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# Remarks on Priestley Duality for Distributive Lattices

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**Abstract.** The notion of a *Priestley relation* between Priestley spaces is introduced, and it is shown that there is a duality between the category of bounded distributive lattices and 0-preserving join-homomorphisms and the category of Priestley spaces and Priestley relations. When restricted to the category of bounded distributive lattices and 0–1-preserving homomorphisms, this duality yields essentially Priestley duality, and when restricted to the subcategory of Boolean algebras and 0-preserving join-homomorphisms, it coincides with the Halmos–Wright duality. It is also established a duality between 0–1-sublattices of a bounded distributive lattice and certain preorder relations on its Priestley space, which are called *lattice preorders*. This duality is a natural generalization of the Boolean case, and is strongly related to one considered by M. E. Adams. Connections between both kinds of dualities are studied, obtaining dualities for closure operators and quantifiers. Some results on the existence of homomorphisms lying between meet and join homomorphisms are given in the Appendix.

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## Introduction

In a classical paper, B. Jónsson and A. Tarski [12] showed that a Boolean algebra endowed with a family of join-preserving operations can be represented as a subalgebra of a power set $\mathfrak{P}(X)$, in such a way that the operations are in correspondence with certain relations defined on the set $X$. Later on, P. R. Halmos [9], [10] characterized the relations between Boolean spaces (i.e., totally disconnected and compact topological spaces) which correspond to 0-preserving join-homomorphisms between the corresponding algebras of clopen (i.e., closed and open) sets. These relations were called *Boolean relations*. F. B. Wright [21] completed these results by showing that the classical Stone duality between Boolean algebras and homomorphisms and Boolean spaces and continuous functions can be extended to a duality between Boolean algebras and 0-preserving join-homomorphisms and Boolean spaces and Boolean relations (see also [7]).

On the other hand, there is a duality between the subalgebras of a Boolean algebra and certain equivalence relations defined on its Boolean space (see, for instance, [14]. Note that in [14] the duals of the subalgebras are called *Boolean equivalences*, a nomenclature which is in conflict with the one previously established by Halmos).

The connections between the duality for join-homomorphisms and that for subalgebras is given by the quantifiers. A *quantifier* on a Boolean algebra $A$ is a closure operator on $A$ such that its range is a subalgebra of $A$. The dual of a quantifier $Q$ considered as a 0-preserving join-homomorphism is an equivalence relation which is also the dual of the range of $Q$.

Stone duality for Boolean algebras has been generalized by H. A. Priestley ([16], [17], see also the survey paper [18]) to a duality between bounded distributive lattices and certain ordered topological spaces, which are known as *Priestley spaces* (see §1). It was shown in [5] that the Boolean duality between quantifier ranges and equivalence relations can be extended to a duality between quantifiers on distributive lattices and certain equivalence relations on the corresponding Priestley spaces. Moreover, L. Vrancken-Mawet [20] established a duality between sublattices which are closed under relative complementation and certain equivalence relations on the corresponding Priestley spaces, while M. E. Adams [1] discovered a duality between sublattices and certain subsets of Priestley spaces. On the other hand, G. Hansoul [11] extended some of the results of [12] to bounded distributive lattices.

The aim of this paper is to consider in a systematic way the extension of both kinds of Boolean dualities to Priestley duality. In Section 1, after recalling some definitions and the main facts about Priestley duality, we introduce the notion of a *Priestley relation* between Priestley spaces, and we show that there is a duality between the category of bounded distributive lattices and 0-preserving join-homomorphisms and the category of Priestley spaces and Priestley relations. When restricted to the category of bounded distributive lattices and 0–1-preserving homomorphisms, this duality yields essentially Priestley duality, and when restricted to the subcategory of Boolean algebras and 0-preserving join-homomorphisms, it coincides with the Halmos–Wright duality. In Section 2 we establish a duality between 0–1-sublattices of a bounded distributive lattice and certain preorder relations on its Priestley space, which are called *lattice preorders*. This duality is a natural generalization of the Boolean case, and is strongly related to that of Adams [1]. Finally the relations among this duality and those developed in [20] and [5] are considered.

P. D. Bacsich [2] showed that a theorem of A. Monteiro [15] on the extension of Boolean homomorphisms dominated by join-homomorphisms is effectively equivalent to the well known theorem of R. Sikorski on the extension of Boolean homomorphisms. S. Graf [7] derived a slightly more general form of Monteiro's theorem from a general selection theorem for Boolean relations between Boolean spaces. Since we do not have in mind any interesting application, we do not try to generalize this selection theorem, but in the Appendix we show that the distributive lattice version of a theorem on the existence of certain minimal join-homomorphisms, which plays an important role in the proof of the selection theorem in [7], is in fact equivalent to the axiom of choice. We also establish the equivalence of a distributive lattice version of Monteiro's theorem given in [4] with a theorem on the existence of some homomorphisms on bounded distributive lattices given in [2]. The equivalence between the main theorem of [4] and Sikorski extension theorem was also considered in [8] (but note that in the statement of the lattice extension theorem on p. 51 of [8], the fundamental condition (C) (see statement (iii) in the Appendix) is missing).

## 1. Join-Homomorphisms and Priestley Relations

In this section $L, M$ will denote bounded distributive lattices. By a join-homomor phism from $L$ into $M$ we understand a mapping $j\colon L \to M$ such that $j(0) = 0$ and $j(a \vee b) = j(a) \vee j(b)$. The meet-homomorphisms are defined dually. A mapping $h\colon L \to M$ is a homomorphism if and only if it is both a join-homomorphism and a meet-homomorphism. The category of bounded distributive lattices and join-(meet-)homomorphisms will be denoted by $\mathscr{J}(\mathscr{M})$, and $\mathscr{D}$ will denote the subcategory of bounded distributive lattices and homomorphisms. Note that the isomorphisms in these categories are the same: the one-to-one and onto homomorphisms.

Given a relation $R \subseteq X \times Y$, for each $Z \subseteq X$, $R(Z)$ will denote the image of $Z$ by $R$, i.e.,

$$R(Z) = \{y \in Y \mid \text{there is } x \in Z \text{ such that } (x, y) \in R\}$$

and for each $Z \subseteq Y$, $R^{-1}(Z)$ will denote the inverse image of $Z$ by $R$, i.e.,

$$R^{-1}(Z) = \{x \in X \mid R(\{x\}) \cap Z \neq \emptyset\}.$$

Note that the domain of $R$ is $R^{-1}(Y)$, in symbols, $\mathrm{dom}(R) = R^{-1}(Y)$. When $x \in X$ ($y \in Y$), we are going to write $R(x)(R^{-1}(y))$ instead of $R(\{x\})(R^{-1}(\{y\}))$.

Let $X$ be a poset (=partially ordered set) and $Y \subseteq X$. We shall denote by $(Y]([Y))$ the set of all $x$ in $X$ such that $x \leqslant y$ ($y \leqslant x$) for some $y \in Y$. $Y$ is *increasing* (*decreasing*) if $Y = [Y)$ ($Y = (Y]$).

A *totally order-disconnected topological space* is a triple $(X, \leqslant_X, \tau)$ such that $(X, \leqslant_X)$ is a poset, $(X, \tau)$ is a topological space and given $x, y$ in $X$ such that $x \nleqslant y$, there is a clopen (=closed and open) increasing set $U$ such that $x \in U$ and $y \notin U$. A *Priestley space* is a compact totally order-disconnected topological space. Given a Priestley space $X$, $\mathbf{D}(X)$ will denote the lattice of increasing clopen subsets of $X$.

