# *Surface Interaction Potentials in Elasticity*

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*Dedicated to Bernard D. Coleman on his sixtieth birthday*

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## 0. Introduction

The problem of modelling body–environment interactions is a rather formidable one: neither is there a wealth of inspiring examples nor is the *corpus* of knowledge in mathematical analysis such as to allow indiscriminate generalization of the few well understood cases.

Primarily, the problem has a *constitutive* nature. In the traditional view, the object of a constitutive theory is the response of the body to deformation processes: modelling body–environment interactions demands *per se* that we assign the constitutive theory the extended task of formalizing our prejudices on the body and the environment under examination, both separately and together.

However, the modelling process is rather inextricably conditioned and guided by the type of initial-value and boundary-value problems that one wishes to formulate, as well as by the mathematical techniques that one wishes to employ, or has to. For example, it is shown in [1] how the choice of admissible interactions is contained by the needs of a local bifurcation analysis by use of formal perturbation methods; on the other hand, paper [2] exemplifies well how that choice may be guided by mathematically reasonable requirements of well-posedness.

In elasticity, the primarily constitutive nature of the modelling process is perhaps best brought about, as done in [3], by confining attention to equilibrium problems with conditions of traction on the entire boundary, in a variational format. The formulation of those equilibrium problems may be "formal", in the sense that a function space setting sufficient to clarify constitutive issues need not allow for, say, existence theorems. In this paper we adopt precisely such a "formal" variational format to arrive at well motivated choices of surface potentials describing the interactions taking place at the common boundary of a body and its environment.

In elastic equilibrium problems, body–environment interactions are accounted for by prescribing a system of loads which are usually of the *dead* type, *i.e.*, they do not depend on the displacement that the body undergoes. More realistic prescriptions of *live*, as opposed to dead, loads were first proposed for study of SEWELL [4], [5], under the heading of "configuration-dependent" loadings (*vid.* also [6], Chapters Q and R).

Although it can be argued with only slight exaggeration that most loadings encountered in applications are of the live type, our subject has not been extensively studied so far. In the context of elasticity linearized about a given equilibrium placement under stress, live loads were considered by CAPRIZ & PODIO-GUIDUGLI [1], PODIO-GUIDUGLI & VERGARA CAFFARELLI [7], PODIO-GUIDUGLI, VERGARA CAFFARELLI & VIRGA [8] and VERGARA CAFFARELLI [9]. In finite elasticity, some results of uniqueness and continuous dependence in the presence of live loads have been obtained by SPECTOR [10], [11]. Remarkably, SEWELL's paper [4], as well as the other papers just cited, have the "formal" character typical of the constitutively oriented approach to the matters alluded to in the preceding paragraph. At variance with this, the results of VALENT [12] concerning local existence, uniqueness and continuous dependence for the case of hydrostatic loading are obtained in a precise function space setting.

From a variational point of view, some body–environment interactions are described by surface potentials, others by volume potentials, others by surface or volume potentials, alternatively.

We here do not deal with genuine volume interaction potentials, for two reasons: first, they are sufficiently well understood from the mechanical point of view and, second, their presence usually does not bring in any mathematical novelty, and sometimes even renders the matters mathematically easier.

The *surface interaction potentials* we consider accomodate many types of live loads,[1] among which the two important examples of *pressure loading* and *membrane loading*; they have the form

$$T\{f\} = \int_{\partial\Omega} \hat{\tau}(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{f}, \nabla \boldsymbol{f})\, \mathrm{d}\,(\mathrm{Srf}), \tag{0.1}$$

with $\tau(\boldsymbol{x}) := \hat{\tau}(\boldsymbol{x}, \boldsymbol{n}(\boldsymbol{x}), \boldsymbol{f}(\boldsymbol{x}), \nabla \boldsymbol{f}(\boldsymbol{x}))$ interpreted as the energy stored per unit area, when a body undergoes a displacement $\boldsymbol{f}$ from a reference placement $\Omega$, at a point $\boldsymbol{x}$ of the boundary $\partial\Omega$ where $\boldsymbol{n}$ is the outward unit normal field.

Our main concern here is precisely the constitutive problem of selecting the density mapping $\hat{\tau}$ for $T$; our approach is the same as in [3]:

(i) We introduce the *total potential*

$$E\{f\} = S\{f\} + T\{f\}, \tag{0.2}$$

with

$$S\{f\} = \int_{\Omega} \hat{\sigma}(\boldsymbol{x}, \nabla \boldsymbol{f})\, \mathrm{d}\,(\mathrm{Vol}), \tag{0.3}$$

where the density $\sigma(\boldsymbol{x}) := \hat{\sigma}(\boldsymbol{x}, \nabla \boldsymbol{f}(\boldsymbol{x}))$ of the *body potential* $S$ is interpreted as the elastic energy stored per unit volume at a point of $\Omega$. For $\boldsymbol{S}$ the stress field over $\bar{\Omega}$ and $\boldsymbol{s}$ the surface load field over $\partial\Omega$, we say that the body has a *conservative interaction* with its environment (with respect to the reference placement $\Omega$) if the field pairs $(\boldsymbol{S}, \boldsymbol{s})$ and $(\sigma, \tau)$ satisfy the variational condition

$$\delta S\{f\}[\boldsymbol{h}] = \int_{\Omega} \boldsymbol{S} \cdot \nabla \boldsymbol{h}, \qquad \delta T\{f\} = -\int_{\partial\Omega} \boldsymbol{s} \cdot \boldsymbol{h} \tag{0.4}$$

for each $\boldsymbol{f} \in \mathscr{D}$, the collection of all admissible displacements, and for each $\boldsymbol{h} \in \mathscr{H}$, the space of all admissible variations.
(ii) We pose the variational problem

$$\mathrm{extr}\,\{E\{f\} \mid \boldsymbol{f} \in \mathscr{D}\}, \tag{0.5}$$

and derive necessary conditions to be obeyed by each extremum $\boldsymbol{f}^{(0)}$.
(iii) As a selection criterion for $\hat{\tau}$, we choose to regard $S$ as given and require that $\hat{\tau}$ be such as to obey the necessary conditions for an extremum identically on the domain $\mathscr{D}$ of $E$.

Our work is divided into two parts. Part I has three sections in which a number of preliminary results of algebra, analysis and mechanics having specific bearing

[1] Some interesting examples of live loadings, such as the pressure loading exerted on the inner wall of a deformable gas container [13], [14], escape our analysis here because of their *non-local* nature.

to the developments to come are collected; of these results, some are included so as to explain our notation carefully and make our paper reasonably self-contained; others are new or proved in a new way. Part II begins with a general formulation of elastic equilibria as extremum problems (Section 4); it continues with a variational study of the two important cases of pressure loadings and membrane loadings (Subsections 5.1 and 5.2, respectively)[2], with the purpose of exemplifying the kinds of behavior we shall later incorporate in our general model of interactions among a body, its surface, and its environment, and ends with two other sections, the bulk of this paper, of the contents of which we now give an extensive account.

Section 6 is devoted to obtaining and discussing the equilibrium conditions, *i.e.*, the necessary conditions on the extrema of the total potential $E$.

We point out that the regions $\Omega$ we consider have boundaries which consist either of one smooth surface $\Sigma$ with no self-intersections, or of two such surfaces $\Sigma_1$ and $\Sigma_2$, with $\Sigma_1 \cap \Sigma_2 = \emptyset$, and their common boundary $\Gamma$, a closed simple curve with tangent $\boldsymbol{t}$; in the second case, the outward normal field has a jump at $\Gamma$, and we further distinguish two subcases: (i) $\boldsymbol{n}_1(\boldsymbol{x}) \neq \boldsymbol{n}_2(\boldsymbol{x})$ for all $\boldsymbol{x} \in \Gamma$; (ii) (the cuspidal subcase) $\boldsymbol{n}_1(\boldsymbol{x}) \equiv -\boldsymbol{n}_2(\boldsymbol{x})$ for all $\boldsymbol{x} \in \Gamma$. The necessary conditions for extrema take different forms on the various parts of the body, namely, its interior part $\Omega$ and its regular and singular boundary parts $\Sigma_1 \cup \Sigma_2$ and $\Gamma$, respectively. Summarizing from Proposition 6.1, we have, as usual, that

$$\operatorname{Div} \hat{\boldsymbol{S}}^{(0)} = \boldsymbol{0}, \qquad \hat{\boldsymbol{S}}^{(0)} := \partial_{\boldsymbol{F}}\hat{\sigma}^{(0)},^{3} \tag{0.6}$$

in the interior part, while on the regular part of the boundary both

$$\hat{\boldsymbol{S}}^{(0)}\boldsymbol{n} = \boldsymbol{s}^{(0)}, \qquad \boldsymbol{s}^{(0)} := -\partial_{\boldsymbol{f}}\hat{\tau}^{(0)} + {}^{s}\operatorname{Div}\,(\partial_{\boldsymbol{F}}\hat{\tau}^{(0)}) \tag{0.7}$$

and

$$(\partial_{\boldsymbol{F}}\hat{\tau}^{(0)})\,\boldsymbol{n} = \boldsymbol{0} \tag{0.8}$$

prevail; moreover, on the singular part of the boundary, where $\hat{\tau}_\alpha^{(0)}$ and $\nabla_\alpha \boldsymbol{f}^{(0)}$ ($\alpha = 1, 2$) are defined as the limits of $\hat{\tau}^{(0)}|_{\Sigma_\alpha}$ and $\nabla \boldsymbol{f}^{(0)}|_{\Sigma_\alpha}$ for $\boldsymbol{x} \to \Gamma$ from $\Sigma_\alpha$, either

$$\partial_{\boldsymbol{F}}\hat{\tau}_1^{(0)}(\boldsymbol{x}, \boldsymbol{n}_1, \boldsymbol{f}^{(0)}, \nabla_1 \boldsymbol{f}^{(0)})\,\boldsymbol{n}_2 + \partial_{\boldsymbol{F}}\hat{\tau}_2^{(0)}(\boldsymbol{x}, \boldsymbol{n}_2, \boldsymbol{f}^{(0)}, \nabla_2 \boldsymbol{f}^{(0)})\,\boldsymbol{n}_1 = \boldsymbol{0} \tag{0.9}$$

or, in the cuspidal subcase,

$$[\partial_{\boldsymbol{F}}\hat{\tau}_1(\boldsymbol{x}, \boldsymbol{n}_1, \boldsymbol{f}^{(0)}, \nabla_1 \boldsymbol{f}^{(0)}) + \partial_{\boldsymbol{F}}\hat{\tau}_2(\boldsymbol{x}, -\boldsymbol{n}_1, \boldsymbol{f}^{(0)}, \nabla_2 \boldsymbol{f}^{(0)})]\,(\boldsymbol{t} \times \boldsymbol{n}_1) = \boldsymbol{0}. \tag{0.10}$$

