabla \cdot \boldsymbol{v} = 0. \tag{6}$$

It should be emphasized that the quantity κ links together, through the above field equations, the velocity field and the micro-rotation field and for this reason it is sometimes called the coupling constant. If κ vanishes, (4) and (5) are decoupled and the classical Navier-Stokes theory is recovered.

To determine the velocity, micro-rotation and pressure fields, (4), (5) and (6) must be solved subject to the relevant boundary conditions. Here we shall adopt the conditions suggested by Eringen [5] which are

$$\boldsymbol{v} = \boldsymbol{v}_B, \qquad \boldsymbol{\nu} = \boldsymbol{\nu}_B \qquad \text{on B,} \tag{7}$$

where the quantities $\boldsymbol{v}_B$, $\boldsymbol{\nu}_B$ are respectively the prescribed values of the velocity and micro-rotation vector at a point on the boundary **B**. Working in spherical polar co-ordinates (r, θ, ϕ) and assuming symmetry about the axis $\theta = 0, \pi$, we let

$$\boldsymbol{v} = [0,0, v_\phi(r, \theta)] \quad \boldsymbol{\nu} = [\nu_r(r, \theta), \nu_\theta(r, \theta), 0].$$

The continuity equation (6) is then automatically satisfied while (4) and (5) reduce to the following form:

$$\frac{\partial p}{\partial r} = 0, \qquad \frac{\partial p}{\partial \theta} = 0, \tag{8}$$

$$(\mu + \kappa) L v_\phi + \kappa G = 0, \tag{9}$$

$$(\alpha + \beta + \gamma)\frac{\partial F}{\partial r} - \frac{\gamma}{r}\left(\frac{\partial G}{\partial \theta} + G\cot\theta\right) + \\ + \frac{\kappa}{r}\left(\frac{\partial v_\phi}{\partial \theta} + v_\phi \cot\theta\right) - 2\kappa \nu_r = 0, \tag{10}$$

$$(\alpha + \beta + \gamma)\frac{1}{r}\frac{\partial F}{\partial \theta} + \gamma\left(\frac{\partial G}{\partial r} + \frac{G}{r}\right) - \kappa\left(\frac{\partial v_\phi}{\partial r} + \frac{v_\phi}{r}\right) - 2\kappa \nu_\theta = 0, \tag{11}$$

where $F(r, \theta) = \nabla \cdot \boldsymbol{\nu}$, $G(r, \theta) = \nabla \times \boldsymbol{\nu})_\phi$ and $L = \nabla^2 - \dfrac{1}{r^2 \sin^2\theta}$

Equation (8) implies a constant pressure throughout the flow field D while (9), (10) and (11) yield the following system of equations:

$$L(L - \lambda^2) v_\phi = 0, \tag{12}$$

$$(\nabla^2 - b^2) F = 0, \tag{13}$$

$$(L - \lambda^2)G = 0, \tag{14}$$

where $\lambda^2 = \dfrac{\kappa(2\mu + \kappa)}{\gamma(\mu + \kappa)}$ and $b^2 = \dfrac{2\kappa}{\alpha + \beta + \gamma}$.

From (12) it follows that if

$$Lv_{\phi_1} = 0, \qquad (L - \lambda^2)v_{\phi_2} = 0, \tag{15}$$

then

$$v_\phi = v_{\phi_1} + v_{\phi_2} \tag{16}$$

is a solution. Using this and (9) we see that

$$G = -\frac{\lambda^2(\mu + \kappa)}{\kappa} v_{\phi_2} \tag{17}$$

Let

$$u_{\phi i} = v_{\phi i} \sin\phi (i = 1, 2);$$

then it can be easily verified that

$$\nabla^2 u_{\phi i} = \sin\phi L v_{\phi i} \tag{18}$$

Hence (15) becomes

$$\nabla^2 u_{\phi_1} = 0, \qquad (\nabla^2 - \lambda^2)u_{\phi_2} = 0. \tag{19}$$

We have thus reduced the mathematical problem to that of solving Laplace's and Helmholtz's equations whose solutions in most widely used co-ordinate systems are well known. Moreover, if we have any solution of the form $u_{\phi i} = H(r, \theta) \sin\phi$, then we can immediately conclude that the solution to our problem is $v_{\phi i} = H(r, \theta)$. For example, a well known solution of Laplace's equation is given by

$$u_{\phi_1} = \sin\phi \sum_{n=1}^{\infty} [A_n^1 r^n + B_n^1 r^{-n-1}][C_n^1 P_n^1(\xi) + D_n^1 Q_n^1(\xi)],$$

where $\xi = \cos\theta$, $P_n^m(\xi)$, $Q_n^m(\xi)$ are the associated Legendre functions and $A_n^1, B_n^1, C_n^1, D_n^1$ are constants.