Given a bounded distributive lattice $L$, $\mathbf{X}(L)$ will denote the Priestley space of $L$, i.e. $\mathbf{X}(L)$ is the set of prime filters of $L$, ordered by inclusion and with the topology having as a sub-basis the sets of the form $\sigma_{\mathrm{L}}(a) = \{P \in \mathbf{X}(L) \mid a \in P\}$ and $\mathbf{X}(L) \backslash \sigma_{\mathrm{L}}(a)$ for each $a \in L$.

It was shown by H. A. Priestley [16], [17] (see also the survey article [18]) that $\sigma_{\mathbf{L}}: L \to \mathbf{D}(\mathbf{X}(L))$ is a lattice isomorphism and that the mapping $\varepsilon_{\mathbf{x}}: X \to \mathbf{X}(\mathbf{D}(X))$ defined by the prescription $\varepsilon_{\mathbf{X}}(x) = \{U \in \mathbf{D}(X) \mid x \in U\}$ is both a homeomorphism and an order isomorphism.

1.1. LEMMA. *Let $j \in \mathscr{J}(L, M)$, $Q \in \mathbf{X}(M)$ and $T \subseteq L$. If given $t_1, \ldots, t_k$ in $T$, there is an $a \in j^{-1}(Q)$ such that $a \leqslant t_1 \wedge \cdots \wedge t_k$, then there is a $P \in \mathbf{X}(L)$ such that $T \subseteq P \subseteq j^{-1}(Q)$.*

*Proof.* Note first that since $Q$ is a prime filter of $M$, then $I = M \backslash j^{-1}(Q)$ is an ideal of $L$. Let $F = F(T)$ be the filter generated in $L$ by the subset $T$. Since $j^{-1}(Q)$ is an increasing subset of $L$, the hypothesis on $T$ implies that $F \subseteq j^{-1}(Q)$, i.e. $F \cap I = \emptyset$. Therefore, by the Birkhoff–Stone Theorem [3, III.4. Theorem 1], there is a prime filter $P$ of $L$ such that $F \subseteq P$ and $P \cap I = \emptyset$. ∎

1.2. DEFINITION. Let $X$ and $Y$ be Priestley spaces. A relation $R \subseteq X \times Y$ is said to be a *Priestley relation* provided the following conditions are satisfied:

(i) For each $x \in X$, $R(x)$ is a closed and decreasing subset of $Y$.

(ii) For each $V \in \mathbf{D}(Y)$, $R^{-1}(V) \in \mathbf{D}(X)$.

A Priestley relation is said to be *functional* in case $\operatorname{dom}(R) = X$ and $R(x)$ has a greatest element for each $x$ in $X$.

We are going to denote by $\mathscr{P}(X, Y)$ the set of all Priestley relations $R \subseteq X \times Y$, and by $\mathscr{F}(X, Y)$ the subset of functional Priestley relations, where $X$ and $Y$ are Priestley spaces.

REMARKS. Let $R \in \mathscr{P}(X, Y)$. Since $Y \in \mathbf{D}(Y)$ and $\operatorname{dom}(R) = R^{-1}(Y)$, we see that $\operatorname{dom}(R) \in \mathbf{D}(X)$. Moreover, $R$ is a closed subset of the product space $X \times Y$. Indeed, suppose $(x, y) \notin R$. Then $y \notin R(x)$, and by condition (i) in Definition 1.2 there is $V \in \mathbf{D}(Y)$ such that $y \in V$ and $V \cap R(x) = \emptyset$. By condition (ii) in the same definition, $R^{-1}(V) \in \mathbf{D}(X)$. Therefore $(X \backslash R^{-1}(V), V)$ is a neighborhood of $(x, y)$ disjoint from $R$.

1.3. EXAMPLES. (i) The empty relation is trivially a Priestley relation.

(ii) Let $X$, $Y$ be Priestley spaces. If $f: X \to Y$ is a continuous and monotonic function, then $R_f = \{(x, y) \in X \times Y \mid y \leqslant f(x)\} \in \mathscr{F}(X, Y)$. Indeed, $R_f(x) = (f(x)]$, which is a closed [18, Proposition 2.6] and decreasing subset of $Y$ and $R_f^{-1}(V) = f^{-1}(V)$ for all increasing subsets of $Y$. Conversely, with each $R \in \mathscr{F}(X, Y)$ associate the function $f_{\mathbf{R}}: X \to Y$ by defining $f_{\mathbf{R}}(x)$ as the greatest element of $R(x)$. Since for each $U \in \mathbf{D}(Y)$, $f_{\mathbf{R}}^{-1}(U) = R^{-1}(U) \in \mathbf{D}(X)$, it follows that $f_{\mathbf{R}}$ is continuous and monotonic. It is plain that $R = R_{f_{\mathbf{R}}}$ and $f = f_{\mathbf{R}_f}$.

(iii) In particular, for each Priestley space $X$, the dual order $\geqslant_{\mathbf{x}} \subseteq X \times X$ is the functional Priestley relation associated with the identity function on $X$.

(iv) Let $L, M$ be bounded distributive lattices. For each $j \in \mathscr{J}(L, M)$, $j^* = \{(Q, P) \in \mathbf{X}(M) \times \mathbf{X}(L) \mid P \subseteq j^{-1}(Q)\}$ is a Priestley relation, and $\operatorname{dom}(j^*) =$

$\{Q \in \mathbf{X}(M) \mid j(1) \in Q\}$. Indeed, it is plain that $j^*(Q)$ is a decreasing closed subset of $\mathbf{X}(M)$ for each $Q \in \mathbf{X}(L)$ and that $j^{*-1}(\sigma_{\mathrm{L}}(a)) \subseteq \sigma_{\mathrm{M}}(j(a))$ for each $a \in L$. Hence to prove that $j^*$ is a Priestley relation we need to show that $\sigma_{\mathrm{M}}(j(a)) \subseteq j^{*-1}(\sigma_{\mathrm{L}}(a))$ for each $a \in L$. Let $Q \in \sigma_{\mathrm{M}}(j(a))$. Then $a \in j^{-1}(Q)$. By Lemma 1.1 with $T = \{a\}$, there is a prime filter $P$ of $M$ such that $a \in P$ and $P \subseteq j^{-1}(Q)$. This implies that $j^*(Q) \cap \sigma_{\mathrm{L}}(a) \neq \emptyset$, i.e. that $Q \in j^{*-1}(\sigma_{\mathrm{L}}(a))$. Finally, $Q \in j^*(\mathbf{X}(L))$ if and only if there is $P \in \mathbf{X}(L)$ such that $P \subseteq j^{-1}(Q)$, and by Lemma 1.1, the last condition holds if and only if $j^{-1}(Q) \neq \emptyset$, i.e. if and only if $j(1) \in Q$.

(v) $j \in \mathscr{J}(L, M)$ is a homomorphism if and only if $j^* \in \mathscr{F}(\mathbf{X}(M), \mathbf{X}(L))$. For, suppose first that $j$ is a homomorphism. For each $Q \in \mathbf{X}(M)$, we have $j^*(Q) = (j^{-1}(Q)]$, because $j^{-1}(Q) \in \mathbf{X}(L)$. Therefore $j^* \in \mathscr{F}(\mathbf{X}(M), \mathbf{X}(L))$. To prove the converse, suppose now that $j^* \in \mathscr{F}(\mathbf{X}(M), \mathbf{X}(L))$. Note first that this implies $\mathbf{X}(M) = \mathrm{dom}(j^*) = \{Q \in \mathbf{X}(M) \mid j(1) \in Q\}$, i.e. $j(1) = 1$. Let $Q \in \mathbf{X}(M)$. There is $P_0 \in \mathbf{X}(L)$ such that $P \subseteq j^{-1}(Q)$ if and only if $P \subseteq P_0$. In particular, $P_0 \subseteq j^{-1}(Q)$. Suppose there is $a \in j^{-1}(Q) \backslash P_0$. Then we could apply Lemma 1.1 with $T = P_0 \cup \{a\}$ to prove the existence of a $P \in \mathbf{X}(L)$ such that $P_0 \subset P \subseteq j^{-1}(Q)$, a contradiction. Therefore $j^{-1}(Q) \in \mathbf{X}(L)$ for each $Q \in \mathbf{X}(M)$, and it is well known that this implies that $j$ is a lattice homomorphism. ∎

REMARK. When $X$ and $Y$ are Boolean spaces (i.e. Priestley spaces in which the order relation is the identity), Priestley relations coincide with the *Boolean relations* defined by Halmos [9]. Note that a Boolean relation is functional if and only if $R(x)$ is a singleton for each $x$ in $X$, i.e., if and only if $R$ is a function from $X$ into $Y$.

