Arch. Rational Mech. Anal. 125 (1993) 271–287. © Springer-Verlag 1993

A Global Existence Theorem for the Initial-Boundary-Value Problem for the Boltzmann Equation when the Boundaries are not Isothermal

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Abstract

We extend the existence theorem recently proved by HAMDACHE for the initial-boundary-value problem for the nonlinear Boltzmann equation in a vessel with isothermal boundaries to more general situations including the case when the boundaries are not isothermal. In the latter case a cut-off for large speeds is introduced in the collision term of the Boltzmann equation.

1. Introduction

In this paper, we deal with the initial-boundary-value problem which arises when we consider the time evolution of a rarefied gas in a vessel Ω whose boundaries are not kept at the same temperature throughout (though, for simplicity, we assume this temperature to be constant in time). The case of an isothermal boundary has been treated by HAMDACHE [1] and is based on a suitable modification of the method used by DIPERNA & LIONS [2] to deal with the pure initial-value problem. We also assume that Ω is a bounded open set of $\mathbf{R}^3$ with a sufficiently smooth boundary $\partial\Omega$. On $\partial\Omega$ we impose a linear boundary condition of a rather standard form [3, 4]:

$$\gamma_D^+ f(t, x, \xi) = \int_{\xi' \cdot n < 0} K(\xi' \to \xi; x, t)\, \gamma_D^- f(t, x, \xi')\, d\xi' \equiv K\gamma_D^- f \qquad (x \in \partial\Omega,\ \xi \cdot n > 0)\,, \tag{1.1}$$

$$K(\xi' \to \xi; x, t) \geqq 0\,, \tag{1.2}$$

$$\int_{\xi \cdot n > 0} K(\xi' \to \xi; x, t)\, |\xi \cdot n|\, d\xi = |\xi' \cdot n|\,, \tag{1.3}$$

$$M_w(\xi) = \int_{\xi' \cdot n < 0} K(\xi' \to \xi; x, t)\, M_w(\xi')\, d\xi' \tag{1.4}$$

where M_w is the wall Maxwellian and $\gamma_D^{\pm}$ are the so-called trace operators on $E^{\pm} = \{(t, x, \xi) \in (0, T) \times \partial\Omega \times \mathbf{R}^3 |\ \pm\xi \cdot n(x) > 0\}$. These operators permit us to define "the values taken on the boundary" $\gamma_D^{\pm} f$ (a.e. in $\xi \in \mathbf{R}^3$ and $x \in \partial\Omega$) by a function f for which in our context this concept is not *a priori* defined. Of course, one must show that these operators are well defined, as discussed in Sec. 2.

In the case of non-isothermal data along $\partial\Omega$ these initial-boundary-value problems possess boundary data which are compatible, not with a Maxwellian, but rather with one of those steady solutions, whose theory is still in its infancy (for an example see the recent paper by ARKERYD *et al.* [5]); thus one cannot expect the solution to tend toward a Maxwellian when $t \to \infty$, as has been recently shown for other kinds of boundary conditions [6, 7, 8]. The main difficulties in tackling this problem seem to lie with large velocities. In fact, the only case that has been treated so far is due to KAWASHIMA [9] and refers to discrete-velocity models.

A central observation for the proof below is

Lemma 1.1 [3, 4, 10–12]. *If Eqs.* (1.1), (1.2), (1.3) *and* (1.4) *hold, then*

$$\int \xi \cdot n\gamma_D f \log \gamma_D f \, d\xi \leqq -\beta_w \int \xi \cdot n |\xi|^2 \gamma_D f \, d\xi \qquad (a.e.\ in\ t\ and\ x \in \partial\Omega) \quad (1.5)$$

where β_w is the inverse temperature evaluated at the point $x \in \partial\Omega$. Unless the kernel in Eq. (1.1) *is a delta function, equality holds in Eq.* (1.9) *if and only if the trace $\gamma_D f$ of f on $\partial\Omega$ coincides with M_w (the wall Maxwellian).*

In the case of non-isothermal $\partial\Omega$, before attacking the problem we must deal with the difficulty related to large speeds. To this end, we shall introduce a modified Boltzmann equation in which we cut off all the collisions such that the sum of the squares of the velocities of two colliding molecules is larger than m^2 where m is an assigned positive constant:

$$\frac{\partial f}{\partial t} + \xi \cdot \frac{\partial f}{\partial x} = Q_m(f, f) \quad (1.6)$$

where

$$Q_m(f, f)(x, \xi, t) = \int_{\mathbf{R}^3} \int_{\mathscr{B}} (f' f'_* - f f_*) B(V, n)\, \Theta(m^2 - |\xi|^2 - |\xi_*|^2)\, d\xi_*\, dn, \quad (1.7)$$

where Θ denotes Heaviside's step function and $\mathscr{B}$ the unit sphere. This means that we can neglect the molecules with speeds larger than m because they never interact with the others. In fact, in order to avoid formal complications we shall also assume that the kernel K in Eq. (1.1) vanishes when $\xi > m$ and $\xi' < m$ or when $\xi < m$ and $\xi' > m$; in this way the two sets of molecules with speeds in $\Xi_m = \{\xi |\ \xi \leqq m\}$ and in $\mathbf{R}^3 \setminus \Xi_m$ evolve independently of each other.

We remark that the only place where we use the cut-off is the entropy estimate (4.8) in Sec. 4; the need for the cut-off disappears when the temperature or its inverse β is constant. Thus the present paper also contains a slightly different proof of HAMDACHE's theorem, with an extension to more general boundary conditions, to a more detailed study of the boundary behavior, and for the full class of collision operators of the existence context of DiPerna & Lions [2]. (Note that even the cut-off assumption on the kernel K made above is not needed when β is constant.) For convenience we omit the index m in $Q_m(f,f)$ since we never try to remove the cut-off for the non-isothermal case in this paper; also, we denote by $G(f,f)$ the gain part of $Q(f,f)$ and by $fL(f)$ the loss part.

2. Results on the traces

Before proving the existence theorem we recall some trace results giving the L^1 regularity of the trace of f on the boundary and study the semigroup generated by the free-streaming operator. This will be done in this section and the next one, respectively.

Let us first review the general results of Ukai [13] on the traces of the solutions. To this end we define

$$\Lambda f = \frac{\partial f}{\partial t} + \xi \cdot \frac{\partial f}{\partial x} \tag{2.1}$$

and assume that $\partial\Omega$ is piecewise C^1.

