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Random Orders of Dimension 2*

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Abstract. A relationship is established between (partially) ordered sets of dimension 2 chosen randomly on a labelled set, chosen randomly by isomorphism type, or generated by pairs of random linear orderings. As a consequence we are able to determine the limiting probability (in each of the above sample spaces) that a two-dimensional order is rigid, is uniquely realizable, or has uniquely orientable comparability graph; all these probabilities lie strictly between 0 and 1. Finally, we show that the number of 2-dimensional (partial) orderings of a labelled n-element set is $(1+o(1))n!^2/(2\sqrt{e})$.

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I. Introduction

In 1975 Kleitman and Rothschild [9] determined the asymptotic number of partial orderings of a labelled n-element set; their approach was to show that for large n nearly all such orderings look alike, then to count the look-alikes. In fact, Compton [2] recently showed that these partial orderings obey the "0–1 law", that is, every first-order property holds either in nearly all orderings or in hardly any.

The situation for two-dimensional orders is radically different. Already in [10] and [11] it was observed that when such orders are generated via random linear orderings, there are simple first-order properties with interesting limiting probabilities like $1-1/e$ and $3/4$. (One consequence of the results below is that the 0–1 law fails also when each 2-dimensional order is weighted equally.) Hence the Kleitman–Rothschild approach is unworkable here; fortunately, the theory of random orders, plus a number of special properties of 2-dimensional orders, make the calculation feasible nonetheless.

The number of *un*labelled 2-dimensional orders has recently been computed by El-Zahar and Sauer [5]; unfortunately, since a significant fraction of these orders fail to be rigid, the labelled count cannot be directly deduced. We include our (independent) proof of the theorem of El-Zahar and Sauer along with the results listed in the abstract.

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II. Two-Dimensional Orderings of a Labelled Set

A (partially) ordered set P, with underlying set S, is said to be *2-dimensional* (see Dushnik and Miller [4]) if it is not linearly ordered, but there are two linear orderings L and M of S whose intersection yields the order on P. Thus, $x < y$ in P iff $x < y$ in both L and M. The pair (L, M) constitute a *realization* of P. (A fine source for information on dimension is Kelly and Trotter [8].)

Since L and M can be switched, it is immediate that the asymptotic number of 2-dimensional orders on $S = \{1, 2, \ldots, n\}$ is bounded by $n!^2/2$. However, the limiting probability that a large 2-dimensional random order has just the two realizations (L, M) and (M, L) turns out to be neither zero nor one. The somewhat surprising result is that the asymptotic number of 2-dimensional orders is a smaller constant multiple of $n!^2$.

THEOREM 2.1. *For every $\varepsilon > 0$ there is a number $n(\varepsilon)$ such that if $n > n(\varepsilon)$, then the number of 2-dimensional orderings of $S = \{1, 2, \ldots, n\}$ lies between $(1 - \varepsilon)n!^2/2\sqrt{e}$ and $(1 + \varepsilon)n!^2/2\sqrt{e}$.*

Proof. We begin with some definitions and general observations concerning 2-dimensional ordered sets.

For any ordered set P let $G(P)$ be the *comparability graph* of P, that is, the graph whose vertices are the points of P and whose edges are the pairs $\{x, y\}$ such that $x < y$ or $y < x$ in P. If a graph G is $G(P)$ for some ordered set P, it is said to be *transitively orientable*; if in addition it is $G(Q)$ *only* for $Q = P$ or $Q = P^d$ (the dual of P) then G is said to be *uniquely* transitively orientable, or "UTO" for short. (See [7] for an excellent survey on the subject of comparability graphs.)

We need only a few simple observations about comparability graphs. Let P be a (partial) ordering of $S = \{1, 2, \ldots, n\}$, with comparability graph $G = G(P)$. A subset A of S with $1 < |A| < n$ is said to be *properly autonomous* (in P) if

(1) every x in $S - A$ is either greater than every element of A, less than every element of A, or incomparable to every element of A;
(2) the subgraph of $G(P)$ induced by A is connected; and
(3) $S - A$ does not consist only of isolated points.

Note that if A is properly autonomous then $2 \leqslant |A| \leqslant n - 2$. The following lemma is derivable from results in [7] or directly from Gallai's original paper [6].

