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Decay Theorems for the Broadwell Equations

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1. Introduction

In this note we establish boundedness and decay theorems for finitely supported solutions to the Broadwell equations with four velocity components. There is a vast literature on these and similar equations (for details see the survey of PLATKOWSKI & ILLNER [1] and the references contained therein). The particular variant of them we examine seems new; though nonlinear, they are homogeneous of degree one.

The basic quantities of interest are particle densities $\tilde{r}$, $\tilde{l}$, $\tilde{u}$, and $\tilde{d}$. Specifically $\tilde{r}(\tilde{x}, \tilde{y}, \tilde{t})$ represents the number of particles per unit area at $(\tilde{x}, \tilde{y})$ at time $\tilde{t}$ travelling with velocity we_1. The densities $\tilde{l}$, $\tilde{u}$, and $\tilde{d}$ have a similar interpretation except that the particles travel with velocities $-we_1$, we_2, and $-we_2$ respectively. The evolution equations for the densities are

$$\begin{aligned} \tilde{r}_{\tilde{t}} + w\tilde{r}_{\tilde{x}} &= \mathscr{C}, \\ \tilde{l}_{\tilde{t}} - w\tilde{l}_{\tilde{x}} &= \mathscr{C}, \\ \tilde{u}_{\tilde{t}} + w\tilde{u}_{\tilde{y}} &= -\mathscr{C}, \\ \tilde{d}_{\tilde{t}} - w\tilde{d}_{\tilde{y}} &= -\mathscr{C} \end{aligned} \tag{1.1}$$

where the collision term $\mathscr{C}$ is given by

$$\mathscr{C} = K(\tilde{u}\,\tilde{d} - \tilde{r}\,\tilde{l}). \tag{1.2}$$

Most of the authors dealing with (1.2), or one of its generalizations, treat K as a constant. In what follows we let

$$\tilde{\varrho}(\tilde{x}, \tilde{y}, \tilde{t}) = (\tilde{r} + \tilde{l} + \tilde{u} + \tilde{d})(\tilde{x}, \tilde{y}, \tilde{t}) \tag{1.3}$$

be the number of particles per unit area at $(\tilde{x}, \tilde{y})$ at time $\tilde{t}$ and model K by

$$K = \frac{1}{\varepsilon\tilde{\varrho}(\tilde{x}, \tilde{y}, \tilde{t})} \tag{1.4}$$

where ε is a fixed relaxation time. With this choice of K, solutions of the full system and solutions of a reduced system analogous to the Euler equations have the same asymptotic structure as time proceeds to infinity. This connection is lost if one models K as a constant. For the details of this connection see Section 3.

Under the scaling

$$x = \frac{\tilde{x}}{w\varepsilon}, \quad y = \frac{\tilde{y}}{w\varepsilon}, \quad t = \frac{\tilde{t}}{\varepsilon}, \quad (r, l, u, d) = (w\varepsilon)^2 (\tilde{r}, \tilde{l}, \tilde{u}, \tilde{d}), \tag{1.5}$$

the system (1.1), (1.2), (1.4) and (1.5) transforms to

$$\begin{aligned} r_t + r_x &= (u\,d - rl)/\varrho, \\ l_t - l_x &= (u\,d - rl)/\varrho, \\ u_t + u_y &= (rl - u\,d)/\varrho, \\ d_t - d_y &= (rl - u\,d)/\varrho \end{aligned} \tag{1.6}$$

where

$$\varrho = r + l + u + d \tag{1.7}$$

is the number density of particles at (x, y) at time t. Our results deal with solutions which are independent of y and thus satisfy

$$\begin{aligned} r_t + r_x &= (u\,d - rl)/\varrho, \\ l_t - l_x &= (u\,d - rl)/\varrho, \\ u_t &= (rl - u\,d)/\varrho, \\ d_t &= (rl - u\,d)/\varrho \end{aligned} \tag{1.8}$$

where again ϱ is given by (1.7). Our interest is in nonnegative solutions of (1.7) and (1.8).