Hence

$$v_{\phi_1} = \sum_{n=1}^{\infty} [A_n^1 r^n + B_n^1 r^{-n-1}][C_n^1 P_n^1(\xi) + C_n^1 Q_n^1(\xi)]. \tag{20}$$

Similarly it can be shown that

$$v_{\phi 2} = \sum_{n=1}^{\infty} r^{-\frac{1}{2}}[A_n^2 I_{n+\frac{1}{2}}(\lambda r) + + B_n^2 K_{n+\frac{1}{2}}(\lambda r)][C_n^2 P_n^1(\xi) + D_n^2 Q_n^1(\xi)], \tag{21}$$

$$F(r, \xi) = \sum_{n=1}^{\infty} r^{-\frac{1}{2}}[A_n^3 I_{n+\frac{1}{2}}(br) + + B_n^3 K_{n+\frac{1}{2}}(br)][C_n^3 P_n(\xi) + D_n^3 Q_n(\xi)], \tag{22}$$

where $I_{n+\frac{1}{2}}$, $K_{n+\frac{1}{2}}$ are the modified Bessel's functions. Hence the velocity field is given by

$$v_\phi = v_{\phi 1} + v_{\phi 2},$$

where the expressions for $v_{\phi 1}$ and $v_{\phi 2}$ are explicitly given above. To obtain the micro-rotation components, we observe from (10), (11) and (17) that

$$\nu_r = \frac{1}{b^2}\frac{\partial F}{\partial r} + \frac{1}{2r}\left(\frac{\partial v_{\phi 1}}{\partial \theta} + v_{\phi 1}\cot\theta\right) + + \frac{\mu+\kappa}{\kappa}\frac{1}{r}\left(\frac{\partial v_{\phi 2}}{\partial \theta} + v_{\phi 2}\cot\theta\right), \tag{23}$$

$$\nu_\theta = \frac{1}{b^2}\frac{\partial F}{r\partial \theta} - \frac{1}{2}\left(\frac{\partial v_{\phi 1}}{\partial r} + \frac{v_{\phi 1}}{r}\right) - \frac{\mu+\kappa}{\kappa}\left(\frac{\partial v_{\phi 2}}{\partial r} + \frac{v_{\phi 2}}{r}\right). \tag{24}$$

Substituting (20), (21) and (22) into the above equations produces after some simplification, the following expressions:

$$\nu_r = \sum_{n=1}^{\infty} \frac{1}{b^2}[C_n^3 P_n(\xi) + D_n^3 Q_n(\xi)][A_n^3\{br^{-\frac{1}{2}} I_{n-\frac{1}{2}}(br) - - (n+1)r^{-\frac{3}{2}} I_{n+\frac{1}{2}}(br)\} - B_n^3\{br^{-\frac{1}{2}} K_{n-\frac{1}{2}}(br) + + (n+1)r^{-\frac{3}{2}} K_{n+\frac{1}{2}}(br)\}] + \frac{(n+1)}{\sqrt{1-\xi^2}}\frac{1}{2r}[A_n^1 r^n + + B_n^1 r^{-n-1}][C_n^1\{\xi P_n^1(\xi) - P_{n-1}^1(\xi)\} + D_n^1\{\xi Q_n^1(\xi) - - Q_{n-1}^1(\xi)\}] + \frac{(n+1)}{\sqrt{1-\xi^2}}\frac{\mu+\kappa}{\kappa} r^{-\frac{3}{2}}[A_n^2 I_{n+\frac{1}{2}}(\lambda r) +$$