We denote by $S^t(x, \xi)$ the pair $(x + \xi t, \xi)$ which gives the position and velocity of a molecule initially located at (x, ξ) as long as $x + \xi t$ stays in Ω. Denote the forward $(t > 0)$ stay time in Ω by $t^+(x, \xi)$ and the backward one by $t^-(x, \xi)$. We recall that $t^\pm$ are lower semicontinuous. Then $S^t \in \Omega \times \mathbf{R}^3$ for $-t^-(x, \xi) < t < t^+(x, \xi)$ and $S^t \in \partial\Omega \times \mathbf{R}^3$ for $t = t^\pm(x, \xi)$ if $t^\pm(x, \xi) < \infty$. Let us also define $\Sigma^\pm = \{(x, \xi) \in \partial\Omega \times \mathbf{R}^3 | \pm \xi \cdot n(x) > 0\}$ and remark that S^t exists for $(x, \xi) \in \Sigma^\mp$ with $t^\pm = 0$, $t^\mp > 0$. In any case, $S^t \in \overline{\Sigma^\mp}$ at $t = \pm t^\pm$. It is now convenient to write $r = (t, x, \xi)$ and for $T > 0$ to define

$$\begin{gathered} D = (0, T) \times \Omega \times \mathbf{R}^3, \quad V^\pm = \{T^\pm\} \times \Omega \times \mathbf{R}^3 \quad (\text{where } T^+ = 0,\ T^- = T), \\ E^\pm = (0, T) \times \Sigma^\pm, \quad \partial D^\pm = E^\pm \cup V^\pm \quad (\text{same sign throughout}). \end{gathered} \tag{2.2}$$

The world line of a molecule passing through $r \in D \cup \partial D^+ \cup \partial D^-$ is given by

$$R^s(r) = (t + s, x + \xi s, \xi), \quad -s^-(r) \leqq s \leqq s^+(r) \tag{2.3}$$

where $s^\pm(r) = \min(T^\mp \mp t, t^\pm(x, \xi))$ and $T^\mp$ are defined as in Eq. (2.2). Obviously,

$$R^s(r) \in D \ \ (-s^-(r) < s < s^+(r)), \quad R^s(r) \in \overline{D^\mp} \ \ (s = s^\pm(r)). \tag{2.4}$$

Clearly, if $f \in L^1(D)$, then $f(R^s(r))$ as a function of s is in $L^1(-s^-(r), s^+(r))$ for almost all $r \in \partial D^\pm$ and

$$\int_D f(r)\, dr = \int_{\partial D^\pm} \int_{s^-(r)}^{s^+(r)} f(R^s(r))\, ds\, d\sigma^\pm \tag{2.5}$$

holds, where

$$dr = dt\, dx\, d\xi, \quad d\sigma^\pm = |n(x) \cdot \xi|\, dt\, d\sigma\, d\xi \ (\text{on } E^\pm), \quad d\sigma^\pm = dx\, d\xi \ (\text{on } V^\pm), \tag{2.6}$$

$n(x)$ being the inward normal to $\partial\Omega$ and $d\sigma$ the usual measure on $\partial\Omega$. We set

$$(f, g) = \int_D fg\, dr, \quad \langle \phi, \psi \rangle_{\pm} = \langle \phi, \psi \rangle_{E^\pm} + \langle \phi, \psi \rangle_{V^\pm} = \int_{\partial D^\pm} \phi\psi\, d\sigma^\pm \tag{2.7}$$

where, of course,

$$\langle \phi, \psi \rangle_{E^\pm} = \int_{E^\pm} \phi\psi |n(x) \cdot \xi|\, dt\, d\sigma\, d\xi, \quad \langle \phi, \psi \rangle_{V^\pm} = \int_V \phi\psi\, dx\, d\xi \quad (\text{for } t = T^\pm). \tag{2.8}$$

The first result on traces to be recalled is due to UKAI [13] and holds between the spaces

$$W^p = \{f \in L^p(D) \mid \Lambda f \in L^p(D)\}, \quad L_\theta^{p,\pm} = L^p(\partial D^\pm; \theta\, d\sigma^\pm),$$
$$\theta = \theta(r) = \min(1, s^+(r) + s^-(r)) \tag{2.9}$$

where Λf is defined in the distributional sense.

The trace operators $\gamma_D^\pm$ are first defined on $C_0^1(\bar{D})$ by

$$\gamma_D^\pm f = f|_{\partial D^\pm} \quad f \in C_0^1(\bar{D}). \tag{2.10}$$

Then, the following result holds

Theorem 2.1 [13, 14]. *Let $p \in [1, \infty]$. $\gamma_D^\pm$ have extensions in $\mathbf{B}(W^p, L_\theta^{p,\pm})$, the spaces of bounded linear operators from W^p to $L_\theta^{p,\pm}$, which are also denoted by $\gamma_D^\pm$. Thus*

$$\| \gamma_D^\pm f \|_{L_\theta^{p,\pm}} \leqq C \| f \|_{W^p} = C(\| f \|_{L^p(D)} + \| \Lambda f \|_{L^p(D)}). \tag{2.11}$$

We cannot remove the weight function θ if $p < \infty$ in Eq. (2.11). For this reason, some authors [15, 16] have obtained just $L_{\text{loc}}^{p,\pm}$-traces. In order to solve the initial-boundary-value problem, however, the $L_\theta^{p,\pm}$-traces are not adequate. We need $L^{p,\pm}$ traces defined by

$$L^{p,\pm} = L^p(\partial D^\pm; d\sigma^\pm). \tag{2.12}$$

We remark that $L^{p,\pm} = L_\theta^{p,\pm}$ for $p = \infty$ but $L^{p,\pm} \subseteq L_\theta^{p,\pm}$ if $p < \infty$. Let us also define, for future use,

$$\hat{W}_p = \{f \in W^p(D) \mid \gamma_D^\pm f \in L^{p,\pm}\} \subset W^p. \tag{2.13}$$

If we impose suitable boundary conditions, then we can make some progress in the direction of proving that $f \in \hat{W}_p$. To this end it is expedient to prove the following

Theorem 2.2 [13, 14]. *Let $f \in W^p$, $p \in [1, \infty)$. If $\gamma_D^\pm f \in L^{p,\pm}$ (only one sign throughout), then $\gamma_D^\mp f \in L^{p,\mp}$. In this case, the following relation holds:*

$$\| \gamma_D^- f(r) \|_{L^{p,-}}^p = \| \gamma_D^+ f(r) \|_{L^{p,+}}^p + p \int_D |f|^{p-2} f \Lambda f\, dr. \tag{2.14}$$

This theorem immediately allows us to deduce the existence of the traces when f is assigned on ∂D^+, as a function of $L^{p,+}$. The situation is more complicated if the boundary conditions are less trivial. We shall assume that boundary conditions of the form (1.1) are satisfied and prove the existence of the traces on the boundary, under suitable assumptions.