We take condition $(0.7)_2$ to define the surface load mapping $\hat{\boldsymbol{s}}$ in terms of $\hat{\tau}$ for conservative interactions, just as condition $(0.6)_2$ defines the stress mapping $\hat{\boldsymbol{S}}$ in terms of $\hat{\sigma}$ for all $\boldsymbol{f} \in \mathcal{D}$. The three remaining conditions (0.8)–(0.10) involve $\hat{\tau}$ only through the "membrane stress" $\partial_{\boldsymbol{F}}\hat{\tau}$. In Section 7 we find for what classes of surface potential densities $\hat{\tau}$ one can expect the associated membrane stress

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[2] Our present analysis of pressure and membrane loadings has characters of greater generality and detail than the one presented in [3]; in particular, the pressure function may here depend on the displacement, as is the case, *e.g.*, for hydrostatic loading.

[3] We use an affixed $^{(0)}$ as a reminder for the operation of evaluation at $\boldsymbol{f}^{(0)}$.

to satisfy (0.8), alone or else together with (0.9) and (0.10), for all admissible displacements $f$ and all reference placements $\Omega$ occupied by the body.

We observe that (0.8) is identically satisfied in $\mathscr{D}$ whenever the membrane stress is tangential, and we show by Proposition 7.1 that this is the case if and only if $\hat{\tau}$ does not depend on normal derivatives of $f$:

$$\hat{\tau}(x, n(x), f(x), \nabla f(x)) = \hat{\tau}(x, n(x), f(x), {}^s\nabla f(x)), \quad {}^s\nabla f := (\nabla f)(I - n \otimes n), \tag{0.11}$$

for all $(x, f) \in \partial\Omega \times \mathscr{D}$ (here ${}^s\nabla f$ is by definition the tangential displacement gradient). We call those surface potentials whose densities satisfy (0.11) *tangential.* Given a surface potential $T$ as in (0.1), we introduce the associated tangential potential $T_{\text{sub}}$ with density

$$\hat{\tau}_{\text{sub}}(x, n, f, {}^s\nabla f) := \inf\{\hat{\tau}(x, n, f, {}^s\nabla f + y \otimes n) \mid y \in \mathscr{V}\}, \tag{0.12}$$

and show that under reasonable hypotheses on $\hat{\tau}$ (*e.g.*, when $\hat{\tau}$ is rank-one convex, as in Proposition 7.5) the extremals of the total potential $E = S + T$ are elements of the extremal set of the associated *subpotential* $E_{\text{sub}} = S + T_{\text{sub}}$. This result, together with the fact that both pressure loading and membrane loading admit a tangential potential, motivates us to restrict attention to tangential potentials for the rest of our paper.

We then turn to the question of characterizing, among tangential potentials, those which are *simple, i.e.*, correspond to surface load mappings $\hat{s}$ depending at most on tangential derivatives of the first order.[4] Restricting here attention to regular regions $\Omega$, and therefore dropping the explicit dependence on the normal field $n$, we first obtain a representation formula for the density of a simple tangential potential as a rank-one affine function of the tangential displacement gradient:

$$\hat{\tau}(x, f, {}^s\nabla f) = \hat{\gamma}(x, f) + \hat{C}_1(x, f) \cdot {}^s\nabla f^* + \hat{C}_2(x, f) \cdot {}^s\nabla f, \tag{0.13}$$

where ${}^s\nabla f^*$ denotes the cofactor of ${}^s\nabla f$, while $\hat{\gamma}$ and $\hat{C}_1$, $\hat{C}_2$ are arbitrary scalar and tensor-valued mappings, respectively (*cf.* Proposition 7.7). Secondly, we characterize the class of surface loadings alternatively described by surface or volume interaction potentials by showing that, if $T$ is a simple tangential potential, then there is a volume potential

$$F\{f\} = \int_\Omega \hat{\varphi}(x, f, F)\, d\,(\text{Vol}), \tag{0.14}$$

which is a *null Lagrangian* in the sense of [15], [16], such that

$$T\{f\} = F\{f\} \tag{0.15}$$

for all displacements $f \in \mathscr{D}$ (and conversely; *cf.* Proposition 7.9).

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[4] Simple surface loadings were introduced by SPECTOR [10], [11]; they are easier to handle than nonsimple ones, basically because they lead to a boundary operator $\mathfrak{B}(f) := \hat{S}(\nabla f)\, n - \hat{s}(f, {}^s\nabla f)$ which, given that the field operator $\mathfrak{F}(f) := -\text{Div}\, \hat{S}(\nabla f)$ is of order two, has the "right" order one (*cf.* [3], [7], [8], [9]).

Finally, we return to the extremum conditions (0.9), (0.10), to be obeyed on the singular boundary part $\Gamma$ of $\Omega$ by the pair $(\hat{\tau}_1, \hat{\tau}_2)$, with $\hat{\tau}_\alpha$ the (necessarily homogeneous) surface potential density on the boundary part $\Sigma_\alpha \cup \Gamma$. We interpret (0.9), (0.10) as *joint tangentiality* conditions on the membrane stresses $\partial_F \hat{\tau}_\alpha$ and we seek a class of surface interactions such as to satisfy (0.9), (0.10), with $\nabla_1 f^{(0)} \equiv \nabla_2 f^{(0)}$, identically in $\mathcal{O}$, the collection of all admissible domains.[5] By Proposition 7.13 we then show that, for two mappings $\hat{\tau}_1$, $\hat{\tau}_2$ to compose a jointly tangential surface density pair, they must have the following representation of the rank-one affine type:

$$\hat{\tau}_\alpha(\boldsymbol{n}, \boldsymbol{f}, {}^s\nabla \boldsymbol{f}) - \hat{\gamma}_\alpha(\boldsymbol{n}, \boldsymbol{f}) = \hat{\boldsymbol{c}}(\boldsymbol{f}) \otimes \boldsymbol{n} \cdot {}^s\nabla \boldsymbol{f}^* + \hat{\boldsymbol{C}}(\boldsymbol{f})\, \boldsymbol{N} \cdot {}^s\nabla \boldsymbol{f}, \qquad \alpha = 1, 2, \qquad (0.16)$$

where $\boldsymbol{N}$ is the skew tensor associated with $\boldsymbol{n}$; thus, not only do they induce a simple loading throughout the boundary, but also they are associated with an energetically equivalent null Lagrangian. In particular, in the important special case of *regular* interactions, *i.e.*, when $\hat{\tau}_1 = \hat{\tau}_2 = \hat{\tau}$, we find (Proposition 7.16) that the associated null Lagrangian $F$ has density

$$\hat{\varphi}(\boldsymbol{f}, \boldsymbol{F}) = \hat{\delta}(\boldsymbol{f}) \det \boldsymbol{F} + \hat{\boldsymbol{D}}(\boldsymbol{f}) \cdot \boldsymbol{F}^*, \qquad (0.17)$$

with $\hat{\delta}$ and $\hat{\boldsymbol{D}}$ depending, respectively, on $\hat{\boldsymbol{c}}$ and $\hat{\boldsymbol{C}}$ only,[6] and that (0.15) is paralleled by

$$T\{\boldsymbol{f}\} = F\{\boldsymbol{f}\} + \int_{\partial\Omega} \hat{\gamma}(\boldsymbol{n}, \boldsymbol{f})\, \mathrm{d}\,(\mathrm{Srf}). \qquad (0.18)$$

# Part I. Preliminaries

## 1. Algebra

In this section we collect, mostly without proof, various results to be of use later; of these results, many are well known, the rest are taken freely from [17], where proofs here omitted may be found.

Our notations are essentially those of [18], with minor variants and additions already introduced in [17] and [3]. As in [18], bold-face type denote vectors, if minuscule and second-order tensors if majuscule; greek letters are reserved for scalar quantities if minuscule, and for regions of space, if majuscule. Our use of parentheses, brackets and braces is completely standard, but we also use brackets to stress linear dependence on the enclosed variable, and braces to enclose the argument of a functional.

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[5] That is to say, for each admissible choice of a curve $\Gamma$ and two unit vector fields $\boldsymbol{n}_\alpha$ defined over $\Gamma$ itself, the latter interpreted as the limits of the outward normal field on the regular part of the boundary on going toward $\Gamma$ starting from $\Sigma_\alpha$.

[6] For pressure loading, a regular interaction, the densities are $\hat{\tau}_p(\boldsymbol{n}, \boldsymbol{f}, {}^s\nabla \boldsymbol{f}) = \hat{\pi}(\boldsymbol{f})\,(\boldsymbol{f} \otimes \boldsymbol{n}) \cdot {}^s\nabla \boldsymbol{f}^*$ for the surface and $\hat{\varphi}_p(\boldsymbol{f}, \nabla \boldsymbol{f}) = \hat{\pi}(\boldsymbol{f}) \det(\nabla \boldsymbol{f})$ for the volume potential (*vid.* Proposition 5.1).

