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On Stability of Navier-Stokes Flows in Exterior Domains

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Dedicated to Professor Yoshio Kato on his sixtieth birthday

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Introduction

Let Ω be an exterior domain in $\mathbf{R}^n$ ($n \geqq 3$), i.e., a domain having a compact complement $\mathbf{R}^n \backslash \Omega$, and assume that the boundary $\partial\Omega$ is of class $C^{2+\mu}$ ($0 < \mu < 1$). The motion of an incompressible fluid occupying Ω is governed by the Navier-Stokes equations:

$$\text{(S)} \qquad \begin{aligned} -\Delta w + w \cdot \nabla w + \nabla \pi = f \quad & \text{in } \Omega, \\ \operatorname{div} w = 0 \quad & \text{in } \Omega, \\ w = 0 \quad & \text{on } \partial\Omega, \\ w(x) \to 0 \quad & \text{as } |x| \to \infty, \end{aligned}$$

where $w = w(x) = (w^1(x), \ldots, w^n(x))$ and $\pi = \pi(x)$ denote the velocity vector and the pressure of the fluid at point $x \in \Omega$, respectively, while $f = f(x) = (f^1(x), \ldots, f^n(x))$ is the external force. In [21], KOZONO & SOHR estabished the existence and uniqueness of solutions to the linearized equations of (S), i.e., the stationary Stokes equations having a finite L^r-gradient $\int_\Omega |\nabla w(x)|^r \, dx < \infty$ for $n/(n-1) < r < n$. Based on their results with the aid of the implicit function theorem, one can easily show that (S) has a smooth solution with

$$\text{(CL)} \qquad w \in L^n(\Omega), \quad \nabla w \in L^{n/2}(\Omega)$$

if $n \geqq 4$, provided the prescribed force f is sufficiently small and decays rapidly at infinity. If $n = 3$, some investigation into the existence of solutions of (S) within the class (CL) has been made by several authors (see, e.g., [9, 22]).

The purpose of this paper is to show the stability in L^r of solutions of (S) in the class (CL). If w is perturbed by a, then the perturbed flow $v(x, t)$ is governed by the following *non-stationary* Navier-Stokes equations:

$$\text{(N-S)}\qquad \begin{aligned} &\frac{\partial v}{\partial t} - \Delta v + v\cdot\nabla v + \nabla q = f && \text{in } \Omega,\ t>0,\\ &\operatorname{div} v = 0 && \text{in } \Omega,\ t>0,\\ &v = 0 && \text{on } \partial\Omega,\ t>0,\\ &v(x,t)\to 0 && \text{as } |x|\to\infty,\\ &v(x,0) = w(x) + a(x) && \text{for } x\in\Omega. \end{aligned}$$

In this paper we shall show that if the stationary flow w and the initial disturbance a are both small enough in the class (CL) and in $L^n(\Omega)$, respectively, then there is a unique *global strong solution* v of (N-S) such that the integrals

$$\text{(D}_r\text{)}\qquad \begin{aligned} &\int_\Omega |v(x,t) - w(x)|^r\,dx \quad \text{for } 1 < r < \infty,\\ &\int_\Omega |\nabla v(x,t) - \nabla w(x)|^r\,dx \quad \text{for } 1 < r < n \end{aligned}$$

converge to zero with *definite decay rates* as $t\to\infty$. Let w and v be solutions of (S) and (N-S), respectively. Then the pair of functions $u \equiv v - w$, $p \equiv q - \pi$ satisfies

$$\text{(N-S}'\text{)}\qquad \begin{aligned} &\frac{\partial u}{\partial t} - \Delta u + w\cdot\nabla u + u\cdot\nabla w + u\cdot\nabla u + \nabla p = 0 \quad \text{in } \Omega,\ t>0,\\ &\operatorname{div} u = 0 \quad \text{in } \Omega,\ t>0,\\ &u = 0 \quad \text{on } \partial\Omega,\ t>0,\quad u(x,t)\to 0 \quad \text{as } |x|\to\infty,\\ &u|_{t=0} = a. \end{aligned}$$

Hence our problem on the stability for (S) can now be reduced to an investigation into the asymptotic behavior of the solution u of (N-S′). For a three-dimensional exterior domain, HEYWOOD [14, 15] and MASUDA [25] considered an inhomogeneous boundary condition at infinity like $w(x)\to w^\infty$ as $|x|\to\infty$, where w^∞ is a prescribed non-zero constant vector in $\mathbf{R}^3$. They showed the stability for such solutions in L^2-spaces. On account of the parabolical wake region behind obstacles, their decay rates are slower than those of our solutions. To obtain sharper decay rates in L^r-spaces of the solutions of (N-S′), we need to establish L^p–L^r-estimates for the semigroup e^{-tL_r}, where L_r is the operator defined by

$$L_r u \equiv A_r u + P_r(w\cdot\nabla u + u\cdot\nabla w).$$

Here P_r is the projection operator from $L^r(\Omega)$ onto $L^r_\sigma(\Omega)$ and $A_r \equiv -P_r\Delta$ denotes the Stokes operator in $L^r_\sigma(\Omega)$.

In case $w\equiv 0$, we have $L_r = A_r$ and hence our problem reduces to obtaining a global strong solution and its decay properties of the Navier-Stokes equations in exterior domains. Since the pioneer work of KATO [19] and UKAI [35], many efforts have been made to get L^p–L^r-estimates for the Stokes semigroup e^{-tA_r} in *exterior* domains. There are mainly two methods: The first, due to GIGA [11], GIGA & SOHR [13] and BORCHERS & MIYAKAWA [3] is to characterize the domain $D(A_r^\alpha)$ of fractional powers A_r^α $(0 < \alpha < 1)$ and the second, due to IWASHITA [16], is to obtain asymptotic expansion of the resolvent $(A_r + \lambda)^{-1}$ near $\lambda = 0$. In our case, since L_r is an operator with *variable* coefficients, it seems difficult to apply either of these methods to show that the asymptotic behavior of e^{-tL_r} is the same as that of e^{-tA_r} as $t \to \infty$. If we restrict our attention to the case $1 < r < n/2$, however, then L_r can be treated as a perturbation of A_r, and for such r, we can get satisfactory L^p–L^r-estimates of e^{-tL_r}, which are enough for the global strong solution of (N-S′). Our proof needs neither estimates of the purely imaginary powers $L_r^{is} (s \in \mathbf{R})$ of L_r nor an asymptotic expansion of $(L_r + \lambda)^{-1}$ near $\lambda = 0$; we need only a resolvent estimate of elliptic differential operators such as AGMON's [2].

Because of the restriction $1 < r < n/2$, we cannot construct the strong solution directly in the same way as GIGA & MIYAKAWA [12] and KATO [19]. Therefore, we first need to introduce a *mild solution* which is intermediate between weak and strong solutions (see Definition 3.1 below). Then we show the existence and uniqueness of the *global* mild solution u of (N-S′) in the class $C([0, \infty); L_\sigma^n(\Omega))$ with the decay property $\|u(t)\|_q = O(t^{-1/2+n/2q})$ as $t \to \infty$ for $n \leq q < \infty$. Using a uniqueness criterion similar to that of SERRIN [30] and MASUDA [26], we may identify the mild solution with the strong solution. As a result, it will be clear that the restriction on r causes no obstruction for our purpose. Moreover, if we assume more rapid decay in space of the initial disturbance, such as $a \in L^r(\Omega) \cap L^n(\Omega)$ for $1 < r < n/2$, then we also get $\|\nabla u(t)\|_r = O(t^{-1/2})$ as $t \to \infty$.

In Section 1, we state our main results. Section 2 is devoted to L^p–L^r-estimates of e^{-tL_r} and ∇e^{-tL_r}. The existence and uniqueness of the global mild solution is established in Section 3. Finally in Section 4, we prove our theorems.

§1. Results

Before stating our results, we introduce some notations and function spaces and then give our definition of strong solutions of (N-S′). Let $C_{0,\sigma}^\infty$ denote the set of all C^∞ real vector functions $\phi = (\phi^1, \ldots, \phi^n)$ with compact support in Ω, such that $\operatorname{div} \phi = 0$. L_σ^r is the closure of $C_{0,\sigma}^\infty$, with respect to the L^r-norm $\| \ \|_r$. $(\cdot,\cdot)$ denotes the L^2 inner product and the duality pairing between L^r and $L^{r'}$, where $1/r + 1/r' = 1$. L^r stands for the usual (vector-valued) L^r-space over Ω, $1 < r < \infty$. $H_{0;\sigma}^{1;r}$ denotes the closure of $C_{0,\sigma}^\infty$ with respect to the norm

$$\|\phi\|_{H^{1,r}} = \|\phi\|_r + \|\nabla\phi\|_r,$$

where $\nabla\phi = (\partial\phi^i/\partial x_j;\ i, j = 1, \ldots, n)$. When X is a Banach space, its norm is denoted by $\|\cdot\|_X$. Then $C^m([t_1, t_2);\ X)$ is the usual Banach space, where $m = 0, 1, 2, \ldots$, and t_1 and t_2 are real numbers such that $t_1 < t_2$. $BC^m([t_1, t_2); X)$ is the set of all functions $u \in C^m([t_1, t_2); X)$ such that $\sup_{t_1 < t < t_2} \|d^m u(t)/dt^m\|_X < \infty$. In this paper, we denote various constants by C. In particular, $C = C(*, \ldots, *)$ denotes a constant depending only on the quantities appearing in the parentheses.

Let us recall the Helmholtz decomposition:

$$L^r = L^r_\sigma \oplus G^r \quad \text{(direct sum)}, \quad 1 < r < \infty,$$

where $G^r = \{\nabla p \in L^r;\, p \in L^r_{\mathrm{loc}}(\bar{\Omega})\}$. For the proof, see FUJIWARA & MORIMOTO [8], MIYAKAWA [27] and SIMADER & SOHR [31]. P_r denotes the projection operator from L^r onto L^r_σ along G^r. The Stokes operator A_r on L^r_σ is then defined by $A_r = -P_r\Delta$ with domain $D(A_r) = \{u \in H^{2,r}(\Omega);\, u|_{\partial\Omega} = 0\} \cap L^r_\sigma$. It is known that

$$(L^r_\sigma)^* \text{ (the dual space of } L^r_\sigma) = L^{r'}_\sigma, \quad A^*_r \text{ (the adjoint operator of } A_r) = A_{r'},$$

where $1/r + 1/r' = 1$.

Let us introduce the operator L_r in L^r_σ. To this end, we make the following assumption on w.

Assumption. *w is a smooth solenoidal vector function on $\bar{\Omega}$ in the class $w \in L^n_\sigma$ and $\nabla w \in L^{n/2}$ with $w|_{\partial\Omega} = 0$.*

For the existence of such solutions w of (S), see KOZONO & SOHR [22] and GALDI & PADULA [9]. Under this assumption, we define the operator B_r on L^r_σ by

$$B_r u \equiv P_r(w \cdot \nabla u + u \cdot \nabla w) \quad \text{with domain } D(B_r) = H^{1,r}_{0,\sigma}.$$

L_r is now defined by

$$D(L_r) = D(A_r), \quad L_r \equiv A_r + B_r.$$

Applying the projection operator P_r to the both sides of (N-S′), we get formally

$$\text{(E)} \qquad \begin{aligned} &\frac{du}{dt} + L_r u + P_r(u \cdot \nabla u) = 0, \quad t > 0, \\ &u(0) = a. \end{aligned}$$

Our definition of a strong solution of (N-S′) is as follows:

Definition. Let $a \in L^n_\sigma$ and let w satisfy the Assumption. A measurable function u defined on $\Omega \times (0, T)$ is called a *strong solution* of (N-S′) on $(0, T)$ if

(1) $u \in C([0, T); L^n_\sigma) \cap C^1((0, T); L^n_\sigma)$,
(2) $u(t) \in D(L_n)$ for $t \in (0, T)$ and $L_n u \in C((0, T); L^n_\sigma)$,
(3) u satisfies (E) in L^n_σ on $(0, T)$.

Our results now read:

Theorem 1. *Let $a \in L^n_\sigma$ and let w satisfy the Assumption. Then there is a positive number $\lambda = \lambda(n)$ such that if*

$$(1.1) \qquad \|a\|_n \leqq \lambda, \quad \|w\|_n + \|\nabla w\|_{n/2} \leqq \lambda,$$

there exists a unique strong solution u of (N-S′) *on* $(0, \infty)$ *with* $\lim_{t\to+0} t^{1/4}\|u(t)\|_{2n} = 0$.

Moreover, for every $n < r < \infty$, there is a positive number $\eta = \eta(n, r)$ such that if

$$\|w\|_n + \|\nabla w\|_{n/2} \leqq \eta, \tag{1.2}$$

then the solution u has the following asymptotic properties:

(1) (*uniform estimate*) $\|u(t)\|_l \leqq Ct^{-(n/2)(1/n-1/l)}$ *for* $n \leqq l \leqq r$ *with* $C = C(n, r, l)$ *independent of* $t > 0$;
(2) (*behavior near* $t = 0$) $\lim_{t\to+0} t^{(n/2)(1/n-1/r)} \|u(t)\|_r = 0$.