We can now introduce the operator P which reflects ξ and the operator λ_D^+ that carries $\gamma_D^- \phi(r)$ into $\gamma_D^+ \phi(r)$ (via $\Lambda\phi = 0$; *i.e.*, λ_D^+ deposits the value $\gamma_D^- \phi(r)$ taken at a point x of the boundary as a value for $\gamma_D^+ \phi(r)$ on the next intersection with the boundary of the half straight line through x directed and oriented as $-\xi$). Let us then consider for any function ϕ of $L^{\infty,-}$ the projector P_O defined as follows

$$P_O \phi = \phi - \frac{\langle \phi, M_w^- \rangle_-}{\langle 1, M_w^- \rangle}. \tag{2.15}$$

We can assume that the operator $I - (PK)^\dagger P\lambda_D^+$ has a bounded inverse in the subspace O of the functions having the form $P_O\phi$. Then one can prove the following

Theorem 2.3 [14]. *Let* $f \in W^1$, $|\xi|^2 f \in L^1$, $|\xi|^2 \Lambda f \in L^1(D)$. *If the boundary condition* (1.1) *applies and* $I - (PK)^\dagger P\lambda_D^+$ *has a bounded inverse in the subspace* O *of* $L^{\infty,-}$, *then* $\gamma_D^\mp f \in L^{1,\mp}$.

Theorem 2.3 is the result that is needed in order to deal with sufficiently general boundary operators K; HAMDACHE's [2] results refer, apart from the deterministic conditions of specular and reverse reflection, only to operators with kernels having compact support in $\mathbf{R}^3 \times \mathbf{R}^3$ for almost any $\{x, t\} \in \partial\Omega \times [0, T]$, which excludes practically all the typical cases. In particular, this property seems to be incompatible with the preservation of equilibrium, since a Maxwellian does not have compact support!

One must, of course, prove that the criterion in Theorem 2.3 is actually satisfied by any reasonable boundary condition for sufficiently smooth boundaries. So far, an explicit proof has been given [17] for the important case of a sufficiently smooth boundary diffusing the particles according to a Maxwellian distribution.

3. Properties of the free-streaming operator

As in the previous sections, we assume that f is assigned at $t = 0$ and satisfies the boundary condition (1.1). We first study the problem

$$(\Lambda + \lambda) f = 0 \quad \text{in } D \quad (\lambda \in \mathbf{R}), \tag{3.1}$$

$$\gamma_D^+ f(x, \xi, t) = K\gamma_D^- f \quad \text{on } \partial D, \tag{3.2}$$

$$f(x, \xi, 0) = f_0(x, \xi). \tag{3.3}$$

The parameter λ is introduced for the sake of more flexibility when obtaining the estimates; in fact if f satisfies Eqs. (3.1)–(3.3), then $\hat{f} = e^{\lambda t} f$ satisfies Eqs. (3.1)–(3.3) with $\lambda = 0$. If the norm of K is (in some space) less than unity, then we can use iteration methods to solve the problem; since, however, we assume that (1.3) is true, the right assumption is $\|K\| = 1$. The boundary is assumed to be piecewise C^1. We use the notation of Sec. 2 and set $Y^{p,\pm} = L^p(\Sigma^{\pm} | \, |n(x)\cdot\xi| \, d\sigma \, d\xi)$. In addition, we assume that $\|K\| \leqq 1$ in $\boldsymbol{B}(Y^{p,-}, Y^{p,+})$ because in this way we can obtain intermediate results, which are useful in the case $\|K\| = 1$. Denote the dual of K by $K^\dagger$. Then for $p \in [1, \infty)$ we have, automatically, $\|K^\dagger\| \leqq 1$ in $B(Y^{q,-}, Y^{q,+})$ with $p^{-1} + q^{-1} = 1$. For $p = \infty$ this is an extra assumption (always true in the physically interesting cases). We also assume that K does not act on t; hence we may replace $Y^{p,\pm}$ by $L^{p,\pm} = L^p(E^{\pm} | \, |n(x)\cdot\xi| \, dt \, d\sigma \, d\xi)$.

The weak solution is defined, as usual, through a sort of Green's formula, which can be easily established [13]:

$$\langle \gamma_D^- f(r), \gamma_D^- \phi(r)\rangle_- - \langle \gamma_D^+ f(r), \gamma_D^+ \phi(r)\rangle_+ = \big(f, (\Lambda - \lambda)\,\phi\big) + \big((\Lambda + \lambda)\, f, \phi\big) \tag{3.4}$$

where $f \in \hat{W}^p$ and $\phi \in \hat{W}^q$ (with $p^{-1} + q^{-1} = 1$) and $\langle \phi, \psi\rangle_\pm$ are defined as in Eq. (2.7).

Theorem 3.1. *If $p \in [1, \infty]$ and $f_0 \in L^p(\Omega \times \mathbf{R}^3)$, then a mild solution $f \in L^p(D)$ exists for $\lambda \geqq 0$ if $\|K\| < 1$. If K carries nonnegative functions into functions of the same kind and f_0 is nonnegative, then f is also nonnegative.*

Proof. This theorem can be proved in many ways. UKAI [13] gives a proof that is valid only if $p \in (1, \infty]$. Here we follow a different strategy. We first consider the case when Eq. (3.2) is replaced by

$$\gamma_D^+ f(x, \xi, t) = f^+ \quad \text{on } \partial D^+ \tag{3.5}$$

where $f^+ \in L^{p,+}$ is a given function. Then the solution can be written in an explicit way:

$$\hat{f}(s, r) = \hat{g}(r)\, e^{-\lambda(s + s^-(r))} \tag{3.6}$$

where $\hat{g}(r)$ equals f_0 or f^+, according to whether $s = -s^-(r)$ corresponds to $t = 0$ or a point of the boundary of the space domain. It is clear, thanks to Eq. (2.5), that the solution constructed in this way is in L^p. If we now go back to the original boundary condition (3.2), we find a solution for that problem provided there exists a function $\hat{g}$ such that

$$\hat{g}(r) = \mathscr{K}[\hat{g}(r^*)\, e^{-\lambda s^+(r^*)}] + g_0 \quad \text{for } r \in \partial D^+ \tag{3.7}$$

where $\mathscr{K}$ is 0 on $t = 0$ and K on the boundary of the space domain, while g_0 is f_0 for $t = 0$ and 0 on the boundary of the space domain. r^* is the other point where the relevant world line intersects ∂D. Since $\|K\| < 1$ (and hence $\|\mathscr{K}\| < 1$), Eq. (3.2) can be solved explicitly by means of a perturbation series. The part on nonnegativity is obvious by glancing at the details of this constructive proof. □

We can now provide some estimates for the mild solutions. This can be done with

Theorem 3.2. *When* $\|K\| < 1$, *the mild solution is the unique solution satisfying the estimate*

$$\lambda p \| f \|^p_{L^p(D)} + (1 - \|K\|^p) \| \gamma^- f \|^p_{L^{p,-}} + \| f(T) \|^p_{L^p(\Omega \times \mathbf{R}^3)} \leqq \| f_0 \|^p_{L^p(\Omega \times \mathbf{R}^3)} \tag{3.8}$$

for $p \in [1, \infty)$.