$$+ B_n^2 K_{n+\frac{1}{2}}(\lambda r)][C_n^2\{\xi P_n^1(\xi) - P_{n-1}^1(\xi)\} + D_n^2\{\xi Q_n^1(\xi) - Q_{n-1}^1(\xi)\}], \tag{25}$$

$$v_\theta = \sum_{n=1}^{\infty} \frac{nr^{-\frac{3}{2}}}{b^2\sqrt{1-\xi^2}}[A_n^3 I_{n+\frac{1}{2}}(br) + B_n^3 K_{n+\frac{1}{2}}(br)][C_n^3\{\xi P_n(\xi) - P_{n-1}(\xi)\} + D_n^3\{\xi Q_n(\xi) - Q_{n-1}(\xi)\}] - \frac{1}{2}[C_n^1 P_n^1(\xi) + D_n^1 Q_n^1(\xi)][(n+1)A_n^1 r^{n-1} - nB_n^1 r^{-n-2}] - \frac{\mu+\kappa}{\kappa}[C_n^2 P_n^1(\xi) + D_n^2 Q_n^1(\xi)] \times [A_n^2\{\lambda r^{-\frac{1}{2}} I_{n-\frac{1}{2}}(\lambda r) - nr^{-\frac{3}{2}} I_{n+\frac{1}{2}}(\lambda r)\} - B_n^2\{\lambda r^{-\frac{1}{2}} K_{n-\frac{1}{2}}(\lambda r) + nr^{-\frac{3}{2}} K_{n+\frac{1}{2}}(\lambda r)\}]. \tag{26}$$

We have thus generated the complete general solution for slow steady axisymmetrical rotational flow. In our particular case of flow due to the rotation of an axially symmetric body in an unbounded micropolar medium, the condition at infinity demands that for all n

$$A_n^1 = A_n^2 = A_n^3 = 0,$$

while

$$D_n^1 = D_n^2 = D_n^3 = O,$$

for all n because of the singularity of the functions $Q_n(\xi)$ and $Q_n^1(\xi)$ along the axis $\xi = \pm 1$. Hence our solution will be of the form

$$v_\phi = \sum_{n=1}^{\infty} P_n^1(\xi)[A_n r^{-n-1} + B_n r^{-\frac{1}{2}} K_{n+\frac{1}{2}}(\lambda r)], \tag{27}$$

$$v_r = \sum_{n=1}^{\infty} \frac{C_n P_n(\xi)}{b^2}[br^{-\frac{1}{2}} K_{n-\frac{1}{2}}(br) + (n+1)r^{-\frac{3}{2}} k_{n+\frac{1}{2}}(br)] + \frac{(n+1)}{\sqrt{1-\xi^2}}[\xi P_n^1(\xi) - P_{n-1}^1(\xi)] \times \left[\frac{A_n}{2} r^{-n-2} + \frac{\mu+\kappa}{\kappa} B_n r^{-\frac{3}{2}} K_{n+\frac{1}{2}}(\lambda r)\right], \tag{28}$$

$$v_\theta = \sum_{n=1}^{\infty} \frac{nC_n r^{-\frac{3}{2}}}{b^2\sqrt{1-\xi^2}} K_{n+\frac{1}{2}}(br)[P_{n-1}(\xi) - \xi P_n(\xi)] + $$

$$+ P_n^1(\xi)\left[\frac{nA_n}{2} r^{-n-2} + \frac{\mu+\kappa}{\kappa} B_n\{\lambda r^{-\frac{1}{2}} K_{n-\frac{1}{2}}(\lambda r) + \right.$$

$$\left. + nr^{-\frac{3}{2}} K_{n+\frac{1}{2}}(\lambda r)\}\right], \tag{29}$$

where A_n, B_n, C_n are constants to be determined from the boundary conditions. We now make the following observations:

(i) The above solutions will also be valid for the case of an axially symmetric system of finite bodies which slowly rotates about its axis of symmetry.

(ii) As $\kappa \to 0$, $v_\phi \to \sum_{n=1}^{\infty} A_n P_n^1(\xi) r^{-n-1}$ which is the classical result [7].