1.4. LEMMA. *Let $X$ and $Y$ be Priestley spaces. For each $R \in \mathscr{P}(X, Y)$ and elements $s, t$ of $X$, $s \leqslant t$ implies $R(s) \subseteq R(t)$.*

*Proof.* Suppose $s \leqslant t$ and take $y \notin R(t)$. Since $R(t)$ is closed and decreasing in $Y$, there is $V \in \mathbf{D}(Y)$ such that $y \in V$ and $V \cap R(t) = \emptyset$. This last equality means that $t \notin R^{-1}(V)$, and since by condition (i) in Definition 1.2, $R^{-1}(V)$ is an increasing set, we also have $s \notin R^{-1}(V)$, that is, $V \cap R(s) = \emptyset$, therefore $y \notin R(s)$. ∎

Note that the *composition of Priestley relations is a Priestley relation.* Indeed, suppose $R \in \mathscr{P}(X, Y)$ and $S \in \mathscr{P}(Y, Z)$. It is plain that the composite relation $SR$ satisfies property (ii) in Definition 1.2. For each $x \in X$,

$$SR(x) = \pi_2((R(x) \times Z) \cap S) = \bigcup \{S(y) \mid y \in R(x)\}.$$

Since by the remark following Definition 1.2, $S$ is a closed subset of $Y \times Z$, the first equality implies that $SR(x)$ is closed in $Z$ (cf. [10, (a) p. 165]), and the second one that it is decreasing. Therefore $SR \in \mathscr{P}(X, Z)$. On the other hand, for each $R \in \mathscr{P}(X, Y)$, we have that $R \geqslant_{\mathrm{x}} = R$ and $\geqslant_{\mathrm{y}} R = R$. Indeed, for each $x$ in $X$, by Lemma 1.4 $R \geqslant_{\mathrm{x}} (x) = \bigcup \{R(t) \mid x \geqslant t\} = R(x)$, and by (i) in Definition 1.2 $\geqslant_{\mathrm{y}} R(x) = \bigcup \{(s] \mid s \in R(x)\} = R(x)$. From these remarks we see that we can define the category $\mathscr{P}$ whose objects are the Priestley spaces and whose morphisms are the Priestley relations. The subcategory formed by the same objects but having the functional Priestley relations as morphisms will be denoted by $\mathscr{P}_0$. It follows from Example 1.3 (ii) that *$\mathscr{P}_0$ is equivalent to the category of Priestley spaces and continuous monotonic functions.*

Our next task will be to show that $\mathscr{P}$ is naturally equivalent to the opposite of the category $\mathscr{J}$.

1.5. LEMMA. (i) *For each $R \in \mathscr{P}(X, Y)$, the correspondence $U \mapsto R^{-1}(U)$ defines a join-homomorphism $R^*: \mathbf{D}(Y) \to \mathbf{D}(X)$, and $R^*(Y) = dom(R)$.*

(ii) *If $j \in \mathscr{J}(L, M)$ and $k \in \mathscr{J}(M, N)$, then $(kj)^* = j^*k^*$.*

(iii) *If $R \in \mathscr{P}(X, Y)$ and $S \in \mathscr{P}(Y, Z)$, then $(SR)^* = R^*S^*$.*

(iv) *If $j \in \mathscr{J}(L, M)$, then for each $a \in L$, $j^{**}(\sigma_{\mathbf{L}}(a)) = \sigma_{\mathbf{M}}(j(a))$.*

(v) *If $R \in \mathscr{R}(X, Y)$, then for each $x \in X$ and each $y \in Y$, $(x, y) \in R$ if and only if $(\varepsilon_{\mathbf{X}}(x), \varepsilon_{\mathbf{Y}}(y)) \in R^{**}$.*

*Proof.* (i) It is an immediate consequence of property (ii) in Definition 1.2.

(ii) It is easy to check that $j^*k^* \subseteq (kj)^*$. To prove the other inclusion, suppose $(S, P) \in (kj)^*$. Hence $P \subseteq j^{-1}(k^{-1}(S))$, i.e. $j(P) \subseteq k^{-1}(S)$. Let $b_1, \ldots, b_k$ be in $j(P)$. Then there are $a_1, \ldots, a_k$ in $P$ such that $b_i = j(a_i)$, $i = 1, \ldots, k$. Since $a = a_1 \wedge \cdots \wedge a_k \in P$, we can apply Lemma 1.1 with $T = j(P)$ to conclude the existence of $Q \in \mathbf{X}(L)$ such that $j(P) \subseteq Q \subseteq k^{-1}(S)$, and this implies that $(S, P) \in j^*k^*$.

(iii) It follows from the well known fact that $(SK)^{-1}(W) = K^{-1}(S^{-1}(W))$ for each $W \subseteq Z$.

(iv) By Lemma 1.1 with $T = \{a\}$, $Q \in \sigma_{\mathbf{M}}(j(a))$ is equivalent to the existence of a $P \in \mathbf{X}(L)$ such that $a \in P \subseteq j^{-1}(Q)$. This condition is equivalent to $j^*(Q) \cap \sigma_{\mathbf{L}}(a) \neq \emptyset$, which in turn is equivalent to $Q \in j^{**}(\sigma_{\mathbf{L}}(a))$.

(v) By condition (i) in Definition 1.2, $(x, y) \notin R$ if and only if there is $V \in \mathbf{D}(Y)$ such that $y \in V$ and $V \cap R(x) = \emptyset$. This is equivalent to $V \in \varepsilon_{\mathbf{Y}}(y)$ and $R^*(V) = R^{-1}(V) \notin \varepsilon_{\mathbf{X}}(x)$, which is equivalent to $V \in \varepsilon_{\mathbf{Y}}(y)$ and $V \notin R^{*-1}(\varepsilon_{\mathbf{X}}(x))$, which is equivalent to $(\varepsilon_{\mathbf{X}}(x), \varepsilon_{\mathbf{Y}}(y)) \notin R^{**}$. ■

It follows from Example 1.3 (iv) and properties (i)–(iii) in the above lemma that we can define contravariant functors $\mathbf{X}: \mathscr{J} \to \mathscr{P}$ and $\mathbf{D}: \mathscr{P} \to \mathscr{J}$ by defining $\mathbf{X}(j) = j^*$ for each join-homomorphism and $\mathbf{D}(R) = R^*$ for each Priestley relation. Moreover, since for each bounded distributive lattice $L$, $\sigma_{\mathbf{L}}: L \to \mathbf{D}(\mathbf{X}(L))$ is an isomorphism in $\mathscr{J}$, property (iv) means that the composite functor $\mathbf{DX}: \mathscr{J} \to \mathscr{J}$ is naturally equivalent to the identity functor, the natural equivalence being given by the isomorphisms $\sigma_{\mathbf{L}}$. On the other hand, since for each Priestley space $X$, $\varepsilon_{\mathbf{X}}$ is both a homeomorphism and an order isomorphism from $X$ onto $\mathbf{X}(\mathbf{D}(X))$, it follows that $\rho_{\mathbf{X}} = R_{\varepsilon_{\mathbf{X}}}$ is an isomorphism in $\mathscr{P}$. Let us see that these isomorphisms define a natural equivalence from the composite functor $\mathbf{XD}$ to the identity functor in $\mathscr{P}$. We need to prove that for each pair of Priestley spaces $X$ and $Y$ and each $R \in \mathscr{P}(X, Y)$,