The characteristic identities

$$\begin{aligned} r(x, t) &= \exp\left(-\int_0^t \frac{l}{\varrho}(x - t + \eta, \eta)\, d\eta\right) r(x - t, 0) \\ &\quad + \int_0^t \exp\left(-\int_s^t \frac{l}{\varrho}(x - t + \eta, \eta)\, d\eta\right) \frac{u\,d}{\varrho}(x - t + s, s)\, ds, \\ l(x, t) &= \exp\left(-\int_0^t \frac{r}{\varrho}(x + t - \eta, \eta)\, d\eta\right) l(x + t, 0) \\ &\quad + \int_0^t \exp\left(-\int_s^t \frac{r}{\varrho}(x + t - \eta, \eta)\, d\eta\right) \frac{u\,d}{\varrho}(x + t - s, s)\, ds, \\ u(x, t) &= \exp\left(-\int_0^t \frac{d}{\varrho}(x, \eta)\, d\eta\right) u(x, 0) + \int_0^t \exp\left(-\int_s^t \frac{d}{\varrho}(x, \eta)\, d\eta\right) \frac{rl}{\varrho}(x, s)\, ds, \\ d(x, t) &= \exp\left(-\int_0^t \frac{u}{\varrho}(x, \eta)\, d\eta\right) d(x, 0) + \int_0^t \exp\left(-\int_s^t \frac{u}{\varrho}(x, \eta)\, d\eta\right) \frac{rl}{\varrho}(x, s)\, ds \end{aligned} \tag{1.9}$$

together with the fact that the integral operators defined by the right-hand side of (1.9) take nonnegative functions to nonnegative functions guarantee that nonnegative initial data produce nonnegative solutions. This can be most easily seen by solving (1.9) iteratively, using the initial data as the starting guess.

Our results deal with solutions to (1.7) and (1.8) generated by bounded, compactly supported initial data. If the support of the data is the interval $(-a, a)$, then the procedure by which the solution is constructed guarantees that

$$\begin{gathered} r \equiv u \equiv d \equiv 0 \quad \text{for } x < -a, \\ l_t - l_x = 0 \quad \text{for } -(a+t) < x < -a \quad \text{and } t > 0, \end{gathered} \tag{1.10}$$

and

$$l \equiv u \equiv d \equiv 0 \text{ for } x > a, \quad r_t + r_x = 0 \text{ for } a < x < a + t \text{ and } t > 0, \tag{1.11}$$

and thus the interaction region is confined to the strip $-a < x < a$ and $t > 0$. In general, $x = -(a + t)$ is a jump discontinuity of the density l, $x = a + t$ is a jump discontinuity of the density r, and the lines $x = \pm a$ are contact discontinuities across which u and d jump, but r and l are continuous[1].

In Section 2 we outline the general theory of (1.7) and (1.8) for data of the type described above and discuss the corresponding long-time behavior. Our principal results are that in the interval $-a < x < a$:

$$\lim_{t \to \infty} (r(x, t), l(x, t), (u\,d)(x, t)) = (0, 0, 0), \tag{1.12}$$

$$\lim_{t \to \infty} u(x, t) = \max (u(x, 0) - d(x, 0), 0), \tag{1.13}$$

$$\lim_{t \to \infty} d(x, t) = \max (0, d(x, 0) - u(x, 0)). \tag{1.14}$$

In Section 3 we treat the more customary form of the Broadwell equations

$$\begin{aligned} r_t + r_x &= (c^2 - rl)/\varrho, \\ l_t - l_x &= (c^2 - rl)/\varrho, \\ c_t &= (rl - c^2)/\varrho \end{aligned} \tag{1.15}$$

where now

$$\varrho = r + 2c + l. \tag{1.16}$$

The equations (1.7) and (1.8) reduce to this system on the manifold $u \equiv d \stackrel{\text{def}}{=} c$. We produce a one-parameter family of exact solutions to the system (1.15) and (1.16) with the structure described in (1.10) and (1.11). These solutions decay as $e^{-\lambda t}$ uniformly on $(-a, a)$; the decay rate λ is completely determined by the length of the interval and is independent of the initial amplitude of the solution. We are also able to show that if a is less than two, the solutions to (1.15) and (1.16)

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[1] For details see Section 2.

decay to zero as $e^{-\lambda t}$ uniformly on $(-a, a)$ for some $0 < \lambda < \frac{1}{2}$ which depends only on $a > 0$. For this result to be valid, the initial data must satisfy certain shape constraints but the amplitude of the data is arbitrary.

2. A-Priori Estimates and Decay Theorems for (1.7) and (1.8)

In this section we sketch the general theory of the system (1.7) and (1.8) supplemented with the initial conditions

$$(r, l, u, d)(x, 0) = (r_0, l_0, u_0, d_0)(x). \tag{2.1}$$

The data r_0, l_0, u_0 and d_0 are nonnegative, bounded, and finitely supported in $(-a, a)$ and $\varrho = r + l + u + d$. The procedure we use to solve this problem is outlined below. We first solve the initial boundary value problem (1.8) and (2.1) on the strip $(-a, a) \times [0, \infty)$ subject to the boundary conditions

$$r(-a^+, t) = l(a^-, t) = 0. \tag{2.2}$$

Granting for the moment that (1.7), (1.8), (2.1), and (2.2) have a solution, we extend it to the rest of $t > 0$ as follows: For $x < -a$, we set