Proof. Let us consider Eq. (2.14) for functions of, say, C_0^1, not necessarily solutions of $\Lambda f = -\lambda f$; if $\gamma_D^+ f = K \gamma_D^- f$, with $\|K\| < 1$, then

$$(1 - \|K\|^p) \| \gamma^- f \|^p_{L^{p,-}} + \| f(\cdot, T) \|^p_{L^p(\Omega \times \mathbf{R}^3)} \leqq \| f_0 \|^p_{L^p(\Omega \times \mathbf{R}^3)} + p \int | f |^{p-2} f (\Lambda f) \, dr. \tag{3.9}$$

If we now take a sequence $\{f_n\}$ of C_0^1 functions which approach a solution f in W^p, the limit as $n \to \infty$ of Eq. (3.9) (with f_n in place of f) is Eq. (3.8). This relation also proves that $f \in \hat{W}_p$. The uniqueness of the solution now follows from linearity and the estimate (3.8). □

We can now consider the case $\|K\| = 1$ and prove

Theorem 3.3. *When* $\|K\| = 1$, *if* K *carries nonnegative functions into functions of the same kind and* f_0 *is nonnegative, then* (3.1)–(3.3) *for* $p \in [1, \infty)$ *have a mild nonnegative solution* $f \in L^p(D)$, *with the estimate* $(\lambda > 0)$

$$\| f(\cdot, T) \|_{L^p(\Omega \times \mathbf{R}^3)} \leqq \| f_0 \|_{L^p(\Omega \times \mathbf{R}^3)} . \tag{3.10}$$

Remark. The problem of uniqueness for $\|K\| = 1$ has been solved [16] only with additional conditions on K.

Proof. Let us replace K by μK with $\mu \in (0, 1)$ in (3.2); then, by the previous theorem, we have a unique mild solution f^μ satisfying (3.8) $(p \in [1, \infty))$, which gives uniform estimates (in μ) for f^μ in $L^p(D)$ and $f^\mu(\cdot, T)$ in $L^p(\Omega \times \mathbf{R}^3)$. Hence, taking a nondecreasing sequence of μ's we obtain a nondecreasing sequence $f^\mu \to f$ in $L^p(D)$ and $f^\mu(\cdot, T) \to h$ in $L^p(\Omega \times \mathbf{R}^3)$, pointwise a.e. and strongly. This f is clearly a mild solution with $f(\cdot, T) = h$ and going to the limit in (3.8) $(p \in [1, \infty))$, we obtain (3.10). □

So far we have assumed that $\lambda > 0$. We have already remarked, however, that the constant λ can be removed and thus all the results remain true with some changes in the estimates. In particular,

Corollary 3.4. *Theorem 3.3 is true for* $\lambda = 0$ *as well.*

Since $f(\cdot, T) \in L^p(\Omega \times \mathbf{R}^3)$ by (3.10) and since $T > 0$ may be arbitrary, we can introduce the solution operator $U(t)$ $(t \in R_+)$ which carries $f_0 = f(\cdot, 0)$

into $f(\cdot, t)$:

$$U(t)\, f_0 = f(\cdot, t)\,. \tag{3.11}$$

Then it is not hard to prove

Theorem 3.5. *If $p \in [1, \infty)$, then $U(t)$ is a C_0-semigroup on $L^p(\Omega \times \mathbf{R}^3)$.*

In the sequel we shall need a generalization of these results to the case when the parameter λ in Eq. (3.1) is replaced by a nonnegative function $l(t, x, \xi) \in L^1((0, T) \times \Omega \times \mathbf{R}^3_{\text{loc}})$. Then the above treatment carries through. The main difference arises in the definition of the spaces W^p and in the proof of the analogue of Theorem 3.1. In fact, W^p is now replaced by W^p_l, such that $f \in L^p$ and $(\Lambda + l)\, f \in L^p$, while Eq. (3.6) must be replaced by

$$\hat{f}(s, r) = \hat{g}(r)\, e^{-\int_{-s^-(r)}^{s} l(R^{s'}(r))\, ds'} \tag{3.12}$$

and Eq. (3.7) by

$$\hat{g}(r) = \mathscr{K}\Big[\hat{g}(r^*)\, e^{-\int_{s^-(r_*)}^{s^+(r_*)} l(R^{s'}(r_*))\, ds'}\Big] + g_0 \quad \text{for } r \in \partial D. \tag{3.13}$$

Thus we may conclude that

Theorem 3.6. *When $\|K\| = 1$, if K carries nonnegative functions into functions of the same kind and f_0 is nonnegative, then the problem*

$$(\Lambda + l)\, f = 0 \quad \text{in } D \tag{3.14}$$

(where $0 \leqq l = l(t, x, \xi) \in L^1((0, T) \times \Omega \times \mathbf{R}^3_{\text{loc}})$) with the boundary and initial conditions (3.2), (3.3) *has a mild nonnegative solution $f \in L^p(D)$, with the estimate*

$$\| f(\cdot, T)\|_{L^p(\Omega \times \mathbf{R}^3)} \leqq \| f_0 \|_{L^p(\Omega \times \mathbf{R}^3)}\,. \tag{3.15}$$

Then the solution can be written as $U_l(t)\, f_0$, where $U_l(t)$ is a C_0-semigroup on $L^p(\Omega \times \mathbf{R}^3)$.

We remark that (3.15) does not follow from the limiting procedure (that would give a constant C_{0T} in front of the norm of f_0), but directly from the fact that Λ with these boundary conditions is contractive.

We shall also have to deal with sequences of nonnegative functions $l_k \in L^1((0, T) \times \Omega \times \mathbf{R}^3_{\text{loc}})$. In this case, if $\{l_k\}$ converges to l in $L^1((0, T) \times \Omega \times \mathbf{R}^3_{\text{loc}})$, then $\{F_k\}$, where

$$F_k = \int_{-s^-(r)}^{0} l_k(s, x - \xi(t + s), \xi)\, ds \tag{3.16}$$

is a bounded sequence in $C([0, T], L^1(\Omega \times \mathbf{R}^3_{\text{loc}}))$ and converges for any $t \in R_+$, a.e. in (x, ξ) to

$$F = \int_{-s^-(r)}^{0} l(s, x - \xi(t + s), \xi)\, ds\,. \tag{3.17}$$

Associated with the sequence $\{l_k\}$ we now have the sequence of solutions $\{U_{l_k}(t) f_0\}$ (for the sake of simplicity we now restrict our attention to the case $p = 1$), which is pointwise dominated by $U(t) f_0$. Thus $\{U_{l_k}(t) f_0\}$ converges to $U_l(t) f_0$ because of the dominated convergence theorem, thanks to the fact that all the relations that we need, such as (3.12) and (3.13), enable us to pass to the limit, when we replace l by l_k and let k go to ∞ (in $[0, T]$ for almost every (x, ξ)).