LEMMA 2.2. *G is UTO if and only if there is no subset A of S which is property autonomous.*

Fix $S = \{1, 2, \ldots, n\}$ and let (L, M) be a pair of linear orderings of S. Put $P = L \cap M$ and $P^* = L \cap M^d$; note that the comparability graphs $G(P)$ and $G(P^*)$ are complementary. We say that P is *uniquely realizable* (UR for short) if $P = L' \cap M'$ implies that either $L' = L$ and $M' = M$, or $L' = M$ and $M' = L$. The relevance of unique transitive orientability to our counting problem is made clear by the following lemma:

LEMMA 2.3. *There is a one-to-one correspondence between realizations of P and transitive orientations of the complement of its comparability graph. In particular, P is UR if and only if $G(P^*)$ is UTO.*

Proof. If $P = L' \cap M'$ is a realization of P then $L' \cap M'^d$ will be a transitive orientation of $G(P^*)$; conversely if Q is a transitive orientation of $G(P^*)$ then the linear orders $P \cup Q$ and $P \cup Q^d$ constitute a realization of P. Since these transformations are inverses of each other, the lemma follows. □

The *random orders* $\mathbf{P}(n)$, introduced for general dimension in [10], are random variables defined by choosing L and M randomly and independently from among the $n!$ linear orderings of S, and letting $\mathbf{P}(n) = L \cap M$. For each $k \geqslant 1$, let $R_k(n)$ be the probability that $\mathbf{P}(n)$ has exactly k realizations; then, for example, $R_1(n) = 1/n!$ since just $n!$ of the $n!^2$ pairs (L, M) have $L = M$.

For $k > n!$ we have $R_k(n) = 0$, since $L \cap M = L \cap M'$ implies $M = M'$. Hence, the number of two-dimensional orderings of S is exactly

$$\sum_{k=2}^{n!} R_k(n) n!^2/k.$$

Since the random variable $\mathbf{P}^*(n)$ given by intersecting L with M^d has the same distribution as $\mathbf{P}(n)$, Lemma 2.3 above implies that $R_k(n)$ is *also* definable as the probability that $G(\mathbf{P}(n))$ has precisely k transitive orientations. In view of Lemma 2.2, therefore, the next order of business is to characterize the properly autonomous subsets of S.

A suborder Q' of an ordered set Q is *convex* if whenever x and z are in Q' and $x < y < z$, then y is also in Q'. A convex suborder of a *linear* order is simply an interval.

LEMMA 2.4. *If a suborder A of $P = L \cap M$ is properly autonomous then A is convex both in L and in M.*

Proof. Let $x \in P - A$. If $x > A$ then x lies above all of A in both L and M, and dually if $x < A$. If x is incomparable to the elements of A, then there must be a partition $A = A_1 \cup A_2$ such that x lies above A_1 and below A_2 in L, but below A_1 and above A_2 in M. But then everything in A_1 is incomparable to everything in A_2, so by connectivity, either A_1 or A_2 is empty. It follows that every $x \in P - A$ lies above or below all of A in L, and similarly in M, proving the lemma. □

To select a pair (L, M) containing a properly autonomous subset of size t we may thus first select t elements of S, then order them in both L and M, then choose their position in L and in M, and finally order the rest of S in L and in M. Hence, the number of such pairs is at most

$$\binom{n}{t} t!^2 (n-t+1)^2 (n-t)!^2.$$

It follows that if $C_t(n)$ is the probability that $\mathbf{P}(n)$ has an autonomous set of size t, then $C_t(n) \leqslant C'_t(n)$, where

$$C'_t(n) = (n-t+1)^2 t!(n-t)!/n!.$$

We are now in a position to show that we need not worry about properly autonomous sets of size greater than 2.

LEMMA 2.5. *With probability approaching* 1 *as* $n \to \infty$, *there are no properly autonomous subsets* A *of* $\mathbf{P}(n)$ *with* $|A| > 2$.