(iii) The far field solution for the velocity field is similar to that for classical fluids since (27) implies that

$$v_\phi = 0(r^{-2}) \text{ as } r \to \infty.$$

§ 3. Flow about a rotating sphere

As a particular case of the solution given by (27)–(29), we have for $n = 1$,

$$v_\phi = [A_1 r^{-2} + B_1 r^{-\frac{1}{2}} K_{\frac{3}{2}}(\lambda r)] \sin\theta, \tag{30}$$

$$v_r = \left[A_1 r^{-3} + 2\frac{(\mu+\kappa)}{\kappa} B_1 r^{-\frac{3}{2}} K_{\frac{3}{2}}(\lambda r) + \frac{C_1}{b^2}\{br^{-\frac{1}{2}} K_{\frac{1}{2}}(br) + \right.$$

$$\left. + 2r^{-\frac{3}{2}} K_{\frac{3}{2}}(br)\}\right] \cos\theta, \tag{31}$$

$$v_\theta = \left[\frac{A_1}{2} r^{-3} + \frac{(\mu+\kappa)}{\kappa} B_1 \{\lambda r^{-\frac{1}{2}} K_{\frac{1}{2}}(\lambda r) + r^{-\frac{3}{2}} K_{\frac{3}{2}}(\lambda r) + \right.$$

$$\left. + \frac{C_1 r^{-\frac{3}{2}}}{b^2} K_{\frac{3}{2}}(br)\right] \sin\theta, \tag{32}$$

which are identical to that obtained by Rao et al. [8] for the case of slow steady rotation of a sphere. If the sphere rotates with a constant angular velocity Ω_0, then the boundary conditions of no-slip and no-spin lead to the following equations on the surface $r = a$:

$$v_\phi = \Omega_0 a \sin\theta, \qquad v_r = v_\theta = 0. \tag{33}$$

Substitution of these boundary conditions (33) into (30), (31) and (32) determines the arbitrary constants A_1, B_1 and C_1 whose values are noted here for future reference:

$$A_1 = \frac{2\Omega_0 a^3(\mu+\kappa)}{A}[(2+2ab+a^2b^2)\lambda^2 + b^2(1+a\lambda)],$$

$$B_1 = -\frac{\Omega_0 \lambda \kappa a^{\frac{5}{2}} b^2}{A K_{\frac{1}{2}}(\lambda a)}, \tag{34}$$

$$C_1 = \frac{2\Omega_0 \lambda^2 a^{\frac{5}{2}} b^3(\mu+\kappa)}{A K_{\frac{1}{2}}(ab)},$$

where $A = 2\lambda^2(\mu+\kappa)(2+2ab+a^2b^2) + b^2(2\mu+\kappa)(1+a\lambda)$.

Evaluation of the couple is often a troublesome problem even more so in the case of micropolar fluids where the couple has contributions from the Cauchy stress t_{kl} as well as the couple stress m_{kl} which these fluids can sustain. In the relatively simple case of a sphere, the expression for the couple N has been calculated [8] and is found to be

$$N = 4\pi(2\mu+\kappa)A_1, \tag{35}$$

where A is given by (34). Note that in the limit as $\kappa \to 0$, we recover the well-known classical result of Stokes [9], i.e.,

$$N_c = 8\pi\mu a^3 \Omega_0. \tag{36}$$

To calculate the couple for any other body would entail cumbersome working and so it would be useful to have a simple formula for this couple.

Kanwal [4] derived such a relation between the couple experienced by an axially symmetric body and the angular velocity of the flow for Newtonian fluids. With the aid of a point or concentrated couple, we now proceed to establish a similar relation for micropolar fluids.

§ 4. Concentrated couple

The mathematical technique devised in our companion paper [1] to obtain the fundamental singular solution due to a point force, consisted of a Helmholtz decomposition followed by a three-dimensional Fourier transform. Here however, the method of associated matrices [10] shall be utilized as an alternative.