$$r \equiv u \equiv d \tag{2.3}$$

and extend l as the solution of

$$l_t - l_x = 0 \quad \text{for } x < -a \text{ and } t > 0, \tag{2.4}$$

$$l(x, 0) = 0, \text{ for } x < -a, \ l(-a^-, t) = l(-a^+, t) \text{ for } t > 0 \tag{2.5}$$

where $l(-a^+, t)$ is the solution to (1.7), (1.8), (2.1) and (2.2) previously constructed. For $x > a$, we set

$$l \equiv u \equiv d \tag{2.6}$$

and extend r as the solution of

$$r_t + r_x = 0 \quad \text{for } x > a \text{ and } t > 0, \tag{2.7}$$

$$r(x, 0) = 0 \quad \text{for } x > a, \quad r(a^+, t) = r(a^-, t) \quad \text{for } t > 0 \tag{2.8}$$

where now $r(a^-, t)$ is the solution to (1.7), (1.8), (2.1) and (2.2).

The resulting functions r, l, u and d satisfy (1.7) and (1.8) in the upper half space $t > 0$ and assume the initial data defined in (2.1). In general, the lines $x = \pm a$ are contact discontinuities across which u and d experience jump discontinuities. The procedure outlined above guarantees that r and l are continuous across these lines. Lines of discontinuity of r are rays $x = x_0 + t$ and those of l are the rays $x = x_0 - t$.

Our primary goal is to obtain global estimates for solutions of (1.7), (1.8), (2.1) and (2.2). These estimates are not only valid for the above problem but also for the regularized problem

$$\begin{aligned} r_t + r_x &= \frac{(u\,d - rl)}{\delta + r + l + u + d}, \\ l_t - l_x &= \frac{(u\,d - rl)}{\delta + r + l + u + d}, \\ u_t = d_t &= \frac{(rl - u\,d)}{\delta + r + l + u + d} \end{aligned} \tag{1.8$_\delta$}$$

together with (2.1) and (2.2). These estimates (which are independent of δ), together with local existence results for $(1.8)_\delta$, (2.1) and (2.2), and standard compactness results are sufficient to guarantee a global existence theorem for (1.7), (1.8), (2.1) and (2.2).

The conservation identities

$$\begin{aligned} (r + u)_t + r_x &= 0, \\ (r + d)_t + r_x &= 0, \\ (l + u)_t - l_x &= 0, \\ (l + d)_t - l_x &= 0 \end{aligned} \tag{2.9}$$

together with the boundary conditions

$$r(-a^+, t) = r(-a^-, t) = 0, \qquad l(a^-, t) = l(a^+, t) = 0 \tag{2.10}$$

imply that

$$\int_{-a}^{x} (r + u)(\xi, t)\, d\xi + \int_0^t r(x, s)\, ds = \int_{-a}^{x} (r_0 + u_0)(\xi)\, d\xi, \qquad -a < x, \tag{2.11}$$

$$\int_{-a}^{x} (r + d)(\xi, t)\, d\xi + \int_0^t r(x, s)\, ds = \int_{-a}^{x} (r_0 + d_0)(\xi)\, d\xi, \qquad -a < x, \tag{2.12}$$

$$\int_{x}^{a} (l + u)(\xi, t)\, d\xi + \int_0^t l(x, s)\, ds = \int_{x}^{a} (l_0 + u_0)(\xi)\, d\xi, \qquad x < a, \tag{2.13}$$

$$\int_{x}^{a} (l + d)(\xi, t)\, d\xi + \int_0^t l(x, s)\, ds = \int_{x}^{a} (l_0 + d_0)(\xi)\, d\xi, \qquad x < a, \tag{2.14}$$

$$\int_{\min(0, t-(a+x))}^{t} u(x - t + s, s)\, ds \leqq \int_{-a}^{x} (r + u)(\xi, t)\, d\xi, \qquad -a < x, \tag{2.15}$$

$$\int_{\min(0, t-(a+x))}^{t} d(x - t + s, s)\, ds \leqq \int_{-a}^{x} (r + d)(\xi, t)\, d\xi, \qquad -a < x, \tag{2.16}$$

$$\int_{\min(0, t-(a-x))}^{t} u(x + t - s, s)\, ds \leqq \int_{x}^{a} (l + u)(\xi, t)\, d\xi, \qquad x < a, \tag{2.17}$$

$$\int_{\min(0, t-(a-x))}^{t} d(x + t - s, s)\, ds \leqq \int_{x}^{a} (l + d)(\xi, t)\, d\xi, \qquad x < a. \tag{2.18}$$