We remark that we can also solve

$$(\Lambda + l) f = g \quad \text{in } D \tag{3.18}$$

with initial and boundary conditions (3.2), (3.3), when $g \in L^1((0, T) \times \Omega \times \mathbf{R}^3_{\text{loc}})$. The solutions is

$$f = U_l(t) f_0 + \int_0^t U_l(t-s)\, g\, ds. \tag{3.19}$$

We remark that the traces do exist and satisfy Eq. (1.1) almost everywhere in $[0, T] \times \partial\Omega \times \mathbf{R}^3$, because this is true of any function of the form $U_l(\tau) g$, $\tau > 0$.

We also notice that $\{U_l g^\nu\}$ is an increasing sequence when $\{g^\nu\}$ is an increasing sequence.

4. Existence in a vessel with a nonisothermal boundary

In order to deal with the existence theorem in a vessel at rest, with a temperature that varies from one point to another, it is convenient to remark that there is a Maxwellian naturally associated with the problem at each point of the boundary, *i.e.*, the wall Maxwellian M_w; an exception is offered by specular reflecting boundaries, which will not be considered in this paper because they have no temperature associated with the boundary. Equation (1.5) gives

$$\int \xi \cdot n \gamma_D f \log \gamma_D f \, d\xi + \beta_w \int \xi \cdot n |\xi|^2 \gamma_D f \, d\xi \leqq 0 \quad \text{(a.e. in } t \text{ and } x \in \partial\Omega). \tag{4.1}$$

As a consequence of this, it is convenient to consider an inverse temperature $\beta(x)$ with $\inf \beta(x) > 0$ which reduces to β_w at each point of the boundary and otherwise depends smoothly on x. It is then convenient to consider the modified H-functional:

$$H = \int f \log f \, d\xi \, dx + \int \beta(x) \, |\xi|^2 f \, d\xi \, dx. \tag{4.2}$$

In general, H does not decrease in time, as a consequence of the Boltzmann equation and inequality (4.1), because a simple calculation shows that

$$\frac{dH}{dt} \leqq - \int_\Omega \int_{\Xi_m} \xi \cdot \frac{\partial \beta}{\partial x} |\xi|^2 f \, d\xi \, dx. \tag{4.3}$$

The right-hand side of inequality (4.3) is bounded by a constant C given by

$$C = m^3 \left\| \frac{\partial \beta}{\partial x} \right\|_{L^\infty} \int f_0 \, d\xi \, dx. \tag{4.4}$$

Thus H is bounded by $H_0 + CT$ on $[0, T]$ if it is bounded initially by H_0.

Let us divide the subset of $\Omega \times \mathbf{R}^3$ where $f < 1$ into two subsets $\triangle^\pm = \{(x, \xi) \mid \pm \log f < \mp \beta(x) \xi^2/2\}$. Then (since $-f \log f$ is a growing function in $(0, e^{-1})$ and less than f for $f > e^{-1}$)

$$-\int_{\triangle^+} f \log f \, d\xi \, dx \leqq \int f \, d\xi \, dx + \int [\beta(x)/2] \, \xi^2 \exp[-\beta(x) \, \xi^2/2] \, d\xi \, dx \leqq C, \tag{4.5}$$

and in $\triangle^-$

$$-\int_{\triangle^-} f \log f \, d\xi \, dx \leqq \int [\beta(x)/2] \, \xi^2 f \, d\xi \, dx. \tag{4.6}$$

Then Eq. (4.2) implies that both $\int f |\log f| \, d\xi \, dx$ and $\int |\xi|^2 f \, d\xi \, dx$ are separately bounded in terms of the initial data. It is now easy to prove that the mass and entropy relations take on the form

$$\int f(\cdot, t) \, d\xi \, dx \leqq \int f(\cdot, 0) \, d\xi \, dx, \tag{4.7}$$

$$\int f \log f(\cdot, t) \, d\xi \, dx + \int \beta(x) |\xi|^2 f(\cdot, t) \, d\xi \, dx + \int_0^t \int e(f)(\cdot, s) \, d\xi \, dx \, ds$$

$$\leqq \int f \log f(\cdot, 0) \, d\xi \, dx + \int \beta(x) |\xi|^2 f(\cdot, 0) \, d\xi \, dx + m^3 + \left\| \frac{\partial \beta}{\partial x} \right\|_{L^\infty} \int f_0 \, d\xi \, dx \tag{4.8}$$

where

$e(f)(x, \xi, t) =$

$$\frac{1}{4} \int_{\mathbf{R}^3} \int_{\mathscr{B}} (f' f'_* - f f_*) \log(f' f'_* / f f_*) \, B(V, n) \, \Theta(m^2 - |\xi|^2 - |\xi_*|^2) \, d\xi_* \, dn \tag{4.9}$$

(with $m = \infty$, *i.e.*, $\Theta = 1$ in the isothermal case).

As hinted at in Sec. 1, we use the equivalent concepts of exponential, mild, and renormalized solutions as defined by DIPERNA & LIONS [2]. Such solutions will be found as limits of functions solving truncated equations.

The existence theorem to be proved reads as follows:

Theorem 4.1. *Let $f^0 \in L^1(\Omega \times \mathbf{R}^3)$ be such that*

$$\int f^0 (1 + |\xi|^2) \, d\xi \, dx < \infty, \quad \int f^0 |\log f^0| \, d\xi \, dx < \infty. \tag{4.10}$$

Then there is a solution $f \in C(\mathbf{R}_+, L^1(\Omega \times \mathbf{R}^3))$ of the Boltzmann equation such that $f(\cdot, 0) = f^0$, which also satisfies relations (4.7) *and* (4.8).

Proof. We only sketch a proof of the theorem, since the argument is rather similar to that presented by DIPERNA & LIONS [2] for the case of $\mathbf{R}^3$, the main differences being the necessity to have trace estimates and the fact that we do not have separate energy and entropy estimates; both aspects have been dealt with above. We shall mention any important modification in the course of the proof. We first introduce a smooth truncation of the Boltzmann equation and prove an existence theorem for the truncated equation. We choose as truncated collision term

$$Q^k(f,f) = \left(1 + k^{-1} \int f\, d\xi\right)^{-1} \int_{\mathbf{R}^3} \int_{\mathscr{B}} d\xi_*\, dn (f' f'_* - f f_*)\, B_k \tag{4.11}$$

where

$$B_k = \begin{cases} B \wedge k & \text{for } \xi^2 + \xi_*^2 \leqq k^2, \\ 0 & \text{otherwise}. \end{cases} \tag{4.12}$$

We also set

$$l_k = \int d\xi_*\, dn\, B_k f_{k*} \left(1 + k^{-1} \int f_k\, d\xi\right)^{-1}, \tag{4.13}$$

$$\tilde{U}_k^{\pm}(r, s) = \exp\left(\pm \int_{-s^-(r)}^{s} l_k(R^\tau(r))\, d\tau \right). \tag{4.14}$$

Further, $l = l_\infty$ and $\tilde{U}^{\pm} = \tilde{U}_\infty^{\pm}$ are analogously defined with $f_\infty = f = \lim f_k$. Subsequences $\{k_j\}$ of $\{k\}$, sometimes necessary from one step to the next, will still be denoted by $\{k\}$.

