A BETTI-RAYLEIGH THEOREM FOR ELASTIC BODIES EXHIBITING TEMPERATURE DEPENDENT PROPERTIES *)

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Summary

The reciprocal theorem of Betti and Rayleigh is extended to thermoelastic problems concerning temperature dependent properties of the bodies. Illustrative examples are solved concerning extension and flexural rotation of a bar, change of volume of a body without and with a cavity, and the thermoelastic displacement.

§ 1. *Introduction.* The reciprocal theorem of Betti and Rayleigh has been extended to thermoeleastic problems for isotropic bodies by Maisel [1]) and for anisotropic bodies by Nowacki [2]). Recently Goodier gave a new derivation of thermoelastic reciprocal theorem and deduced several simple formulas for overall thermoelastic deformation [3]). In these papers the elastic coefficients have been postulated as temperature independent.

However, recent years have seen a rising interest in temperature fields of such intensity that the variation of mechanical and thermal properties of bodies, with changing temperature, cannot be ignored. Inasmuch as temperature dependence of elastic constants leads to differential field equations with variable coefficients, thus far only few solutions involving axially or polarly symmetrical case **) have been given.

In view of the mathematical difficulties concerned, it seems therefore expedient to generalize the integral theorem under consideration to elastic bodies exhibiting temperature dependent properties. Of

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**) See, e.g., [4]) or [5]), where further references may also be found.

course, this provides information for the intensity of the elastic field, in general, of an imperfect character, still in some cases it may prove of considerable service.

§ 2. *Basic equation.* Consider an elastic body occupying a region V with the boundary S. Let $T(Q)$, $\boldsymbol{F}(Q)$ and $\boldsymbol{P}(Q)$ be the temperature field and the intensities of the body forces and surface tractions to which the body is subjected, with Q as a point in $V + S$ and on S, respectively. Assume that the shear modulus $\mu = \mu(T)$, Poisson's ratio $\nu = \nu(T)$ and the coefficient of thermal expansion $\alpha = \alpha(T)$ of the body are functions of temperature and, consequently, of position in $V + S$. In indicial notation, and with reference to a rectangular cartesian reference frame x_i, $i = 1, 2, 3$, we have throughout the body

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}),$$

$$\tau_{ij} = 2\mu\left[e_{ij} + \left(\frac{\nu}{1-2\nu}e_{kk} - \frac{1+\nu}{1-2\nu}\int_0^T \alpha(\vartheta)\,\mathrm{d}\vartheta\right)\delta_{ij}\right], \quad (1)$$

$$\tau_{ij,j} + F_i = 0,$$

and on the boundary S

$$\tau_{ij}n_j = P_i.$$

Here u_i, e_{ij}, τ_{ij} and n_i are the cartesian components of the displacement, strain, stress and the unit vector normal to S, respectively. Let u_i be twice continuously differentiable functions of Q in the region V including its boundary. With no loss in generality suppose that μ, ν and α are continuously differentiable functions of T or Q in $V + S$. Clearly, for $T = 0$, the quantities μ, ν, α take their usual (constant) values in ambient conditions. In this case the body becomes homogeneous. For $T \neq 0$, owing to the temperature dependence of the properties, the substance of the body becomes heterogeneous, with μ, ν, and α being functions of position.

Suppose, for the time being, that the thermal heterogeneity of the substance of the body represents its permanent property provoked by any factor, not necessarily by the temperature. In other words, in the absence of the temperature rise, the body is supposed to possess an "inherent" heterogeneity identical with the thermal heterogeneity stated above.

Consider two states of equilibrium of the heterogeneous body concerned: one thermoelastic state with displacements u_i due to the body forces F_i, surface tractions P_i and temperature rise T, and the other an elastic athermal state with displacements u'_i due to the body forces F'_i and surface tractions P'_i, with $T = 0$. Name

$$\frac{E}{1-2\nu}\int_0^T \alpha(\vartheta)\,\mathrm{d}\vartheta$$

the "hydrostatic thermal tension" in the thermoelastic state of the body (E denotes Young's modulus).