$R^{**}\rho_x = \rho_y R$. Suppose $(x, \varepsilon_Y(y)) \in R^{**}\rho_X$. This means that there is $t \in X$ such that $(x, \varepsilon_X(t)) \in \rho_X$ and $(\varepsilon_X(t), \varepsilon_Y(y)) \in R^{**}$. It follows from the definition of $\rho_X$ and Lemma 1.5 (v) that these conditions are equivalent to $\varepsilon_X(t) \leqslant \varepsilon_X(x)$ and $(t, y) \in R$, and by taking into account that $\varepsilon_X$ is an order isomorphism and Lemma 1.4 we obtain that $(x, y) \in R$. Since obviously $(y, \varepsilon_Y(y)) \in \rho_y$, we have $(x, \varepsilon_Y(y)) \in \rho_Y R$. Therefore $R^{**}\rho_X \subseteq \rho_Y R$. A similar argument (which uses property (i) in Definition 1.2 instead of Lemma 1.4) shows that $\rho_Y R \subseteq R^{**}\rho_X$. Thus we have proved the following:

1.6. THEOREM. *The categories $\mathscr{J}$ and $\mathscr{P}^{op}$ are naturally equivalent. More precisely, the composite functors* **DX** *and* **XD** *are naturally equivalent to the identity functors of $\mathscr{J}$ and $\mathscr{P}$, respectively. The corresponding natural equivalences are $\sigma$ and $\rho$.* ■

In the above proof we have used the fact that $R_f$ is an isomorphism in $\mathscr{P}$ provided that $f: X \to Y$ is both a homeomorphism and an order isomorphism. In the next proposition we show that all isomorphisms in $\mathscr{P}$ are of this form.

1.7. PROPOSITION. *$R \in \mathscr{P}(X, Y)$ is an isomorphism in the category $\mathscr{P}$ if and only if there is a homeomorphism $f$ from $X$ onto $Y$ which is also an order isomorphism and such that $R = R_f$.*

*Proof.* Let $R \in \mathscr{P}(X, Y)$ be an isomorphism. Since **D** is a (contravariant) functor, $\mathbf{D}(R) = R^*$ is an isomorphism in $\mathscr{J}$, i.e., a one-to-one homomorphism from $\mathbf{D}(Y)$ onto $\mathbf{D}(X)$, and by Examples 1.3 (v) and (ii), there is a continuous and monotonic $g: \mathbf{X}(\mathbf{D}(X)) \to \mathbf{X}(\mathbf{D}(Y))$ such that $R^{**} = R_g$. It is easy to check that $g$ is in fact an order isomorphism and a homeomorphism. Therefore $f = \varepsilon_Y^{-1} g \varepsilon_X$ is an order isomorphism and a homeomorphism from $X$ onto $Y$. By Lemma 1.5 (v), $(x, y) \in R$ if and only if $(\varepsilon_X(x), \varepsilon_Y(y)) \in R_g$. This last condition is equivalent to $y \leqslant \varepsilon_Y^{-1}(g(\varepsilon_X(x)))$, i.e., equivalent to $(x, y) \in R_f$. The converse implication is obvious. ■

REMARK. By Example 1.3 (v), the restriction $\mathbf{X}_0$ of the functor $\mathbf{X}$ to the subcategory $\mathscr{D}$ is a contravariant functor of this category into $\mathscr{P}_0$, and it is easy to check that the functors **D** and $\mathbf{X}_0$ establish a natural duality between the categories $D$ and $\mathscr{P}_0$. Since, as noted above, $\mathscr{P}_0$ is equivalent to the category of Priestley spaces and continuous order-preserving functions, in this way we obtain essentially Priestley duality.

We now turn our attention to the case in which $Y = X$.

1.8. LEMMA. *The following properties hold true for each $R \in \mathscr{R}(X, X)$ and $U \in \mathbf{D}(X)$:*

(i) *$U \subseteq R^*(U)$ if and only if $R$ is reflexive.*

(ii) *$R^*(R^*(U)) \subseteq R^*(U)$ if and only if $R$ is transitive.*

*Proof.* (i) Each of the following conditions is equivalent to the next one: (1) $(x, x) \notin R$, (2) $x \notin R(x)$, (3) there is $U \in \mathbf{D}(X)$ such that $x \in U$ and $U \cap R(x) = \emptyset$ and (4) there is $U \in \mathbf{D}(X)$ such that $U \nsubseteq R^{-1}(U)$.

(ii) It is obvious. ■

## 2. Relations Associated with Sublattices

In this section we continue to denote by $L$ a bounded distributive lattice, and $X$ will denote a Priestley space.

For each $M \subseteq L$, define $M^{\#} = \{(Q, P) \in \mathbf{X}(L) \times \mathbf{X}(L) \mid P \cap M \subseteq Q\}$ and for each $R \subseteq X \times X$, define $R^{\#} = \{U \in \mathbf{D}(X) \mid R^{-1}(U) \subseteq U\}$.

REMARK. By Lemma 1.8 (i), if $R$ is *reflexive*, then $R^{\#} = \{U \in \mathbf{D}(X) \mid R^{-1}(U) = U\}$.

2.1. LEMMA. *The following properties hold true for each $M \subseteq L$:*

(i) *$M^{\#}$ is a preorder ( = reflexive and transitive) relation on $\mathbf{X}(L)$.*

(ii) *For each $a \in M$, $M^{\# -1}(\sigma_{\mathrm{L}}(a)) \subseteq \sigma_{\mathrm{L}}(a)$.*

(iii) *If $M$ is a 0–1-sublattice of $L$, then $M^{\# -1}(\sigma_{\mathrm{L}}(a)) \subseteq \sigma_{\mathrm{L}}(a)$ implies $a \in M$.*

*Proof.* (i) is obvious. To prove (ii), let $a \in M$. If $P \in M^{\# -1}(\sigma_{\mathrm{L}}(a))$, then there is $Q \in \sigma_{\mathrm{L}}(a)$ such that $Q \cap M \subseteq P$. Since $a \in Q \cap M$, we have $a \in P$, i.e., $P \in \sigma_{\mathrm{L}}(a)$. To prove (iii), take $a \notin M$. Let $F$ be the filter of $L$ generated by $[a) \cap M$. Since $(a] \cap F = \emptyset$, by the Birkhoff–Stone theorem there is a $P \in \mathbf{X}(L)$ such that $[a) \cap M \subseteq P$ and $a \notin P$. Let $I$ be the ideal of $L$ generated by $(L \backslash P) \cap M$. Since $I \cap [a) = \emptyset$, again by the Birkhoff–Stone theorem there is $Q \in \mathbf{X}(L)$ such that $a \in Q$ and $Q \cap M \cap (L \backslash P) = \emptyset$. This shows that $P \in M^{\#*}(\sigma_{\mathrm{L}}(a))$ and $P \notin \sigma_{\mathrm{L}}(a)$. ■

2.2. LEMMA. *The following properties hold true for each $R \subseteq X \times X$:*

(i) *$R^{\#}$ is a 0–1-sublattice of $\mathbf{D}(X)$.*

(ii) *If $(x, y) \in R$, then $(\varepsilon_{\mathrm{X}}(x), \varepsilon_{\mathrm{X}}(y)) \in R^{\#\#}$.*

(iii) *If $R$ satisfies the condition: $(l)$ Given $x, y$ in $X$ such that $(x, y) \notin R$, there is $U \in R^{\#}$ such that $y \in U$ and $x \notin U$, then $(\varepsilon_{\mathrm{X}}(x), \varepsilon_{\mathrm{X}}(y)) \in R^{\#\#}$ implies $(x, y) \in R$.*

*Proof.* (i) is very easy to check. To complete the proof, note first that $(\varepsilon_{\mathrm{X}}(x), \varepsilon_{\mathrm{X}}(y)) \notin R^{\#\#}$ if and only if there is $U \in R^{\#}$ such that $y \in U$ and $x \notin U$. Since this last condition implies $y \notin R(x)$, we have (ii), and (iii) is now obvious. ■

Motivated by (iii) in the above lemma, we introduce the following:

2.3. DEFINITION. A *lattice preorder* on $X$ is a preorder relation defined on $X$ which satisfies property $(l)$.

2.4. THEOREM. (i) *For each* $M \subseteq L$, $M^{\#\#} = \sigma_{\mathbf{L}}(M)$ *if and only if* $M$ *is a* $0-1$*-sublattice of* $L$.