Theorem 2. (1) (i) *Let* $1 < p < n$ *and let* $a \in L^p_\sigma \cap L^n_\sigma$. *There is a positive number* $\lambda' = \lambda'(n, p) \leqq \lambda$ *such that if*

$$\|a\|_n \leqq \lambda', \ \|w\|_n + \|\nabla w\|_{\frac{n}{2}} \leqq \lambda', \tag{1.3}$$

then the strong solution u given in Theorem 1 satisfies

$$u \in BC([0, \infty); L^p_\sigma \cap L^n_\sigma).$$

(ii) *In particular, if* $1 < p < \frac{n}{2}$ *for* $n \geqq 5$ *and if* $1 < p \leqq 2$ *for* $n = 3, 4$, *then under the condition* (1.3),

$$t^{1/2}\nabla u(\cdot) \in BC([0, \infty); L^p).$$

(2) (i) *Let* $n \geqq 3$ *and* $1 < p < n$. *Assume* (1.3). *Then for every r with* $p \leqq r < \infty$, *there is a positive number* $\eta' = \eta'(n, p, r) \leqq \eta$ *such that if*

$$\|w\|_n + \|\nabla w\|_{n/2} \leqq \eta', \tag{1.4}$$

then u has the decay property

$$\|u(t)\|_l = O(t^{-(n/2)(1/p-1/l)}) \quad \text{for } p \leqq l \leqq r \tag{1.5}$$

as $t \to \infty$.

(ii) *Let* $n \geqq 3$ *and* $1 < p < \frac{n}{2}$, $p \leqq r < n$. (*In case* $n = 3, 4$, *we may let also* $1 < p \leqq r \leqq 2$.) *Assume* (1.3). *Then under the condition* (1.4),

$$\|\nabla u(t)\|_l = O(t^{-(n/2)(1/p-1/l)-1/2}) \quad \text{for } p \leqq l \leqq r \tag{1.6}$$

as $t \to \infty$.

Corollary. *Let the conditions* (1.1) *and* (1.2) *hold. Then the solution u given in Theorem 1 has the sharper decay property*

$$\|u(t)\|_l = o(t^{-(n/2)(1/p-1/l)}) \quad \text{for } p \leqq l \leqq r \tag{1.7}$$

as $t \to \infty$.

Remark. (1) The behavior near $t = 0$ in Theorem 1 is necessary for uniqueness of mild solutions.

(2) In Theorem 2, it may happen that $\lim_{p\to1} \lambda'(n, p) = 0$ and $\lim_{p\to1} \eta'(n, p, r) = 0$.

(3) The most important decay of $\|u(t)\|_r$ is for $r = 2$ and $r = n$. The former is just the energy decay of weak solutions. When $w \equiv 0$, WIEGNER [37] and BORCHERS

& MIYAKAWA [3] obtained the best decay rates in $L^2(\mathbf{R}^n)$ and in $L^2(\Omega)$, respectively. WIEGNER's rate is optimal. On the other hand, the case $r = n$ is closely related to the scaling invariance of solutions. Even when $w \equiv 0$ and when $\Omega = \mathbf{R}^n$ our decay rate (1.7) in $L^n(\Omega)$ is sharper than any other result ([14, 15, 19, 25, 29, 35]).

§2. L^p–L^r-estimates of the semigroup e^{-tL_r}

Let us first recall some previous results on the Stokes operator A_r in L^r_σ due to BORCHERS & SOHR [4] and GIGA & SOHR [13].

Proposition 2.1 (BORCHERS & SOHR [4], GIGA & SOHR [13, Theorem 3.1]). (1) *Let $\frac{\pi}{2} < \omega < \pi$. For every $1 < r < \infty$, the resolvent set $\rho(-A_r)$ of $-A_r$ contains the sector $\Sigma_\omega \equiv \{\lambda \in \mathbf{C};\ |\arg\lambda| < \omega\}$ and there is a constant $M_{r,\omega}$ depending only on r and ω such that*

$$\|(A_r + \lambda)^{-1}\|_{B(L^r_\sigma)} \leqq M_{r,\omega}|\lambda|^{-1} \tag{2.1}$$

holds for all $\lambda \in \Sigma_\omega$.

(2) *If $1 < r < \frac{n}{2}$, the following stronger estimate holds:*

$$|\lambda|\,\|u\|_r + \|D^2 u\|_r \leqq C\,\|(A_r + \lambda)u\|_r \tag{2.2}$$

for all $u \in D(A_r)$ and all $\lambda \in \Sigma_\omega$, where $C = C(r, \omega)$.

Remark. By (2.2) and the interpolation inequality, we have

$$\|D^k(A_r + \lambda)^{-1}u\|_r \leqq C|\lambda|^{-1+k/2}\|u\|_r, \quad 1 < r < \tfrac{n}{2}, \quad k = 0, 1, 2, \tag{2.3}$$

for all $u \in L^r_\sigma$ and all $\lambda \in \Sigma_\omega$, where $C = C(n, r, \omega, k)$.

Let us introduce the operator L_r in L^r_σ. We first define the operators B_r and B'_r by

$$D(B_r) = D(B'_r) = H^{1,r}_{0,\sigma}(= D(A_r^{1/2})),$$

$$B_r u \equiv P_r(w\cdot\nabla u + u\cdot\nabla w), \quad B'_r u \equiv P_r\left(-w\cdot\nabla u + \sum_{j=1}^{n}\nabla w^j u^j\right),$$

where w is the function on $\bar{\Omega}$ satisfying the Assumption. L_r and L'_r are then defined by

$$D(L_r) = D(L'_r) = D(A_r), \quad L_r \equiv A_r + B_r, \quad L'_r = A_r + B'_r.$$

Since $A_r^* = A_{r'}$ $(1/r + 1/r' = 1)$, it is easy to see

$$L_r^* = L'_{r'}, \tag{2.4}$$

where A_r^* and L_r^* denote the adjoint operators of A_r and L_r in L^r_σ, respectively.

Let us first investigate the behavior of semigroups e^{-tL_r} and $e^{-tL'_r}$ near $t = 0$.

Lemma 2.2. *Let w be as in the Assumption.* (1) *For $1 < r < \infty$, $-L_r$ and $-L'_r$ generate quasi-bounded holomorphic semigroups $\{e^{-tL_r}\}_{t \geqq 0}$ and $\{e^{-tL'_r}\}_{t \geqq 0}$ of class C^0 in L^r_σ, respectively. Hence there is a constant $\beta_r > 0$ such that $(L_r + \beta_r)^{-1}$, $(L'_r + \beta_r)^{-1} \in \mathbf{B}(L^r_\sigma)$ and such that the fractional powers $(L_r + \beta_r)^\alpha$, $(L'_r + \beta_r)^\alpha$ $(0 < \alpha < 1)$ are well defined. Moreover, there are continuous imbeddings $D((L_r + \beta_r)^\alpha)$, $D((L'_r + \beta_r)^\alpha) \subset H^{2\alpha, r}$, with*

$$\|u\|_{H^{2\alpha,r}} \leqq \begin{cases} C\|(L_r + \beta_r)^\alpha\|_r \\ C\|(L'_r + \beta_r)^\alpha\|_r, \end{cases} \tag{2.5}$$

for all $u \in D((L_r + \beta_r)^\alpha) \equiv D((L'_r + \beta_r)^\alpha)(0 \leqq \alpha \leqq 1)$, where $C = C(r, \alpha)$ and $H^{2\alpha,r}$ denotes the space of Bessel potentials over Ω.

(2) *For every $1 < p \leqq r < \infty$ and $0 < T < \infty$, there is a constant $M_{p,r,T}$ such that*

$$\|e^{-tL}a\|_r,\ \|e^{-tL'}a\|_r \leqq M_{p,r,T}\, t^{-(n/2)(1/p - 1/r)}\|a\|_p, \tag{2.6}$$

$$\|\nabla e^{-tL}a\|_r,\ \|\nabla e^{-tL'}a\|_r \leqq M_{p,r,T}\, t^{-(n/2)(1/p - 1/r) - 1/2}\|a\|_p \tag{2.7}$$

for all $a \in L^p_\sigma$ and all $t \in (0, T)$.

Proof. (1) It follows from GIGA [11] and GIGA & SOHR [13] that $D(A^\alpha_r)$ is continuously imbedded into $H^{2\alpha,r}$ with

$$\|u\|_{H^{2\alpha,r}} \leqq C\|(A_r + 1)^\alpha\|_r, \quad 1 < r < \infty, \quad 0 \leqq \alpha \leqq 1, \tag{2.8}$$

for all $u \in D(A^\alpha_r)$ with $C = C(r, \alpha)$. Then we have

$$\begin{aligned} \|B_r u\|_r &\leqq \|P_r\|_{\mathbf{B}(L^r, L^r_\sigma)}(\|w\|_\infty \|\nabla u\|_r + \|\nabla w\|_\infty \|u\|_r) \\ &\leqq \|P_r\|_{\mathbf{B}(L^r, L^r_\sigma)}(\|w\|_\infty + \|\nabla w\|_\infty)\|u\|_{H^{1,r}} \\ &\leqq C_r(\|w\|_\infty + \|\nabla w\|_\infty)\|(A_r + 1)^{1/2}u\|_r \end{aligned} \tag{2.9}$$

for all $u \in D(B_r)$. Hence B_r is A_r-bounded with relative bound 0, and perturbation theory (KATO [18, p. 500, Corollary 2.5]) states that $-L_r$ is a generator of a quasi-bounded holomorphic semigroup $\{e^{-tL_r}\}_{t \geqq 0}$. Moreover, it follows from (2.9) and FUJIWARA [7, Theorem A in Appendix] that $D((L_r + \beta_r)^\alpha) = D(A^\alpha_r)$ for $0 \leqq \alpha \leqq 1$; then (2.8) yields the desired estimate (2.5).

(2) By (2.5) and the Sobolev imbedding theorem, we have

$$\|u\|_r \leqq C\|(L_p + \beta_p)^\alpha u\|_p \quad \text{for } u \in D((L_p + \beta_p)^\alpha),$$

$$\|\nabla u\|_r \leqq C\|(L_p + \beta_p)^{\alpha + 1/2}u\|_p \quad \text{for } u \in D((L_p + \beta_p)^{\alpha + 1/2}),$$

where $\alpha = (n/2)(1/p - 1/r)$ and $C = C(p, r)$. Taking $u = e^{-tL}a (a \in L^p_\sigma)$ in these estimates, we get (2.6) and (2.7) by a standard argument for holomorphic semigroups (see, e.g., TANABE [33, Theorem 3.3.3]). Since

$$\|B'_r u\|_r \leqq C_r(\|\nabla w\|_\infty + \|w\|_\infty)\|(A_r + 1)^{1/2}u\|_r$$

for all $u \in D(B'_r)$, the last argument holds also for L'_r, so we get the desired result. This proves Lemma 2.2. □

We next investigate behavior of e^{-tL_r} and ∇e^{-tL_r} as $t \to \infty$. To this end, we need to estimate the resolvent $(L_r + \lambda)^{-1}$ near $\lambda = 0$. In such an estimate, we impose restrictions on r and require the smallness of w in the class (CL).

Lemma 2.3 (Resolvent estimate). *For every r and ω satisfying $1 < r < \infty$ and $\frac{\pi}{2} < \omega < \pi$, there is a positive number $\mu = \mu(r, \omega)$ such that if*

$$\|w\|_n + \|\nabla w\|_{n/2} \leqq \mu, \tag{2.10}$$

then both resolvent sets $\rho(-L_r)$ and $\rho(-L_r')$ contain the sector $\Sigma_\omega \equiv \{\lambda \in \mathbf{C}; |\arg \lambda| < \omega\}$ and the estimates

$$\|(L_r + \lambda)^{-1}\|_{\mathbf{B}(L_\sigma^r)}, \quad \|(L_r' + \lambda)^{-1}\|_{\mathbf{B}(L_\sigma^r)} \leqq \tilde{M}_{r,\omega} |\lambda|^{-1} \tag{2.11}$$

hold for all $\lambda \in \Sigma_\omega$ with a constant $\tilde{M}_{r,\omega}$ depending only on r and ω.