Proof. In view of the above remarks it suffices to show that

$$\lim_{n\to\infty} \sum_{t=3}^{n-1} C'_t(n) = 0.$$

To see this note first that

$$\frac{C'_{t+1}(n)}{C'_t(n)} = \frac{t}{n-t}\left(\frac{n-t}{n-t+1}\right)^2$$

which is less than 1 for $t \leqslant n/2$ and greater than 1 for $t > n/2$. Thus the sequence $C'_3(n), C'_4(n), \ldots, C'_{n-1}(n)$ is upside-down unimodal, i.e. falls and then rises. Now

$$C'_4(n) < 24/n^2$$

and

$$C'_{n-2}(n) < 36/n^2$$

so that

$$\sum_{t=3}^{n-1} C'_t(n) = C'_3(n) + \sum_{t=4}^{n-2} C'_t(n) + C'_{n-1}(n) \leqslant 6/n + (n-5)(36/n^2) + 4/n \to 0$$

proving the lemma. □

A two-element properly autonomous subset of $P = L \cap M$ consists merely of what we call a *reversible edge*, that is, a pair (x, y) such that y covers x both in L and in M. It turns out that the distribution of the number of reversible edges in $\mathbf{P}(n)$ is asymptotically Poisson, with mean 1. We have indicated briefly in [10] how this is proved (en route to a 0–1 law counterexample) but since the result is critical here, an explicit argument is warranted.

LEMMA 2.6. *Let $X(n)$ be the number of reversible edges in* $\mathbf{P}(n)$. *Then for any $\varepsilon > 0$ there is an $n(\varepsilon)$ such that for any $n > n(\varepsilon)$ and any integer $s \geqslant 0$,*

$$\left|\Pr(\mathbf{X}(n) = s) - \frac{1}{s!e}\right| < \varepsilon.$$

Proof. For positive integers m and r let $(m)_r$ denote the product $m(m-1)(m-2)\cdots(m-r+1)$. The rth factorial moment of a random variable $\mathbf{X}$ is defined to be the expected value of $(\mathbf{X})_r$. It follows from classical results (see, e.g., Bollobas [1], Theorem 20, p. 23) that if the factorial moments of a sequence of non-negative integer valued random variables approach the factorial moments of a Poisson random variable $\mathbf{X}$, then the sequence approaches $\mathbf{X}$ in distribution. If in particular $\mathbf{X}$ is Poisson with mean 1, in which case $E_r(\mathbf{X}) = 1$ for every $r \geqslant 0$, it follows that to establish

$$\lim_{n\to\infty} E((\mathbf{X}(n))_r) = 1$$

for each $r \geqslant 0$, suffices to prove the lemma.

Clearly $(\mathbf{X}(n))_r$ counts the number of r-tuples of distinct reversible pairs in $\mathbf{P}(n)$. Since the existence of reversible pairs is not dependent on the labelling of $\mathbf{P}(n)$, we may assume that the linear ordering L is the identity and that M is given by a random permutation σ, in the sense that the height of i in M is $\sigma(i)$. In that case a reversible pair may be identified with a number i, necessarily less than n, such that $\sigma(i+1) = \sigma(i) + 1$. Let us temporarily call such a number *reversible*.

Fix n and select an r-tuple $(i_1, \ldots, i_r)$ with $1 \leqslant i_1 < i_2 < \cdots < i_r < n$. If p is the probability that all these i_j's are reversible in $\mathbf{P}(n)$, then

$$p = \sum_{j=1}^{r} p_j$$

where p_1 is the probability that i_1 is reversible and for each $j > 1$, p_j is the probability that i_j is reversible given that $i_1, \ldots, i_{j-1}$ are all reversible.

Let us imagine that σ is chosen by first selecting $\sigma(i_1)$, then $\sigma(i_1+1)$, then $\sigma(i_2)$ (if not already done), then $\sigma(i_2+1)$, then $\sigma(i_3)$ (if not already done), and so forth up to $\sigma(i_r+1)$; the rest of the values of σ are of course irrelevant. When $\sigma(i_j+1)$ is chosen there is at most one value available which allows i_j to be reversible, and at most $2j-1$ values of σ have already been assigned; hence $p_j \leqslant 1/(n-2j+1)$ and therefore