Then for $f, g \in L^1(\Omega \times \mathbf{R}^3)$,

$$\| Q^k(f,f) - Q^k(g,g) \|_{L^1(\Omega\times\mathbf{R}^3)} \leqq C_k \| f - g \|_{L^1(\Omega\times\mathbf{R}^3)}. \tag{4.15}$$

Hence, for $f^0 \in L^1(\Omega \times \mathbf{R}^3)$, the mild Boltzmann equation

$$f_k = U(t)\, f^0 + \int_0^t U(t-s)\, Q^k(f_k, f_k)\, ds \tag{4.16}$$

with the desired boundary behavior can be solved by a contraction mapping argument.

If (4.10) holds for f^0, then by Green's formula (3.4) f_k satisfies (4.7). Via suitable smooth approximations it can also be shown that (thanks to (4.5) and (4.6)) f_k satisfies (4.8) as well, even with $f_k(t,\cdot) \log f_k(t,\cdot)$ replaced by $|f_k(t,\cdot) \log f_k(t,\cdot)|$, if we add a suitable k-independent constant to the right-hand side. In particular,

$$\forall T > 0, \qquad \sup_{t\in[0,T]} \sup_k \int f_k(\cdot, t)\, d\xi\, dx < \infty, \tag{4.17}$$

$$\forall T > 0, \qquad \sup_{t\in[0,T]} \sup_k \int \xi^2 f_k(\cdot, t)\, d\xi\, dx < \infty, \tag{4.18}$$

$$\forall T > 0, \qquad \sup_{t\in[0,T]} \sup_k \int f_k |\log f_k(\cdot, t)|\, d\xi\, dx < \infty, \tag{4.19}$$

$$\sup_k \int_0^\infty \int e_k(f_k)(\cdot, s)\, d\xi\, dx\, ds < \infty, \tag{4.20}$$

$$e_k(f_k)(x, \xi, t) = \tfrac{1}{4}\left(1 + k^{-1}\int f_k\, d\xi\right)^{-1} \int_{\Xi^m}\int_{\mathscr{B}} (f_k' f_{k*}' - f_k f_{k*})$$

$$\times \log(f_k' f_{k*}'/f_k f_{k*})\, B_k(V, n)\, \Theta(m^2 - |\xi|^2 - |\xi_*|^2)\, d\xi_*\, dn \tag{4.21}$$

with $\Theta(\cdot)$ and Ξ_m replaced by unity and $\mathbf{R}^3$, respectively, in the isothermal case. Via (3.4) with a test function which is a suitable extension to the interior of the function $n(x)$ as defined on the boundary, it then follows that

$$\int_{E^\pm} |\xi \cdot n(x)|^2\, \gamma_D^\pm f_k\, d\sigma\, d\xi\, ds \leqq C_{0T}, \tag{4.22}$$

where C_{0T} only depends on T and the right-hand side of (4.8) (but not on k). Under the conditions of, *e.g.*, Theorem 2.3 these traces of course also belong to $L^{1\pm}$ but that is not known to be true in general.

We can now apply the Dunford-Pettis criterion to our sequence to conclude from (4.17)–(4.19) that $\{f_k\}$ has a subsequence that converges weakly to some function f.

It is then easy to show that the sequences $\left\{\dfrac{G^k(f_k, f_k)}{1+f_k}\right\}, \left\{\dfrac{f_k L^k f_k}{1+f_k}\right\}$ are in a weakly compact set of $L^1((0, T) \times \Omega \times \mathbf{B}_R)$, where $\mathbf{B}_R$ is the ball of radius R in velocity space. This is proved in exactly the same way as in the case [2] of $\mathbf{R}^3$.

We denote by $l = Lf$ the function multiplying f in the loss term of the Boltzmann equation and by $U_l(t)$ the semigroup associated with it according to Theorem 3.6. When we solve the truncated equation, we deal with l_k and $U_{l_k}(t)$. According to Eq. (3.19) we have

$$f_k = U_{l_k}(t) f_k^0 + \int_0^t U_{l_k}(t-s)\, G^k(f_k, f_k)\, ds. \tag{4.23}$$

Using the velocity-averaging lemma of GOLSE *et al.* [18] we shall prove that f is a solution of the Boltzmann equation which retains a fairly weak control of the traces.

As in the case of $\mathbf{R}^3$, one exploits the fact that all the terms are non-negative in order to go to the limit in Eq. (4.23). Let us denote by $\alpha^\nu(s)$ the minimum $\nu \wedge s = \min(\nu, s)$ and put $f_k^\nu = \alpha^\nu(f_k)$. We may assume that f_k^ν tends (weakly) to some f^ν in $L^1((0, T) \times \mathbf{R}^3 \times \mathbf{R}^3)$ and as a consequence f^ν (different from $\alpha^\nu(f)$ in general) converges to f monotonically. Given ν, we can apply the averaging lemma to study the convergence of f_k^ν, which is also bounded by ν; then $G^k(f_k^\nu, f_k)$ converges weakly to $G(f^\nu, f)$ in $L^1((0, T) \times \Omega \times \mathbf{B}^R)$ for any $R > 0$. On the other hand, Eq. (4.23) implies that

$$f_k \geqq U_{l_k}(t) f_k^0 + \int_0^t U_{l_k}(t-s)\, G^k(f_k^\nu, f_k)\, ds. \tag{4.24}$$

For almost every $r \in D$ either the x-component of $R^s(r)$ is nontangential to $\partial\Omega$ at $-s = s^-(r) > 0$ and belongs to an open C^1-component of $\partial\Omega$, or the

x-component belongs to Ω at $-s = s^-(r) = 0$. (The analogous situation for $s = s^+(r)$ will not be discussed.) For such an r, there is a neighborhood $\mathcal{N}$ of $R^{-s^-(r)}(r)$ in ∂D^+, so that the world lines emanating from that neighborhood have the same properties as $R^s(r)$, and have a x-component staying in Ω through a neighborhood of r. Let $\psi_{\mathcal{N}}$, or, for convenience, simply ψ, be the characteristic function of $\mathcal{N}$ prolonged with value unity along the corresponding world lines $R^s(\cdot)$, and with $\psi = 0$ otherwise. For $r' \in \mathcal{N}$ it follows from (4.23) that for $0 < \delta < s < s^-(r)$,

$$f_k(R^s(r')) \geqq \tilde{U}_k^-(r', s)\,\tilde{U}_k^+(r', \delta)\,f_k(R^\delta(r'))$$
$$+ \int_\delta^s \tilde{U}_k^-(r', s)\,\tilde{U}_k^+(r', \tau)\,G^k(f_k^\nu, f_k)\,(R^\tau(r'))\,d\tau. \qquad (4.25)$$