The Betti-Rayleigh reciprocal theorem as generalized to elastic bodies with temperature dependent properties can be now expressed in the following form.

Theorem. If an elastic body with temperature-dependent properties is subjected to two systems of body and surface forces, the first of these systems being, moreover, connected with a temperature rise T, then the work done by the first system of forces F_i, P_i in acting through the displacements u'_i due to the second system of forces, plus the work done by the "hydrostatic thermal tensions" of the first system in acting through the dilatation of the second system, is equal to the work that would be done by the second system of forces F'_i, P'_i in acting through the displacements u_i due to the first system, the scalar fields of μ, ν and α being identical for both systems and related to the temperature rise T:

$$\int_{(V)} F_i u'_i\,\mathrm{d}\tau + \int_{(S)} P_i u'_i\,\mathrm{d}\sigma + \int_{(V)} \frac{E}{1-2\nu}\int_0^T \alpha(\vartheta)\,\mathrm{d}\vartheta\cdot\theta'\,\mathrm{d}\tau = \\ = \int_{(V)} F'_i u_i\,\mathrm{d}\tau + \int_{(S)} P'_i u_i\,\mathrm{d}\sigma, \qquad (3)$$

with an alternative equation

$$\int_{(V)} F'_i u_i\,\mathrm{d}\tau + \int_{(S)} P'_i u_i\,\mathrm{d}\sigma = \int_{(V)} \tau'_{ij} e_{ij}\,\mathrm{d}\tau. \qquad (4)$$

Here $E = 2\mu\,(1+\nu)$ and θ' denote, as before, Young's modulus and dilatation e'_{ii}, respectively.

The proof of the foregoing theorem is rather trivial and can be given most simply by the following formal reasoning. Using (1) write explicitly the identities

$$\tau_{ij}u'_{i,j} = 2\mu\left[\tfrac{1}{2}(u_{i,j}+u_{j,i})u'_{i,j} + \frac{\nu}{1-2\nu}\theta\theta' - \frac{1+\nu}{1-2\nu}\int_0^T \alpha(\vartheta)\,\mathrm{d}\vartheta\theta'\right],$$

$$\tau'_{ij}u_{i,j} = 2\mu'\left[\tfrac{1}{2}(u'_{i,j}+u'_{j,i})\,u_{i,j} + \frac{\nu}{1-2\nu}\,\theta\theta'\right].$$

Since by hypothesis $\mu = \mu'$, we obtain from (4) by integrating over the region V:

$$\int_{(V)} \tau_{ij i,j} u'_{i,j} \,\mathrm{d}\tau + \int_{(V)} \frac{E}{1-2\nu} \int_0^T \alpha(\vartheta)\,\mathrm{d}\vartheta \cdot \theta' \mathrm{d}\tau = \int_{(V)} \tau'_{ij} u_{i,j} \,\mathrm{d}\tau. \quad (5)$$

Bearing in mind that u_i and τ_{ij} (as well as the same quantities with primes) are continuously differentiable functions in $V + S$, we can use the Gauss-Green divergence theorem

$$\int_{(V)} A_{i,i} \,\mathrm{d}\tau = \int_{(S)} A_i u_i \,\mathrm{d}\sigma, \quad (5.1)$$

with A_i as functions of position. Respecting the identity, e.g.,

$$\tau_{ij} u'_{i,j} = (\tau_{ij} u'_i)_{,j} - \tau_{ij,j} u'_i, \quad (5.2)$$

we transform (5) by virtue of the last equation (1) and of (2) into the form (3) which completes the proof of the theorem.

It results clearly from the foregoing development that in order to apply the reciprocal theorem to the thermoelastic problem involving temperature-dependent properties of the body, a related elastic problem for an "inherently" non-homogeneous body must first be solved. Of course, this requires, in general, a special investigation since to the present time only a few solutions to the latter problem have been found. However, some trivial solutions hitherto available will permit us to derive, for illustration, several simple formulae for overall thermoelastic deformation in the case of temperature-dependent properties of bodies. These formulae correspond to some of the known results given in the papers by Chree [7]) and by Goodier [3]) for athermal or temperature-insensitive cases. In what follows we find it expedient to depart from indicial notation and use x, y, z instead of $x_i (i = 1, 2, 3)$.