The identities (2.11)–(2.14) come from integrating (2.9) over the rectangles $(-a, x)\times(0, t)$ or $(x, a)\times(0, t)$. Inequalities (2.15) and (2.16) follow from integrating $(2.9)_{1,2}$ over $\{(\xi, s) \mid x - t < \xi < x - t + s, 0 < s < t\}$ and making use of (2.3), (2.11) and (2.12), while (2.17) and (2.18) follow from integrating $(2.9)_{3,4}$ over $\{(\xi, s) \mid x + t - s < \xi < x + t, 0 < s < t\}$ and making use of (2.6), (2.13) and (2.14). Thus, for data of the type under consideration we see that $r(x, \cdot)$ and $l(x, \cdot)$ are in $L_1(0, \infty)$ and we have the estimates

$$\sup_{-a<x<a} \int_0^\infty r(x, s)\, ds \leqq m_0, \qquad \sup_{-a<x<a} \int_0^\infty l(x, s)\, ds \leqq m_0 \tag{2.19}$$

where

$$m_0 \overset{\text{def}}{=} \int_{-a}^{a} (r_0 + l_0 + u_0 + d_0)(\xi)\, d\xi. \tag{2.20}$$

The evolution equations $(1.18)_\delta$, together with the inequalities $\frac{u\,d - rl}{\delta + r + l + u + d} \leqq \frac{u + d}{2}$ and $\frac{rl}{\delta + r + l + u + d} \leqq \frac{r + l}{2}$, imply that r, l, u and d satisfy

$$r_t + r_x \leqq \frac{u + d}{2}, \tag{2.21}$$

$$l_t + l_x \leqq \frac{u + d}{2}, \tag{2.22}$$

$$u_t + \frac{u\,d}{\delta + r + l + u + d} \leqq \frac{r + l}{2}, \tag{2.23}$$

$$d_t + \frac{u\,d}{\delta + r + l + u + d} \leqq \frac{r + l}{2}. \tag{2.24}$$

The inequalities (2.21) and (2.22) imply that

$$r(x, t) \tag{2.25}$$

$$\leqq \begin{cases} r_0(x - t) + \frac{1}{2}\int_0^t (u + d)(x - t + s, s)\, ds & \text{for } 0 < t < (a + x) \text{ and } x > -a, \\ \frac{1}{2}\int_{t-(a+x)}^{t} (u + d)(x - t + s, s)\, ds & \text{for } (a + x) < t \text{ and } x > -a, \end{cases}$$

$$l(x, t) \tag{2.26}$$

$$\leqq \begin{cases} l_0(x + t) + \frac{1}{2}\int_0^t (u + d)(x + t - s, s)\, ds & \text{for } 0 < t < (a - x) \text{ and } x < a, \\ \frac{1}{2}\int_{t-(a-x)}^{t} (u + d)(x + t - s, s)\, ds & \text{for } (a - x) < t \text{ and } x < a. \end{cases}$$

Moreover, (2.25) and (2.26), together with (2.11)–(2.18), imply that

$$\sup_{-a<x<a} r(x,t) \leqq m_1, \quad \sup_{-a<x<a} l(x,t) \leqq m_1 \tag{2.27}$$

where

$$m_1 = m_0 + \sup_{-a<x<a} [\max (r_0(x), l_0(x), u_0(x), d_0(x))] \tag{2.28}$$

and m_0 is defined in (2.20). The inequalities (2.23) and (2.24) along with (2.19) also imply that

$$u(x,t) + \int_0^t \frac{u\,d}{\delta+\varrho}(x,s)\,ds \leqq m_1, \tag{2.29}$$

$$d(x,t) + \int_0^t \frac{u\,d}{\delta+\varrho}(x,s)\,ds \leqq m_1, \tag{2.30}$$

and these in turn yield the following L_∞-estimates for u, d, u_t and d_t:

$$u \leqq m_1, \quad d \leqq m_1, \quad |u_t| \leqq 2m_1, \ |d_t| \leqq 2m_1. \tag{2.31}$$

The fact that $u\,d = (\delta+\varrho)\,u\,d/(\delta+\varrho) \leqq (4m_1+\delta)\,u\,d/(\delta+\varrho)$, together with (2.29), guarantees that for each x, $(u\,d)\,(x,\cdot)$ is in $L_1(0,\infty)$ and this, combined with (2.31), implies that $(u\,d)_t\,(x,\cdot) \in L_\infty(0,\infty)$ and thus that

$$\lim_{t\to\infty} (u\,d)\,(x,t) = 0. \tag{2.32}$$

L_∞ estimates for the partial derivatives r_t and l_t are obtained by differentiating the first two equations of the system $(1.8)_\delta$ with respect to t and by exploiting the previously obtained L_∞ estimates for r, l, u and d and the boundary conditions $r_t(-a^+,t) = 0$ and $l_t(a^-,t) = 0$. These estimates, along with (2.19), then yield the limit relations

$$\lim_{t\to\infty} (r(x,t), l(x,t)) = (0,0) \quad \text{for } -a<x<a. \tag{2.33}$$