For $R^{-s^-(r)}(r) = (t, x, \xi) \in E^+$ pick a product neighborhood of (t, x, ξ): $I_t \times \mathcal{N}_x \times \mathcal{N}_\xi \subset \mathcal{N}$. For $0 \leqq s \leqq s^-(r)$ let $\mathcal{N}_s$ be the projection in $[0, T] \times \Omega$ of $R^s(I_t \times \mathcal{N}_x \times \{\xi\})$ Take a smaller product neighborhood of (t, x, ξ):

$$\mathcal{N}' = I_t' \times \mathcal{N}_x' \times \mathcal{N}_\xi' \subset I_t \times \mathcal{N}_x \times \mathcal{N}_\xi \qquad (4.26)$$

so that

$$\{R^s(r') \mid r' \in \mathcal{N}',\ 0 \leqq s \leqq S^-(r)\} \subset \bigcup_s \mathcal{N}_s \times \mathcal{N}_\xi'. \qquad (4.27)$$

Denote by $\mathcal{N}_{\delta s}$ the subset

$$\mathcal{N}_{\delta s} = \left\{ R^{s'}(r') \,\middle|\, r' \in \mathcal{N}',\ 0 \leqq s' \leqq s^-(r),\ R^{s'}(r') \in \bigcup_{\delta \leqq s'' \leqq s} \mathcal{N}_{s''} \times \mathcal{N}_\xi' \right\}. \qquad (4.28)$$

Set $\psi_{\mathcal{N}_{\delta s}} = \psi$. It is a consequence of the averaging lemma and the estimates of f_k that for the integral over $\mathcal{N}_{\delta s}$,

$$\lim_{\nu \to \infty} \lim_{k \to \infty} \int \left| \int (f_k^\nu \psi - f\psi)\,d\xi \right| dx\,dt = 0. \qquad (4.29)$$

It follows that for almost every $0 < s'' < s^-(r)$,

$$\lim_{\nu \to \infty} \lim_{k \to \infty} \int_{\mathcal{N}_{s''}} \left| \int_{\mathcal{N}_\xi'} (f_k^\nu \psi - f\psi)\,d\xi \right| dx\,dt = 0. \qquad (4.30)$$

We are now going to use a technique that was explained in a previous paper [19]. Multiplying (4.25) by ψ and integrating, we get, by averaging and using (4.30) for almost every δ and s with the same double limit and after letting the support of ψ shrink to a Lebesgue world line, that

$$f(R^s(r')) \geqq \tilde{U}^-(r', s)\,\tilde{U}^+(r', \delta)\,f(R^\delta(r'))$$
$$+ \int_\delta^s \tilde{U}^-(r', s)\,\tilde{U}^+(r', \tau)\,G(f, f)\,(R^\tau(r'))\,d\tau \qquad (4.31)$$

for almost every $r' \in E^+$ and almost every $0 < \delta < s < s^+(r')$. An analogous reasoning gives (4.31) in the case $R^{-s^-(r)}(r) \in V^+$.

Having obtained the last inequality, we now prove that the opposite inequality also holds, in order to be able to conclude that the equality sign applies in (4.31). To this end, let us now set $f_k^\nu = \nu \log(1 + f_k/\nu)$ so that

$$f_k^\nu = U_{l_k}(t) f_k^{0\nu} + \int_0^t U_{l_k}(t-s) \frac{G^k(f_k, f_k)}{1 + f_k/\nu} ds + \int_0^t U_{l_k}(t-s) \left[l_k \left(f_k^\nu - \frac{f_k}{1 + f_k/\nu} \right) \right] ds. \tag{4.32}$$

Rewriting (4.32) like (4.25) and arguing as for (4.31) we get for almost every $r' \in E^+$ and almost every $0 < \delta < s < s^+(r')$ that

$$f(R^s(r')) \leqq \tilde{U}^-(r', s)\, \tilde{U}^+(r', \delta) f(R^\delta(r')) + \int_\delta^s \tilde{U}^-(r', s)\, \tilde{U}^+(r', \tau)\, G(f, f)\, (R^\tau(r'))\, d\tau, \tag{4.33}$$

which together with (4.31) implies that the equality sign holds in (4.31). Thus when $\delta \to 0$,

$$f(r) = \tilde{U}^-(r, 0) f(R^{-s^-(r)}(r)) + \int_{-s^-(r)}^0 \tilde{U}^-(r, 0)\, \tilde{U}^+(r, s)\, G(f, f)\, (R^s(r))\, ds \tag{4.34}$$

for any r on almost every world line in D. Equation (4.34) allows us to conclude that f is a solution of the Boltzmann equation.

Finally the entropy inequality can also be proved, by starting from the truncated equation and arguing as in the case of $\mathbf{R}^3$. □

We are now in a position to study the boundary condition satisfied by these solutions and prove

Theorem 4.2. *There is a solution as in Theorem 4.1, which satisfies*

$$\gamma_D^+(f) \geqq K(\gamma_D^- f) \quad \textit{a.e. on } E^+. \tag{4.35}$$

Proof. It follows from (4.22) that

$$\int_{E^+} |\xi \cdot n|^2\, \gamma_D^+ f_k^\nu\, d\sigma\, d\xi'\, ds \leqq C_{0T}, \tag{4.36}$$

$$\int_{E^-} |\xi \cdot n|^2\, \gamma_D^- f_k^\nu\, d\sigma\, d\xi'\, ds \leqq C_{0T}. \tag{4.37}$$

Given $\varepsilon > 0$ consider the subset $E_\varepsilon^\pm \subset E^\pm$ where the $\partial\Omega$-projection is in the open C^1 part of $\partial\Omega$ and $s^+ + s^- > \varepsilon$. ψ_ε or, for convenience, simply ψ, will hereafter denote the characteristic function of a bounded Borel set in $E_\varepsilon^\pm$ prolonged with value unity along the world lines and equal to zero otherwise.

The particular Borel sets actually considered will also be required to be contained in the intersection of $E_\varepsilon^\pm$ with a product neighborhood $I_t \times \mathcal{N}_x \times \overline{\mathcal{N}_\xi}$, where $\mathcal{N}_x$ is contained in a C^1 piece of $\partial\Omega$ and $\xi' \cdot n(x') \neq 0$ for $x' \in \mathcal{N}_x$, $\xi' \in \overline{\mathcal{N}_\xi}$. We finally require that $f(R(\cdot)) \to f(\cdot)$ uniformly when $\delta \to 0$ on the Borel sets considered.