§ 3. *Applications.* (a) Extension of a bar. Suppose that the z-axis of the x, y, z system is parallel to the generators of a cylindrical bar and that there exists only a transverse temperature variation $T = T(x, y)$, T being independent of the coordinate z. Hence $E = E(x, y)$ and $E' = E'(x, y)$. Denote by A the area of a cross-section of the bar and define as the "reduced centroid" C^* of a cross-section the point in that cross-section for which

$$\int_{(A)} E'(x, y)\, x \,\mathrm{d}A = \int_{(A)} E'(x, y) y \,\mathrm{d}A = 0. \quad (6.1)$$

With the notation

$$E^* = \int_{(A)} E'(x, y) \, dA/A,$$

the expression

$$\tau'_{zz} = P'E'/E^*A \tag{6}$$

represents the tensile stress induced by a force P' applied at C^* in the athermal case [8]). In fact, for other stress components equal to zero the stress system (6) satisfies the differential equations of internal equilibrium and is statically equivalent to longitudinal force P' operative at C^*. Furthermore, since

$$e'_{zz} = P'/E^*A \text{ and } e'_{xx} = e'_{yy} = -\nu'e'_{zz} \tag{7}$$

are constant, the compatibility conditions are satisfied. Consequently

$$\theta' = \frac{P'}{E^*A}(1 - 2\nu), \tag{8}$$

and

$$u'_x = -\frac{\nu'P'}{E^*A}x, \quad u'_y = -\frac{\nu'P'}{E^*A}y, \quad u'_z = \frac{P'}{E^*A}z. \tag{9}$$

The tensile force P' is formed from a distribution of tensile stress τ'_{zz} (6) and the right hand member of (3) becomes the work of P' on the end displacement induced by the system F_i, P_i and T. If this work is written $P'\Delta l$, then Δl is a mean thermoelastic elongation of the bar of length l. Thus (3) with (8) and (9) yields

$$\Delta l = \frac{1}{E^*A}\left\{\int_{(V)} [-\nu(F_x x + F_y y) + F_z z] \, d\tau + \right.$$
$$\left. + \int_{(S)} [-\nu(P_x x + P_y y) + P_z z] \, d\sigma + \int_{(V)} E \int_0^T \alpha(\vartheta) \, d\vartheta \cdot d\tau \right\}. \tag{10}$$

Since for a given temperature field ν, E, E^* and α are known functions of position in $V + S$, Δl can be found by simple quadratures.

(b) Flexural rotation of a bar. Choose the locus of the reduced centroid of a uniform bar with transverse heterogeneity $E' = E'(x, y)$ as the z-axis of the coordinate system, the x- and y-axes being the "principal reduced" axes of a cross-section. This has to mean that for these axes

$$\int_{(A)} E'(x, y) \, xy \, dA = 0, \tag{11}$$

by definition. Then

$$\tau'_{zz} = -\frac{M'_y E'(x, y)}{J_y{}^*}\, x \tag{12}$$

represents the only non-vanishing component of stress in the bar induced by a pair of bending moments M'_y acting in the plane zx. Here

$$J^*_y = \int_{(A)} E'(x, y)\, x^2 \, \mathrm{d}A$$

is the "reduced moment of inertia" with respect to the "reduced principal" y-axis through C^*. In fact, τ'_{zz} in (12) does not provide any resultant force in the z direction since x is a reduced axis passing through C^* and for such an axis Eq. (6.1) holds. Again by virtue of (11) M'_x vanishes, and the moment of τ'_{zz} around an axis paralell to y yields $-M'_y$. Furthermore, the differential equations of internal equilibrium are satisfied as well as the compatibility conditions, since the longitudinal elongation

$$e'_{zz} = -\frac{M_y}{J^*_y}\, x, \tag{13}$$

and two other unit elongations are linear functions of the coordinates. It is trivially seen that

$$\theta' = (2\nu - 1)\,\frac{M_y}{J^*_y}\, x. \tag{14}$$

Hence for a mean, purely thermal, flexural rotation φ_y in the plane zx of one end of the bar relative to the other, we obtain from (3)

$$\varphi_y = \frac{1}{J^*_y} \int_{(V)} E \int_0^T \alpha(\vartheta)\, \mathrm{d}\vartheta \cdot x \, \mathrm{d}\tau. \tag{15}$$