### 1.1 Collections of Second-Order Tensors

In this paper, $\mathscr{V}$ is an oriented three-dimensional vector space over the real field $\mathbb{R}$, equipped with an inner product denoted by $\cdot$ and a vector product denoted by $\times$.

Lin is the set of all linear transformations of $\mathscr{V}$ into itself (the space of all second-order tensors). In particular, $\boldsymbol{I}$ is the identity transformation; for $\boldsymbol{A} \in \text{Lin}$, $\boldsymbol{A}^{\mathrm{T}}$ is the transpose of $\boldsymbol{A}$; for $\boldsymbol{a}, \boldsymbol{b} \in \mathscr{V}$, $\boldsymbol{a} \otimes \boldsymbol{b} \in \text{Lin}$, the tensor product of $\boldsymbol{a}$ and $\boldsymbol{b}$, is defined by

$$(\boldsymbol{a} \otimes \boldsymbol{b})\, \boldsymbol{v} := (\boldsymbol{b} \cdot \boldsymbol{v})\, \boldsymbol{a} \quad \text{for all } \boldsymbol{v} \in \mathscr{V}. \tag{1.1}$$

The symmetric and skew subcollections of elements of Lin are denoted by Sym and Skw, respectively, so that

$$\text{Lin} = \text{Sym} \oplus \text{Skw} \tag{1.2}$$

and, for all $\boldsymbol{A} \in \text{Lin}$,

$$\begin{gathered} \boldsymbol{A} = \operatorname{sym} \boldsymbol{A} + \operatorname{skw} \boldsymbol{A}, \\ 2 \operatorname{sym} \boldsymbol{A} := (\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}) \in \text{Sym}, \quad 2 \operatorname{skw} \boldsymbol{A} := (\boldsymbol{A} - \boldsymbol{A}^{\mathrm{T}}) \in \text{Skw}. \end{gathered} \tag{1.3}$$

Beside Sym and Skw, other subcollections of Lin to be considered here are

$$\text{Lin}^+ := \{\boldsymbol{F} \in \text{Lin} \mid \det \boldsymbol{F} > 0\}, \quad \text{Rot} := \{\boldsymbol{R} \in \text{Lin} \mid \boldsymbol{R}\boldsymbol{R}^{\mathrm{T}} = \boldsymbol{R}^{\mathrm{T}}\boldsymbol{R} = \boldsymbol{I}, \det \boldsymbol{R} = 1\}. \tag{1.4}$$

As is well known, Skw and $\mathscr{V}$ can be set in one-to-one correspondence:

$$\text{Skw} \ni \boldsymbol{W} \leftrightarrow \boldsymbol{w} \in \mathscr{V}, \quad \boldsymbol{W}\boldsymbol{v} = \boldsymbol{w} \times \boldsymbol{v} \text{ for all } \boldsymbol{v} \in \mathscr{V}; \tag{1.5}$$

it follows that

$$|\boldsymbol{W}|^2 = 2\,|\boldsymbol{w}|^2,$$

and, more generally, for $\boldsymbol{w}, \boldsymbol{z} \in \mathscr{V}$ and for $\boldsymbol{W}, \boldsymbol{Z}$ the corresponding elements of Skw,

$$\boldsymbol{W} \cdot \boldsymbol{Z} = 2\boldsymbol{w} \cdot \boldsymbol{z}, \tag{1.6}$$

where we have used the inner product of Lin defined in terms of the trace function by

$$\boldsymbol{A} \cdot \boldsymbol{B} := \operatorname{tr}(\boldsymbol{A}\boldsymbol{B}^{\mathrm{T}}) \tag{1.7}$$

and two bars $|\ |$ denote the norm associated with the inner product of either $\mathscr{V}$ or Lin, so that $\mathscr{S}(1)$, the unit sphere of $\mathscr{V}$, is defined by

$$\mathscr{S}(1) := \{\boldsymbol{v} \in \mathscr{V} \mid \boldsymbol{v} \cdot \boldsymbol{v} = |\boldsymbol{v}|^2 = 1\}. \tag{1.8}$$

For $\boldsymbol{w} \in \mathscr{S}(1)$ and $\boldsymbol{W}$ the element of Skw associated to $\boldsymbol{w}$ by (1.5), we have

$$-\boldsymbol{W}^2 = \boldsymbol{I} - \boldsymbol{P}(\boldsymbol{w}), \tag{1.9}$$

with

$$\boldsymbol{P}(\boldsymbol{w}) := \boldsymbol{w} \otimes \boldsymbol{w} \tag{1.10}$$

the orthogonal projector on the axis of $\boldsymbol{w}$. We record here for future use the following easy result.

**1.1. Proposition.** *Let* $A \in \mathrm{Lin}$ *and* $\boldsymbol{w} \in \mathcal{S}(1)$ *be such that* $A\boldsymbol{w} = \boldsymbol{0}$. *Then, for* $W$ *the skew tensor associated with* $\boldsymbol{w}$, *we have*

$$A = BW, \quad \text{with } B = -AW \tag{1.11}$$

(*and conversely*).

**Proof.** It suffices to note that the following chain of implications holds:

$$A\boldsymbol{w} = \boldsymbol{0} \quad \Rightarrow \quad A = A(I - \boldsymbol{w} \otimes \boldsymbol{w}) \quad \Rightarrow \quad A = -AW^2. \quad \square \tag{1.12}$$

*1.2 Cofactor and Determinant*

Let $\boldsymbol{k}$ be arbitrarily chosen in $\mathcal{V}$, and let $K$ be the element of Skw associated to $\boldsymbol{k}$ by (1.5). The *cofactor* $A^*$ of $A$ is the unique element of Lin such that $A^*\boldsymbol{k}$ is the vector associated to $AKA^{\mathrm{T}}$ by (1.5). It follows from this definition that, for $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ an orthonormal basis for $\mathcal{V}$,

$$A^*\boldsymbol{k} = A\boldsymbol{i} \times A\boldsymbol{j}. \tag{1.13}$$

A related definition of the *determinant* of $A$ is

$$\det A := A\boldsymbol{i} \times A\boldsymbol{j} \cdot A\boldsymbol{k}. \tag{1.14}$$

We now list some straightforward consequences of the first definition alone (*vid.* also [19] and Section 1.1 of [20]).

**1.2. Proposition.** *Let* $\alpha \in \mathbb{R}$; $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{V}$; $A \in \mathrm{Lin}$; $W \in \mathrm{Skw}$ *be arbitrarily chosen, Then,*

$$(\alpha A)^* = \alpha^2 A^* \ (\textit{in particular, } (-A)^* = A^*); \tag{1.15}$$

$$I^* = I; \quad (\boldsymbol{a} \otimes \boldsymbol{b})^* = \boldsymbol{0}; \quad (I - \boldsymbol{a} \otimes \boldsymbol{a})^* = \boldsymbol{a} \otimes \boldsymbol{a}; \tag{1.16}$$

$$W^* = \boldsymbol{w} \otimes \boldsymbol{w}. \tag{1.17}$$

Less elementary properties of the cofactor mapping follow by using also the definition of determinant.

**1.3. Proposition.** *Let* $A$, $B$ *be arbitrarily chosen in* Lin. *Then*

$$A^*A^{\mathrm{T}} = A^{\mathrm{T}}A^* = (\det A)\, I; \tag{1.18}$$

$$\det(A + B) = \det A + A^* \cdot B + A \cdot B^* + \det B. \tag{1.19}$$

Note that it follows from (1.17) and (1.18) that

$$\det W = 0 \quad \text{for all } W \in \mathrm{Skw}. \tag{1.20}$$

It also follows from (1.18) that

$$A^* \cdot A = 3 \det A \qquad \text{for all } A \in \text{Lin}, \tag{1.21}$$

and, moreover, that

$$F^* = (\det F)\, F^{-T} \qquad \text{for all } F \in \text{Lin}^+, \tag{1.22}$$

so that, in particular,

$$R^* = R \qquad \text{for all } R \in \text{Rot}. \tag{1.23}$$

Other properties of the cofactor result from (1.19) and some well known properties of the determinant, namely,

$$\det A = \det A^T \qquad \text{for all } A \in \text{Lin} \tag{1.24}$$

and

$$\det(AB) = (\det A)(\det B) \qquad \text{for all } A, B \in \text{Lin}.^7 \tag{1.25}$$

We list these properties in the next proposition.

**1.4. Proposition.** *Let* $A$, $B$ *be arbitrarily chosen in* Lin. *Then,*

$$(AB)^* = A^*B^*; \tag{1.26}$$

$$(A^T)^* = (A^*)^T. \tag{1.27}$$

With (1.26), $(1.16)_3$ yields a result that will be of use later, when we deal with pressure interactions: for all $A \in \text{Lin}$ and $v \in \mathscr{V}$,

$$(A(I - v \otimes v))^* = A^*v \otimes v. \tag{1.28}$$

We now prove two formulae for the derivatives of the cofactor and the determinant mappings.

**1.5. Proposition.** *For all* $A, B \in \text{Lin}$,

$$\partial_A \det A = A^*; \tag{1.29}$$

*in addition, if* $A \in \text{Lin}^+$,

$$(\det A)\, \partial_A A^*[B] = (A^* \cdot B)\, A^* - A^*B^TA^*.^8 \tag{1.30}$$

**Proof.** A straightforward consequence of definition (1.14) is

$$\det(\alpha A) = \alpha^3 \det A \qquad \text{for all } \alpha \in \mathbb{R} \text{ and } A \in \text{Lin}. \tag{1.31}$$

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[7] These properties of the determinant are best proved by first restricting attention to invertible tensors and then appealing to a continuity argument to establish the desired result for all of Lin.

[8] Notice that, for all $A \in \text{Lin}$, $(\det A)\, \partial_A A^*$ is a *symmetric* mapping from Lin into itself, in the sense that, for all $B, C \in \text{Lin}$,

$$C \cdot (\det A)\, \partial_A A^*[B] = B \cdot (\det A)\, \partial_A A^*[C]. \tag{\#}$$

Of course, the symmetry of $(\det A)\, \partial_A A^*$ is a consequence of the fact that, due to (1.29), $\partial_A A^*$ is the second derivative of the determinant mapping.