An immediate consequence of Lemma 2.3 is

Corollary 2.4. Let $\omega \in (\frac{\pi}{2}, \pi)$ *be fixed arbitrarily. Under the condition* (2.10), $-L_r$ *and* $-L_r'$ *respectively generate uniformly bounded holomorphic semigroups* $\{e^{-tL_r}\}_{t \geqq 0}$ *and* $\{e^{-tL_r'}\}_{t \geqq 0}$ *of class C^0 in L_σ^r. Hence the fractional powers L_r^α and $(L_r')^\alpha (0 \leqq \alpha \leqq 1)$ of L_r and L_r', respectively, can be defined.*

Proof of Lemma 2.3. (i) Let us first consider the case $1 < r < \frac{n}{2}$, for which it follows from GIGA & SOHR [13, Corollary 2.2, Theorem 3.1] that

$$\|u\|_{nr/(n-2r)} \leqq C\|\nabla u\|_{nr/(n-r)} \leqq C\|D^2 u\|_r \leqq C\|A_r u\|_r \tag{2.12}$$

for all $u \in D(A_r)$, where $C = C(n, r)$. By Proposition 2.1, we have

$$L_r + \lambda = A_r + B_r + \lambda = (1 + B_r(A_r + \lambda)^{-1})(A_r + \lambda) \tag{2.13}$$

for all $\lambda \in \Sigma_\omega$. By (2.1), (2.12) and the Hölder inequality,

$$\begin{aligned}
&\|B_r(A_r + \lambda)^{-1} u\|_r \\
&\quad \leqq \|P_r(w \cdot \nabla (A_r + \lambda)^{-1} u)\|_r + \|P_r((A_r + \lambda)^{-1} u \cdot \nabla w)\|_r \\
&\quad \leqq \|P\|_{\mathbf{B}(L^r, L_\sigma^r)} (\|w\|_n \|\nabla (A_r + \lambda)^{-1} u\|_{nr/(n-r)} \\
&\qquad + \|(A_r + \lambda)^{-1} u\|_{nr/(n-2r)} \|\nabla w\|_{n/2}) \\
&\quad \leqq C(\|w\|_n + \|\nabla w\|_{n/2}) \|A_r(A_r + \lambda)^{-1} u\|_r \\
&\quad \leqq C(1 + M_{r,\omega})(\|w\|_n + \|\nabla w\|_{n/2}) \|u\|_r
\end{aligned}$$

for all $u \in L_\sigma^r$ and all $\lambda \in \Sigma_\omega$, where $C = C(n, r)$. Hence, by taking $\mu = \mu(r, \omega) \equiv 1/2C(1 + M_{r,\omega})$, under the condition (2.10), we have

$$\|B_r(A_r + \lambda)^{-1}\|_{\mathbf{B}(L_\sigma^r)} \leqq \tfrac{1}{2} \quad \text{for all } \lambda \in \Sigma_\omega. \tag{2.14}$$

Now an elementary consideration of the Neumann series yields (2.11).

(ii) We next consider the case $(n/2)' \equiv n/(n-2) < r < \infty$. In this case we have $1 < r' \equiv r/(r-1) < \frac{n}{2}$, and (2.4) yields

$$(L_r + \lambda)^* = L'_{r'} + \bar{\lambda} = A_{r'} + B'_{r'} + \bar{\lambda} = (1 + B'_{r'}(A_{r'} + \bar{\lambda})^{-1})(A_{r'} + \bar{\lambda}) \tag{2.15}$$

for all $\lambda \in \Sigma_\omega$. Since $1 < r' < \frac{n}{2}$, the same argument as above works for $A_{r'}$ and $B'_{r'}$ and hence we can choose a positive number $\mu = \mu(r, \omega)$ such that the condition (2.10) yields

$$\| B'_{r'}(A_{r'} + \bar{\lambda})^{-1} \|_{\mathbf{B}(L^{r'}_\sigma)} \leqq \tfrac{1}{2} \quad \text{for all } \lambda \in \Sigma_\omega.$$

By (2.15) and this estimate, we find

$$\Sigma_\omega \subset \rho(-L_r), \quad \|(L_r + \lambda)^{-1}\|_{\mathbf{B}(L^r_\sigma)} = \|((L_r + \lambda)^*)^{-1}\|_{\mathbf{B}(L^{r'}_\sigma)} \leqq 2M_{r',\omega}|\lambda|^{-1}$$

for all $\lambda \in \Sigma_\omega$, which shows (2.11).

(iii) Now it remains to treat the case $\frac{n}{2} \leqq r \leqq (\frac{n}{2})'$ for $n = 3, 4$. Take $1 < r_1 < \frac{n}{2}$ and $(\frac{n}{2})' < r_2 < \infty$. We have $1/r = (1-\theta)/r_1 + \theta/r_2$ for some $0 < \theta < 1$. Let $\mu(r, \omega) \equiv \min\{\mu(r_1, \omega), \mu(r_2, \omega)\}$. Now the above results (i), (ii) and interpolation yield that

$$\Sigma_\omega \subset \rho(-L_r), \quad \|(L_r + \lambda)^{-1}\|_{\mathbf{B}(L^r_\sigma)} \leqq 2M^{1-\theta}_{r_1,\omega} M^{\theta}_{r_2,\omega} |\lambda|^{-1}, \quad \lambda \in \Sigma_\omega,$$

from which we obtain the desired result on L_r for all $1 < r < \infty$. It is easy to see that the proof for L'_r is quite similar to that for L_r, so we may omit it. This proves Lemma 2.3. □

If we impose a restriction on r, we also get the estimates of derivatives for the resolvent $(L_r + \lambda)^{-1}$ near $\lambda = 0$.

Lemma 2.5. (1) *Let $n \geqq 3$ and $1 < r < n/2$. Then under the condition* (2.10),

$$\|D^k(L_r + \lambda)^{-1}u\|_r, \|D^k(L'_r + \lambda)^{-1}u\|_r \leqq C|\lambda|^{-1+k/2}\|u\|_r, \quad k = 1, 2, \tag{2.16}$$

for all $u \in L^r_\sigma$ and all $\lambda \in \Sigma_\omega$, where $C = C(n, r, \omega)$.

(2) *Let $n = 3, 4$ and $1 < r \leqq 2$ and let $\frac{\pi}{2} < \omega < \pi$. There is a positive number $\mu' = \mu'(r, \omega)$ such that if*

$$\|w\|_n + \|\nabla w\|_{n/2} \leqq \mu', \tag{2.10'}$$

then the estimates

$$\|\nabla(L_r + \lambda)^{-1}u\|_r, \|\nabla(L'_r + \lambda)^{-1}u\|_r \leqq C|\lambda|^{-1/2}\|u\|_r \tag{2.17}$$

hold for all $u \in L^r_\sigma$ and all $\lambda \in \Sigma_\omega$, where $C = C(n, r, \omega)$.

Proof. We only prove this lemma for L_r because the proof for L'_r is quite similar.

(1) By (2.14) the operator $1 + B_r(A_r + \lambda)^{-1}$ is invertible in L^r_σ with bound

$$\|(1 + B_r(A_r + \lambda)^{-1})^{-1}\|_{\mathbf{B}(L^r_\sigma)} \leqq 2 \quad \text{for all } \lambda \in \Sigma_\omega.$$

Hence (2.3) and (2.13) yield

$$\|D^k(L_r+\lambda)^{-1}u\|_r = \|D^k(A_r+\lambda)^{-1}(1+B_r(A_r+\lambda)^{-1})^{-1}u\|_r \leqq C|\lambda|^{-1+k/2}\|u\|_r$$

for all $u \in L^r_\sigma$ and all $\lambda \in \Sigma_\omega$ where $C = C(n, r, \omega)$, and we obtain (2.16).

(2) If $n = 3, 4$, we have $\frac{n}{2} \leqq 2$ and make use of the quadratic form $(L_2 u, u)$ on L^2_σ. By the Sobolev inequality $\|u\|_{2n/(n-2)} \leqq C\|\nabla u\|_2 (u \in H^{1,2}_{0,\sigma})$, we have

$$\begin{aligned}|(B_2 u, u)| &\leqq |(w\cdot\nabla u, u)| + |(u\cdot\nabla w, u)|\\ &\leqq \|w\|_n\|\nabla u\|_2\|u\|_{2n/(n-2)} + \|\nabla w\|_{n/2}\|u\|^2_{2n/(n-2)}\\ &\leqq C_*(\|w\|_n + \|\nabla w\|_{n/2})\|\nabla u\|_2^2\end{aligned}$$

for all $u \in H^{1,2}_{0,\sigma}$, where $C_* = C_*(n)$. Now take $\mu'(2,\omega) \equiv \min\{1/2C_*, \mu(2,\omega)\}$, where μ is the same number as in (2.10). Then under the condition (2.10′), we have

$$\begin{aligned}(L_2 u, u) &= (A_2 u, u) + (B_2 u, u)\\ &\geqq \{1 - C_*(\|w\|_n + \|\nabla w\|_{n/2})\}\|\nabla u\|_2^2 \geqq \tfrac{1}{2}\|\nabla u\|_2^2\end{aligned}$$

for all $u \in D(L_2)$. Hence (2.11) with $r = 2$ yields

$$\begin{aligned}\|\nabla(L_2+\lambda)^{-1}u\|_2^2 &\leqq 2(L_2(L_2+\lambda)^{-1}u, (L_2+\lambda)^{-1}u)\\ &\leqq 2(\tilde{M}_{2,\omega}+1)\tilde{M}_{2,\omega}|\lambda|^{-1}\|u\|_2^2\end{aligned}$$

for all $u \in L^2_\sigma$ and all $\lambda \in \Sigma_\omega$, from which we obtain (2.17) for $r = 2$.

For $\frac{n}{2} \leqq r \leqq 2$, we take $1 < r_1 < \frac{n}{2}$ and $0 < \theta \leqq 1$ such that $1/r = (1-\theta)/r_1 + \theta/2$. Since (2.17) is true for r_1, implied by (1), we may define $\mu'(r,\omega)$ as $\mu'(r,\omega) = \min\{\mu(r_1,\omega), \mu(2,\omega)\}$. Then under condition (2.10′), the interpolation inequality yields the desired result. □

In what follows, we fix $\omega \in (\frac{\pi}{2}, \pi)$ and regard μ and μ' in (2.10) and (2.10′) as constants depending only on r.

Lemma 2.6. (1) *Let* $1 < r < \infty$. *Under condition* (2.10),

$$\|e^{-tL_r}a\|_r,\ \|e^{-tL'_r}a\|_r \leqq M_r\|a\|_r \tag{2.18}$$

for all $a \in L^r_\sigma$ *and all* $t > 0$ *with a constant* M_r *depending only on* r.

(2) (i) *Let* $n \geqq 3$ and $1 < r < \frac{n}{2}$. Under condition (2.10),

$$\|D^k e^{-tL_r}a\|_r,\ \|D^k e^{-tL'_r}a\|_r \leqq M'_r t^{-k/2}\|a\|_r, \quad k = 1, 2, \tag{2.19}$$

for all $a \in L^r_\sigma$ *and all* $t > 0$ *with a constant* M'_r *depending only on* r.

(ii) *Let* $n = 3, 4$ *and* $1 < r \leqq 2$. *Under condition* (2.10′),

$$\|\nabla e^{-tL_r}a\|_r,\ \|\nabla e^{-tL'_r}a\|_r \leqq M'_r t^{-1/2}\|a\|_r \tag{2.19′}$$

for all $a \in L^r_\sigma$ *and all* $t > 0$ *with a constant* M'_r *depending only on* r.

Proof. Take β such that $0 < \beta < \omega - \pi/2$. Then under the conditions (2.10), (2.10′), we have

$$e^{-tL_r}a = \frac{1}{2\pi i}\int_\Gamma e^{t\lambda}(L_r + \lambda)^{-1}\, a\, d\lambda,$$

where $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$; $\Gamma_1 = \{\lambda = \rho e^{i(\pi/2+\beta)};\ 1/t\sin\beta < \rho < \infty\}$, $\Gamma_2 = \{\lambda = e^{i\theta}/t\sin\beta;\ -\frac{\pi}{2} - \beta < \theta < \frac{\pi}{2} + \beta\}$ and $\Gamma_3 = \{\lambda = \rho e^{-i(\pi/2+\beta)};\ 1/t\sin\beta < \rho < \infty\}$. Then it follows from (2.11) and (2.16) that

$$\begin{aligned}(2.20)\qquad &\left\| D^k \frac{1}{2\pi i}\int_{\Gamma_1} e^{t\lambda}(L_r + \lambda)^{-1} a\, d\lambda \right\|_r \\ &\leqq \frac{1}{2\pi}\int_{\Gamma_1} |e^{\lambda t}|\, \|D^k(L_r + \lambda)^{-1} a\|_r d|\lambda| \\ &\leqq C\int_{\Gamma_1} e^{\operatorname{Re}\lambda t}|\lambda|^{-1+k/2} d|\lambda|\, \|a\|_r \\ &= C\int_{1/t\sin\beta}^{\infty} e^{-\rho t\sin\beta}\rho^{-1+k/2} d\rho\, \|a\|_r \\ &\qquad \text{(by changing the variable } \rho \to s = \rho t\sin\beta) \\ &= C(t\sin\beta)^{-k/2}\int_1^\infty e^{-s}s^{k/2-1} ds\, \|a\|_r \\ &\leqq Ct^{-k/2}\|a\|_r;\end{aligned}$$

$$\begin{aligned}(2.21)\qquad &\left\| D^k \frac{1}{2\pi i}\int_{\Gamma_2} e^{t\lambda}(L_r + \lambda)^{-1} a d\lambda \right\|_r \\ &\leqq C\int_{-\pi/2-\beta}^{\pi/2+\beta} e^{\cos\theta/\sin\beta}(t\sin\beta)^{1-k/2}\frac{d\theta}{t\sin\beta}\|a\|_r \\ &\leqq Ct^{-k/2}\|a\|_r\end{aligned}$$

for all $a \in L^r_\sigma$ and all $t > 0$, where $k = 0, 1, 2$ and $C = C(n, r, \beta)$. As in (2.20), we obtain the estimate of the integral along Γ_3:

$$(2.22)\qquad \left\| D^k \frac{1}{2\pi i}\int_{\Gamma_3} e^{t\lambda}(L_r + \lambda)^{-1} a d\lambda \right\|_r \leqq Ct^{-k/2}\|a\|_r.$$

Now (2.20)–(2.22) yield the desired estimates (2.18), (2.19). Based on (2.17), we can prove (2.19′) in the same way as above. This proves Lemma 2.6. □

The following L^p–L^r-estimates play an important role for our purpose.