Let

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix},$$

$$Y = \begin{bmatrix} 0 & -X_3 & X_2 \\ X_3 & 0 & -X_1 \\ -X_2 & X_1 & 0 \end{bmatrix}, \qquad Z = \begin{bmatrix} X_1^2 & X_1X_2 & X_1X_3 \\ X_2X_1 & X_2^2 & X_2X_3 \\ X_3X_1 & X_3X_2 & X_3^2 \end{bmatrix}$$

be matrices with elements as real numbers. If $d^2 = X_1^2 + X_2^2 + X_3^2$ and the superscript "t" over a matrix denotes its transpose, then the following can easily be verified:

$$X^tX = d^2, \quad X^tY = 0, \quad X^tZ = d^2X^t,$$

$$YX = 0, \quad Y^2 = -d^2I + Z, \quad YZ = 0,$$

$$ZX = d^2X, \quad ZY = 0, \quad Z^2 = d^2Z,$$

$$XX^t = Z.$$

Denoting matrix operators A, X, Y, Z etc. by capitals and working in Cartesian co-ordinates (x_1, x_2, x_3), we let $\partial/\partial x_i = X_i$ $(i = 1, 2, 3)$. It then follows that

$$\nabla^2 = d^2, \quad \nabla \times \boldsymbol{u} = Y\boldsymbol{u} \quad \text{and} \quad \nabla\nabla \cdot \boldsymbol{u} = Z\boldsymbol{u},$$

where a vector $\boldsymbol{u}$ is represented by the column matrix $\{u_1, u_2, u_3]^t$. The system of equations given by (4), (5) and (6) can now be represented in matrix form as

$$A \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\nu} \\ p \end{bmatrix} = \begin{bmatrix} -\boldsymbol{f} \\ -\boldsymbol{l} \\ o \end{bmatrix}, \tag{37}$$

where the matrix A is given by

$$A = \begin{bmatrix} L_1I & \kappa Y & -X \\ \kappa Y & L_2I + (\alpha + \beta)Z & o \\ X^t & o & o \end{bmatrix}, \tag{38}$$

where $L_1 = (\mu + \kappa)d^2$ and $L_2 = \gamma d^2 - 2\kappa$.

The solution of (37) *takes the form*

$$\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\nu} \\ p \end{bmatrix} = A^{-1} \begin{bmatrix} -\boldsymbol{f} \\ -\boldsymbol{l} \\ o \end{bmatrix}, \tag{39}$$

and so the problem reduces to finding the inverse matrix A^{-1} of (38).

After some working this is found to be

$$A^{-1} = \begin{bmatrix} \dfrac{L_2 d^2 I - L_2 Z}{L_3 d^2} & -\dfrac{\kappa Y}{L_3} & \dfrac{X}{d^2} \\ -\dfrac{\kappa Y}{L_3} & \dfrac{L_1 L_4 I + \{\kappa^2 - (\alpha+\beta)L_1\}Z}{L_3 L_4} & o \\ -\dfrac{X^t}{d^2} & o & \mu + \kappa \end{bmatrix}, \tag{40}$$

where $L_3 = L_1 L_2 + \kappa^2 d^2$ and $L_4 = (\alpha + \beta + \gamma)d^2 - 2\kappa$. Substitution of (40) into (39) now produces the following Galerkin-type representations for the field parameters $\boldsymbol{v}$, $\boldsymbol{\nu}$, and p:

$$\boldsymbol{v} = \nabla^2[\gamma\nabla^2 - 2\kappa]\boldsymbol{\Psi} - [\gamma\nabla^2 - 2\kappa]\nabla\nabla\cdot\boldsymbol{\Psi} - \\ - \kappa[\alpha + \beta + \gamma)\nabla^2 - 2\kappa]\nabla \times \boldsymbol{\varphi} + \nabla\Phi, \tag{41}$$

$$\boldsymbol{\nu} = -\kappa\nabla^2(\nabla \times \boldsymbol{\Psi}) + (\mu + \kappa)\nabla^2[(\alpha + \beta + \gamma)\nabla^2 - 2\kappa]\boldsymbol{\varphi} + \\ + [\kappa^2 - (\alpha + \beta)(\mu + \kappa)\nabla^2]\nabla\nabla\cdot\boldsymbol{\varphi}, \tag{42}$$

$$p = -\nabla^2[\gamma(\mu + \kappa)\nabla^2 - \kappa(2\mu + \kappa)]\nabla\cdot\boldsymbol{\Psi}, \tag{43}$$

where $\boldsymbol{\Psi}$, $\boldsymbol{\varphi}$ and Φ satisfy the equations

$$\nabla^4[\gamma(\mu + \kappa)\nabla^2 - \kappa(2\mu + \kappa)]\boldsymbol{\Psi} = -\boldsymbol{f}, \tag{44}$$