With this and (1.15) we obtain from (1.19) that

$$\det(A + \alpha B) = \det A + \alpha A^* \cdot B + \alpha^2 A \cdot B^* + \alpha^3 \det B; \tag{1.32}$$

differentiation of the function

$$\alpha \mapsto \det(A + \alpha B)$$

yields (1.29). Next, we differentiate $(1.18)_1$ and, using (1.29), get

$$(\partial_A A^*[B])\, A^T + A^* B^T = (A^* \cdot B)\, I;\ ^9 \tag{1.33}$$

multiplying (1.33) on the right by $A^*$ and recalling $(1.18)_2$ we have (1.30). □

We close this section by establishing a formula for the cofactor of the sum of two tensors.

**1.6. Proposition.** *For all* $A, B \in \mathrm{Lin}$,

$$(A + B)^* = A^* + B^* + \partial_A A^*[B], \quad \textit{with} \quad \partial_A A^*[B] = \partial_B B^*[A]. \tag{1.34}$$

**Proof.** Having (1.29), for all $A, B \in \mathrm{Lin}$ we obtain

$$\partial_A(\det(A + B)) = \partial_B(\det(A + B)) = (A + B)^*,$$

so that, on recalling (1.19), we arrive at (1.34). □

## 2. Analysis

Let $\mathscr{E}$ be a three-dimensional euclidean point space, whose associated translation space we shall identify with the vector space $\mathscr{V}$ introduced in Section 1.1.[10]

Throughout this paper, $\Omega$ is a domain, *i.e.*, an open bounded set of $\mathscr{E}$. We shall interpret $\Omega$ as the region occupied by a continuous body, and study surface interactions between that body and its environment. Therefore, it is important for us to describe precisely what type of boundary $\partial\Omega$ we shall assign to $\Omega$. Although many of our developments do hold (oftentimes in an obvious fashion) under more general hypotheses, for simplicity and concreteness we shall assume hereafter that one of the following two situations occurs:

(i) $\partial\Omega$ consists of a single surface of class $C^3$, with no self-intersections;

(ii) $\partial\Omega = \Sigma_1 \cup \Sigma_2 \cup \Gamma$, where both $\Sigma_1$ and $\Sigma_2$ are $C^3$-surfaces with no self-intersections, such that $\Sigma_1 \cap \Sigma_2 = \emptyset$ and that their common boundary $\Gamma$ is a closed simple curve of class $C^3$; on $\Gamma$, the vector fields

$$\Gamma \ni x \mapsto n_\alpha(x) \in \mathscr{V}, \quad \text{with } n_\alpha(x) \in \mathscr{S}(1) \quad \text{for } \alpha = 1, 2, \tag{2.1}$$

[9] Notice that we write $\partial_A A^*[B]$ for the derivative of the cofactor mapping evaluated at $A$, a linear transformation of Lin into itself, acting on $B$.

[10] On occasion, having chosen an origin $o \in \mathscr{E}$ once and for all, we shall also tacitly identify a point $x \in \mathscr{E}$ with the corresponding position vector $(x - o) \in \mathscr{V}$.

are well-defined as the limits of the outward unit normal to $\Sigma_\alpha$ on approaching $\boldsymbol{x} \in \Gamma$ from $\Sigma_\alpha$, and it is assumed that

$$\boldsymbol{n}_1(\boldsymbol{x}) \neq \boldsymbol{n}_2(\boldsymbol{x}) \quad \text{for all } \boldsymbol{x} \in \Gamma. \tag{2.2}$$

In case (ii) we shall call $\Sigma_1 \cup \Sigma_2$ and $\Gamma$ the *regular* and *singular* parts of the boundary of $\Omega$, respectively; in case (i) we shall consistenly say that $\Omega$ has a regular boundary, or else that the domain $\Omega$ is regular. If at a point $\boldsymbol{x} \in \Gamma$

$$\boldsymbol{n}_1(\boldsymbol{x}) \neq -\boldsymbol{n}_2(\boldsymbol{x}), \tag{2.3}$$

then the unit tangent vector $\boldsymbol{t}(\mathrm{x})$ at $\boldsymbol{x}$ can be chosen so as to coincide with

$$\boldsymbol{t}(\boldsymbol{x}) = \frac{\boldsymbol{n}_1(\boldsymbol{x}) \times \boldsymbol{n}_2(\boldsymbol{x})}{|\boldsymbol{n}_1(\boldsymbol{x}) \times \boldsymbol{n}_2(\boldsymbol{x})|}. \tag{2.4}$$

When condition (2.3) applies throughout $\Gamma$ (so that there are no cuspidal situations anywhere), we let the tangent field on $\Gamma$ be defined by (2.4) and choose the corresponding positive path orientation.

**2.1. Remark.** In the following, when we write $\partial\Omega$ it will be always clear from the context whether we refer exclusively to a regular region or rather to a region with either one of the two types of boundary above; when we write $\Sigma_1 \cup \Sigma_2$, we do so to stress exclusion of the singular part $\Gamma$ of $\partial\Omega$, but the statement just made will also apply as a rule, perhaps *modulo* some trivial changes, to cases when $\Omega$ is regular. □

## *2.1 Differential Operators on a Surface*

When the situation *sub* (i) above occurs, it is well known that any scalar or vector field of class $C^2$ over $\Omega$ can be extended to an equally smooth field defined over an open set containing $\bar{\Omega}$. Likewise, in the situation *sub* (ii), for $\alpha = 1$ or 2 one can regard $\Sigma_\alpha$ as a part of the complete boundary of a regular domain $\Omega_\alpha$, and extend any field defined over $\Sigma_\alpha$ to a corresponding field over a neighborhood of $\Sigma_\alpha$. Consequently, the definitions of intrinsic differential operators on the regular part of $\partial\Omega$, such as the surface gradient and the surface divergence, can be obtained by restricting to $\Sigma_\alpha$ the corresponding differential constructs relative to appropriately extended fields defined over an open set containing $\Sigma_\alpha$.

For all points of $\Omega$ where the outward unit normal $\boldsymbol{n}$ is uniquely defined, let $\boldsymbol{N}$, $\boldsymbol{P}(\boldsymbol{n})$ be the skew tensor and the orthogonal projector associated to $\boldsymbol{n}$ by (1.5) and (1.10), respectively (note that

$$\boldsymbol{N}\boldsymbol{P}(\boldsymbol{n}) = \boldsymbol{P}(\boldsymbol{n})\,\boldsymbol{N} = \boldsymbol{0}). \tag{2.5}$$

The *surface gradient* ${}^{s}\nabla \boldsymbol{v}$ of a vector field $\boldsymbol{v}$ over $\partial\Omega$ is the tensor field defined over $\partial\Omega$ as

$${}^{s}\nabla \boldsymbol{v} := (\nabla \boldsymbol{v})\,(\boldsymbol{I} - \boldsymbol{P}(\boldsymbol{n})) = -(\nabla \boldsymbol{v})\,\boldsymbol{N}^2; \tag{2.6}$$

the *surface divergence* is

$$^{s}\mathrm{Div}\,\boldsymbol{v} := \mathrm{tr}\,(^{s}\nabla\boldsymbol{v}) = (\nabla\boldsymbol{v})\cdot(\boldsymbol{I} - \boldsymbol{P}(\boldsymbol{n})) = -(\nabla\boldsymbol{v})\cdot\boldsymbol{N}^2. \tag{2.7}$$

Moreover, for consistency with definition (2.6), the *normal gradient* of $\boldsymbol{v}$ is the tensor field $^{n}\nabla\boldsymbol{v}$ over $\partial\Omega$ defined by

$$^{n}\nabla\boldsymbol{v} := \nabla\boldsymbol{v} - {}^{s}\nabla\boldsymbol{v} = (\nabla\boldsymbol{v})\,\boldsymbol{P}(\boldsymbol{n}) = (^{n}\partial\boldsymbol{v})\otimes\boldsymbol{n}, \tag{2.8}$$

where

$$^{n}\partial\boldsymbol{v} := (\nabla\boldsymbol{v})\,\boldsymbol{n} \tag{2.9}$$

is the directional derivative of $\boldsymbol{v}$ with respect to the normal $\boldsymbol{n}$.

**2.2. Remark.** In anticipation we remark here that, when we consider the gradient and divergence of fields defined over the image $\boldsymbol{f}(\Omega)$ of $\Omega$ (or $\boldsymbol{f}(\partial\Omega)$ of $\partial\Omega$) under a deformation $\boldsymbol{f}$ of $\overline{\Omega}$, we shall denote the gradient by grad (or $^{s}$grad) and the divergence by div (or $^{s}$div). □

### *2.2 Some Properties of Curl*

The value at $\boldsymbol{x}\in\Omega$ of the *curl* of a vector field $\boldsymbol{v}$ over $\Omega$ is defined to be the vector associated to the skew tensor field $(\nabla\boldsymbol{v} - (\nabla\boldsymbol{v})^{\mathrm{T}})$ evaluated at $\boldsymbol{x}$:

$$(\mathrm{Curl}\,\boldsymbol{v}(\boldsymbol{x}))\times\boldsymbol{u} := [\nabla\boldsymbol{v}(\boldsymbol{x}) - (\nabla\boldsymbol{v}(\boldsymbol{x}))^{\mathrm{T}}]\,\boldsymbol{u} \quad \text{for all } \boldsymbol{u}\in\mathscr{V}. \tag{2.10}$$

**2.3. Proposition.** *Let $\boldsymbol{v}$ be a smooth vector field over $\Omega$. Then,*

(i) *if $\boldsymbol{v}$ denotes the skew tensor field associated with $\boldsymbol{v}$,*

$$\mathrm{Curl}\,\boldsymbol{v} = -\mathrm{Div}\,\boldsymbol{V}; \tag{2.11}$$

(ii) *if $\boldsymbol{v}(\boldsymbol{x})\in\mathscr{S}(1)$ for $\boldsymbol{x}\in\Omega$,*

$$\boldsymbol{v}\times\mathrm{Curl}\,\boldsymbol{v} = -(\nabla\boldsymbol{v})\,\boldsymbol{v}. \tag{2.12}$$

We omit the proof of this proposition, which amounts to straightforward computations. The proof of the next proposition is less trivial.