$$p \leqslant \frac{1}{(n-1)(n-3)\cdots(n-2r+1)}.$$

On the other hand, let $i_{j(1)} < i_{j(2)} < \cdots < i_{j(v)}$ be the subsequence consisting exactly of those i_j's, including i_1, for which $i_j \neq i_{j-1} + 1$. Let us first choose $\sigma(i_{j(1)}), \cdots, \sigma(i_{j(v)})$; then with probability at least $(n-r)(n-3r)(n-5r)\cdots(n-(2v-1)r)/(n)_v$, no range $[k, k+r-1]$ contains more than one of the values $\sigma(i_{j(u)})$ and the range $[n-r+1, n]$ contains none at all. Should this occur, when we now choose the values $\sigma(i_j + 1)$, $1 \leqslant j \leqslant r$, each is guaranteed to have a value available – out of the at most $n - j$ unchosen values – which makes i_j reversible. It follows that

$$p \geqslant \frac{(n-r)(n-3r)(n-5r)\cdots(n-(2r-1)r)}{(n)_r(n-1)_r}.$$

Now the number of r-tuples of distinct elements of $\{1, 2, \ldots, n-1\}$ is of course $(n-1)_r$, hence

$$\frac{(n-r)(n-3r)(n-5r)\cdots(n-(2r-1)r)}{(n)_r}$$

$$\leqslant E((\mathbf{X}(n))_r) \leqslant \frac{(n-1)_r}{(n-1)(n-3)\cdots(n-2r+1)}.$$

Since for fixed r the bounding quantitites approach 1 as $n \to \infty$, we have $E((\mathbf{X}(n))_r) \to 1$ as desired, and the lemma is proved. □

Let us suppose now that $P = \mathbf{P}(n)$ has exactly s reversible edges, but no properly autonomous subset of size greater than 2. Then the points involved in the reversible edges are all distinct, so each such edge can be oriented independently; hence P has precisely 2^{s+1} transitive orientations. This means, of course, that P^* has just 2^{s+1} realizations. It follows that if $N(n)$ is the total number of two-dimensional orderings of S, then

$$\lim_{n\to\infty} \frac{N(n)}{n!^2} = \sum_{s=0}^{\infty} \left(\left(\frac{1}{s!e}\right)\left(\frac{1}{2^{s+1}}\right) \right) = \frac{1}{2\sqrt{e}}$$

proving the theorem. □

III. Probabilities among 2-Dimensional Orders

It is now useful to define an additional ordered-set-valued random variable $\mathbf{Q}(n)$ in such a way that each 2-dimensional ordering of S occurs with equal probability, and all other orders with zero probability. If Φ is any statement about orderings of S, then we let $\Phi[\mathbf{P}]$ (respectively $\Phi[\mathbf{Q}]$) represent the limiting probability (if it exists) that Φ holds in $\mathbf{P}(n)$ (respectively in $\mathbf{Q}(n)$).

We can now move from probabilities involving $\mathbf{P}(n)$, which are often easy to compute, to probabilities involving $\mathbf{Q}(n)$. (We regard the random variable $\mathbf{P}(n)$ as the more natural one, but the mere fact that $\mathbf{Q}(n)$ has uniform distribution justifies interest.)

Note first that a reversible edge in P^* corresponds to what we shall call a "twin pair" in P, that is, a pair (x, y) such that y covers x in L but x covers y in M. It is easy to see directly (without considering P^*) that twin pairs create extra realizations for P, since the order of x and y can be switched in the two linear orderings.

THEOREM 3.1. *Let* Φ *be any statement about ordered sets, let* $\mathbf{Y}(n)$ *be the number of twin pairs in* $\mathbf{P}^*(n)$, *and suppose that for each* $s \geqslant 0$ *there exists a real number* r_s *such that*

$$r_s = \lim_{n \to \infty} \Pr(\mathbf{P}(n) \text{ satisfies } \Phi \text{ given } \mathbf{Y}(n) = s).$$

Then

$$\Phi[\mathbf{Q}] = \frac{1}{\sqrt{e}} \sum_{s=0}^{\infty} \frac{r_s}{2^s s!}.$$

Proof. From the previous section, we have that the number of two-dimensional orderings of S which satisfy Φ is $c(n)n!^2$, where $c(n)$ is asymptotic to

$$\frac{1}{2} r_0 \frac{1}{e} + \frac{1}{4} r_1 \frac{1}{e} + \frac{1}{8} r_2 \frac{1}{2e} + \cdots = \sum_{s=0}^{\infty} \frac{r_s}{2^{s+1} s! e};$$

dividing $n!^2 c(n)$ by the total number $n!^2/2\sqrt{e}$ of two-dimensional orderings gives the result. □

COROLLARY 3.2. *Any statement with limiting probability strictly between* 0 *and* 1 *in* $\mathbf{P}(n)$ *also has nontrivial limiting probability in* $\mathbf{Q}(n)$; *in particular, the* $0-1$ *law fails also in* $\mathbf{Q}(n)$. *Further, any statement with limiting probability* 0 *or* 1 *in* $\mathbf{P}(n)$ *has the same limiting probability in* $\mathbf{Q}(n)$.