Theorem 2.7 (L^p–L^q-estimates). (1) *Let $n \geqq 3$ and $1 < p \leqq r < \infty$. There is a positive number $\kappa = \kappa(p, r)$ such that if*

$$(2.23)\qquad \|w\|_n + \|\nabla w\|_{n/2} \leqq \kappa(p, r),$$

then

$$\|e^{-tL}a\|_r,\ \|e^{-tL'}a\|_r \leqq M_{p,r}t^{-(n/p-n/r)/2}\|a\|_p \tag{2.24}$$

for all $a \in L^p_\sigma$ and all $t > 0$ with a constant $M_{p,r}$ depending only on p and r.

(2) *Let $n \geqq 3$ and let $1 < p < \frac{n}{2}$, $p \leqq r < n$. (In case $n = 3$, 4, we may let also $1 < p \leqq r \leqq 2$.) There is a positive number $\kappa' = \kappa'(p,r)$ such that if*

$$\|w\|_n + \|\nabla w\|_{n/2} \leqq \kappa'(p,r), \tag{2.25}$$

then

$$\|\nabla e^{-tL}a\|_r,\ \|\nabla e^{-tL'}a\|_r \leqq M'_{p,r}t^{-(n/p-n/r)/2-1/2}\|a\|_p \tag{2.26}$$

for all $a \in L^p_\sigma$ and all $t > 0$ with a constant $M'_{p,r}$ depending only on p and r.

Proof. (1) *Step 1.* We first prove (2.24) for $1 < r < \frac{n}{2}$. Consider the case $\frac{1}{p} - \frac{1}{n} \leqq \frac{1}{r} \leqq \frac{1}{p}$. Then by (2.19) and the Sobolev inequality we have

$$\|e^{-tL}a\|_{(1/p-1/n)^{-1}} \leqq C\|\nabla e^{-tL}a\|_p \leqq CM'_p t^{-1/2}\|a\|_p, \quad a \in L^p_\sigma,\ t > 0$$

with $C = C(n,p)$, provided that $\|w\|_n + \|\nabla w\|_{n/2} \leqq \mu(p)$. Under the same condition on w and ∇w, we have by (2.18) that $\|e^{-tL}a\|_p \leqq M_p\|a\|_p$. Since $\frac{1}{p} - \frac{1}{n} \leqq \frac{1}{r} \leqq \frac{1}{p}$, we have $\frac{1}{r} = \frac{1-\theta}{p} + \theta(\frac{1}{p} - \frac{1}{n})$, where $\theta = n(\frac{1}{p} - \frac{1}{r})$. Hence if $\|w\|_n + \|\nabla w\|_{n/2} \leqq \mu(p)$, then by interpolation we obtain $e^{-tL} \in \mathbf{B}(L^p_\sigma, L^r_\sigma)$ with the bound

$$\|e^{-tL}\|_{\mathbf{B}(L^p_\sigma, L^r_\sigma)} \leqq (CM'_p t^{-1/2})^\theta M_p^{1-\theta} \leqq Ct^{-(n/p-n/r)/2}.$$

We next proceed to the case that $\frac{2}{n} < \frac{1}{p} - \frac{2}{n} \leqq \frac{1}{r} < \frac{1}{p} - \frac{1}{n}$. Taking $1 < p_1 < \frac{n}{2}$ as $\frac{1}{p_1} = \frac{1}{p} - \frac{1}{n}$, we have $\frac{1}{p_1} - \frac{1}{n} \leqq \frac{1}{r} < \frac{1}{p_1}$. Hence if

$$\|w\|_n + \|\nabla w\|_{n/2} \leqq \min\{\kappa(p,p_1), \kappa(r,p_1)\} \equiv \kappa(p,r),$$

then the above argument with p replaced by p_1 yields

$$\begin{aligned}
\|e^{-tL}a\|_r &= \|e^{-\frac{t}{2}L}(e^{-\frac{t}{2}L}a)\|_r \\
&\leqq M_{p_1,r}t^{-(n/p_1-n/r)/2}\|e^{-\frac{t}{2}L}a\|_{p_1} \\
&\leqq M_{p_1,r}M_{p,p_1}t^{-(n/p_1-n/r)/2}t^{-(n/p-n/p_1)/2}\|a\|_p \\
&= M_{p_1,r}M_{p,p_1}t^{-(n/p-n/r)/2}\|a\|_p
\end{aligned}$$

for all $a \in L^p_\sigma$ and all $t > 0$.

Proceeding in the case $\frac{2}{n} < \frac{1}{r} < \frac{1}{p} - \frac{2}{n}$ as above, within a finite number of steps, we obtain (2.24) for $1 < p \leqq r < \frac{n}{2}$.

Step 2. We next prove (2.24) for $\frac{n}{2} \leqq r < \infty$. Let us take $\tilde{r}$ and q such that $1 < \tilde{r} < \frac{n}{2} \leqq r < q < \infty$. Then we have $\frac{1}{r} = \frac{1-\theta}{\tilde{r}} + \frac{\theta}{q}$ *for some* $0 < \theta < 1$. *Defining* $1 < \tilde{p} < p$ by the relation $\frac{1}{p} = \frac{1-\theta}{\tilde{p}} + \frac{\theta}{q}$, we get $1 < \tilde{p} \leqq \tilde{r} < \frac{n}{2}$. Hence the result of Step 1 states that if

$$\|w\|_n + \|\nabla w\|_{n/2} \leqq \kappa(\tilde{p}, \tilde{r}),$$

then

$$\|e^{-tL}a\|_{\tilde{r}} \leqq M_{\tilde{p},\tilde{r}} t^{-(n/\tilde{p}-n/\tilde{r})/2} \|a\|_{\tilde{p}}, \quad a \in L_\sigma^{\tilde{p}}.$$

On the other hand, it follows from (2.18) that if

$$\|w\|_n + \|\nabla w\|_{n/2} \leqq \mu(q),$$

then

$$\|e^{-tL}a\|_q \leqq M_q \|a\|_q, \quad a \in L_\sigma^q.$$

Hence under the condition that $\|w\|_n + \|\nabla w\|_{n/2} \leqq \min\{\kappa(\tilde{p}, \tilde{r}), \mu(q)\}$, by interpolation we have $e^{-tL} \in \mathbf{B}(L_\sigma^p, L_\sigma^r)$ with the bound

$$\|e^{-tL}\|_{\mathbf{B}(L_\sigma^p, L_\sigma^r)} \leqq (M_{\tilde{p},\tilde{r}} t^{-(n/\tilde{p}-n/\tilde{r})/2})^{1-\theta} M_q^\theta = M_{\tilde{p},\tilde{r}}^{1-\theta} M_q^\theta t^{-(n/p-n/r)/2},$$

which yields the desired estimate (2.24) also for $\frac{n}{2} \leqq r < \infty$.

(2) (i) We first consider the case $n \geqq 3$ and $1 < p < \frac{n}{2}$, $p \leqq r < n$.

Step 1: $\frac{1}{p} - \frac{1}{n} \leqq \frac{1}{r} \leqq \frac{1}{p}$. It follows from (2.19) with $k = 2$ and the Sobolev inequality (2.12) that if $\|w\|_n + \|\nabla w\|_{n/2} \leqq \mu(p)$, then

$$\|\nabla e^{-tL}a\|_{(1/p-1/n)^{-1}} \leqq C\|D^2 e^{-tL}a\|_p \leqq CM'_p t^{-1}\|a\|_p, \quad a \in L_\sigma^p, t > 0,$$

where $C = C(p)$. Moreover, under the same condition on w, we have by (2.19) with $k = 1$ that

$$\|\nabla e^{-tL}a\|_p \leqq M'_p t^{-1/2}\|a\|_p, \quad a \in L_\sigma^p, t > 0.$$

Taking $0 < \theta < 1$ as $\theta \equiv n(\frac{1}{p} - \frac{1}{r})$, we have $\frac{1}{r} = \frac{1-\theta}{p} + \theta(\frac{1}{p} - \frac{1}{n})$, so the above estimates and interpolation yield that $\nabla e^{-tL} \in \mathbf{B}(L_\sigma^p, L^r)$ with bound

$$\|\nabla e^{-tL}\|_{\mathbf{B}(L_\sigma^p, L^r)} \leqq (CM'_p t^{-1})^\theta (M'_p t^{-1/2})^{1-\theta} = C^\theta M'_p t^{-(n/p-n/r)/2-1/2},$$

provided $\|w\|_n + \|\nabla w\|_{n/2} \leqq \mu(p)$. This implies (2.26).

Step 2: $\frac{1}{n} < \frac{1}{r} < \frac{1}{p} - \frac{1}{n}$. Choosing s with $\frac{1}{s} = \frac{1}{r} + \frac{1}{n}$, we have by assumption that $p < s < \frac{n}{2}$. Hence it follows from (2.19) with $k = 2$, (2.24) and the Sobolev inequality (2.12) that if

$$\|w\|_n + \|\nabla w\|_{n/2} \leqq \min\{\mu(s), \kappa(p, s)\} \equiv \kappa'(p, r),$$

then

$$\begin{aligned}\|\nabla e^{-tL}a\|_r &\leqq C\|D^2 e^{-tL}a\|_s = C\|D^2 e^{-t/2L}(e^{-t/2L}a)\|_s \\ &\leqq CM_s t^{-1}\|e^{-t/2L}a\|_s \\ &\leqq CM_s M_{p,s} t^{-1} t^{-(n/p-n/s)/2}\|a\|_p \\ &= CM_s M_{p,s} t^{-(n/p-n/r)/2-1/2}\|a\|_p, \quad a \in L_\sigma^p, t > 0\end{aligned}$$

with $C = C(n, r)$, which implies (2.26).

(ii) We next consider the case $n = 3, 4$ and $1 < p \leqq r \leqq 2$. For $1 < \tilde{p} < \frac{n}{2}$, there is a $\theta \in (0, 1]$ such that $\frac{1}{p} = \frac{1-\theta}{\tilde{p}} + \frac{\theta}{2}$. We also choose $1 < \tilde{r} < r$ with $\frac{1}{r} = \frac{1-\theta}{\tilde{r}} + \frac{\theta}{2}$. For such $\tilde{p}$ and $\tilde{r}$, we have $1 < \tilde{p} < \frac{n}{2}$, $\tilde{p} \leqq \tilde{r} < n$, so that (i) yields

$$\|\nabla e^{-tL}a\|_{\tilde{r}} \leqq M'_{\tilde{p},\tilde{r}} t^{-(n/\tilde{p}-n/\tilde{r})/2-1/2}\|a\|_{\tilde{p}}, \quad a \in L_\sigma^{\tilde{p}}, t > 0,$$

provided that $\|w\|_n + \|\nabla w\|_{n/2} \leqq \kappa'(\tilde{p}, \tilde{r})$. On the other hand, if $\|w\|_n + \|\nabla w\|_{n/2} \leqq \mu'(2)$, then (2.19′) implies that

$$\|\nabla e^{-tL} a\|_2 \leqq M'_2 t^{-1/2} \|a\|_2, \quad a \in L^2_\sigma, t > 0.$$

Hence under the condition that $\|w\|_n + \|\nabla w\|_{n/2} \leqq \min\{\kappa'(\tilde{p}, \tilde{r}), \mu'(2)\} \equiv \kappa'(p, r)$, we have by the above estimates and by interpolation that $\nabla e^{-tL} \in \mathbf{B}(L^p_\sigma, L^r)$ with bound

$$\begin{aligned}\|\nabla e^{-tL}\|_{\mathbf{B}(L^p_\sigma, L^r)} &\leqq (M'_{\tilde{p},\tilde{r}} t^{-(n/\tilde{p} - n/\tilde{r})/2 - 1/2})^{1-\theta} (M'_2 t^{-1/2})^\theta \\ &= (M'_{\tilde{p},\tilde{r}})^{1-\theta} (M'_2)^\theta t^{-(n/p - n/r)/2 - 1/2}, \quad t > 0,\end{aligned}$$

which yields (2.26). This proves Theorem 2.7. □

Lemma 2.8. *Let* $1 < r \leqq 2$ *for* $n = 3, 4$ *and let* $1 < r < \frac{n}{2}$ *for* $n \geqq 5$. *Then under conditions* (2.10) *and* (2.10′),

$$(2.27) \quad \|\nabla L^\alpha_r e^{-tL_r} a\|_r, \ \|\nabla (L'_r)^\alpha e^{-tL'_r} a\|_r \leqq \tilde{M}_{r,\alpha} t^{-\alpha - 1/2} \|a\|_r, \quad 0 \leqq \alpha < 1$$

for all $a \in L^r_\sigma$ *and all* $t > 0$ *with a constant* $\tilde{M}_{r,\alpha}$ *depending only on* r *and* α.