**2.4. Proposition.** *Let $\boldsymbol{w}$, $\boldsymbol{z}$ be two smooth vector fields over $\Omega$, with $|\boldsymbol{w}|\equiv 1$, and let $\boldsymbol{W}$ be the skew tensor field associated with $\boldsymbol{w}$. Then*

$$\boldsymbol{W}\cdot\nabla(\boldsymbol{W}\boldsymbol{z}) = -\boldsymbol{W}^2\cdot\nabla\boldsymbol{z} - (\mathrm{Div}\,\boldsymbol{w})\,(\boldsymbol{w}\cdot\boldsymbol{z}). \tag{2.13}$$

**Proof.** Firstly, apply the identity

$$\boldsymbol{A}\cdot\nabla(\boldsymbol{A}\boldsymbol{a}) = (\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})\cdot\nabla\boldsymbol{a} + (\mathrm{Div}\,(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}) - \boldsymbol{A}^{\mathrm{T}}\,\mathrm{Div}\,\boldsymbol{A})\cdot\boldsymbol{a}, \tag{2.14}$$

which holds for all sufficiently smooth tensor fields $\boldsymbol{A}$ and vector fields $\boldsymbol{a}$, to the skew tensor field $\boldsymbol{W}$ and the vector field $\boldsymbol{z}$, and obtain

$$\boldsymbol{W}\cdot\nabla(\boldsymbol{W}\boldsymbol{z}) = -\boldsymbol{W}^2\cdot\nabla\boldsymbol{z} - (\mathrm{Div}\,\boldsymbol{W}^2 - \boldsymbol{W}\,\mathrm{Div}\,\boldsymbol{W})\cdot\boldsymbol{z}. \tag{2.15}$$

Secondly, observe that (1.9), (1.10), (2.11) and (2.12) imply that

$$\begin{aligned}\operatorname{Div} \boldsymbol{W}^2 - \boldsymbol{W} \operatorname{Div} \boldsymbol{W} &= \operatorname{Div}(\boldsymbol{w} \otimes \boldsymbol{w}) + \boldsymbol{W} \operatorname{Curl} \boldsymbol{w} \\ &= (\operatorname{Div} \boldsymbol{w})\, \boldsymbol{w} + (\nabla \boldsymbol{w})\, \boldsymbol{w} + \boldsymbol{w} \times \operatorname{Curl} \boldsymbol{w} = (\operatorname{Div} \boldsymbol{w})\, \boldsymbol{w}. \quad (2.16)\end{aligned}$$

(2.15) and (2.16) together yield the desired result. □

### *2.3 The Surface Divergence Theorem*

Let $\Sigma$ be a smooth oriented simple surface, with boundary $\partial\Sigma$ a smooth, consistently oriented, closed simple curve: in our present context, typically, $\Sigma \equiv \Sigma_\alpha$ and $\partial\Sigma \equiv \Gamma$. As is well known, for $\boldsymbol{v}$ a vector field over a neighborhood of $\partial\Sigma$, *Stokes' formula* holds, namely,

$$\int_\Sigma (\operatorname{Curl} \boldsymbol{v}) \cdot \boldsymbol{n} = \int_{\partial\Sigma} \boldsymbol{v} \cdot \boldsymbol{t}, \tag{2.17}$$

where of course $\boldsymbol{n}$ is the normal to $\Sigma$, and $\boldsymbol{t}$ is the tangent to $\partial\Sigma$.

**2.5. Remark.** Only tangential derivatives of $\boldsymbol{v}$ evaluated on $\Sigma$ occur in the left-hand side of Stokes' formula. Indeed, on the one hand by (1.5) and the definition (2.10),

$$2\,(\operatorname{Curl} \boldsymbol{v}) \cdot \boldsymbol{n} = (\nabla \boldsymbol{v} - (\nabla \boldsymbol{v})^T) \cdot \boldsymbol{N} = 2 \nabla \boldsymbol{v} \cdot \boldsymbol{N}; \tag{2.18}$$

on the other hand, (2.5) and the definition (2.6) yield

$$\nabla \boldsymbol{v} \cdot \boldsymbol{N} = (\nabla \boldsymbol{v}(\boldsymbol{I} - \boldsymbol{P}(\boldsymbol{n})) \cdot \boldsymbol{N} = {}^s\nabla \boldsymbol{v} \cdot \boldsymbol{N}; \tag{2.19}$$

in conclusion,

$$(\operatorname{Curl} \boldsymbol{v}) \cdot \boldsymbol{n} = {}^s\nabla \boldsymbol{v} \cdot \boldsymbol{N}. \quad \square \tag{2.20}$$

Our next proposition, the *surface divergence theorem*, has manifold applications in the following, *e.g.*, when we deal with membrane loading.

**2.6. Proposition.** *Let $\Sigma$ be a smooth oriented simple surface of normal $\boldsymbol{n}$, having a smooth oriented closed simple curve of tangent $\boldsymbol{t}$ as its boundary $\partial\Sigma$. Furthermore, let $\boldsymbol{u}$ be a vector field of class* $C^1$, *defined and* tangential *over $\Sigma$*:

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on } \Sigma. \tag{2.21}$$

*Then*

$$\int_\Sigma {}^s\operatorname{Div} \boldsymbol{u} = \int_{\partial\Sigma} \boldsymbol{u} \cdot \boldsymbol{t} \times \boldsymbol{n}. \tag{2.22}$$

**Proof.** Let $\boldsymbol{u}$ and $\boldsymbol{n}$ be interpreted as smooth extensions to a neighborhood of $\Sigma$ of the corresponding vector fields $\boldsymbol{u}$ and $\boldsymbol{n}$ defined over $\Sigma$. With a view towards applying Stokes' formula to the vector field

$$\boldsymbol{v} = \boldsymbol{N}\boldsymbol{u}, \tag{2.23}$$

we observe that (2.20), (2.5), (2.13), (2.21) and (2.7) imply that

$$\begin{aligned}\operatorname{Curl}(\boldsymbol{N}\boldsymbol{u})\cdot\boldsymbol{n} &= {}^{s}\nabla(\boldsymbol{N}\boldsymbol{u})\cdot\boldsymbol{N} = \nabla(\boldsymbol{N}\boldsymbol{u})\cdot\boldsymbol{N}\\ &= -\boldsymbol{N}^2\cdot\nabla\boldsymbol{u} - (\operatorname{Div}\boldsymbol{n})(\boldsymbol{n}\cdot\boldsymbol{u}) = {}^{s}\operatorname{Div}\boldsymbol{u}.\end{aligned} \tag{2.24}$$

But,

$$\boldsymbol{v}\cdot\boldsymbol{t} = \boldsymbol{N}\boldsymbol{u}\cdot\boldsymbol{t} = \boldsymbol{u}\cdot\boldsymbol{t}\times\boldsymbol{n}, \tag{2.25}$$

so that Stokes' formula directly yields the desired result. □

Let $\Omega$ be a domain as described under (i) in the beginning of this section. Firstly, for $\alpha$ and $\boldsymbol{v}$ two sufficiently smooth scalar and vector fields over $\partial\Omega$, we have from (2.7)

$${}^{s}\operatorname{Div}(\alpha\boldsymbol{v}) = \alpha\,{}^{s}\operatorname{Div}\boldsymbol{v} + ((\boldsymbol{I} - \boldsymbol{P}(\boldsymbol{n}))\,\boldsymbol{v})\otimes\nabla\alpha.$$

Then, choosing $\boldsymbol{v}\equiv\boldsymbol{n}$, we see that

$${}^{s}\operatorname{Div}(\alpha\boldsymbol{n}) = \alpha\,{}^{s}\operatorname{Div}\boldsymbol{n}. \tag{2.26}$$

Writing now

$$\boldsymbol{v} = \boldsymbol{P}(\boldsymbol{n})\,\boldsymbol{v} + (\boldsymbol{I} - \boldsymbol{P}(\boldsymbol{n}))\,\boldsymbol{v},$$

we deduce from (2.22) and (2.26) that

$$\int_{\partial\Omega} {}^{s}\operatorname{Div}\boldsymbol{v} = \int_{\partial\Omega} ({}^{s}\operatorname{Div}\boldsymbol{n})\,\boldsymbol{n}\cdot\boldsymbol{v}. \tag{2.27}$$

Secondly, a tensor field $\boldsymbol{A}$ over $\partial\Omega$ is *tangential* if

$$\boldsymbol{A}\boldsymbol{n} = \boldsymbol{0} \quad \text{on } \partial\Omega; \tag{2.28}$$

using (2.7), we define the *surface divergence* of such a field as

$${}^{s}\operatorname{Div}\boldsymbol{A}\cdot\boldsymbol{v} := {}^{s}\operatorname{Div}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{v}) - \boldsymbol{A}\cdot{}^{s}\nabla\boldsymbol{v}, \tag{2.29}$$

for each sufficiently smooth vector field $\boldsymbol{v}$ over $\partial\Omega$. It follows from (2.28) that the vector field $\boldsymbol{u} := \boldsymbol{A}^{\mathrm{T}}\boldsymbol{v}$ is tangential on $\partial\Omega$ for each $\boldsymbol{v}\in\mathscr{V}$. Thus, for each sufficiently smooth vector field $\boldsymbol{v}$ over $\partial\Omega$, the surface divergence theorem implies that

$$\int_{\partial\Omega} {}^{s}\operatorname{Div}\boldsymbol{A}\cdot\boldsymbol{v} = -\int_{\partial\Omega} \boldsymbol{A}\cdot{}^{s}\nabla\boldsymbol{v}. \tag{2.30}$$

Finally, let $\partial\Omega$ consist of two regular pieces $\Sigma_1, \Sigma_2$ with a common boundary $\Gamma$. Then (2.30) must be replaced by

$$\int_{\Sigma_1\cup\Sigma_2} {}^{s}\operatorname{Div}\boldsymbol{A}\cdot\boldsymbol{v} = -\int_{\Sigma_1\cup\Sigma_2} \boldsymbol{A}\cdot{}^{s}\nabla\boldsymbol{v} + \int_{\Gamma} [\boldsymbol{A}_1(\boldsymbol{t}\times\boldsymbol{n}_1) - \boldsymbol{A}_2(\boldsymbol{t}\times\boldsymbol{n}_2)]\cdot\boldsymbol{v}, \tag{2.31}$$

where $\boldsymbol{A}_\alpha$ ($\alpha = 1, 2$) denotes the limit value of the tensor field $\boldsymbol{A}$ on approaching $\Gamma$ from $\Sigma_\alpha$.