Noting that

$$(u-d)\,(x,t) \equiv (u_0-d_0)\,(x) \quad \text{for } -a<x<a, \tag{2.34}$$

and that

$$u\,d = \frac{(u+d)^2\,(x,t) - (u_0-d_0)^2\,(x)}{4} \tag{2.35}$$

converges to zero as $t\to\infty$, we see that

$$\lim_{t\to\infty} u(x,t) = \tfrac{1}{2}\,[|u_0-d_0|\,(x) + (u_0-d_0)\,(x)], \tag{2.36}$$

$$\lim_{t\to\infty} d(x,t) = \tfrac{1}{2}\,[|u_0-d_0|\,(x) - (u_0-d_0)\,(x)] \tag{2.37}$$

for $-a<x<a$.

3. The Reduced System (1.15) and (1.16)

We start with the observation that solving the system (1.15) and (1.16) is equivalent to solving

$$\varrho_t + (\varrho v)_x = 0,$$
$$(\varrho v)_t + (\varrho m)_x = 0, \qquad (3.1)$$
$$(\varrho m)_t + (\varrho v)_x = \varrho\left(\frac{1+v^2}{2} - m\right)$$

where

$$\varrho = r + 2c + l, \quad \varrho v = r - l, \quad \text{and} \quad \varrho m = r + l. \qquad (3.2)$$

Moreover, solutions of (3.1) satisfying

$$\varrho \geqq 0, \quad -m \leqq v \leqq m \quad \text{and} \quad 0 \leqq m \leqq 1 \qquad (3.3)$$

generate nonnegative solutions of (1.15) and (1.16). The densities r, l and c are related to ϱ, m and v by

$$r = \frac{\varrho(m+v)}{2}, \quad l = \frac{\varrho(m-v)}{2}, \quad c = \frac{\varrho(1-m)}{2}. \qquad (3.4)$$

The system (3.1) is hyperbolic and supports solutions with jump discontinuities. If $x = X(t)$ is the locus of such a discontinuity and $\alpha = \frac{dX}{dt}$, then across this curve

$$\alpha[\varrho] = [\varrho v], \quad \alpha[\varrho v] = [\varrho m], \quad \alpha[\varrho m] = [\varrho v] \qquad (3.5)$$

where for any function f, $[f](t) = \lim_{x \to X(t), x < X(t)} f(x,t) - \lim_{x \to X(t), x > X(t)} f(x,t)$. These are the Rankine-Hugoniot equations for the system (3.1). The identity (3.5) implies that $\alpha = -1, 0$ or $+1$. When $\alpha = -1$, we have $[\varrho m] = [\varrho] = -[\varrho v]$; when $\alpha = 0$, we have $[\varrho m] = [\varrho v] = 0$; and when $\alpha = 1$, we have $[\varrho m] = [\varrho v] = [\varrho]$.

The same procedure which led to (2.3)–(2.8) implies that if the initial data satisfy (3.3) and vanish outside of $(-a, a)$, then for $a < x < a + t$ they satisfy

$$v \equiv m \equiv 1, \quad \text{and} \quad \varrho_t + \varrho_x = 0, \qquad (3.6)$$

whereas for $-(a+t) < x < -a$ they satisfy

$$v \equiv -1, \quad m \equiv 1, \quad \text{and} \quad \varrho_t - \varrho_x = 0. \qquad (3.7)$$

Across the lines $x = \pm a$, the jump conditions (3.5) imply that

$$\varrho(-a^-, t) = -\varrho(-a^+, t)\, v(-a^+, t), \quad m(-a^+, t) + v(-a^+, t) = 0, \qquad (3.8)$$
$$\varrho(a^+, t) = \varrho(a^-, t)\, v(a^-, t), \quad m(a^-, t) - v(a^-, t) = 0, \qquad (3.9)$$

and that $\varrho \equiv 0$ for $x \notin (-a-t, a+t)$.