(ii) *For each* $R \subseteq X \times X$, $R^{\#\#} = \{(\varepsilon_{\mathbf{X}}(x), \varepsilon_{\mathbf{X}}(y)) \mid (x, y) \in R\}$ *if and only if* $R$ *is a lattice preorder on* $X$.

*Proof.* (i) If $M$ is a $0-1$-sublattice of $L$, it follows from (ii) and (iii) in Lemma 2.1 that $M^{\#\#} = \sigma_{\mathbf{L}}(M)$. On the other hand, by (i) in Lemma 2.1, $M^{\#}$ is a preorder on $\mathbf{X}(L)$, and then by (i) in Lemma 2.2 it follows that $M^{\#\#}$ is a $0-1$-sublattice of $\mathbf{D}(\mathbf{X}(L))$. Since $\sigma_{\mathbf{L}}$ is an isomorphism, if $M^{\#\#} = \sigma_{\mathbf{L}}(M)$, then $M$ is a $0-1$-sublattice of $L$.

(ii) If $R$ is a lattice preorder on $X$, it follows from (ii) and (iii) in Lemma 2.2 that $R^{\#\#} = \{(\varepsilon_{\mathbf{X}}(x), \varepsilon_{\mathbf{X}}(y)) \mid (x, y) \in R\}$. On the other hand, by (i) in Lemma 2.2, $R^{\#}$ is a sublattice of $\mathbf{D}(X)$, and then by (i) in Lemma 2.1 $R^{\#\#}$ is a preorder on $\mathbf{X}(\mathbf{D}(X))$. Therefore, $R^{\#\#} = \{(\varepsilon_{\mathbf{X}}(x), \varepsilon_{\mathbf{X}}(y)) \mid (x, y) \in R\}$ implies that $R$ is a preorder on $X$, and since $R^{\#\#}$ satisfies ($l$) (see the proof of Lemma 2.2), it follows that $R$ also satisfies ($l$). ∎

Since for each family $\{R_i : i \in I\}$ of relations on $X$, $\bigcup \{R_i^{\#} : i \in I\} \subseteq (\bigcap \{R_i : i \in I\})^{\#}$, it follows that *the lattice preorders on* $X$, *ordered by inclusion, form a complete lattice*, which will be denoted by $\mathcal{O}(X)$. We will denote by $\mathcal{S}(L)$ the lattice of $0-1$-sublattices of $L$.

Let $M, N$ be in $\mathcal{S}(L)$. It is obvious that $M \subseteq N$ implies $N^{\#} \subseteq M^{\#}$. Suppose now that $N^{\#} \subseteq M^{\#}$ and let $a \in M$. By (ii) in Lemma 2.1 $M^{\#-1}(\sigma_{\mathbf{L}}(a)) \subseteq \sigma_{\mathbf{L}}(a)$, and since $N^{\#-1}(\sigma_{\mathbf{L}}(a)) \subseteq M^{\#-1}(\sigma_{\mathbf{L}}(a))$, by (iii) in the same lemma $a \in N$. Hence $M \subseteq N$ if and only if $N^{\#} \subseteq M^{\#}$. On the other hand, let $R$ be a lattice preorder on $\mathbf{X}(L)$, and let $M = \sigma_{\mathbf{L}}^{-1}(R^{\#}) = \{a \in L \mid R^{-1}(\sigma_{\mathbf{L}}(a)) \subseteq \sigma_{\mathbf{L}}(a)\}$. Since $\sigma_{\mathbf{L}}$ is an isomorphism, by (i) in Lemma 2.2 $M \in \mathcal{S}(L)$, and by condition ($l$), $M^{\#} = R$. Therefore we have proved the following:

2.5. THEOREM. *The correspondence* $M \mapsto M^{\#}$ *establishes an anti-isomorphism from the lattice* $\mathcal{S}(L)$ *onto the lattice* $\mathcal{O}(L)$. ∎

Note that

$$S(M) = M^{\#-1} \setminus \leqslant_{\mathbf{X}(L)} = \{(P, Q) \in \mathbf{X}(L) \times \mathbf{X}(L) \mid P \cap M \subseteq Q \text{ and } P \nsubseteq Q\}$$

is the *separating set of the subspace* $M$ introduced by Adams [1]. The connection between $S(M)$ and $M$ was established by considering essentially the equivalence $M^e = M^{\#} \cap M^{\#-1}$ (which is the kernel of the dual mapping of the inclusion $M \hookrightarrow L$) and showing directly that the quotient set $\mathbf{X}(L)/M^e$, endowed with the quotient topology and with the order induced on the equivalence classes by $M^{\#-1}$, is a Priestley space, order isomorphic and homeomorphic to $\mathbf{X}(M)$ (see [18, p. 50]).

As particular cases of lattice preorders we can consider *lattice orders* and *lattice equivalences*.

2.6. THEOREM. *The following propositions hold true for each $M \in \mathscr{S}(L)$:*

(i) *$M^{\#}$ is an order if and only if for each prime filter $P$ of $M$ there is exactly one prime filter $Q$ of $L$ such that $P = Q \cap M$.*

(ii) *$M^{\#}$ is an equivalence if and only if $M$ is a Boolean sublattice of the center of $L$.*

*Proof.* (i) is obvious. To prove (ii) observe first that $M$ is a Boolean sublattice of the center of $L$ if and only if it is a Boolean algebra. Suppose now that $M$ is a Boolean algebra and that $(P, Q) \in M^{\#}$. Since $Q \cap M$ and $P \cap M$ are maximal filters of $M$, we must have $Q \cap M = P \cap M$, and $(Q, P) \in M^{\#}$. Suppose now that $M$ is not a Boolean algebra. Then by a well known theorem of Nachbin [3, III.6 Theorem 3] there are prime filters $p$, $q$ of $M$ such that $p \subset q$. If $P$, $Q$ are prime filters of $L$ such that $P \cap M = p$ and $Q \cap M = q$ (see [3, III.6 Theorem 5(ii)], we have that $(P, Q) \in M^{\#}$ and $(Q, P) \notin M^{\#}$. ∎

REMARK. From (ii) in the above Theorem we obtain the well known correspondence between subalgebras of a Boolean algebra and equivalences on its Stone space satisfying property ($l$) (see [14, §8.2]).

For each $j \in \mathscr{J}(L, L)$, let $M_j = \{a \in L \mid j(a) \leqslant a\}$. It is easy to check that $M_j \in \mathscr{S}(L)$, and then $j^{\#} = M_j^{\#}$ is a lattice preorder associated with $j$. The mapping $j \mapsto j^{\#}$ is neither one-to-one nor onto. Indeed, if $k(a) = a \vee j(a)$, then $M_k = \{a \in L \mid k(a) = a\}$ and $M_j = M_{k^n}$ for each $n \geqslant 1$. If $L = \{0\} \amalg \{a\} \amalg Z^-$, where $Z^-$ denotes the negative integers, $a \notin Z$ and $\amalg$ indicates ordinal sum, then $M = L \backslash \{a\} \in \mathscr{S}(L)$ and there is no $j \in \mathscr{J}(L, L)$ such that $M = M_j$.