*2.4 The Weingarten Tensor*

For $\Sigma$ a smooth oriented simple surface of normal $\boldsymbol{n}$, the *Weingarten tensor* field $\boldsymbol{L}$ over $\Sigma$ is defined to be the negative of the surface gradient of the normal field:

$$\boldsymbol{L} := -{}^{s}\nabla\boldsymbol{n}; \tag{2.32}$$

the trace of $\boldsymbol{L}$ is proportional to the *mean curvature* $\varkappa$ of $\Sigma$:

$$\varkappa := \tfrac{1}{2}\,{}^{s}\mathrm{Div}\,\boldsymbol{n}. \tag{2.33}$$

We now prove that the Weingarten tensor field has symmetric values, a classical result in the differential geometry of surfaces obtained here, in a fashion different from usual, as an easy consequence of Stokes' formula and the surface divergence theorem.

**2.7. Proposition.** *Let $\Sigma$ be a smooth oriented simple surface of normal $\boldsymbol{n}$, and let $\Sigma$ be regarded as part of the complete boundary $\partial\Omega$ of a regular domain $\Omega$. Then*

$${}^{s}\nabla\boldsymbol{n}(\boldsymbol{x}) \in \mathrm{Sym} \quad \textit{for all } \boldsymbol{x} \in \Sigma. \tag{2.34}$$

**Proof.** In place of (2.34), we shall prove the equivalent statement that, for every skew tensor field $\boldsymbol{W}$ of class $C^1$ over $\partial\Omega$

$$\int_{\partial\Omega} {}^{s}\nabla\boldsymbol{n} \cdot \boldsymbol{W} = 0. \tag{2.35}$$

To this end, we consider two smooth vector fields $\boldsymbol{w}$ and $\boldsymbol{z}$ over $\Sigma$ and the associated skew tensor fields $\boldsymbol{W}$ and $\boldsymbol{Z}$, and observe that the differential identity

$$\nabla(\boldsymbol{W}\boldsymbol{z}) = \boldsymbol{W}\,\nabla\boldsymbol{z} - \boldsymbol{Z}\,\nabla\boldsymbol{w}$$

yields

$${}^{s}\mathrm{Div}\,(\boldsymbol{W}\boldsymbol{z}) = \boldsymbol{W} \cdot {}^{s}\nabla\boldsymbol{z} - \boldsymbol{Z} \cdot {}^{s}\nabla\boldsymbol{w}. \tag{2.36}$$

Choosing $\boldsymbol{z} = \boldsymbol{n}$ in (2.36) and integrating over $\partial\Omega$ we obtain

$$\int_{\partial\Omega} {}^{s}\nabla\boldsymbol{n} \cdot \boldsymbol{W} = \int_{\partial\Omega} {}^{s}\nabla\boldsymbol{w} \cdot \boldsymbol{N} + \int_{\partial\Omega} {}^{s}\mathrm{Div}\,(\boldsymbol{W}\boldsymbol{n}). \tag{2.37}$$

On the right-hand side of (2.37), the first integral is null by Stokes' formula and (2.20); the second one, as the vector field $\boldsymbol{W}\boldsymbol{n}$ is tangential, by the surface divergence theorem. □

## 3. Mechanics

Let a regular domain $\Omega \subset \mathscr{E}$ as described in the opening of Section 2 be pointwise identified with a continuous body. Classically, a deformation of the body is the assignment of a *displacement*

$$\boldsymbol{f}: \Omega \to \mathscr{E}, \quad \boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x}), \tag{3.1}$$

from the *reference placement* $\Omega$ into the *current placement* $f(\Omega)$, with $f$ smooth (say, of class $C^2(\Omega) \cap C^1(\bar{\Omega})$), injective and locally orientation-preserving in the sense that

$$\det \nabla f(x) > 0 \quad \text{a.e. in } \Omega. \tag{3.2}$$

**3.1. Remark.** This notion of displacement fits our present needs for a quick, explicit account of the mechanical aspects of elastic equilibrium problems; of course, neither a general existence theory based on such a notion is available for such problems, nor do we propose it as a goal. □

Given a displacement $f$, we denote by

$$F := \nabla f \quad \text{and} \quad F^* := (\nabla f)^* \tag{3.3}$$

the displacement gradient and its cofactor (*cf.* (1.22)). The ratios of the current to the reference volume and surface measures are expressed in terms of $F$ and $F^*$ by

$$\det F = \frac{\mathrm{d}\,(\mathrm{vol})}{\mathrm{d}\,(\mathrm{Vol})} \quad \text{and} \quad |F^* n| = \frac{\mathrm{d}\,(\mathrm{srf})}{\mathrm{d}\,(\mathrm{Srf})}, \tag{3.4}$$

respectively, $(3.4)_2$ being a consequence of the formula of Nanson for the unit normal $m$ to a point of the boundary $\partial f(\Omega)$[11] of the current placement:

$$m\,\mathrm{d}\,(\mathrm{srf}) = F^* n\,\mathrm{d}\,(\mathrm{Srf}). \tag{3.5}$$

Finally, at a regular point $x \in \partial\Omega$, we shall denote by ${}^sF$ and ${}^nF$ the superficial and normal parts of the displacement gradient $F$, with ${}^sF$ and ${}^nF$ defined by (2.6) and (2.8), respectively.

For a given displacement $f$, the *Piola-Kirchhoff* stress field $S$ over $\Omega$ and the corresponding *Cauchy* stress field $T$ over $f(\Omega)$ are related by

$$S(x) = T(y)\,F^*(x), \quad \text{with } y = f(x). \tag{3.6}$$

In terms of these stress measures the equilibrium equations are written as

$$\operatorname{Div} S = \mathbf{0} \quad \text{in } \Omega \quad \text{and} \quad \operatorname{div} T = \mathbf{0} \quad \text{in } f(\Omega), \tag{3.7}$$

respectively. The traction boundary condition is

$$Sn = s \quad \text{on } \partial\Omega \tag{3.8}$$

in terms of the Piola-Kirchhoff stress.

**3.2. Remark.** In (3.8), we have left unspecified the functional dependence, to be chosen later, of the surface load $s$ on the displacement $f$; consistently, we refrain from writing the corresponding formula in terms of the Cauchy stress, which would be, rather than generic, vague at this stage. Suffice it to record here the aspect of such a formula in the easy case of dead loading, *i.e.*, when

$$\partial\Omega \ni x \mapsto \hat{s}(x) \in \mathcal{V}, \tag{3.9}$$

[11] Note that, under the present hypotheses, $\partial f(\Omega) = f(\partial\Omega)$.

and we have

$$Tm = |F^*n|^{-1} s, \quad \text{with } s = \hat{s}(f^{-1}(y)), \quad \text{on } f(\Sigma_1) \cup f(\Sigma_2), \tag{3.10}$$

where use has been made of (3.5). □

Like $s$, the Piola-Kirchhoff stress $S$ at a point $x \in \Omega$ will be soon the subject of a constitutive prescription, whose general aspect is

$$\hat{S}: \Omega \times \text{Lin}^+ \to \text{Lin}, \quad S = \hat{S}(x, A), \tag{3.11}$$

with

$$\hat{S}(x, A)\, A^T \in \text{Sym} \quad \text{identically in } \Omega \times \text{Lin}^+. \tag{3.12}$$

# Part II. Surface Interactions

## 4. Variational Equilibrium Problems

A stress field $S$ over $\overline{\Omega}$ and a surface load field $s$ over $\partial\Omega$ are in equilibrium when they obey $(3.7)_1$ and (3.8). Formally, those two equations are equivalent to the single condition

$$\int_\Omega S \cdot \nabla h - \int_{\partial\Omega} s \cdot h = 0 \quad \text{for all test functions } h. \tag{4.1}$$

The question is now to choose constitutive prescriptions for both $S$ and $s$ such as to yield $(3.7)_1$–(3.8) and (4.1) as the strong and weak forms, respectively, of the Euler-Lagrange equation associated to a suitable total energy functional.

We assume that such total energy functional $E$ or, as we also say, the *total potential*, has two parts: a *body potential* $S$ of density $\sigma$ and a *surface interaction potential* $T$ of density $\tau$:

$$f \mapsto E\{f\} = S\{f\} + T\{f\}, \tag{4.2$_1$}$$

with

$$S = \int_\Omega \sigma \,\mathrm{d}\,(\text{Vol}) \quad \text{and} \quad T = \int_{\partial\Omega} \tau \,\mathrm{d}\,(\text{Srf}). \tag{4.2$_{2,3}$}$$

When the body undergoes a displacement $f$ from the reference placement $\Omega$, $\sigma$ is interpreted as the elastic energy stored per unit volume at a point of $\Omega$; $\tau$ is interpreted as the elastic energy stored per unit area at a point of $\partial\Omega$.