Associated with the system (3.1) is the reduced system obtained by solving

$$\begin{aligned} \varrho_t + (\varrho v)_x &= 0, \\ (\varrho v)_t + (\varrho m)_x &= 0 \end{aligned} \tag{3.10}$$

and replacing $(3.1)_3$ with

$$m = \frac{1+v^2}{2}. \tag{3.11}$$

This system with the quasistatic approximation (3.11) bears the same relation to the full system as the Euler equations of hydrodynamics with an equation of state based on a local Maxwellian do to the Boltzmann equation. Moreover, this reduced system is the same if the right-hand side of $(3.1)_3$ is replaced by $\varrho^2\left(\frac{1+v^2}{2} - m\right)$. The latter case is what obtains if we model the factor K in (1.2) as a constant. The identities (3.6)–(3.9) also obtain for the reduced system, but in this latter case (3.8) and (3.9) imply that

$$m(-a^+, t) = -v(-a^+, t) = 1, \tag{3.12}$$

$$m(a^-, t) = v(a^-, t) = 1. \tag{3.13}$$

We note that the reduced system has a particularly simple set of exponentially decaying solutions

$$\begin{aligned} \varrho(x, t) &= \varrho_0 e^{-t/a}, \\ v(x, t) &= \frac{x}{a}, \\ m(x, t) &= \frac{1+(x/a)^2}{2} \end{aligned} \tag{3.14}$$

for $-a < x < a$ and $t > 0$.

Our goal is to show that the full system supports solutions which are separable on $(-a, a)$ and are of the form:

$$\varrho = e^{-\lambda t} s(x), \quad v = v(x), \quad \text{and} \quad m = m(x), \quad -a < x < a. \tag{3.15}$$

For $a < x < a + t$ they satisfy

$$v \equiv m \equiv 1 \quad \text{and} \quad \varrho_t + \varrho_x = 0, \tag{3.16}$$

whereas for $-(a+t) < x < -a$ they satisfy

$$v \equiv -1, \quad m \equiv 1 \quad \text{and} \quad \varrho_t - \varrho_x = 0. \tag{3.17}$$

Across the lines $x = \pm a$, the jump conditions (3.5) imply that

$$\varrho(-a^-, t) = -e^{-\lambda t} s(-a^+)\, v(-a^+), \quad m(-a^+) + v(-a^+) = 0, \tag{3.18}$$

$$\varrho(a^+, t) = e^{-\lambda t} s(a^-)\, v(a^-) \quad \text{and} \quad m(a^-) - v(a^-) = 0, \tag{3.19}$$

and these, when combined with (3.16) and (3.17), imply that

$$\varrho(x,t)=\begin{cases} s(a^-)\,v(a^-)\,e^{-\lambda(t+a-x)} & \text{for } a<x<a+t,\\ -s(-a^+)\,v(-a^+)\,e^{-\lambda(t+a+x)} & \text{for } -(a+t)<x<-a.\end{cases} \tag{3.20}$$

For $x \notin (-(a+t), a+t)$, the density ϱ is identically zero.

We note that no such solutions exist if the interaction term $\varrho^2\left(\frac{1+v^2}{2}-m\right)$ is used in $(3.1)_3$, and recall that this is the term we obtain if we model K in (1.2) as a constant. This observation is an *a-posteriori* justification for our choice of K in (1.3).

From (3.1) we find that in $(-a, a)$ the functions s, v and m must satisfy

$$\frac{d}{dx}(sv)=\lambda s, \tag{3.21}$$

$$\frac{d}{dx}(sm)=\lambda sv, \tag{3.22}$$

$$\frac{d}{dx}(sv)=\lambda sm+s\left(\frac{1+v^2}{2}-m\right). \tag{3.23}$$

Moreover, (3.21) and (3.23) combine to yield

$$m=\frac{(1-2\lambda+v^2)}{2(1-\lambda)}. \tag{3.24}$$

The boundary condition (3.19), when combined with (3.24), yields

$$v^2(a^-)-2(1-\lambda)\,v(a^-)+(1-2\lambda)=(v(a^-)-1)\,(v(a^-)-(1-2\lambda))=0.$$

We choose the boundary condition $v(a^-)=(1-2\lambda)$. A similar argument also yields the boundary condition $v(-a^+)=-(1-2\lambda)$. Using the algebraic relation (3.24) to eliminate m, we find that (3.21) and (3.22) reduce to the following equations for v and s:

$$\frac{dv}{dx}=\frac{\lambda(1-2\lambda)\,(1-v^2)}{(1-v^2-2\lambda)}, \qquad -a<x<a, \tag{3.25}$$

$$\frac{ds}{dx}=\frac{-2\lambda^2 vs}{(1-v^2-2\lambda)}, \qquad -a<x<a. \tag{3.26}$$

Equation (3.25), along with the boundary conditions, may be integrated to obtain

$$v-\lambda\log\left[\frac{1+v}{1-v}\right]=\lambda(1-2\lambda)\,x, \qquad -a<x<a \tag{3.27}$$

where λ and a satisfy

$$1-2\lambda-\lambda\log\left(\frac{1-\lambda}{\lambda}\right)=\lambda(1-2\lambda)\,a. \tag{3.28}$$