In general $j^* \subseteq j^{\#}$. We are going to investigate under which conditions $j^{\#} \subseteq j^*$ holds. We start by the following:

REMARK. Let $M \in \mathscr{S}(L)$. It is easy to check that $M^{\#}(P)$ is a closed decreasing subset for each $P \in \mathrm{X}(L)$. Moreover, a simple compactness argument shows that $M^{\#-1}(K)$ is a closed increasing set for each closed $K \subseteq \mathrm{X}(L)$. Therefore *$M^{\#}$ is a Priestley relation if and only if $M^{\#-1}(\sigma_L(a))$ is an open subset of $\mathrm{X}(L)$ for each $a \in L$.*

Recall that an *additive closure* on $L$ is a $j \in \mathscr{J}(L)$ such that $a \leqslant j(a)$ and $j(j(a)) = j(a)$ for every $a \in L$. The image of $j$, $j(L)$, is in $\mathscr{S}(L)$, and for each $a \in L$, $j(a)$ is the smallest element in the set $[a) \cap j(L)$ (see [3, II.4 Theorem 11]). The set of all additive closures on $L$ will be denoted by $\mathscr{C}(L)$.

A *quantifier* on $L$ ([19], [5]) is an additive closure $j$ such that $j(j(a) \wedge b) = j(a) \wedge j(b)$ for all $a$, $b$ in $L$. The set of quantifiers on $L$ will be denoted by $\mathscr{Q}(L)$.

2.7. THEOREM. *The following are equivalent conditions for each $j \in \mathscr{J}(L, L)$:*

(i) $j \in \mathscr{C}(L)$.

(ii) $j^{\#} \subseteq j^*$.

(iii) $j^{\#} = j^*$.

(iv) *$j^*$ is a preorder.*

*Proof.* Suppose $j \in \mathscr{C}(L)$ and let $P \cap M_j \subseteq Q$. Since $a \leqslant j(a) \in j(L) = M_j$, $P \subseteq j^{-1}(Q)$. Therefore (i) implies (ii). Since it was already observed that in general $j^* \subseteq j^{\#}$, (ii) implies (iii), and it is obvious that (iii) implies (iv). Finally, suppose $j^*$ is a preorder on $\mathbf{X}(L)$. By Lemma 1.8, $j^{**} \in \mathscr{C}(\mathbf{D}(\mathbf{X}(L)))$, and $j = \sigma_{\mathrm{L}} j^{**} \sigma_{\mathrm{L}}^{-1} \in \mathscr{C}(L)$. Hence (iv) implies (i). ∎

2.8. COROLLARY. *The correspondence $j \mapsto j^*$ defines an anti-isomorphism from the lattice $\mathscr{C}(L)$ onto $\mathscr{O}(\mathbf{X}(L)) \cap \mathscr{P}(\mathbf{X}(L), \mathbf{X}(L))$, considered as a sublattice of $\mathscr{O}(\mathbf{X}(L))$.* ∎

2.9. THEOREM. *Let $M \in \mathscr{S}(L)$. There exists $j \in \mathscr{C}(L)$ such that $M = M_j$ if and only if $M^{\#}$ is a Priestley relation.*

*Proof.* If $M = M_j$, with $j \in \mathscr{C}(L)$, then by Theorem 2.7 $j^* = j^{\#} = M^{\#}$. Hence $M^{\#} \in \mathscr{P}(\mathbf{X}(L), \mathbf{X}(L))$. Suppose now that $M^{\#}$ is a Priestley relation. Since it is a preorder, it follows from Lemma 1.8 that $M^{\#*} \in \mathscr{C}(\mathbf{D}(\mathbf{X}(L)))$. Therefore the composition $\sigma_{\mathrm{L}}^{-1} M^{\#*} \sigma_{\mathrm{L}} = j \in \mathscr{C}(L)$. Moreover, by (ii) and (iii) in Lemma 2.1, $M^{\#*}(\sigma_{\mathrm{L}}(a)) \subseteq \sigma_{\mathrm{L}}(a)$ if and only if $a \in M$. Hence $M = M_j$. ∎

REMARK. Let $L = \{0\} \amalg Z^+ \amalg Z^-$ and $j: L \to L$ be defined by $j(x) = x$ for $x \in M = \{0\} \amalg Z^-$ and $j(x) = x + 1$ for $x \in Z^+$. Then $j \in \mathscr{J}(L, L)$, $M = M_j$, but $j^{\#}$ is not a Priestley relation on $\mathbf{X}(L)$.

By taking into account Theorem 2.6(ii) we have:

2.10. COROLLARY. *Let $M \in \mathscr{S}(L)$. Then $M^{\#}$ is a Priestley equivalence if and only if $M$ is a Boolean sublattice of the center of $L$ and there is $j \in \mathscr{C}(L)$ such that $j(L) = M$.* ∎

Note that a lattice equivalence $E$ satisfies the condition: *If $(x, y) \notin E$, then there are $U, V$ in $E^{\#}$ such that $x \in U$, $y \notin U$, $x \notin V$ and $y \in V$.* An interesting class of equivalences satisfying a weaker condition was identified by Vrancken-Mawet [20] by the following:

2.11. DEFINITION. An equivalence relation $E$ on $X$ is said to be a *congruence* provided $(x, y) \notin E$ implies that there is $U \in E^{\#}$ such that $x \in U$ and $y \notin U$ or there is $V \in E^{\#}$ such that $x \notin V$ and $y \in V$.

Recall that $M \in \mathscr{S}(L)$ is *closed under relative complementation* provided for each $b \in L$, if there is $a \in M$ such that $a \vee b \in M$ and $a \wedge b \in M$, then $b \in M$.

2.12. LEMMA. *If $E$ is a congruence on a Priestley space $X$, then $E^{\#}$ is closed under relative complementation.*

*Proof.* By Lemma 2.2(i), $E^{\#} \in \mathscr{S}(\mathbf{D}(X))$. Let $V \in \mathbf{D}(X)$ and suppose there is $U \in E^{\#}$ such that $U \cup V \in E^{\#}$ and $U \cap V \in E^{\#}$. Since $A \mapsto E(A)$ is a quantifier on the lattice of all subsets of $\mathbf{X}(L)$, we have:

$$U \cup V = E(U) \cup E(V) = U \cup E(V) \tag{1}$$

and

$$U \cap V = E(E(U) \cap V) = E(U) \cap E(V) = U \cap E(V) \tag{2}$$

and from (1) and (2) it follows $V = E(V)$, i.e. $V \in E^{\#}$. ■

2.13. LEMMA. *If $M \in \mathscr{S}(L)$ is closed under relative complementation, then $M^e = M^{\#} \cap M^{\# -1}$ is a congruence on $\mathbf{X}(L)$, and $M^{e\#} = \sigma_{\mathbf{L}}(M)$.*