As a domain for the total potential (the collection of all admissible displacements from $\Omega$), we select

$$\mathscr{D} := \{f \in C^2(\Omega \cup \Sigma_1 \cup \Sigma_2) \cap C^0(\overline{\Omega}) \mid f \text{ injective \& locally orientation-preserving; } \nabla f \text{ bounded in } \Omega \text{ \& } \lim_{x \to x_0} \nabla f(x) = \nabla_\alpha f(x_0) \text{ for } x \to x_0 \in \Gamma \text{ from } \Sigma_\alpha\, (\alpha = 1, 2)\}. \tag{4.3}$$

We remark that the classical smoothness requirement on $f$, namely, $f \in C^2(\Omega) \cap C^1(\overline{\Omega})$, which guarantees the possibility of writing in its strong form the Euler-Lagrange equation associated to $E$ in the interior part, is here adapted because in some applications (*e.g.*, membrane interactions) the surface load $s$ depends on certain second derivatives of $f$ on the regular part of the boundary, specifically on the tangential derivatives of $\nabla f$, while on the singular part of the boundary equilibrium may require that $f$ not be continuously differentiable.

Obviously, the domain $\mathscr{D}$ is neither a flat space nor a convex set. However, not only does $\mathscr{D}$ agree with the classical notion of admissible displacement recalled in the beginning of this section, but also, and crucially for our further developments, it allows for a calculus of variations in a sense made precise by the following proposition.

**4.1. Proposition.** *For $\mathscr{D}$ defined as in* (4.3), *let* $\mathscr{H} := C^2(\overline{\Omega})$ *be the associated space of admissible variations. Then, for each $f \in \mathscr{D}$ and $h \in \mathscr{H}$, there is a strictly positive number $\varepsilon_0$, depending in general on $\Omega$, $f$ and $h$, such that*

$$(f + \varepsilon h) \in \mathscr{D} \quad \text{for all } \varepsilon \in ]-\varepsilon_0, \varepsilon_0[. \tag{4.4}$$

We omit the proof, which amounts to a straightforward adaptation of part (b) in the proof of Theorem 5.5.1 on p. 223 of [20].

We now lay down the following

**4.2. Definition.** *A continuous body has a* conservative interaction *with its environment (with respect to a reference placement $\Omega$) if the field pairs $(S, s)$ and $(\sigma, \tau)$ satisfy the variational condition*

$$\delta S\{f\}[h] = \int_{\Omega} S \cdot \nabla h, \quad \delta T\{f\}[h] = -\int_{\partial\Omega} s \cdot h \quad \text{for all } f \in \mathscr{D} \text{ and all } h \in \mathscr{H}. \tag{4.5}$$

Whenever, given $(S, s)$, densities $(\sigma, \tau)$ can be found for the functionals $S$ and $T$ such that (4.5) prevails, we say that the stress field $S$ and the surface load field $s$ admit, respectively, the body potential $S$ and the surface interaction potential $T$, defined by $(4.2)_2$ and $(4.2)_3$. Given $(\sigma, \tau)$, we regard (4.5) as a constitutive prescription for $(S, s)$. Indeed, with (4.2), (4.5), the differential identity

$$\operatorname{Div}(A^T v) - \operatorname{Div} A \cdot v = A \cdot \nabla v, \tag{4.6}$$

which holds for all smooth tensor fields $A$ and vector fields $v$, and, finally, the divergence theorem, we conclude that

$$\delta E[h] = \int_{\Omega} S \cdot \nabla h - \int_{\partial\Omega} s \cdot h = -\int_{\Omega} \operatorname{Div} S \cdot h + \int_{\partial\Omega} (Sn - s) \cdot h \tag{4.7}$$

for all $h \in \mathscr{H}$. Therefore, all solutions of the variational problem

$$\operatorname{extr}\{E(f) \mid f \in \mathscr{D}\} \tag{4.8}$$

necessarily satisfy the associated Euler-Lagrange equation, in either one of its forms $(3.7)_1$–(3.8) and (4.1).

As to the density of the body potential, we adhere to common practice in elasticity and choose a smooth mapping

$$\hat{\sigma} : \Omega \times \text{Lin}^+ \to \mathbb{R}^+, \quad \sigma = \hat{\sigma}(x, A) \tag{4.9}$$

which is *spatially symmetric, i.e.*, such that

$$\hat{\sigma}(\boldsymbol{x}, \boldsymbol{R}\boldsymbol{A}) = \hat{\sigma}(\boldsymbol{x}, \boldsymbol{A}) \quad \text{for all } \boldsymbol{R} \in \text{Rot and all } (\boldsymbol{x}, \boldsymbol{A}) \in \Omega \times \text{Lin}^+. \tag{4.10}$$

Setting then

$$\boldsymbol{S} = \hat{\boldsymbol{S}}(\boldsymbol{x}, \boldsymbol{A}) = \partial_{\boldsymbol{A}}\hat{\sigma}(\boldsymbol{x}, \boldsymbol{A}) \tag{4.11}$$

we see that $(4.5)_1$ is satisfied identically in $\mathscr{D}$.

**4.3. Remark.** As is well known, (4.10) yields

$$(\partial_{\boldsymbol{A}}\hat{\sigma}(\boldsymbol{x}, \boldsymbol{A}))\, \boldsymbol{A}^{\mathrm{T}} \in \text{Sym} \quad \text{for all } (\boldsymbol{x}, \boldsymbol{A}) \in \Omega \times \text{Lin}^+. \tag{4.12}$$

Thus, with (4.11), $\hat{\boldsymbol{S}}$ satisfies (3.12). □

We defer our choice of a surface interaction potential density of generality comparable with (4.9), (4.10) until after the discussion of the examples of pressure and membrane loadings. We here recall, however, that a dead interaction (*cf.* (3.9)) admits the potential density

$$\hat{\tau} : \partial\Omega \times \mathscr{V} \to \mathbb{R}, \quad \tau = \hat{\tau}(\boldsymbol{x}, \boldsymbol{v}) = \hat{\boldsymbol{s}}(\boldsymbol{x}) \cdot (\boldsymbol{v} - \boldsymbol{x}); \tag{4.13}$$

with

$$\boldsymbol{s} = \hat{\boldsymbol{s}}(\boldsymbol{x}) = -\partial_{\boldsymbol{v}}\hat{\tau}(\boldsymbol{x}, \boldsymbol{v}), \tag{4.14}$$

$(4.5)_2$ is satisfied identically in $\mathscr{D}$.

## 5. Examples

In this section we take $\Omega$ to be a regular region in the sense of Section 2.

### *5.1 Pressure Interaction*

By definition, a body has a *pressure interaction* with its environment when, given a *pressure function*

$$\hat{\pi} : \mathscr{E} \to \mathbb{R}^+, \quad \pi = \hat{\pi}(\boldsymbol{y}), \tag{5.1}$$

the traction boundary condition in terms of the Cauchy stress is

$$\boldsymbol{T}\boldsymbol{m} = -\pi\boldsymbol{m} \quad \text{on } \partial\boldsymbol{f}(\Omega) \tag{5.2}$$

for all $\boldsymbol{f} \in \mathscr{D}$. In the reference placement, the corresponding boundary condition is

$$\boldsymbol{S}\boldsymbol{n} = -\pi\boldsymbol{F}^*\boldsymbol{n}, \quad \text{with } \pi = \hat{\pi}(\boldsymbol{f}(\boldsymbol{x})), \quad \text{on } \partial\Omega, \tag{5.3}$$

where both (3.5) and (3.6) have been used; we shall call *pressure loading* the vector field

$$\boldsymbol{x} \mapsto \boldsymbol{s}_p(\boldsymbol{x}) := -\hat{\pi}(\boldsymbol{f}(\boldsymbol{x}))\, \boldsymbol{F}^*(\boldsymbol{x})\, \boldsymbol{n}(\boldsymbol{x}) \tag{5.4}$$

over $\partial\Omega$.

**5.1. Proposition.** *Pressure loading admits the surface interaction potential*

$$T_p\{f\} = \int_{\partial\Omega} \bar{\pi}(f)\, f \cdot (\nabla f)^* n \, \mathrm{d}\,(\mathrm{Sfr}), \tag{5.5$_1$}$$

*with the mapping* $\bar{\pi}$ *from* $\mathscr{V}$ *into* $\mathbb{R}$ *defined by*

$$v \mapsto \bar{\pi}(v) := \int_0^1 \hat{\pi}(\gamma v)\, \gamma^2 \, \mathrm{d}\gamma \tag{5.5$_2$}$$

*Moreover, the volume potential*

$$P_p\{f\} = \int_{\Omega} \hat{\pi}(f) \det(\nabla f) \, \mathrm{d}\,(\mathrm{Vol}) \tag{5.6}$$

*is such that*

$$T_p\{f\} = P_p\{f\} \quad \textit{for all } f \in \mathscr{D}. \tag{5.7}$$

**Proof.**[12] We find it expedient to prove (5.7) first, and then to show that

$$\delta T_p\{f\}\,[h] = \delta P_p\{f\}\,[h] = \int_{\partial\Omega} s_p \cdot h. \tag{5.8}$$

We begin by observing that the successive use of the divergence theorem, the differential identity (4.6) and the fact known to Euler that the cofactor of the gradient is divergenceless:

$$\operatorname{Div} F^* = \mathbf{0}, \tag{5.9}$$

imply that

$$\int_{\partial\Omega} \bar{\pi} f \cdot F^* n = \int_{\Omega} F^* \cdot \nabla(\bar{\pi} f). \tag{5.10}$$

Moreover,

$$\nabla(\bar{\pi} f) = \bar{\pi} F + f \otimes \nabla\bar{\pi} \quad \text{and} \quad \nabla\bar{\pi} = F^{\mathrm{T}} \operatorname{grad} \bar{\pi}, \tag{5.11}$$

so that, on recalling (1.18) and its consequence (1.21), we have

$$F^* \cdot \nabla(\bar{\pi} f) = (3\bar{\pi} + f \cdot \operatorname{grad} \bar{\pi}) \det F. \tag{5.12}$$

Finally, differentiation of $(5.5)_2$ yields

$$\operatorname{grad} \bar{\pi} = (-3\bar{\pi} + \hat{\pi})\, |f|^{-2} f, \tag{5.13}$$

---

[12] In view of $(3.4)_1$ and (3.5), for pressure loading the volume potential may be written as

$$P_p\{f\} = \int_{f(\Omega)} \hat{\pi}(y) \, \mathrm{d}\,(\mathrm{vol}), \tag{\#}$$

and the surface potential as

$$T_p\{f\} = \int_{\partial f(\Omega)} \bar{\pi}(y)\, y \cdot m(y) \, \mathrm{d}\,(\mathrm{srf}), \tag{\#\#}$$

with $\bar{\pi}$ and $\hat{\pi}$ related by $(5.5)_2$. Of course, the appealing greater simplicity of these formulae is counterbalanced by the need of taking variations of integrals with variable domain. Notice also that (5.1) and (5.7) imply that $T_p\{f\} > 0$ throughout $\mathscr{D}$.

so that, in particular,

$$3\bar{\pi} + \boldsymbol{f} \cdot \operatorname{grad} \bar{\pi} = \hat{\pi}. \tag{5.14}$$

With the use of (5.12), and (5.14), (5.10) yields (5.7).