(c) Change of volume. Let us find the change of volume of the solid material in an arbitrary body with cavities, under the action of any temperature field. Choose as the second system the uniform normal traction p' distributed over all bounding surfaces of the body. Then

$$\sigma'_x = \sigma'_y = \sigma'_z = p' \text{ and } \theta' = 3(1 - 2\nu)\, p'/E.$$

It results now from the theorem (3) that the change of volume V of the solid materials equals

$$\Delta V = 3 \int_{(V)} \int_0^T \alpha(\vartheta)\, \mathrm{d}\vartheta \, \mathrm{d}\tau, \tag{16}$$

the contribution from the thermal stress being nil. This completes the results obtained by Goodier, Hieke and Nowacki.

(d) Change of volume of a cavity cannot be found in a like trivial manner. To take a special case, consider a hollow sphere of radius $a \leqslant r \leqslant b$ undergoing any polarly symmetrical temperature rise $T(r)$. As the auxiliary system we take a non-homogeneous sphere acted upon by constant inner normal tractions p'_i. Denoting by u' the respective radial displacement we obtain from the second eq. (1) the radial and the hoop stresses

$$\begin{aligned} \sigma_r &= E'(r)\left[(\beta-\alpha)\,u'_{,r}+2\alpha\,\frac{u'}{r}\right], \\ \sigma_r &= E'(r)\left(\alpha u'_{,r}+\beta\,\frac{u'}{r}\right). \end{aligned} \tag{17}$$

Here

$$\alpha = \nu/(1+\nu)(1-2\nu) \text{ and } \beta = \alpha/\nu,$$

where Poisson's ratio is supposed to be temperature-independent in order not to complicate the resulting equations. From the divergence equation of stress we find the basic equation of the problem

$$u'_{,rr}+\left(\frac{2}{r}+\frac{1}{E'}\,E'_{,r}\right)u'_{,r}+2\left(\frac{\nu}{1-\nu}\,E'_{,r}-\frac{1}{r}\right)\frac{u}{r}=0, \tag{18}$$

which can be solved in closed form for specific forms of $E'(r)$ *) only. Suppose that $u'(r)$ is known and that $\theta' = u'_{,r} + 2u'/r$ can be found. Then from (3) we find the change of volume of the spherical cavity

$$\Delta V_c = \frac{4\pi}{1-2\nu}\int_a^b E(r)\int_0^{T(r)} \alpha(\vartheta)\,\mathrm{d}\vartheta\cdot(r^2u')_{,r}\,\mathrm{d}r. \tag{19}$$

(e) Thermoelastic displacement of a point of a temperature sensitive body can be directly obtained from the theorem (3) assuming $F_i = P_i = 0$ and taking F'_i or P'_i (for a definite value of i) equal to unity (the remaining components of F_i, P_i being equal to zero). Hence for any point (ξ, η, ζ) in the region $V + S$ we get a general formula

$$u_i(\xi,\eta,\zeta) = \int_{(V)} \frac{E}{1-2\nu}\int_0^T \alpha(\vartheta)\,\mathrm{d}\vartheta\cdot\theta'\,\mathrm{d}\tau, \tag{20}$$

*) See, e.g., ref. [10]).

in which $\theta'(x, y, z)$, however, has to be found solving the athermal problem for a non-homogeneous body acted upon by a concentrated unit force operative at the point ξ, η, ζ of $V + S$. Some particular solutions of the latter problem are known.

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