*Proof.* Let $M \in \mathscr{S}(L)$. Since $M^{e-1}(A) \subseteq M^{\# -1}(A)$ for each $A \subseteq \mathbf{X}(L)$, by Lemma 2.1(ii), $M^{e-1}(\sigma_{\mathbf{L}}(a)) \subseteq \sigma_{\mathbf{L}}(a)$ for each $a \in M$. Suppose now that $a \in L \backslash M$ and let $F = \{b \in M \mid a \vee b \in M\}$. Since $F$ is a proper filter of $M$, the set $\mathbf{F} = \{Q \in \mathbf{X}(M) \mid F \subseteq Q\}$ ordered by reverse inclusion is inductive and non-empty, and by Zorn's lemma there is a maximal element $R$ in $\mathbf{F}$. We are going to show that the following two properties hold, where $[A)$ and $(A]$ denote respectively the filter and the ideal of $L$ generated by $A \subseteq L$:

$$[F) \cap ((M \backslash R) \cup \{a\}] = \emptyset \tag{1}$$

and

$$[F \cup \{a\}) \cap (M \backslash R] = \emptyset. \tag{2}$$

Suppose (1) does not hold, and let $x \in [F) \cap ((M\backslash R) \cup \{a\}]$. Then there are $f \in F$ and $m \in M \backslash R$ such that $f \leqslant x \leqslant a \vee m$. This implies that $f \wedge (a \vee m) = f \in M$, and since $f \vee a \vee m \in M$, we have $a \vee m \in M$, i.e. $m \in F \subseteq R$, a contradiction. The proof of (2) is similar. It follows from (1) and the Birkhoff–Stone theorem that there is $P \in \mathbf{X}(L)$ such that $F \subseteq P$, $a \notin P$ and $P \cap M \subseteq R$. Since $F \subseteq P$ implies $F \cap M \in \mathbf{F}$, we have $P \cap M = R$. Analogously, from (2) it follows that there is $Q \in \mathbf{X}(L)$ such that $a \in Q$ and $Q \cap M = R$. Hence we have shown the existence of a $P \in \sigma_{\mathbf{L}}(a)$ and a $Q \in \mathbf{X}(L) \backslash \sigma_{\mathbf{L}}(a)$ such that $(P, Q) \in M^e$, which means that $M^{e-1}(\sigma_{\mathbf{L}}(a)) \nsubseteq \sigma_{\mathbf{L}}(a)$. Consequently, if $M$ is closed under relative complementation, then $M^{e\#} = \sigma_{\mathbf{L}}(M)$, and this equality implies that $M^e$ is a congruence on $\mathbf{X}(L)$. ■

It follows from Lemma 2.13 that for members $M, N$ of $\mathscr{S}(L)$ closed under relative complementation, $M \subseteq N$ if and only if $N^e \subseteq M^e$. Moreover, it follows from Lemma 2.12 that if $E$ is a congruence on $\mathbf{X}(L)$, then $M = \sigma_{\mathbf{L}}^{-1}(E^{\#})$ is closed under relative complementation in $L$, and that $M^e = E$. Therefore we have proved the following theorem, first established by L. Vrancken-Mawet in [20] by a different method:

2.14. THEOREM. *The correspondence* $M \mapsto M^e$ *establishes an antiisomorphism from the sublattice of* $\mathcal{S}(L)$ *formed by the elements closed under relative complementation, onto the lattice of congruences on* $\mathbf{X}(L)$.

2.15. DEFINITION. A *Q-congruence* on a Priestley space $X$ is a congruence $E$ such that $E(U) \in \mathbf{D}(X)$ for all $U \in \mathbf{D}(X)$.

It follows from [15, Lemma 2.5] that *an equivalence relation E on a Priestley space X such that* $E(U) \in \mathbf{D}(X)$ *for all* $U \in \mathbf{D}(X)$ *is a Q-equivalence if and only if* $E(x)$ *is closed for each* $x \in X$. Therefore, we have that *a Q-congruence is a Priestley equivalence if and only if* $E(x)$ *is a decreasing subset of X for each* $x \in X$.

The next theorem follows at once from §2 in [5]:

2.16. THEOREM. *Let* $M \in \mathcal{S}(L)$. *There is* $j \in Q(L)$ *such that* $j(L) = M$ *if and only if* $M^e$ *is a Q-congruence on* $\mathbf{X}(L)$. ∎

REMARK. Let $M \in \mathcal{S}(L)$ be closed under relative complementation. The existence of $j \in \mathscr{C}(L)$ such that $j(L) = M$ does not imply that $M^e$ is a $Q$-congruence, as the following example shows: Let $K$ be the three-element chain $0 < a < 1$, $L = K \times K$ and $M = \{(0, 0), (a, 0), (1, a), (1, 1)\}$. ∎

## Appendix

It was shown in [7, Proposition 3.3] that *when L and M are Boolean algebras, the homomorphisms from L to M are the minimal elements in the set* $\mathscr{J}(L, M)$ *with the pointwise order.* This result plays an important role in the proof of a general selection theorem for Boolean relations [7, Theorem 6.2], which in turn is used [7, Corollary 9.10] to prove a generalized version of a theorem of A. Monteiro [15] on extension of homomorphisms on Boolean algebras (see also [2]).

In general, for distributive lattices $L$ and $M$ there are no connections between homomorphisms and minimal elements in $\mathscr{J}(L, M)$. We are going to present in this Appendix a particular case, which is connected with a generalization of Monteiro's theorem to distributive lattices given in [4].

In what follows, $L$ will denote a bounded distributive lattice and $C$ a complete Boolean algebra. For each $m \in \mathscr{M}(L, C)$, $\mathscr{J}_m(L, C) = \{j \in \mathscr{J}(L, C) \mid m(a) \leqslant j(a)$ for each $a \in L\}$, endowed with the pointwise order, and $\mathscr{J}_{m,j}(L, C) = \{k \in \mathscr{J}_m(L, C) \mid k \leqslant j\}$.

A.1. LEMMA. *Each minimal element in* $\mathscr{J}_m(L, C)$ *is a homomorphism.*

*Proof.* For each $j \in \mathscr{J}_m(L, C)$ and each $c \in C$, define mappings $j_c$ and $j^c$ from $L$ into $C$ by the prescriptions:

$$j_c(a) = (j(a) \wedge \neg\, m(c)) \vee j(a \wedge c)$$

and

$$j^c(a) = (j(a) \wedge \neg\, j(c)) \vee j(a \wedge c).$$

It is plain that $j_c \in \mathcal{J}_m(L, C)$ and that $j_c \leqslant j$. Hence $j_c = j$ for each minimal $j$ in $\mathcal{J}_m(L, C)$. In particular, the following holds true for each minimal $j$ in $\mathcal{J}_m(L, C)$ and each pair $a, c$ of elements in $L$:

$$j(a) \wedge m(c) = j(a \wedge c) \wedge m(c). \tag{1}$$

It is also plain that $j^c \in \mathcal{J}(L, C)$ and that $j^c \leqslant j$. Let $j$ be minimal in $\mathcal{J}_m(L, C)$. It follows from (1) that $j^c(a) \wedge m(a) = m(a)$. Therefore $j^c \in \mathcal{J}_m(L, C)$ and $j^c = j$. Consequently, given a minimal $j$ in $\mathcal{J}_m(L, C)$ and elements $a, b$ in $L$:

$$j(a) \wedge j(b) = ((j(a) \wedge \neg j(b)) \vee j(a \wedge b)) \wedge j(b) = j(a \wedge b)$$

and since $1 = m(1) \leqslant j(1)$, $j$ is a homomorphism. ∎

REMARK. When $C$ is the two-element Boolean algebra, the above lemma reduces to the well known fact that *the maximal elements in the set of ideals of L which are disjoint from a given filter are prime.*

Let $p \in \mathcal{M}(L, C)$ be defined by the prescription $p(1) = 1$ and $p(a) = 0$ for $a \in L \backslash \{1\}$. From the above remark and Nachbin's theorem we can easily obtain:

A.2. PROPOSITION. *If every $h \in \mathcal{D}(L, C)$ is a minimal element in $\mathcal{J}_p(L, C) = \mathcal{J}(L, C)$, then L is a Boolean algebra.*

Consider the following three statements:

(i) Given $m \in \mathcal{M}(L, C)$ and $j \in \mathcal{J}_m(L, C)$, there are minimal elements in $\mathcal{J}_{m,j}(L, C)$.
(ii) Given $m \in \mathcal{M}(L, C)$ and $j \in \mathcal{J}_m(L, C)$, there is $h \in \mathcal{D}(L, C) \cap \mathcal{J}_{m,j}(L, C)$.
(iii) Given $m \in \mathcal{M}(L, C)$, $j \in \mathcal{J}(L, C)$, $S \in \mathcal{S}(L)$ and $h \in \mathcal{D}(S, C)$ such that the following condition holds:
(C) For $a, b$ in $S$ and $d, e$ in $L$, if $a \wedge d \leqslant b \vee e$, then $h(a) \wedge m(d) \leqslant h(b) \vee j(e)$,
there exists $h_1 \in \mathcal{D}(L, C) \cap \mathcal{J}_{m,j}(L, C)$ such that $h_1(a) = h(a)$ for all $a \in S$.