EXAMPLE 3.3. Let Φ be the statement "there exists a pair (x, y) of elements with x maximal, y minimal, and x and y incomparable." This statement is shown in [11] to have limiting probability 3/4 in $\mathbf{P}(n)$; moreover, its truth depends entirely on the behavior of the (at most) four elements at the top and bottom of the two linear orderings. Thus it is asymptotically independent of the statement "$\mathbf{Y}(n) = s$" for any fixed s, and so $\Phi[\mathbf{Q}] = 3/4$ also.

EXAMPLE 3.4. Let Φ_1 be the statement "the order is uniquely realizable" and let Φ_2 be the statement "the comparability graph is uniquely transitively orientable." The two statements are dual with respect to the correspondence $\mathbf{P}(n) \leftrightarrow \mathbf{P}^*(n)$ and each has $\Phi_i[\mathbf{P}] = 1/e \approx .368$. For Φ_1, we have that $r_0 = 1$ and $r_s = 0$ for all $s > 0$, so $\Phi_1[\mathbf{Q}] = 1/2\sqrt{e} \approx .303$. However, since reversible edges are not twin pairs, Φ_2 is asymptotically independent from the events $\mathbf{Y}(n) = s$ and we have $\Phi_2[\mathbf{Q}] = 1/e$.

IV. Isomorphism Types

The isomorphism types of two-dimensional orderings of an n-element set were counted (asymptotically) by El-Zahar and Sauer [5]; we give an independent computation which, in combination with previous results, gives additional information about probabilities.

To study the isomorphism types of two-dimensional orderings of S, we introduce the random variable $\mathbf{U}(n)$ whose values are these isomorphism types, each taken with equal likelihood. We denote the isomorphism class of a particular ordering P of S by $[P]$.

An order P is said to be *rigid* if it has no non-trivial automorphisms; in that case the class $[P]$ contains a full complement of $n!$ orderings of S. If P is not rigid then one of its non-trivial automorphisms σ can be applied to a realization (L, M) of P, yielding a new realization $(\sigma L, \sigma M)$; hence there would appear to be a relation between rigidity and unique realizability. The following examples, however, show that this relationship is perhaps more subtle than expected.

EXAMPLE 4.1. Let $n = 4$ and take P to consist just of the relations $1 < 2$ and $3 < 4$. Then P has a non-trivial automorphism, namely $\sigma = (1, 3)(2, 4)$, but is uniquely realizable (by the orders $1 < 2 < 3 < 4$ and $3 < 4 < 1 < 2$).

EXAMPLE 4.2. Let $n = 5$ and let P consist of the relations $1 < 2$, $3 < 4$ and $1 < 4$. Then P is rigid but nonetheless has two essentially different realizations, $(1 < 2 < 3 < 4 < 5, 5 < 3 < 1 < 4 < 2)$ and $(5 < 1 < 2 < 3 < 4, 3 < 1 < 4 < 2 < 5)$.

Hasse diagrams for the two examples are pictured in Figure 1 below. Notice that Example 4.2 incorporates a properly autonomous set of size greater than two, a rare occurrence. In Example 4.1, the reason that the automorphism σ fails to produce a new realization is that it merely switches the orders L and M, i.e. $\sigma L = M$ and $\sigma M = L$. We show that this is also a rare event.

THEOREM 4.3. *With probability approaching* 1 *the number of realizations of* $\mathbf{P}(n)$ *is exactly twice the cardinality of its automorphism group.*

Proof. Let the automorphism group of $\mathbf{P}(n)$ consist of the permutations $\sigma_1, \sigma_2, \ldots, \sigma_k$ with σ_1 equal to the identity. We begin by showing that with probability approaching 1, the realizations $(\sigma_i L, \sigma_i M)$ and $(\sigma_i M, \sigma_i L)$, $1 \leqslant i \leqslant k$, are all different.