The fact that the map $\lambda \to \tilde{a}(\lambda) \stackrel{\text{def}}{=} \frac{1}{\lambda} - \frac{1}{(1-2\lambda)} \log\left(\frac{1-\lambda}{\lambda}\right)$ is monotone decreasing on $(0, \frac{1}{2})$ and satisfies

$$\tilde{a}(0^+) = +\infty, \quad \tilde{a}(\tfrac{1}{2}^-) = 0 \tag{3.29}$$

guarantees that for each $a \in (0, \infty)$ there is a unique $\lambda \in (0, \frac{1}{2})$ such that (3.28) holds. It is this number we choose to obtain the separable solution.

The density s is expressible in terms of v by

$$s(x) = s_0(1 - v^2(x))^{\frac{\lambda}{1-2\lambda}} \tag{3.30}$$

where $s_0 > 0$ is arbitrary. There are no solutions to (3.25) consistent with the boundary conditions $v(-a^+) = -1$ and $v(a^-) = 1$. We also note that $r = e^{-\lambda t}\mathscr{R} = e^{-\lambda t}\frac{s(m+v)}{2}$, $l = e^{-\lambda t}\mathscr{L} = e^{-\lambda t}\frac{s(m-v)}{2}$ and $c = e^{-\lambda t}\mathscr{C} = e^{-\lambda t} \times \frac{s(1-m)}{2}$ are given by

$$\begin{aligned}
\mathscr{R} &= s_0 \frac{(1-v^2(x))^{\frac{\lambda}{1-2\lambda}}}{4(1-\lambda)} (v(x)+1)\,(v(x)+(1-2\lambda)),\\
\mathscr{L} &= s_0 \frac{(1-v^2(x))^{\frac{\lambda}{1-2\lambda}}}{4(1-\lambda)} (v(x)-1)\,(v(x)-(1-2\lambda)),\\
\mathscr{C} &= s_0 \frac{(1-v^2(x))^{\frac{1-\lambda}{1-2\lambda}}}{4(1-\lambda)}.
\end{aligned} \tag{3.31}$$

On the basis of numerical evidence we conjecture, but have not succeeded in showing, that the separable solutions are the limiting wave forms on $(-a, a)$ where the interactions take place.

We now prove the result on exponential decay alluded to in the introduction. Equation (1.15), along with $\frac{c^2 - rl}{r + 2c + l} \leqq \frac{c}{2}$ implies that r and l satisfy

$$r_t + r_x \leqq \frac{c}{2} \quad \text{and} \quad l_t - l_x \leqq \frac{c}{2}. \tag{3.32}$$

Since

$$\frac{rl - c^2}{r + 2c + l} = \frac{r+l}{4} - \frac{c}{2} - \frac{(r-l)^2}{4(r+2c+l)}, \tag{3.33}$$

we also have

$$c_t \leqq \frac{r+l}{4} - \frac{c}{2}. \tag{3.34}$$

Lemma. *Suppose $\mathscr{R}$, $\mathscr{L}$ and $\mathscr{C}$ satisfy*

$$\mathscr{R}_t + \mathscr{R}_x = \frac{\mathscr{C}}{2}, \quad \mathscr{L}_t - \mathscr{L}_x = \frac{\mathscr{C}}{2}, \quad \text{and} \quad \mathscr{C}_t = \frac{\mathscr{R} + \mathscr{L}}{4} - \frac{\mathscr{C}}{2} \tag{3.35}$$

in $-a < x < a$ *and* $t > 0$, *the boundary conditions* $\mathscr{R}(-a^+, t) = 0$ *and* $\mathscr{L}(a^-, t) = 0$, *and the same initial conditions as the solution to* (1.15) *and* (1.16). *Then,* $\mathscr{R} \geqq r$, $\mathscr{L} \geqq l$ *and* $\mathscr{C} \geqq c$ *in* $-a < x < a$ *and* $t > 0$.