To prove the trace statement it is enough to prove for such ψ's that

$$\langle \gamma_D^+ f, \gamma_D^+ \psi\rangle_{E_+} \geqq \langle K\gamma_D^- f, \gamma_D^+ \psi\rangle_{E_+}. \tag{4.38}$$

With f_k^ν as in (4.32) and $f^\nu = \text{w-lim}_{k\to\infty} f_k^\nu$ in $L^1(D)$, we have $f^\nu \uparrow f$ pointwise a.e. and strongly in $L^1(D)$ when $\nu \to \infty$. So for almost every (small) $\delta > 0$, $\psi f^\nu \uparrow \psi f$ strongly in $L^1(\mathscr{N}_{\delta \times \mathscr{N}_\xi})$. By averaging, as in (4.29), (4.30) above, we obtain

$$\text{w-}\lim_{k\to\infty} \psi f_k^\nu = \psi f_\nu \quad \text{in } L^1_{\mathscr{N}_\delta}, \tag{4.39}$$

and outside a set of arbitrarily small measure

$$\lim_{k\to\infty} \psi(r) \int_0^s l_k(R^\tau(r))\, d\tau = \psi(r) \int_0^s l(R^\tau(r))\, d\tau \tag{4.40}$$

uniformly over $r \in E_+ \cap \operatorname{supp} \psi$, and with the right-hand side uniformly bounded. Let $\bar{\psi} = \psi$ for the remaining set of world lines in supp ψ, and $\bar{\psi} = 0$ otherwise.

For $\delta > \delta' > \delta'' > 0$ it follows from (4.32) that

$$\begin{aligned}
0 &\leqq \int_{\delta'}^{\delta''} \tilde{U}_k^+(r,\tau)\left[\bar{\psi} l_k \left(f_k^\nu - \frac{f_k}{1+f_k/\nu}\right)(R^\tau(r))\right] d\tau \\
&\leqq \tilde{U}_k^+(r,\delta')[\bar{\psi} f_k^\nu(R^{\delta'}(r))] - \tilde{U}_k^+(r,\delta'')[\bar{\psi} f_k^\nu(R^{\delta''}(r))] \\
&\leqq \tilde{U}_k^+(r,\delta)[\bar{\psi} f_k(R^{\delta}(r))] - \tilde{U}_k^+(r,\delta'')[\bar{\psi} f_k(R^{\delta''}(r))] \\
&\quad + \int_{\delta''}^{\delta'} \tilde{U}_k^+(r,\tau)[l_k f_k(R^\tau(r))]\, d\tau \\
&\leqq (j+1)\{\tilde{U}_k^+(r,\delta)[\bar{\psi} f_k(R^{\delta}(r))] - \tilde{U}_k^+(r,\delta'')[\bar{\psi} f_k(R^{\delta''}(r))]\} \\
&\quad + \frac{2}{\log j} \int_0^\delta e(f_k)(R^\tau)\, d\tau.
\end{aligned} \tag{4.41}$$

In the limit $k \to \infty$, the first two inequalities give

$$0 \leqq \tilde{U}^+(r,\delta')[\bar{\psi} f^\nu(R^{\delta'}(r))] - \tilde{U}^+(r,\delta'')[\bar{\psi} f^\nu(R^{\delta''}(r))]. \tag{4.42}$$

The first two terms in the last member of (4.41) give in the limit when $k \to \infty$

$$(j+1)\{\tilde{U}^+(r,\delta)[\bar{\psi} f(R^{\delta}(r))] - \tilde{U}^+(r,\delta'')[\bar{\psi} f(R^{\delta''}(r))]\},$$

which is bounded by

$$(j+1)\{\tilde{U}^+(r,\delta)[\bar{\psi} f(R^{\delta}(r))] - \bar{\psi} f(r)\}.$$

Recalling that f_k satisfies (4.20), and the uniform convergence $f \circ R^\delta \to f$, we conclude that

$$\lim_{\delta \to 0} \int | \tilde{U}^+ [\overline{\psi} f^\nu_{|\mathcal{N}_\delta}] - \overline{\psi} f^\nu_{|E_+} | \, dt \, d\sigma \, d\xi = 0 \tag{4.43}$$

uniformly with respect to ν. This together with

$$\text{s-}\lim_{\nu \to \infty} \overline{\psi} f^\nu = \overline{\psi} f \quad \text{in } L^1(\mathcal{N}'_\delta) \tag{4.44}$$

implies that

$$\text{s-}\lim_{\nu \to \infty} \gamma_D^+ [\overline{\psi} f^\nu] = \gamma_D^+ [\psi f] \quad \text{in } L^{1+} . \tag{4.45}$$

An analogous result holds for L^{1-}.

By concavity and Jensen's inequality, (1.1) implies

$$\gamma_D^+ f_k^\nu \geqq K(\gamma_D^- f_k^\nu) . \tag{4.46}$$

Now in $L^{1\pm}$

$$\text{w-}\lim_{k \to \infty} \gamma_D^\pm [\overline{\psi} f_k^\nu] = \gamma_D^\pm [\overline{\psi} f^\nu] \tag{4.47}$$

and so

$$\langle \gamma_D^+ f^\nu, \gamma_D^+ \overline{\psi} \rangle_{E_+} \geqq \langle K \gamma_D^- f^\nu, \gamma_D^+ \overline{\psi} \rangle_{E_+} . \tag{4.48}$$

By (4.45) this gives

$$\langle \gamma_D^+ f, \gamma_D^+ \overline{\psi} \rangle_{E_+} \geqq \langle K \gamma_D^- f, \gamma_D^+ \overline{\psi} \rangle_{E_+} \tag{4.49}$$

and from here finally

$$\langle \gamma_D^+ f, \gamma_D^+ \psi \rangle_{E_+} \geqq \langle K \gamma_D^- f, \gamma_D^+ \psi \rangle_{E_+} . \quad \square \tag{4.50}$$

Remark. If the traces of the solutions are in $L^{1\pm}$ (as in the case of Maxwellian diffusion on the boundary), and if $Q(f,f)$ belongs to $L^1(D)$, then there is equality in (4.35).

5. Concluding remarks

As mentioned in the introduction, there is a basic restriction in HAMDACHE's theorem, *i.e.*, that the Maxwellian M_w has constant temperature along $\partial\Omega$; in other words M_w is the same at all points of $\partial\Omega$. In this paper we have removed this restriction in the case of a gas, whose large velocities have been cut off in the collision term. At the same time we have given a different proof of HAMDACHE's result. The extension is of interest for the study of the solutions of the Boltzmann equation when the boundaries drive the gas out of equilibrium. Further developments lie in the direction of removing the cut-off and studying the asymptotic trend of the solution.

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(Accepted May 10, 1993)