Proof. We make use of the representation

$$\nabla L^\alpha_r e^{-tL_r} a = \nabla \frac{1}{2\pi i} \int_\Gamma (-\lambda)^\alpha e^{t\lambda} (L_r + \lambda)^{-1} a \, d\lambda,$$

where Γ is the same path in the complex plane as in the proof of Lemma 2.6. Hence, using the estimates (2.16), (2.17), we obtain the desired result in the same way as (2.19), (2.19′). □

§3. Global mild solution

In this section, we construct a mild solution which is weaker than the strong solution. If u is a strong solution, then u satisfies the integral equation

$$\text{(I.E)} \qquad u(t) = e^{-tL} a - \int_0^t e^{-(t-s)L} P(u \cdot \nabla u)(s) \, ds.$$

Our definition of a mild solution is

Definition 3.1. Let $a \in L^n_\sigma$ and let w be as in the Assumption. Suppose that $n < r < \infty$. A measurable function u on $\Omega \times (0, T)$ is called a *mild solution* of (N-S′) in the class $S_r(0, T)$ if

(1) $u \in BC([0, T); L^n_\sigma)$ and $t^{(1-n/r)/2} u(\cdot) \in BC([0, T); L^r_\sigma)$,

(2) $\lim_{t \downarrow +0} t^{(1-n/r)/2} \|u(t)\|_r = 0$,

(3) $(u(t), \phi) = (e^{-tL} a, \phi) + \int_0^t (u(s) \cdot \nabla e^{-(t-s)L'} \phi, u(s)) \, ds$

for all $\phi \in C^\infty_{0,\sigma}$ and *all* $0 < t < T$.

Taking $\delta = n/r$, we have $0 < \delta < 1$ and by (2), $\sup_{0<\tau<T} \tau^{(1-\delta)/2} \|u(\tau)\|_{n/\delta} < \infty$. Then it follows from (2.7) with $p = n' \equiv n/(n-1)$ and $r = n/(n-1-\delta)$ that

$$\text{(3.1)} \qquad \left| \int_0^t (u(s)\cdot \nabla e^{-(t-s)L'}\phi, u(s))\, ds \right|$$

$$\leqq \int_0^t \|u(s)\|_n \|\nabla e^{-(t-s)L'}\phi\|_{n/(n-1-\delta)} \|u(s)\|_{n/\delta}\, ds$$

$$\leqq M_{n', n/(n-1-\delta), T} \sup_{0<\tau<T} \|u(\tau)\|_n \sup_{0<\tau<T} \tau^{(1-\delta)/2} \|u(\tau)\|_{n/\delta}$$

$$\times \int_0^t (t-s)^{-(1+\delta)/2} s^{-(1-\delta)/2}\, ds \cdot \|\phi\|_{n'}$$

$$= M_{n', n/(n-1-\delta), T} B\left(\tfrac{1-\delta}{2}, \tfrac{1+\delta}{2}\right)$$

$$\times \sup_{0<\tau<T} \|u(\tau)\|_n \sup_{0<\tau<T} \tau^{(1-\delta)/2} \|u(\tau)\|_{n/\delta} \cdot \|\phi\|_{n'}$$

for all $0 < t < T$, where $B(\cdot,\cdot)$ denotes the beta function. Hence if u is in the class $S_r(0, T)$, then the integral on the right-hand side of (3) in Definition 3.1 is well-defined.

Concerning the uniqueness of mild solutions, we have

Lemma 3.2 (Uniqueness). *Let $a \in L^n_\sigma$ and let w be as in the Assumption. Suppose that $n < r < \infty$. Then the mild solution of* (N-S′) *is unique within the class $S_r(0, T)$.*

Proof. Let u and v be mild solutions of (N-S′) in $S_r(0, T)$ with the same initial data a. Then as in (3.1) we have that

$$|(u(t) - v(t), \phi)|$$

$$= \left| \int_0^t \{(u(s)\cdot\nabla e^{-(t-s)L'}\phi, u(s)) - (v(s)\cdot\nabla e^{-(t-s)L'}\phi, v(s))\}\, ds \right|$$

$$\leqq \int_0^t |((u(s) - v(s))\cdot\nabla e^{-(t-s)L'}\phi, u(s))|\, ds$$

$$+ \int_0^t |(v(s)\cdot\nabla e^{-(t-s)L'}\phi, u(s) - v(s))|\, ds$$

$$\leqq M_{n', n/(n-1-\delta), T} B\left(\tfrac{1-\delta}{2}, \tfrac{1+\delta}{2}\right)$$

$$\times \left(\sup_{0<s\leqq t} s^{(1-\delta)/2} \|u(s)\|_{n/\delta} + \sup_{0<s\leqq t} s^{(1-\delta)/2} \|v(s)\|_{n/\delta} \| \right)$$

$$\times \sup_{0<s\leqq t} \|u(s) - v(s)\|_n \cdot \|\phi\|_{n'}$$

for all $\phi \in C_{0,\sigma}^{\infty}$ and all $0 < t < T$, where $\delta = n/r$. Let us define the functions $D(t)$ and $K(t)$ on $(0, T)$ by

$$D(t) \equiv \sup_{0<s\leqq t} \|u(s) - v(s)\|_n,$$

$$K(t) \equiv \sup_{0<s\leqq t} s^{(1-\delta)/2}\|u(s)\|_{n/\delta} + \sup_{0<s\leqq t} s^{(1-\delta)/2}\|v(s)\|_{n/\delta}.$$

By the last estimate and by duality, we have

$$\|u(t) - v(t)\|_n \leqq C_* K(t)\cdot D(t), \quad 0 < t < T,$$

where $C_* = M_{n',n/(n-1-\delta),T} B(\frac{1}{2}(1-\delta), \frac{1}{2}(1+\delta))$. Since $K(t)\cdot D(t)$ is a monotone increasing function of t, we obtain that

$$D(t) \leqq C_* K(t)\cdot D(t) \quad \text{for all } 0 < t < T. \tag{3.2}$$

Since $K(t)$ is a continuous function on $[0, T)$ with $K(+0) = 0$, implied by (2) in Definition 3.1, we can choose a small positive number t_1 such that $C_* K(t_1) < 1$. Hence from (3.2), it follows that $D(t_1) = 0$, which yields

$$u(t) \equiv v(t) \quad \text{for } 0 \leqq t \leqq t_1.$$

Next we show that $u(t) \equiv v(t)$ for $t_1 \leqq t < T$. Since $t^{(1-\delta)/2}u(\cdot), t^{(1-\delta)/2}v(\cdot) \in BC([0, T); L_\sigma^{n/\delta})$, there is a constant K_* such that

$$\sup_{t_1\leqq s<T} \|u(s)\|_{n/\delta} + \sup_{t_1\leqq s<T} \|v(s)\|_{n/\delta} \leqq K_*. \tag{3.3}$$

For our purpose, it suffices to show the following proposition:

Proposition 3.3. *Let τ be any point in $[t_1, T)$ and let ξ be given by*

$$\xi \equiv \left(\frac{1-\delta}{4M_{n',n/(n-1-\delta),T}K_*}\right)^{2/(1-\delta)}. \tag{3.4}$$

If $u \equiv v$ on $[0, \tau]$, then $u \equiv v$ on $[0, \tau + \xi]$.

Proof of Proposition 3.3. Let $D_1(t) \equiv \sup_{\tau\leqq s<t}\|u(s) - v(s)\|_n$. By assumption, we have

$$(u(t) - v(t), \phi) = \int_\tau^t ((u(s) - v(s))\cdot\nabla e^{-(t-s)L'}\phi, u(s))\,ds$$

$$+ \int_\tau^t (v(s)\cdot\nabla e^{-(t-s)L'}\phi, u(s) - v(s))\,ds$$

for all $\phi \in C_{0,\sigma}^{\infty}$ and all $\tau \leqq t < T$. Then it follows from (2.7) and (3.3) that

$$|(u(t) - v(t), \phi)| \leqq \int_{\tau}^{t} \| u(s) - v(s) \|_n \| \nabla e^{-(t-s)L'} \phi \|_{n/(n-1-\delta)} (\| u(s) \|_{n/\delta} + \| v(s) \|_{n/\delta})\, ds$$

$$\leqq M_{n', n/(n-1-\delta), T} K_* D_1(t) \int_{\tau}^{t} (t-s)^{-1/2-\delta/2} ds \cdot \| \phi \|_{n'}$$

$$\leqq \frac{2M_{n', n/(n-1-\delta), T}}{1-\delta} K_* D_1(t)(t-\tau)^{(1-\delta)/2} \cdot \| \phi \|_{n'},$$

$$\phi \in C_{0,\sigma}^{\infty}, \ \tau \leqq t < T.$$

Hence by duality, we have

$$\| u(t) - v(t) \|_n \leqq \frac{2M_{n', n/(n-1-\delta), T}}{1-\delta} K_* D_1(\tau + \xi) \xi^{(1-\delta)/2} \quad \text{for all } t \in [\tau, \tau + \xi],$$

which together with (3.4) implies that $D_1(\tau + \xi) \leqq \frac{1}{2} D_1(\tau + \xi)$. Thus $D_1(\tau + \xi) = 0$ and $u(t) \equiv v(t)$ on $[0, \tau + \xi]$. This proves Proposition 3.3 and the proof of Lemma 3.2 is complete. □

Our existence theorem for mild solutions is

Theorem 3.4 (Global mild solution). (1) *Let $a \in L_\sigma^n$ and let w be as in the Assumption. There is a positive number $\lambda(n)$ such that if*

(3.5) $$\| a \|_n \leqq \lambda(n), \quad \| w \|_n + \| \nabla w \|_{n/2} \leqq \lambda(n),$$

then there exists a unique mild solution u of (N-S′) *in the class $S_{2n}(0, \infty)$ with the property*

(3.6)

$$u(t) \in D(L_n^\alpha) \text{ for } t > 0, \quad t^\alpha L_n^\alpha u(\cdot) \in BC([0, \infty); L_\sigma^n) \text{ with } \lim_{t \downarrow +0} t^\alpha \| L^\alpha u(t) \|_n = 0,$$

where $0 < \alpha < \frac{1}{2}$.

(2) *Moreover, for every $n < r < \infty$, there is a positive number $\eta(n, r)$ such that if*

(3.7) $$\| w \|_n + \| \nabla w \|_{n/2} \leqq \eta(n, r),$$

then the uniform estimate

(3.8) $$\| u(t) \|_l \leqq C t^{-(1-n/l)/2}, \quad n \leqq l \leqq r,$$

holds for all $t > 0$, where $C = C(n, r, l)$.

Remark. For the decay of solution in arbitrary L^r-spaces ($r > n$), the smallness on the initial disturbance a does not depend on r. However, we need to make the stationary flow w relative to r.

Proof of Theorem 3.4. (1) Let us construct the mild solution according to the following scheme:

$$u_0(t) = e^{-tL}a,$$

$$u_{j+1}(t) = u_0(t) - \int_0^t e^{-(t-s)L} P(u_j \cdot \nabla u_j)(s)\, ds, \quad j = 0, 1, \dots .$$

Under the condition

$$\|w\|_n + \|\nabla w\|_{n/2} \leqq \min\left\{\kappa(n, 2n), \kappa'\left(\frac{2n}{2n-1}, \frac{n}{n-1}\right)\right\}, \tag{3.9}$$

we have

$$\sup_{0<t<\infty} t^{1/4} \|u_j(t)\|_{2n} \leqq K_j, \quad j = 0, 1, \dots, \tag{3.10}$$

where κ and κ' are the same constants as in (2.23)–(2.25). Indeed, by (2.24),

$$\|u_0(t)\|_{2n} = \|e^{-tL}a\|_{2n} \leqq M_{n,2n} t^{-1/4} \|a\|_n, \quad t > 0, \tag{3.11}$$

and hence we may take

$$K_0 \equiv M_{2n,n} \|a\|_n. \tag{3.12}$$

Suppose that (3.10) is true provided that condition (3.9) is fulfilled. Then by integration by parts and (2.26),

$$\begin{aligned}
&\left|\left(-\int_0^t e^{-(t-s)L} P(u_j \cdot \nabla u_j)(s)\, ds, \phi \right)\right| \\
&\quad = \left| \int_0^t (u_j(s) \cdot \nabla e^{-(t-s)L'} \phi, u_j(s))\, ds \right| \\
&\quad \leqq \int_0^t \|u_j(s)\|_{2n}^2 \|\nabla e^{-(t-s)L'} \phi\|_{n/(n-1)}\, ds \\
&\quad \leqq M'_{2n/(2n-1), n/(n-1)} K_j^2 \int_0^t (t-s)^{-3/4} s^{-1/2}\, ds \cdot \|\phi\|_{2n/(2n-1)} \\
&\quad = M'_{2n/(2n-1), n/(n-1)} B(\tfrac{1}{4}, \tfrac{1}{2}) K_j^2 t^{-1/4} \cdot \|\phi\|_{2n/(2n-1)}
\end{aligned}$$

for all $\phi \in C_{0,\sigma}^{\infty}$, which by duality implies

$$\left\| \int_0^t e^{-(t-s)L} P(u_j \cdot \nabla u_j)(s)\, ds \right\|_{2n} \leqq M'_{2n/(2n-1), n/(n-1)} B(\tfrac{1}{4}, \tfrac{1}{2}) K_j^2 t^{-1/4}$$

for all $t > 0$. Hence (3.10) is true with j replaced by $j + 1$, with

$$K_{j+1} \equiv K_0 + C_n^{(1)} K_j^2, \tag{3.13}$$

where $C_n^{(1)} \equiv M'_{2n/(2n-1), n/(n-1)} B(\frac{1}{4}, \frac{1}{2})$. If

$$K_0 < \frac{1}{4C_n^{(1)}}, \tag{3.14}$$

then the sequence $\{K_j\}_{j=0}^{\infty}$ is *bounded* with

$$K_j \leqq \frac{1-\sqrt{1-4C_n^{(1)}K_0}}{2C_n^{(1)}} \equiv k < \frac{1}{2C_n^{(1)}}, \quad j=0,1,\dots . \tag{3.15}$$