$$\nabla^2[(\alpha + \beta + \gamma)\nabla^2 - 2\kappa][\gamma(\mu + \kappa)\nabla^2 - \kappa(2\mu + \kappa)]\boldsymbol{\varphi} = -\boldsymbol{l} \tag{45}$$

$$\nabla^2\Phi = 0. \tag{46}$$

It is immediately obvious that

$$\nabla^2 p = \nabla\cdot\boldsymbol{f},$$

so that in the absence of body forces, p is in general a harmonic function. We now examine the particular case of a concentrated couple in the absence of any body force in an infinite unbounded medium otherwise at rest. To enable us to do this, we will require the three-dimensional solutions of the singular equations

$$[\nabla^2, \nabla^4, \nabla^2 + a_0^2, (\nabla^2 + a_1^2)(\nabla^2 + a_2^2), (\nabla^2 + a_1^2)(\nabla^2 + a_2^2)(\nabla^2 + a_3^2)]g = = -\delta(x-y)[1, 1, 1, 1, 1] \tag{47}$$

which are given by

$$g = \frac{1}{4\pi}\left[\frac{1}{r}, \frac{r}{2}, \frac{\exp(ia_0 r)}{r}, \frac{\exp(ia_1 r) - \exp(ia_2 r)}{r(a_1^2 - a_2^2)}, \sum_{s=1}^{3} E_s \frac{\exp(ia_s r)}{r}\right] \tag{48}$$

where

$$x = (x_1, x_2, x_3),$$

$$r^2 = \sum_{i=1}^{3} (x_i - y_i)^2, \quad E_s = [(a_s^2 - a_m^2)(a_s^2 - a_n^2)]^{-1},$$

$s \neq m \neq n$, $s, m, n, = 1, 2, 3$ and $\delta(x-y)$ is the Dirac delta function.

Let

$$\boldsymbol{f} = 0, \quad \boldsymbol{l} = \boldsymbol{N}\delta(\mathrm{x} - \mathrm{y})$$

with $\boldsymbol{N}$ as an arbitrary constant vector acting at an arbitrary point y. It can be easily verified using (47) and (48) that the solutions of the equations (44), (45) and (46) are given respectively by

$$\boldsymbol{\Psi} = 0, \tag{49}$$

$$\boldsymbol{\psi} = \frac{\boldsymbol{N}}{4\pi\gamma(\mu+\kappa)(\alpha+\beta+\gamma)}\left[\frac{1}{\lambda^2 b^2}\frac{1}{r} + \frac{1}{b^2(b^2-\lambda^2)} \times \times \frac{\exp(-br)}{r} + \frac{1}{\lambda^2(\lambda^2-b^2)}\frac{\exp(-\lambda r)}{r}\right], \tag{50}$$

$$\boldsymbol{\Phi} = 0, \tag{51}$$

and consequently (41), (42) and (43) yield

$$\boldsymbol{v} = \frac{1}{4\pi(2\mu+\kappa)} \nabla \times \boldsymbol{N} \left[\frac{1 - \exp(-\lambda r)}{r} \right] \tag{52}$$

$$\begin{aligned} \boldsymbol{v} - = {} & \frac{\boldsymbol{N}}{4\pi\gamma} \frac{\exp(-\lambda r)}{r} + \frac{\nabla\nabla\cdot\boldsymbol{N}}{4\pi\gamma(\mu+\kappa)(\alpha+\beta+\gamma)} \times \\ & \times \left[\frac{\kappa^2}{\lambda^2 b^2} \frac{1}{r} + \frac{\kappa^2 - \lambda^2(\alpha+\beta)(\mu+\kappa)}{\lambda^2(\lambda^2 - b^2)} \frac{\exp(-\lambda r)}{r} + \right. \\ & \left. + \frac{\kappa^2 - b^2(\alpha+\beta)(\mu+\kappa)}{b^2(b^2-\lambda^2)} \frac{\exp(-br)}{r} \right], \end{aligned} \tag{53}$$

$$p = 0. \tag{54}$$

These then, are the fundamental singular solutions describing the axisymmetric flow field due to a point couple and we shall now utilize these results to obtain a general expression for the couple experienced by an axially symmetric body in Stokes' flow.