Proof. The fact that r, l and c satisfy (3.32) and (3.34) along with the fact that $\mathscr{R}$, $\mathscr{L}$ and $\mathscr{C}$ satisfy (3.35) imply that the differences

$$D_1 = \mathscr{R} - r, \quad D_2 = \mathscr{L} - l, \quad D_3 = \mathscr{C} - c \tag{3.36}$$

satisfy

$$D_{1t} + D_{1x} \geqq \frac{D_3}{2}, \quad D_{2t} + D_{2x} \geqq \frac{D_3}{2}, \quad D_{3t} \geqq \frac{D_1 + D_2}{4} - \frac{D_3}{2}, \tag{3.37}$$

$$D_1(-a^+, t) = 0, \quad D_2(a^-, t) = 0, \tag{3.38}$$

$$D_1(x, 0) \geqq 0, \quad D_2(x, 0) \geqq 0, \quad \text{and} \quad D_3(x, 0) \geqq 0, \quad -a < x < a. \tag{3.39}$$

These inequalities are equivalent to

$$D_1(x, t) = p_1 + \begin{cases} D_1(x - t, 0) + \frac{1}{2} \int\limits_0^t D_3(x - t + \eta, \eta)\, d\eta & \text{for } 0 < t < a + x, \\ \frac{1}{2} \int\limits_{t-(a+x)}^{t} D_3(x - t + \eta, \eta)\, d\eta & \text{for } t > a + x, \end{cases} \tag{3.40}$$

$$D_2(x, t) = p_2 + \begin{cases} D_2(x + t, 0) + \frac{1}{2} \int\limits_0^t D_3(x + t - \eta, \eta)\, d\eta & \text{for } t < a - x, \\ \frac{1}{2} \int\limits_{t-(a-x)}^{t} D_3(x + t - \eta, \eta)\, d\eta & \text{for } t > a + x, \end{cases} \tag{3.41}$$

$$D_3(x, t) = p_3 + D_3(x, 0)\, e^{-\frac{t}{2}} + \frac{1}{4} \int\limits_0^t e^{-(t-\eta)/2} (D_1 + D_2)(x, \eta)\, d\eta \tag{3.42}$$

where p_1, p_2 and p_3 are nonnegative functions on $t > 0$. Since these equations are solvable by iteration and since at each stage the iterates are nonnegative we conclude that the limit functions are endowed with the same property and this concludes the proof. □

It is straightforward to verify that for any positive constant δ_0 and $0 < \lambda < \frac{1}{2}$ the following functions are separable, exponentially decaying solutions to (3.25) on $(-a, a)$:

$$\mathscr{R} = \delta_0 e^{-\lambda t} \left(\sqrt{\frac{2\lambda(1-2\lambda)}{1+2\lambda-4\lambda^2}} \cosh \varkappa x + \sinh \varkappa x \right), \tag{3.43}$$

$$\mathscr{L} = \delta_0 e^{-\lambda t} \left(\sqrt{\frac{2\lambda(1-2\lambda)}{1+2\lambda-4\lambda^2}} \cosh \varkappa x - \sinh \varkappa x \right), \tag{3.44}$$

$$\mathscr{C} = \frac{\delta_0 e^{-\lambda t}}{(1-2\lambda)} \sqrt{\frac{2\lambda(1-2\lambda)}{1+2\lambda-4\lambda^2}} \cosh \varkappa x. \tag{3.45}$$

Here

$$\varkappa = \sqrt{\frac{\lambda(1+2\lambda-4\lambda^2)}{2(1-2\lambda)}}, \tag{3.46}$$

and the boundary conditions $\mathscr{R}(-a^+, t) = \mathscr{L}(a^-, t) = 0$ imply that λ must satisfy

$$\sqrt{\frac{2\lambda(1-2\lambda)}{1+2\lambda-4\lambda^2}} = \tanh \left(\sqrt{\frac{\lambda(1+2\lambda-4\lambda^2)}{2(1-2\lambda)}}\, a \right). \tag{3.47}$$

It is easily checked that if $0 < a < 2$, then (3.47) has a unique solution $\lambda = \lambda(a)$ in $(0, \frac{1}{2})$ and $\mathscr{R}$, $\mathscr{L}$, and $\mathscr{C}$ are positive on $(-a, a)$, with this choice of λ.

Our comparison lemma now implies that if for some $\delta_0 > 0$, the data (r_0, l_0, c_0) for (1.15) satisfy

$$\begin{aligned} r_0(x) &\leqq \delta_0 \left(\sqrt{\frac{2\lambda(1-2\lambda)}{1+2\lambda-4\lambda^2}} \cosh \varkappa x + \sinh \varkappa x \right), \\ l_0(x) &\leqq \delta_0 \left(\sqrt{\frac{2\lambda(1-2\lambda)}{1+2\lambda-4\lambda^2}} \cosh \varkappa x - \sinh \varkappa x \right), \\ c_0(x) &\leqq \frac{\delta_0}{(1-2\lambda)} \sqrt{\frac{2\lambda(1-2\lambda)}{1+2\lambda-4\lambda^2}} \cosh \varkappa x \end{aligned} \tag{3.48}$$

with $\varkappa$ given by (3.46) and with λ the unique solution to (3.47), then the solution to (1.15) with this data decays at least as fast as $e^{-\lambda t}$ in $(-a, a)$.

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