Now we see by (3.12) and (3.14) that if $\|a\|_n \leqq 1/8M_{2n,n}C_n^{(1)}$ and if (3.9) holds, then (3.15) holds. Defining $v_j \equiv u_j - u_{j-1} (u_{-1} \equiv 0)$, we obtain from a calculation similar to that above that

$$\|v_j(t)\|_{2n} \leqq k(2C_n^{(1)}k)^j t^{-1/4}, \quad j=0,1,\dots, t>0. \tag{3.16}$$

Since $u_j = \Sigma_{i=0}^{j} v_i$, (3.15) and (3.16) yield a limit $u \in C((0,\infty); L_\sigma^{2n})$ with $t^{1/4}u(\cdot) \in BC([0,\infty); L_\sigma^{2n})$ such that

$$\sup_{0<t<\infty} t^{1/4}\|u_j(t)-u(t)\|_{2n} \to 0 \quad \text{as } j\to\infty. \tag{3.17}$$

Moreover, under the condition (3.9), we have by (2.18) and (2.24) that

$$\begin{aligned} \sup_{0<t<T} t^{1/4}\|e^{-tL}a\|_{2n} & \\ &\leqq \sup_{0<t<T} t^{1/4}\|e^{-tL}(a-\tilde a)\|_{2n} + \sup_{0<t<T} t^{1/4}\|e^{-tL}\tilde a\|_{2n} \\ &\leqq M_{n,2n}\|a-\tilde a\|_n + M_{2n}\|\tilde a\|_{2n}T^{1/4} \end{aligned} \tag{3.18}$$

for all $\tilde a \in L_\sigma^n \cap L_\sigma^{2n}$ and all $0<T<\infty$. Since (3.10)–(3.15) hold with $0<t<\infty$ replaced by $0<t<T$ for arbitrary $T>0$ and since $L_\sigma^n \cap L_\sigma^{2n}$ is dense in L_σ^n, (3.15) with the aid of (3.18) yields

$$\lim_{t\downarrow +0} t^{1/4}\|u(t)\|_{2n} = 0. \tag{3.19}$$

We next show $u \in BC([0,\infty); L_\sigma^n)$ provided

$$\|w\|_n + \|\nabla w\|_{n/2} \leqq \min\left\{\kappa(n), \kappa'\left(\frac{n}{n-1}, \frac{n}{n-1}\right)\right\}. \tag{3.20}$$

Indeed, from (2.18), (2.26) and (3.15), we obtain

$$\|u_0(t)\|_n \leqq M_n\|a\|_n,$$

$$\begin{aligned} &\left|\left(-\int_0^t e^{-(t-s)L}P(u_j\cdot\nabla u_j)(s)\,ds, \phi\right)\right| \\ &= \left|\int_0^t (u_j(s)\cdot\nabla e^{-(t-s)L'}\phi, u_j(s))\,ds\right| \\ &\leqq \int_0^t \|u_j(s)\|_{2n}^2 \|\nabla e^{-(t-s)L'}\phi\|_{n/(n-1)}\,ds \\ &\leqq M'_{n/(n-1),n/(n-1)}k^2\int_0^t (t-s)^{-1/2}s^{-1/2}\,ds\cdot\|\phi\|_{n/(n-1)} \\ &= M'_{n/(n-1),n/(n-1)}B(\tfrac12,\tfrac12)k^2\cdot\|\phi\|_{n/(n-1)}, \quad \phi\in C_{0,\sigma}^\infty,\ t>0, \end{aligned}$$

which yields

$$\sup_{0<t<\infty} \| u_{j+1}(t) \|_n \leqq M_n \| a \|_n + M'_{n/(n-1), n/(n-1)} B(\tfrac{1}{2}, \tfrac{1}{2}) k^2 \quad \text{for all } j .$$

This uniform estimate with respect to j ensures that the limit u satisfies also $u \in BC([0, \infty); L^n_\sigma)$.

To see that such u is a mild solution in the class $S_{2n}(0, \infty)$, we need to prove that

$$(3.21) \qquad \left(-\int_0^t e^{-(t-s)L} P(u_j \cdot \nabla u_j)(s) ds, \phi \right) \to \int_0^t (u(s) \cdot \nabla e^{-(t-s)L'} \phi, u(s)) \, ds$$

for each $\phi \in C^\infty_{0,\sigma}$ as $j \to \infty$. Indeed, by integration by parts, by (3.15) and by (3.17), we have

$$\left| \left(-\int_0^t e^{-(t-s)L} P(u_j \cdot \nabla u_j)(s) ds, \phi \right) - \int_0^t (u(s) \cdot \nabla e^{-(t-s)L'} \phi, u(s)) \, ds \right|$$

$$= \left| \int_0^t \{ (u_j(s) \cdot \nabla e^{-(t-s)L'} \phi, u_j(s)) - (u(s) \cdot \nabla e^{-(t-s)L'} \phi, u(s)) \} \, ds \right|$$

$$\leqq \int_0^t (\| u_j(s) \|_{2n} + \| u(s) \|_{2n}) \| u_j(s) - u(s) \|_{2n} \| \nabla e^{-(t-s)L'} \phi \|_{n/(n-1)} \, ds$$

$$\leqq 2M'_{n/(n-1), n/(n-1)} k \sup_{0<s<\infty} s^{1/4} \| u_j(s) - u(s) \|_{2n}$$

$$\times \int_0^t (t-s)^{-1/2} s^{-1/2} ds \cdot \| \phi \|_{n/(n-1)}$$

$$= 2M'_{n/(n-1), n/(n-1)} B(\tfrac{1}{2}, \tfrac{1}{2}) k \sup_{0<s<\infty} s^{1/4} \| u_j(s) - u(s) \|_{2n} \| \phi \|_{n/(n-1)}$$

$$\to 0 \quad \text{as } j \to \infty \quad (\phi \in C^\infty_{0,\sigma}),$$

which implies (3.21).

Now it remains to show that $t^\alpha L^\alpha u(\cdot) \in BC([0, \infty); L^n_\sigma)$ with (3.6) for $0 < \alpha < \frac{1}{2}$. To this end, assume moreover that

$$(3.22) \qquad \| w \|_n + \| \nabla w \|_{n/2} \leqq \begin{cases} \mu\left(\dfrac{n}{n-1} \right) & \text{for } n \geqq 5, \\ \mu'\left(\dfrac{n}{n-1} \right) & \text{for } n = 3, 4. \end{cases}$$

Then By Lemma 2.8 and by (3.15) we have

$$\begin{aligned}
&\left|\left(-L^\alpha \int_0^t e^{-(t-s)L} P(u_j\cdot\nabla u_j)(s)\,ds,\ \phi\right)\right| \\
&\quad = \left|\int_0^t (u_j(s)\cdot\nabla(L')^\alpha e^{-(t-s)L'}\phi, u_j(s))\,ds\right| \\
&\quad \leqq \int_0^t \|u_j(s)\|_{2n}^2 \|\nabla(L')^\alpha e^{-(t-s)L'}\phi\|_{n/(n-1)}\,ds \\
&\quad \leqq \tilde{M}_{n/(n-1),\alpha} k^2 \int_0^t (t-s)^{-\alpha-1/2} s^{-1/2}\,ds\cdot\|\phi\|_{n/(n-1)} \\
&\quad = \tilde{M}_{n/(n-1),\alpha} k^2 B(\tfrac{1}{2}-\alpha, \tfrac{1}{2}) t^{-\alpha}\cdot\|\phi\|_{n/(n-1)}, \quad t>0, \quad 0<\alpha<\tfrac{1}{2},
\end{aligned}$$

for all $\phi\in C_{0,\sigma}^\infty$ and all j, from which it follows that

$$\begin{aligned}
&\sup_{0<t<\infty} t^\alpha \|L^\alpha u_{j+1}(t)\|_n \\
&\quad \leqq \sup_{0<t<\infty} t^\alpha \|L^\alpha e^{-tL} a\|_n + \tilde{M}_{n/(n-1),\alpha} k^2 B(\tfrac{1}{2}-\alpha, \tfrac{1}{2}), \quad 0<\alpha<\tfrac{1}{2}, \quad j=0,1,\ldots.
\end{aligned}$$

This uniform estimate for j asserts that $u(t)\in D(L_n^\alpha)$ for $t>0$ with $t^\alpha L_n^\alpha u(\cdot)\in BC([0,\infty); L_\sigma^n)$, where $0<\alpha<\frac{1}{2}$. Since $D(L_n^\alpha)$ is dense in L_σ^n, we can prove (3.6) in the same way as (3.18). Now it is easy to see that the constant $\lambda(n)$ in (3.5) can be determined by (3.9), (3.14), (3.20) and (3.22).

(2) We show $u(t)\in L_\sigma^r(t>0)$ for all $n<r<\infty$ with

$$\sup_{0<t<\infty} t^{(1-n/r)/2}\|u_j(t)-u(t)\|_r \to 0 \quad \text{as } j\to\infty,$$

provided that condition (3.7) is fulfilled. Taking $r=n/\beta$, we have $0<\beta\leqq 1$. Assume that w is subject to the estimate:

$$\|w\|_n + \|\nabla w\|_{n/2} \leqq \min\left\{\lambda(n), \kappa\left(n,\frac{n}{\beta}\right), \kappa'\left(\frac{n}{n-1},\frac{n}{n-1}\right)\right\}. \tag{3.23}$$

Then it follows from (2.24)–(2.26) and (3.15) that

$$\|u_0(t)\|_{n/\beta} \leqq M_{n,\frac{n}{\beta}} t^{-(1-\beta)/2}\|a\|_n,$$

$$\begin{aligned}
&\left|\left(-\int_0^t e^{-(t-s)L} P(u_j\cdot\nabla u_j)(s)\,ds,\ \phi\right)\right| \\
&\quad = \left|\int_0^t (u_j(s)\cdot\nabla e^{-(t-s)L'}\phi,\ u_j(s))\,ds\right| \\
&\quad \leqq \int_0^t \|u_j(s)\|_{2n}^2 \|\nabla e^{-(t-s)L'}\phi\|_{n/(n-1)}\,ds \\
&\quad \leqq M'_{n/n-\beta,\,n/n-1} k^2 \int_0^t (t-s)^{\beta/2-1} s^{-1/2}\,ds\cdot\|\phi\|_{n/(n-\beta)} \\
&\quad \leqq M'_{n/n-\beta,\,n/n-1} B(\tfrac{\beta}{2},\tfrac{1}{2}) k^2 t^{-(1-\beta)/2}\cdot\|\phi\|_{n/(n-\beta)}, \quad \phi\in C_{0,\sigma}^\infty,
\end{aligned}$$

from which we obtain

$$(3.24)\qquad \sup_{0<t<\infty} t^{(1-\beta)/2}\|u_{j+1}(t)\|_{n/\beta} \leqq M_{n,\frac{n}{\beta}}\|a\|_n + M'_{n/(n-\beta),\,n/(n-1)}B(\tfrac{\beta}{2},\tfrac{1}{2})k^2 \equiv k_\beta$$

for all j. It is easy to see that this uniform estimate for j ensures that $t^{(1-\beta)/2}u(\cdot)\in BC([0,\ \infty); L^{n/\beta}_\sigma)$. Now the positive number $\eta(n,r)$ in (3.7) can be determined by (3.23), and we get the desired estimate (3.8) by interpolation. This proves Theorem 3.4. □

If we assume a more rapid spatial decay for the initial disturbance, then we obtain the decay of $\nabla u(t)$ as $t\to\infty$:

Theorem 3.5. (1) *Let $n \geqq 3$ and let $1 < p \leqq n$. Suppose that $a \in L^p_\sigma \cap L^n_\sigma$. There is a positive number $\lambda'(n,p)$ with $\lambda'(n,p) \leqq \lambda(n)$ such that if*

$$(3.25)\qquad \|a\|_n \leqq \lambda'(n,p), \quad \|w\|_n + \|\nabla w\|_{n/2} \leqq \lambda'(n,p),$$

then the mild solution u given by Theorem 3.4 has the additional property that

$$(3.26)\qquad u\in BC([0,\ \infty); L^p_\sigma\cap L^n_\sigma).$$

(2) *In particular, if $1 < p < \frac{n}{2}$ for $n \geqq 5$ and if $1 < p \leqq 2$ for $n = 3, 4$, then also*

$$(3.27)\qquad t^{1/2}\nabla u(\cdot)\in BC([0,\ \infty); L^p).$$

Proof. Let us first prove (3.27). Defining $\gamma = \frac{n}{p}$, we have by assumption that $2<\gamma<n$ for $n \geqq 5$ and $\frac{n}{2}\leqq\gamma<n$ for $n = 3, 4$. We return to the approximate solutions $\{u_j(t)\}_{j=0}^{\infty}$ in the proof of Theorem 3.4 and show that

$$(3.28)\qquad \sup_{0<t<\infty} t^{1/2}\|\nabla u_j(t)\|_{n/\gamma} \leqq L_j, \quad j = 0, 1, \ldots\ ,$$

under the condition

$$(3.29)\qquad \|w\|_n + \|\nabla w\|_{n/2} \leqq \min\left\{\kappa'\left(\frac{n}{\gamma},\frac{n}{\gamma}\right),\ \kappa'\left(\frac{n}{\gamma+1/2},\frac{n}{\gamma}\right)\right\},$$

where κ' is the same constant as in (2.25). Indeed, since $a\in L^{n/\gamma}_\sigma\cap L^n_\sigma$, we have by (2.26) that

$$\|\nabla u_0(t)\|_{n/\gamma} = \|\nabla e^{-tL}a\|_{n/\gamma} \leqq M'_{n/\gamma,\,n/\gamma}t^{-1/2}\|a\|_{n/\gamma}$$

for all $t>0$ and we may define $L_0 \equiv M'_{n/\gamma,\,n/\gamma}\|a\|_{n/\gamma}$. Suppose that (3.28) is true for j. Then it follows from (2.26) and (3.15) that

$$\begin{aligned}
&\left\|\nabla\int_0^t e^{-(t-s)L}P(u_j\cdot\nabla u_j)(s)\,ds\right\|_{n/\gamma}\\
&\leqq M'_{n/(\gamma+1/2),\,n/\gamma}\int_0^t (t-s)^{-3/4}\|u_j(s)\|_{2n}\|\nabla u_j(s)\|_{n/\gamma}\,ds\\
&\leqq M'_{n/(\gamma+1/2),\,n/\gamma}kL_j\int_0^t (t-s)^{-3/4}s^{-3/4}\,ds\\
&= M'_{n/(\gamma+1/2),\,n/\gamma}B(\tfrac{1}{4},\tfrac{1}{4})kL_jt^{-1/2}, \quad t>0.
\end{aligned}$$

Hence (3.28) holds with j replaced by $j+1$, and with

$$L_{j+1} \equiv L_0 + C_\gamma^{(2)} k L_j \tag{3.30}$$

where $C_\gamma^{(2)} = M'_{n/(\gamma+1/2), n/\gamma} B(\frac{1}{4}, \frac{1}{4})$. The linear recurrence identity (3.30) shows that if

$$k < \frac{1}{C_\gamma^{(2)}}, \tag{3.31}$$

then the sequence $\{L_j\}_{j=0}^{\infty}$ is *bounded* with

$$L_j \leqq \frac{L_0}{1 - C_\gamma^{(2)} k} \equiv l_\gamma, \quad j = 0, 1, \ldots. \tag{3.32}$$

By the standard argument, such a bound yields $t^{1/2} \nabla u(\cdot) \in BC([0, \infty); L^{n/\gamma})$. Since k is determined by (3.15), we can choose $\tilde{\lambda}(n, p)$ such that the condition $\|a\|_n \leqq \tilde{\lambda}(n, p)$ yields (3.31). Then the positive number $\lambda'(n, p)$ in (3.25) can be determined by (3.29) and this $\tilde{\lambda}(n, p)$, so we obtain (3.27).

We next prove (3.26). Let us first assume that p belongs to the same range as in the case (2) above. Then under the condition

$$\|w\|_n + \|\nabla w\|_{n/2} \leqq \min\left\{\mu\left(\frac{n}{\gamma}\right), \kappa\left(\frac{n}{\gamma+1/2}, \frac{n}{\gamma}\right)\right\}, \tag{3.33}$$

we have by (2.18), (2.24), (3.15) and (3.32) that

$$\begin{aligned}
\|u_{j+1}(t)\|_{n/\gamma} &\leqq \|u_0(t)\|_{n/\gamma} + \int_0^t \|e^{-(t-s)L} P(u_j \cdot \nabla u_j)(s)\|_{n/\gamma}\, ds \\
&\leqq M_{n/\gamma} \|a\|_{n/\gamma} + M_{n/(\gamma+1/2), n/\gamma} \int_0^t (t-s)^{-1/4} \|u_j(s)\|_{2n} \|\nabla u_j(s)\|_{n/\gamma}\, ds \\
&\leqq M_{n/\gamma} \|a\|_{n/\gamma} + M_{n/(\gamma+1/2), n/\gamma} B(\tfrac{3}{4}, \tfrac{1}{4}) k l_\gamma
\end{aligned}$$

for all $j = 0, 1, \ldots$, which yields $u \in BC([0, \infty); L^p_\sigma)$.

It remains to prove (3.26) in case $2 < p \leqq n$ for $n = 3, 4$ and in case $\frac{n}{2} \leqq p \leqq n$ for $n \geqq 5$. In such cases we have $1 \leqq \gamma < \frac{n}{2}$ for $n = 3, 4$ and $1 \leqq \gamma \leqq 2$ for $n \geqq 5$. Then there is β such that $0 < \beta < 1$ and such that $1 < \gamma + \beta \leqq \frac{n}{2}$ for $n = 3$, 4 and $\gamma + \beta < n - 1$ for $n \geqq 5$. Under the condition

$$\|w\|_n + \|\nabla w\|_{n/2} \leqq \min\left\{\mu\left(\frac{n}{\gamma}\right), \kappa'\left(\frac{n}{n-\gamma}, \frac{n}{n-\gamma-\beta}\right)\right\}, \tag{3.34}$$

we have

$$\sup_{0<t<\infty} \|u_j(t)\|_{n/\gamma} \leqq K'_j, \quad j = 0, 1, \ldots. \tag{3.35}$$

Indeed, for $j = 0$, we may define $K'_0 \equiv M_{n/\gamma} \| a \|_{n/\gamma}$. Suppose that (3.35) is true for j. Then it follows from (2.26) and (3.24) that

$$\left| \left(- \int_0^t e^{-(t-s)L} P(u_j \cdot \nabla u_j)(s)\, ds, \phi \right) \right|$$

$$= \left| \int_0^t (u_j(s) \cdot \nabla e^{-(t-s)L'} \phi, u_j(s))\, ds \right|$$

$$\leqq \int_0^t \| u_j(s) \|_{n/\beta} \| u_j(s) \|_{n/\gamma} \| \nabla e^{-(t-s)L'} \phi \|_{n/(n-\gamma-\beta)}\, ds$$

$$\leqq M'_{n/(n-\gamma), n/(n-\gamma-\beta)} k_\beta K'_j \int_0^t (t-s)^{-\beta/2 - 1/2} s^{-(1-\beta)/2} ds \cdot \| \phi \|_{n/(n-\gamma)}$$

$$= M'_{n/(n-\gamma), n/(n-\gamma-\beta)} B(\tfrac{1}{2}(1-\beta), \tfrac{1}{2}(1+\beta)) k_\beta K'_j \cdot \| \phi \|_{n/(n-\gamma)},$$

$$\phi \in C^\infty_{0,\sigma}, \quad t > 0.$$

By duality, we see that (3.35) is true with j replaced by $j+1$, with

$$K'_{j+1} \equiv K'_0 + C^{(3)}_\gamma k_\beta K'_j,$$

where $C^{(3)}_\gamma = M'_{n/(n-\gamma), n/(n-\gamma-\beta)} B(\tfrac{1}{2}(1-\beta), \tfrac{1}{2}(1+\beta))$. This *linear* recurrence identity shows that if $k_\beta < 1/C^{(3)}_\gamma$, then the sequence $\{K'_j\}_{j=0}^\infty$ is *bounded*, so that $u \in BC([0, \infty); L^{n/\gamma}_\sigma)$. Since k_β is controlled by $\| a \|_n$ (see (3.24)), we can also define $\lambda'(n, p)$ in (3.25). This proves Theorem 3.5. □

§4. Proof of the theorems

4.1. Proof of Theorem 1

To identify the mild solution in Theorem 3.4 with a strong solution, we need the following local existence theorem:

Theorem 4.1 (Local existence). *Let $a \in L^n_\sigma$ and let w be as in the Assumption with*

(4.1) $$\| w \|_n + \| \nabla w \|_{n/2} \leqq \mu(n)\,,$$

where μ is the same number as in Lemma 2.3. Then there exist $T_* > 0$ and a unique strong solution u of (N-S′) *on* $(0, T_*)$ *such that*

(4.2) $$\lim_{t \downarrow +0} t^\alpha \| L^\alpha_n u(t) \|_n = 0 \quad \textit{for } 0 < \alpha < 1.$$

If $a \in D(L^\beta_n)$ for $0 < \beta < \tfrac{1}{4}$, then T_* may be chosen as

(4.3) $$T_* = \frac{C}{(\| a \|_n + \| L^\beta_n a \|_n)^{1/\beta}},$$

where $C = C(n, \beta)$.

Remark. In the same way as (2.5), under the condition (4.1), we have the continuous imbedding $D((L_n+1)^\alpha) \subset H^{2\alpha,n}$ with $\|u\|_{H^{2\alpha,n}} \leqq C\|(L_n+1)^\alpha u\|_n$ for all $u \in D((L_n+1)^\alpha)$. Since $H^{2\alpha,n} \subset L^r$ for $1/r = 1/n - 2\alpha/n$ $(0<\alpha<1/2)$, we obtain from (4.2) that

$$\lim_{t\downarrow+0} t^{(1-n/r)/2}\|u(t)\|_r = 0 \quad \text{for } n<r<\infty.$$

Hence Lemma 3.2 assures uniqueness of the strong solution u with property (4.2).

Theorem 4.1 deals only with the local solution, so its proof is standard and may be omitted (see, e.g., MIYAKAWA [27] and KOZONO [20]).

Proof of Theorem 1. Let u be the mild solution of (N-S′) in the class $S_{2n}(0,\infty)$ given by Theorem 3.4. Then it follows from Theorem 4.1 and Lemma 3.2 that u coincides with the strong solution on $(0, T_*)$. Since $\sup_{0<t<\infty}\|u(t)\|_n < \infty$, $\sup_{T_*\leqq t<\infty}\|L_n^\alpha u(t)\|_n < \infty$ for $0<\alpha<\frac{1}{2}$, we conclude from (4.3) by a standard argument that $u(t)$ is also a strong solution on $[T_*, \infty)$. This proves Theorem 1. □

4.2. *Proof of Theorem 2*

By virtue of Theorem 3.5, we need only show the asymptotic behavior (1.5) and (1.6).

(i) Let $\gamma = \frac{n}{p}$ and $\delta = \frac{n}{r}$. Then we have $1<\gamma<n$ and $0<\delta<1$. Without loss of generality, we may assume $0<\delta<\frac{1}{2}$. Let us take ε and γ' such that $0<\varepsilon<\delta$, $1<\gamma'\leqq\gamma$ and $1+\delta-\varepsilon<\gamma'<1+\delta$. Then $1<\gamma'<n-1$ and $\frac{1}{2}(1-\varepsilon)<\frac{1}{2}(\gamma'-\delta)<\frac{1}{2}$. By (3.26) we have

$$\sup_{0<t<\infty}\|u(t)\|_{n/\gamma'} \leqq k_{\gamma'} < \infty, \tag{4.4}$$

where $k_{\gamma'}$ is a constant depending only on γ'. Choose β such that

$$0<\beta<1, \quad \tfrac{1}{2}(\gamma'-\delta)+\tfrac{1}{2}\beta<\tfrac{1}{2}, \quad \gamma'+\beta<n-1.$$

If

$$\|w\|_n + \|\nabla w\|_{n/2} \leqq \min\left\{\lambda'\left(n,\frac{n}{\beta}\right), \kappa'\left(\frac{n}{n-\delta}, \frac{n}{n-\beta-\gamma'}\right)\right\},$$

then it follows from (3.24) and (4.4) that

$$\begin{aligned}
&\left|\int_0^t (u(s)\cdot\nabla e^{-(t-s)L'}\phi, u(s))\,ds\right| \\
&\leqq \int_0^t \|u(s)\|_{n/\gamma'}\|u(s)\|_{n/\beta}\|\nabla e^{-(t-s)L'}\phi\|_{n/(n-\gamma'-\beta)}\,ds \\
&\leqq M'_{n/(n-\delta),\,n/(n-\gamma'-\beta)}k_{\gamma'}k_\beta\int_0^t (t-s)^{-(\gamma'+\beta-\delta)/2-1/2}s^{-(1-\beta)/2}\,ds\cdot\|\phi\|_{n/(n-\delta)} \\
&= M'_{n/(n-\delta),\,n/(n-\gamma'-\beta)}k_{\gamma'}k_\beta B(\tfrac{1}{2}(1+\delta-\gamma'-\beta), \tfrac{1}{2}(1+\beta))t^{-(\gamma'-\delta)/2}\cdot\|\phi\|_{n/(n-\delta)}
\end{aligned}$$

for all $\phi \in C^{\infty}_{0,\sigma}$ and all $t > 0$. Thus by duality we obtain

(4.5) $$\|u(t)\|_{n/\delta} \leqq M_{n/\gamma,n/\delta}\|a\|_{n/\gamma}t^{-(\gamma-\delta)/2} + Ct^{-(\gamma'-\delta)/2} \leqq Ct^{-(1-\varepsilon)/2}$$

for all $t \geqq 1$, where $C = C(n, \gamma, \delta)$.