§ 5. Couple on an axially symmetric body

Introducing cylindrical polar co-ordinates (R, ϕ, z), we let ${}_c\boldsymbol{v}$ represent the velocity field due to a point couple of magnitude N acting at the origin along the z-direction. Hence

$${}_c\boldsymbol{v} = {}_c v \hat{\boldsymbol{e}}_\phi, \quad \boldsymbol{N} = N \hat{\boldsymbol{e}}_z, \tag{55}$$

where $\hat{\boldsymbol{e}}_\phi$, $\hat{\boldsymbol{e}}_z$ are the usual unit vectors.

Substitution of (55) into (52), produces the following simplified expression for ${}_c v$:

$${}_c v = \frac{N}{4\pi(2\mu+\kappa)} \left[\frac{R}{r^3} - \frac{R\exp(-\lambda r)}{r^3} - \frac{R\lambda\exp(-\lambda r)}{r^2} \right],$$

or

$$r^3 {}_c\Omega = \frac{N}{4\pi(2\mu+\kappa)} [1 - \exp(-\lambda r) - r\lambda\exp(-\lambda r)], \tag{56}$$

where ${}_c\Omega$ is the angular velocity.

As $\kappa \to 0$, ${}_c v \to 0$, which is expected for classical fluids.

In order to establish the required formula, an appeal is made to an argument advanced by Brenner [2] and utilized for the drag formula [1]. The argument is the following: If the medium is unbounded, at a sufficiently large distance from the obstacle, the flow field must become identical to that which would be generated by the action of a point couple equal in magnitude to the couple on the obstacle, provided the fluid is at rest at infinity. Hence if N is the couple on any axially symmetric body rotating uniformly about its axis of symmetry and Ω is the angular velocity associated with the flow generated, then

$$\lim_{r\to\infty} r^3{}_c\Omega = \lim_{r\to\infty} r^3\Omega. \tag{57}$$

This, together with (56), produces the desired result

$$N = 4\pi(2\mu + \kappa) \lim_{r\to\infty} r^3\Omega. \tag{58}$$

With the help of this formula, one can evaluate the couple experienced on any axially symmetric body simply from a knowledge of the angular velocity of the flow and a simple limiting process.

If we put $\kappa = 0$, (58) reduces to the classical formula

$$N_0 = 8\pi\mu \lim_{r\to\infty} r^3\Omega \tag{59}$$

obtained by Kanwal [4]. It is of interest to compare (58) and (59) with the corresponding drag formulae given respectively by the author [1] and Payne and Pell [3]. They are:

$$D = 4\pi(2\mu + \kappa) \lim_{r\to\infty} \frac{r\psi}{R^2}$$

$$D_0 = 8\pi\mu \lim_{r\to\infty} \frac{r\psi}{R^2},$$

where ψ is the stream function.

As an application of our main result (58), we now deduce the couple on a rotating sphere. From (30), we see that the angular velocity of the flow field is of the form

$$\Omega = A_1 r^{-3} + B_1 r^{-\frac{3}{2}} K_{\frac{3}{2}}(\lambda r),$$

where the constants A_1 and B_1 are given explicitly by (34). With the aid of (58), the couple experienced by the sphere is easily obtained as

$$N = \lim_{r\to\infty} r^3\Omega = 4\pi(2\mu + \kappa)A_1$$

which is identical to (35).

In conclusion we make the following observations in the case of a sphere and we note that these observations are similar to those for the drag problem [1]:

(i) From (35) and (36), it can be verified that $N/N_0 > 1$. In fact, we can establish the following bounds:

$$1 + \frac{\kappa}{2\mu} < N/N_0 < 1 + \frac{\kappa}{\mu}.$$

(ii) The couple increases with κ/μ, so that in the case of very strong coupling ($\kappa \gg \mu$), the couple becomes infinitely great.

(iii) In comparison to the classical theory, micropolar fluid theory gives rise to an increased couple. Whether an increased couple is experienced by any axisymmetric body in the case of micropolar theory, is a problem worth investigating now that a simple formula for the couple has been obtained.

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