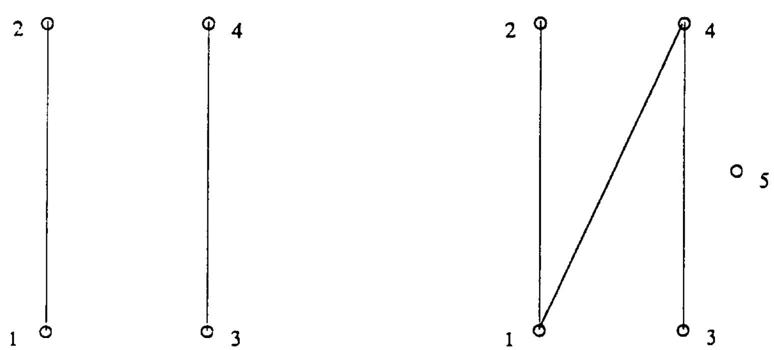

Fig. 1.

If two of these realizations are the same then we must have $\sigma_i L = \sigma_j M$ and $\sigma_j L = \sigma_i M$ for distinct i and j. Let τ be the permutation defined by $M = \tau L$; then $\tau = \sigma_j^{-1}\sigma_i$ and $\tau^2 L = \tau M = L$, so that τ is an involution.

Since L and M are random and independent; τ is *a priori* equally likely to be any of the $n!$ permutations of S. The number of involutions is only

$$\sum_{t=0}^{\lfloor t/2 \rfloor} \frac{n!}{(n-2t)!t!2^t} = o(n!)$$

so we have that with probability approaching 1, there are at least two realizations per automorphism.

On the other hand, from Section II we know that $\mathbf{P}(n)$ nearly always has precisely 2^{s+1} realizations, for some $s \geqslant 0$, traceable to s twin pairs $(x_1, y_1), (x_2, y_2), \ldots, (x_s, y_s)$ where the x_i's and y_i's are all distinct. If σ_i is the automorphism which switches x_i with y_i and leaves the other elements fixed, then the σ_i's generate a subgroup isomorphic to $\mathbf{Z}_2^s$ within the automorphism group of $\mathbf{P}(n)$. Thus with probability approaching 1 there are at least half as many automorphisms as realizations. □

COROLLARY 4.4. *The limiting probability that a random 2-dimensional order is rigid is* $1/e$ *with respect to* $\mathbf{P}(n)$, *and* $1/(2\sqrt{e})$ *with respect to* $\mathbf{Q}(n)$.

Now let $\mathbf{I}(n)$ be the total number of realizations of the *isomorphism class* $[\mathbf{P}(n)]$, i.e. the number of pairs (L, M) such that $L \cap M$ is isomorphic to $\mathbf{P}(n)$. If $\mathbf{P}(n)$ has i automorphisms and j realizations, then $\mathbf{I}(n) = j(n!/i)$ which by Theorem 4.3 above is nearly always equal to $2n!$. From this we can deduce that the number of isomorphism classes is $(1 + o(1))n!/2$.

THEOREM 4.5 [5]. *Fro every* $\varepsilon > 0$ *there is an* $n(\varepsilon)$ *such that if* $n > n(\varepsilon)$, *then the number of isomorphism classes of* n*-element two-dimensional ordered sets lies between* $(1-\varepsilon)n!/2$ *and* $(1+\varepsilon)n!/2$.

From the fact that nearly all pairs (L, M) generate an isomorphism class with exactly $2n!$ realizations, and the fact that *every* isomorphism class has at least $n!$ realizations, we can deduce that nearly all isomorphism classes have $2n!$ realizations. It follows that the random variables $\mathbf{U}(n)$ and $\mathbf{P}(n)$ are asymptotically equivalent with respect to isomorphism-invariant statements. Put formally:

THEOREM 4.6. *Let* Φ *be an isomorphism-invariant statement about two-dimensional orders which has a limiting probability either in* $\mathbf{P}(n)$ *or in* $\mathbf{U}(n)$. *Then a limiting probability exists in the other case as well and the two probabilities are equal.*

Relationships among the random variables $\mathbf{P}^*(n)$, $\mathbf{P}(n)$, $\mathbf{Q}(n)$ and $\mathbf{U}(n)$ are indicated schematically in Figure 2 below.