To obtain sharper decay rates for $\|u(t)\|_{n/\delta}$ as $t \to \infty$, we make use of the representation

(4.6) $$(u(t), \phi) = (e^{-(t-T)L}u(T), \phi) + \int_T^t (u(s)\cdot\nabla e^{-(t-s)L'}\phi, u(s))\,ds, \quad \phi \in C^{\infty}_{0,\sigma}$$

for all $t \geqq T \geqq 0$. By (2.26) and (4.5), we have

(4.7) $$\begin{aligned}&\left|\int_T^t (u(s)\cdot\nabla e^{-(t-s)L'}\phi, u(s))\,ds\right| \\ &\leqq \int_T^t \|u(s)\|^2_{n/\delta}\|\nabla e^{-(t-s)L'}\phi\|_{n/(n-2\delta)}\,ds \\ &\leqq CM'_{n/(n-\delta),n/(n-2\delta)}\int_T^t (t-s)^{-\delta/2-1/2}s^{-2(1/2-\varepsilon/2)}\,ds\cdot\|\phi\|_{n/(n-\delta)} \\ &\leqq CT^{(1-\delta)/2-2(1/2-\varepsilon/2)}\cdot\|\phi\|_{n/(n-\delta)}, \quad \phi \in C^{\infty}_{0,\sigma}\end{aligned}$$

for all $t > T \geqq 1$. Since

$$\|e^{-(t-T)L}u(T)\|_{n/\delta} \leqq M_{n/\gamma,n/\delta}\sup\nolimits_{0<s<\infty}\|u(s)\|_{n/\gamma}(t-T)^{-(\gamma-\delta)/2}$$

for all $t > T$, we have by (4.6), (4.7) with $T = t/2$ that

$$\|u(t)\|_{n/\delta} \leqq C(t^{-(\gamma-\delta)/2} + t^{(1-\delta)/2-2(1/2-\varepsilon/2)})$$

for all $t \geqq 2$. Substituting this decay result into (4.7) again, we have

$$\|u(t)\|_{n/\delta} \leqq C(t^{-(\gamma-\delta)/2} + t^{(1-\delta)/2+(1-\delta)-4(1/2-\varepsilon/2)})$$

for all $t \geqq 4$. Now iterating this procedure m times, we obtain

$$\begin{aligned}\|u(t)\|_{n/\delta} &\leqq C(t^{-(\gamma-\delta)/2} + t^{(1-\delta)\sum_{j=1}^{m}(1/2)2^{j-1} - 2^m(1/2-\varepsilon/2))}) \\ &= C(t^{-(\gamma-\delta)/2} + t^{-2^{m-1}(\delta-\varepsilon)-(1-\delta)/2})\end{aligned}$$

for all $m = 1, 2, \ldots,$ and all $t \geqq 2^m$, where $C = C(n, \gamma, \delta, m)$. Since $\varepsilon < \delta$, this estimate assures (1.5).

(ii) Finally, it remains to prove (1.6). In the same way as above, let us define γ and δ so that $\gamma = n/p$ and $\delta = n/r$, respectively. Then the assumption that $1 < p < n/2$, $p \leqq r < n$ for $n \geqq 3$ is equivalent to $2 < \gamma < n$, $1 < \delta \leqq \gamma (n \geqq 3)$. The alternative assumption that $1 < p \leqq r \leqq 2$ for $n = 3, 4$ is equivalent to $n/2 \leqq \delta \leqq \gamma < n$ $(n = 3, 4)$. We treat the former case. The latter can be handled in the same way.

Let us first assume that $\gamma - 1 < \delta \leqq \gamma$. We choose $0 < \beta < 1$ such that $\gamma - 1 + \beta < \delta \leqq \gamma$. If $\|w\|_n + \|\nabla w\|_{n/2} \leqq \kappa'(n/(\gamma+\beta), n/\delta)$, then it follows from

(2.26), (3.24) and (3.32) that

$$\left\| \nabla \int_0^t e^{-(t-s)L} P(u \cdot \nabla u)(s)\, ds \right\|_{n/\delta}$$

$$\leqq M'_{n/(\gamma+\beta), n/\delta} \int_0^t (t-s)^{-(\gamma+\beta-\delta)/2-1/2} \| u(s) \|_{n/\beta} \| \nabla u(s) \|_{n/\gamma}\, ds$$

$$\leqq M'_{n/(\gamma+\beta), n/\delta} B(\tfrac{1}{2}(\delta + 1 - \gamma - \beta), \tfrac{1}{2}\beta) k_\beta l_\gamma t^{-(\gamma-\delta)/2-1/2}$$

for all $t > 0$, which yields (1.6).

We next proceed to the case that $\gamma - \frac{3}{2} \leqq \delta \leqq \gamma - \frac{1}{2}$. Taking γ' with $\gamma - 1 < \gamma' < \gamma - \frac{1}{2}$, we have by last result that

$$\| \nabla u(t) \|_{n/\gamma'} \leqq C t^{-(\gamma-\gamma')/2-1/2}, \quad t > 0. \tag{4.8}$$

Since $\gamma' - 1 < \delta$, there is a $\beta \in (0, 1)$ such that $\gamma' - 1 + \beta < \delta$. Hence if $\| w \|_n + \| \nabla w \|_{n/2} \leqq \kappa'(n/(\gamma' + \beta), n/\delta)$, then we have by (3.24) and (4.8) that

$$\left\| \nabla \int_T^t e^{-(t-s)L} P(u \cdot \nabla u)(s)\, ds \right\|_{n/\delta}$$

$$\leqq M'_{n/(\gamma'+\beta), n/\delta} \int_T^t (t-s)^{-(\gamma'+\beta-\delta)/2-1/2} \| u(s) \|_{n/\beta} \| \nabla u(s) \|_{n/\gamma'}\, ds$$

$$\leqq C \int_T^t (t-s)^{-(\gamma'+\beta-\delta)/2-1/2} s^{(\beta+\gamma'-\gamma)/2-1}\, ds$$

$$\leqq C T^{-(\gamma-\delta)/2-1/2}$$

for all $t > T \geqq 1$. Since

$$\| \nabla e^{-(t-T)L} u(T) \|_{n/\delta} \leqq M'_{n/\gamma, n/\delta} \sup_{0 < s < \infty} \| u(s) \|_{n/\gamma} (t - T)^{-(\gamma-\delta)/2-1/2}$$

for all $t > T$, we obtain from the last estimate with $T = t/2$ that

$$\| \nabla u(t) \|_{n/\delta} \leqq C t^{-(\gamma-\delta)/2-1/2}, \quad t \geqq 2.$$

Iterating this procedure to the case $\delta \leqq \gamma - 1$, within a finite number of steps we can cover all exponents δ with $1 < \delta \leqq \gamma$ and obtain the estimate $\| \nabla u(t) \|_{n/\delta} \leqq C t^{-(\gamma-\delta)/2-1/2}$ for sufficiently large t. This proves Theorem 2. □

4.3. *Proof of the Corollary*

Let the conditions (1.1) and (1.2) hold. Set $U_\lambda \equiv \{a \in L^n_\sigma; \| a \|_n < \lambda(n)\}$. By Theorem 3.4, we can define a map F by

$$F: a \in U_\lambda \mapsto u = Fa \in BC([0, \infty); L^n_\sigma),$$

where u is the unique mild solution of (N-S′) in the class $S_{2n}(0, \infty)$ with $u(0) = a$. Then we have the following key lemma:

Lemma 4.2. *The mapping F is continuous from U_λ into $BC([0,\infty); L^n_\sigma)$.*

Proof. Let $\|w\|_n + \|\nabla w\|_{n/2} \leqq \min\left\{\kappa(n,n), \kappa'\left(n', \frac{2n}{2n-3}\right)\right\}$. Since the number k in (3.15) is determined by the size of $\|a\|_n$, we may take $\lambda(n)$ so small that

$$\sup_{0<t<\infty} t^{1/4}\|(Fa)(t)\|_{2n} \leqq \frac{1}{4M'_{n,2n/2n-3}B(\frac{1}{4},\frac{3}{4})} \tag{4.9}$$

holds for all $a \in U_\lambda$. Now for $a, b \in U_\lambda$, set $u = Fa$, $v = Fb$ and we have by Definition 3.1 and Theorem 2.7 that

$$\begin{aligned}
&|(u(t)-v(t),\phi)| \\
&\leqq |(e^{-tL}a - e^{-tL}b, \phi)| \\
&\quad + \left|\int_0^t \{(u(s)\cdot\nabla e^{-(t-s)L'}\phi, u(s)) - (v(s)\cdot\nabla e^{-(t-s)L'}\phi, v(s))\}\,ds\right| \\
&\leqq |(e^{-tL}(a-b),\phi)| \\
&\quad + \int_0^t |((u(s)-v(s))\cdot\nabla e^{-(t-s)L'}\phi, u(s))|\,ds \\
&\quad + \int_0^t |(v(s)\cdot\nabla e^{-(t-s)L'}\phi, u(s)-v(s))|\,ds \\
&\leqq \|e^{-tL}(a-b)\|_n\|\phi\|_{n'} \\
&\quad + \int_0^t \|u(s)-v(s)\|_n\|\nabla e^{-(t-s)L'}\phi\|_{2n/(2n-3)}(\|u(s)\|_{2n} + \|v(s)\|_{2n})ds \\
&\leqq M_{n,n}\|a-b\|_n\|\phi\|_{n'} \\
&\quad + M'_{n',2n/(2n-3)}\left(\sup_{0<s<\infty} s^{1/4}\|u(s)\|_{2n} + \sup_{0<s<\infty} s^{1/4}\|v(s)\|_{2n}\right) \\
&\quad \times \sup_{0<s<\infty}\|u(s)-v(s)\|_n \\
&\quad \times \int_0^t (t-s)^{-(n/2)(1/n'-(2n-3)/2n)-1/2}s^{-1/4}ds\cdot\|\phi\|_{n'} \\
&= M_{n,n}\|a-b\|_n\|\phi\|_{n'} \\
&\quad + M'_{n',2n/(2n-3)}B(\tfrac{1}{4},\tfrac{3}{4})\left(\sup_{0<s<\infty} s^{1/4}\|u(s)\|_{2n} + \sup_{0<s<\infty} s^{1/4}\|v(s)\|_{2n}\right) \\
&\quad \times \sup_{0<s<\infty}\|u(s)-v(s)\|_n\|\phi\|_{n'}
\end{aligned}$$

for all $\phi \in C_{0,\sigma}^{\infty}$ and all $0 < t < \infty$. Hence it follows from (4.9) and from a duality argument that

$$\sup_{0<s<\infty} \|u(s) - v(s)\|_n \leqq M_{n,n}\|a - b\|_n + \tfrac{1}{2} \sup_{0<s<\infty} \|u(s) - v(s)\|_n.$$

This implies that $\sup_{0<s<\infty} \|u(s) - v(s)\|_n \leqq 2M_{n,n}\|a - b\|_n$ and we get the desired continuity. □

Proof of Corollary. Let u be the strong solution given by Theorem 1. Since $C_{0,\sigma}^{\infty}$ is dense in L_σ^n and since the mapping F is continuous, for any $\varepsilon > 0$, there is $\tilde{a} \in U_\lambda \cap C_{0,\sigma}^{\infty}$ such that

$$\sup_{0<t<\infty} \|u(t) - (F\tilde{a})(t)\|_n = \sup_{0<t<\infty} \|(Fa)(t) - (F\tilde{a})(t)\|_n < \varepsilon.$$

On the other hand, by (1.5) in Theorem 2, we see

$$\|(F\tilde{a})(t)\|_n \to 0 \quad \text{as } t \to \infty.$$

Then it follows that

$$\limsup_{t\to\infty} \|u(t)\|_n \leqq \limsup_{t\to\infty} \|u(t) - (F\tilde{a})(t)\|_n + \limsup_{t\to\infty} \|(F\tilde{a})(t)\|_n \leqq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude $\lim_{t\to\infty} \|u(t)\|_n = 0$ and the desired result (1.7) is a consequence of the uniform estimate (1) in Theorem 1 and the interpolation between L^n and L^r. This proves the Corollary. □

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