REMARK. It is plain that condition (C) is necessary for the existence of $h_1$ satisfying the requirements given in statement (iii) (cf. [4]).

A.3. PROPOSITION. *Statement* (*i*) *is equivalent to the axiom of choice* (*assuming the other axioms of Zermelo–Fraenkel set theory*).

*Proof.* Suppose first that the axiom of choice holds. Let $\{k_\lambda : \lambda \in \Lambda\}$ be a chain in $\mathcal{J}_{m,j}(L, C)$. For each $a \in L$, $\bigcap \{\sigma_c(k_\lambda(a)) : \lambda \in \Lambda\} = W(a)$ is a closed subset of $\mathbf{X}(C)$, and since $C$ is a complete Boolean algebra, the interior of $W$, $W^0$, is clopen (see, for instance, [3, Chapter V, §1, Theorem 10]). It is proved in [7, Theorem 6.4] that if $S, T$ are closed subsets of $\mathbf{X}(C)$ with closed interior, then $(S \cup T)^0 = S^0 \cup T^0$. Hence if we define $k(a) = \sigma_c^{-1}(W(a)^0)$, then $k \in \mathscr{J}(L, C)$ and $k \leqslant k_\lambda$ for each $\lambda \in \Lambda$. Moreover, since $\sigma_c(m(a)) \subseteq W(a)$ and $\sigma_c(m(a))$ is open, $\sigma_c(m(a)) \subseteq W(a)^0$, and $k$ is a lower bound of $\{k_\lambda : \lambda \in \Lambda\}$ in $\mathscr{J}_{m,j}(L, C)$. Now an application of Zorn's Lemma completes the proof. Suppose now that statement (i) holds for each bounded distributive lattice $L$ and each complete Boolean algebra $C$. As in the remark following Lemma A.1 we can see that this implies the existence of maximal ideals in each bounded distributive lattice, and by a result of Klimovsky [13], this property implies the axiom of choice (see also [3, Chapter III, §4]). ∎

Since a minimal element in $\mathscr{J}_{m,j}(L, C)$ is also minimal in $\mathscr{J}_m(L, C)$, it follows at once from Lemma A.1 that statement (i) implies statement (ii). But it was shown in [2, Theorem 3.1], that statement (ii) is equivalent to Sikorski extension theorem [3, Chapter V, §9], and to our knowledge it is still an open problem whether this theorem implies the axiom of choice. On the other hand we have:

A.4. PROPOSITION. *Statements (ii) and (iii) are effectively equivalent.*

*Proof.* Suppose first that statement (ii) holds true. For each $(a, b, d, e) \in S \times S \times L \times L$ let

$$u(a, b, d, e) = h(a) \wedge m(d) \wedge \neg h(b) \wedge \neg j(e)$$

and

$$v(a, b, d, e) = \neg h(a) \vee \neg m(d) \vee h(b) \vee j(e).$$

Now, for each $c$ in $L$ define

$$U(c) = \{(a, b, d, e) \in S \times S \times L \times L \mid a \wedge d \leqslant b \vee c \vee e\}$$

$$m_1(c) = \bigvee \{u(a, b, d, e) \mid (a, b, d, e) \in U(c)\}$$

$$V(c) = \{(a, b, d, e) \in S \times S \times L \times L \mid a \wedge c \wedge d \leqslant b \vee e\}$$

and

$$j_1(c) = \bigwedge \{v(a, b, d, e) \mid (a, b, d, e) \in V(c)\}.$$

Suppose $(a_i, b_i, d_i, e_i) \in U(c_i)$, $i = 1, 2$. Then:

$$((a_1 \wedge a_2), (b_1 \vee b_2), (d_1 \wedge d_2), (e_1 \vee e_2)) \in U(c_1 \wedge c_2)$$

and

$$u(a_1, b_1, d_1, e_1) \wedge u(a_2, b_2, d_2, e_2) \leqslant m_1(c_1 \wedge c_2).$$

Since $C$ is a complete Boolean algebra we have:

$$\begin{aligned} u(a_1, b_1, d_1, e_1) \wedge m_1(c_2) &= \bigvee \{u(a_1, b_1, d_1, e_1) \\ &\qquad \wedge u(a_2, b_2, d_2, e_2) \mid (a_2, b_2, d_2, e_2) \in U(c_2)\} \\ &\leqslant m_1(c_1 \wedge c_2) \end{aligned}$$

and then

$$m(c_1) \wedge m(c_2) \leqslant m(c_1 \wedge c_2).$$

Moreover, it is easy to check that $m(c_1 \wedge c_2) \leqslant m(c_1) \wedge m(c_2)$ and that $m(1) = 1$. Consequently, $m_1 \in \mathcal{M}(L, C)$ and in an analogous way we can prove that $j_1 \in \mathcal{J}(L, C)$. Let $(a_1, b_1, d_1, e_1) \in U(c)$ and $(a_2, b_2, d_2, e_2) \in V(c)$. Then (cf. [4, (5)–(9)]):

$$\begin{aligned} a_2 \wedge d_2 \wedge a_1 \wedge d_1 &\leqslant a_2 \wedge d_2 \wedge (b_1 \vee e_1 \vee c) \\ &= (a_2 \wedge d_2 \wedge (b_1 \vee e_1)) \vee (a_2 \wedge d_2 \wedge c) \\ &\leqslant b_1 \vee e_1 \vee b_2 \vee e_2 \end{aligned}$$

and by taking into account condition (C), we obtain:

$$u(a_1, b_1, d_1, e_1) \wedge \neg\, v(a_2, b_2, d_2, e_2) = 0$$

and this implies that $m_1(c) \leqslant j_1(c)$. Hence we have shown that $j_1 \in \mathcal{J}_{m_1}(L, C)$, and by Theorem A.2 there is $h_1 \in \mathcal{D}(L, C)$ such that $m_1(c) \leqslant h_1(c) \leqslant j_1(c)$ for each $c \in L$. Let $a \in S$. Since $(a, 0, 1, 0) \in U(a)$ and $(1, a, 1, 0) \in V(a)$, we have

$$h(a) \leqslant m_1(a) \leqslant j_1(a) \leqslant h(a).$$

Therefore $h_1(a) = h(a)$ for all $a \in S$, and to complete the proof of (ii) implies (iii) note that since $(1, 0, c, 0) \in U(c)$ and $(1, 0, 1, c) \in V(c)$, $m(c) \leqslant m_1(c)$ and $j_1(c) \leqslant j(c)$ for all $c \in L$. Suppose now that (iii) holds true, and let $S = \{0, 1\}$. It is easy to check that if $m(a) \leqslant j(a)$ for all $a \in L$, then the only element in $\mathcal{D}(S, C)$ satisfies condition (C). Therefore (iii) implies (ii) (see [4]). ■

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