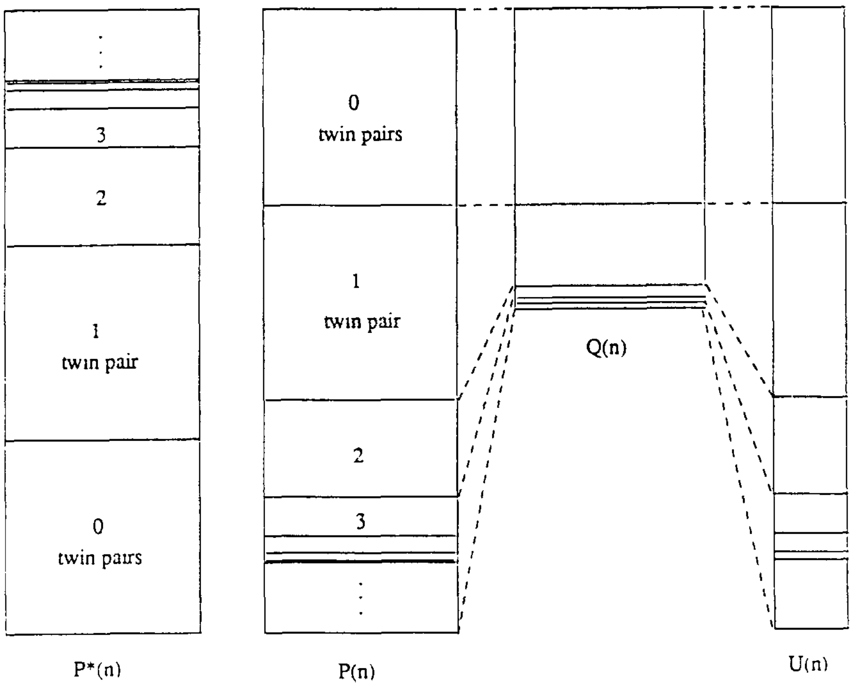

Fig. 2.

V. Problems

The obvious problem is to extend these results in some manner to higher dimension. However, much of the above relies on special properties of dimension 2; at the moment we do not even know how many realizations an antichain has in dimension 3 or higher. (See [3] for some general results on realizations.) The 0–1 law fails also in higher dimensions [12] but determination of probabilities seems much harder.

One further problem is open for all dimensions $d \geqslant 2$: does every first-order statement in the language of partial orders have a limiting probability?

References

[1] B. Bollobas (1985) *Random Graphs*, Academic Press, London.
[2] K. J. Compton (1988) The computational complexity of asymptotic problems. I: Partial orders, *Inform. and Computation* **78**, 103–123.
[3] D. Duffus and P. Winkler (1990) On the number of k-realizations of an ordered set, *Order* **7**, 267–273.
[4] B. Dushnik and E. W. Miller (1941) Partially ordered sets, *Amer. J. Math.* **63**, 600–610.
[5] M. El-Zahar and N. W. Sauer (1987), Asymptotic enumeration of two-dimensional posets, U. of Calgary Dept. of Mathematics and Statistics, Research Paper No. 643.
[6] T. Gallai (1967) Transitiv orientierbare Graphen, *Acta Math. Acad. Sci. Hungar.* **18**, 25–66.
[7] D. Kelly (1985) Comparability graphs, in *Graphs and Order* (ed. I. Rival), D. Reidel, Dordrecht, pp. 3–40.

[8] D. Kelly and W. T. Trotter (1981) Dimension theory for ordered sets, in *Ordered Sets* (ed. I. Rival), D. Reidel, Dordrecht, pp. 171–211.
[9] D. J. Kleitman and B. L. Rothschild (1975) Asymptotic enumeration of partial orders on a finite set, *Trans. Am. Math. Soc.* **205**, 205–210.
[10] P. Winkler (1985) Random orders, *Order* **1**, 317–331.
[11] P. Winkler (1985) Connectedness and diameter for random orders of fixed dimension, *Order* **2**, 165–171.
[12] P. Winkler (1989) A counterexample in the theory of random orders, *Order* **5**, 363–368.