In order to prove $(5.8)_2$, it is sufficient to observe that, due to (4.6) and (5.9),

$$\delta\left(\int_\Omega \hat{\pi} \det \boldsymbol{F}\right)[\boldsymbol{h}] = \int_\Omega ((\operatorname{grad} \hat{\pi} \cdot \boldsymbol{h}) \det \boldsymbol{F} + \hat{\pi} \operatorname{Div} (\boldsymbol{F}^{*\mathrm{T}}\boldsymbol{h})). \tag{5.15}$$

But, by (1.18) and $(5.11)_2$ with $\hat{\pi}$ in place of $\bar{\pi}$,

$$(\operatorname{grad} \hat{\pi} \cdot \boldsymbol{h}) \det \boldsymbol{F} = \boldsymbol{F}^{*\mathrm{T}} \boldsymbol{h} \cdot \boldsymbol{F}^{\mathrm{T}} \operatorname{grad} \hat{\pi} = \boldsymbol{F}^{*\mathrm{T}} \boldsymbol{h} \cdot \nabla \hat{\pi}. \tag{5.16}$$

Hence, in view of the differential identity

$$\operatorname{Div} (\alpha \boldsymbol{v}) = \alpha \operatorname{Div} \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \alpha \tag{5.17}$$

holding for each smooth scalar field $\alpha$ and vector field $\boldsymbol{v}$, we have

$$\delta\left(\int_\Omega \hat{\pi} \det \boldsymbol{F}\right)[\boldsymbol{h}] = \int_\Omega \operatorname{Div} (\hat{\pi} \boldsymbol{F}^{*\mathrm{T}} \boldsymbol{h}) = \int_{\partial\Omega} \hat{\pi} \boldsymbol{F}^* \boldsymbol{n} \cdot \boldsymbol{h}, \tag{5.18}$$

which, by (5.6) and (5.4), establishes $(5.8)_2$. □

As to the last proposition, some comments are in order, for the purpose also of putting our further developments into perspective.

Firstly, pressure loading $\boldsymbol{s}_p$ depends only on the superficial part ${}^s\boldsymbol{F}$ of the displacement gradient $\boldsymbol{F}$: indeed, one immediately sees from (1.28) that

$$\boldsymbol{F}^* \boldsymbol{n} = ({}^s\boldsymbol{F})^* \boldsymbol{n}.\text{[13]} \tag{5.19}$$

This observation suggests that, in generalizing from (5.5), one begin by considering surface interaction potentials of the form[14]

$$T\{f\} = \int_{\partial\Omega} \hat{\tau}(f, {}^s\nabla f) \, \mathrm{d}\,(\mathrm{Srf}). \tag{5.20}$$

Secondly, recall that a *null Lagrangian*[15] is, in our present context, a functional on

$$f \mapsto F\{f\} = \int_\Omega \hat{\varphi}(f, \nabla f) \, \mathrm{d}\,(\mathrm{Vol}) \tag{5.21}$$

$$\bar{\mathscr{D}} := C^2(\Omega) \cap C^1(\bar{\Omega}) \tag{5.22}$$

---

[13] *Vid.* [3] for a different proof of the fact that $\boldsymbol{F}^*\boldsymbol{n}$ depends upon surface derivatives alone. The mechanical relevance of this circumstance was first remarked by Gurtin in [21].

[14] Here and henceforth in this section we silently fail to remark that the functionals we introduce may depend upon $\boldsymbol{x}$.

[15] Null Lagrangians are treated at length in [22] and [23], where appropriate reference to the papers by Ericksen [15], Edelen [16], and others, is also to be found; an interesting paper on the subject, which has just appeared, is [24].

for which a vector field $x \mapsto v_f(x)$ over $\partial\Omega$ can be found, in general depending functionally on the restriction of $f$ to $\partial\Omega$, such that

$$\delta F\{f\}[h] = \int_{\partial\Omega} v_f \cdot h \, \mathrm{d}\,(\mathrm{Srf}) \quad \text{for all } f \in \bar{\mathscr{D}} \text{ and } h \in \mathscr{H}. \tag{5.23}$$

In view of $(5.8)_2$ and (5.6), we recognize in $P_p$ a null Lagrangian of the form (5.21). Our general results in this direction are the contents of Proposition 7.9 below.

**5.2. Remark.** Comparing (5.20) and (5.23) in the light of (5.7) one is led to consider live loadings alternatively described by surface and volume potential densities $\tau$ and $\varphi$ such that

$$\int_{\partial\Pi} \hat{\tau}(f, {}^{s}\nabla f) = \int_{\Pi} \hat{\varphi}(f, \nabla f) \quad \text{for all } f \in \bar{\mathscr{D}} \text{ and parts } \Pi \subset \Omega; \tag{5.24}$$

for such loadings, invariance under change in observer might be assessed by requiring that $\hat{\varphi}$ has the behavior usually postulated for constitutive functionals having the character of volume densities, thus circumventing the nontrivial task of choosing reasonable invariance assumptions for $\tau$ or $s$. This suggestion has been put forward in [3], where it has been also shown that, if $\varphi = \hat{\varphi}(A)$, then (5.24) and the requirement that $\hat{\varphi}$ be spatially symmetric in the sense of (4.10) are compatible (if and) only if

$$\hat{\varphi}(A) = \pi_0 \det A, \quad \text{with } \pi_0 \text{ an arbitrary constant}, \tag{5.25}$$

which is the case for *uniform* pressure loading. □

### *5.2 Membrane Interaction*

We think of a body having a *membrane interaction* with its environment as a body with an elastic membrane glued to it,[16] so that

$$Tm = -2\varepsilon_0 \varkappa m \quad \text{on } \partial f(\Omega), \tag{5.26}$$

where $\varepsilon_0 > 0$ is the material modulus of the membrane and, in view of (2.26) and (3.5),

$$\varkappa = \tfrac{1}{2}\,{}^{s}\mathrm{div}\, m \quad \text{with} \quad m = \frac{(\nabla f)^* n}{|(\nabla f)^* n|}, \tag{5.27}$$

is the mean curvature of the current boundary of the body. In analogy with (5.4), we call shall *membrane loading* the vector field

$$x \mapsto s_m(x) := -2\varepsilon_0 \varkappa(f(x))\, F^*(x)\, n(x) \tag{5.28}$$

over $\partial\Omega$.

[16] Or, alternatively, as a body acted upon by surface tension (*cf.* [3] and [25], p. 153).

**5.3. Proposition.** *Membrane loading $s_m$ admits the surface interaction potential*

$$T_m\{f\} = \int_{\partial\Omega} \hat{\tau}_m(f, \nabla f)\,\mathrm{d}\,(\mathrm{Srf}), \tag{5.29$_1$}$$

*with density*

$$\hat{\tau}_m : \mathscr{S}(1)\times \mathrm{Lin}, \qquad \tau_m = \hat{\tau}_m(u, A) := \varepsilon_0\,|A^* u|,^{17} \tag{5.29$_2$}$$

*and equals the surface divergence of the (referential)* membrane stress $M$:

$$s_m = {}^{\mathrm{s}}\mathrm{Div}\, M, \qquad M(u, A) := \partial_A \hat{\tau}_m(u, A), \tag{5.30}$$

*with the class* $C^1$ *tangential tensor field* $M$ *over* $\partial\Omega$ *appearing in* $(5.30)_1$ *given by*

$$M = \tau_m(I - m \otimes m)\, F^{-\mathrm{T}}, \tag{5.31}$$

*as a consequence of definition* $(5.30)_2$.

**Proof.** As preliminary, we recall that the absolute value mapping on $\mathscr{V}$ has derivative

$$\partial_v\,|v| = |v|^{-1}\,v$$

away from the origin of $\mathscr{V}$. This, and the fact pointed out in Footnote 8 that the derivative of the cofactor mapping is a symmetric transformation of Lin into itself, allow us to write

$$\delta T_m\{f\}\,[h] = \varepsilon_0 \int_{\partial\Omega} \delta\,|(\nabla f)^* n|\,[h] = \varepsilon_0 \int_{\partial\Omega} \partial_F F^* \left[\frac{(\nabla f)^* n}{|(\nabla f)^* n|} \otimes n\right] \cdot \nabla h. \tag{5.32}$$

Next, (1.30) and (5.31) imply that

$$\delta T_m\{f\}\,[h] = \int_{\partial\Omega} M \cdot \nabla h. \tag{5.33}$$

Now, in view of (2.7) and because,

$$\nabla h = (\mathrm{grad}\, h)\, \nabla f, \tag{5.34}$$

we have

$$M \cdot \nabla h = \varepsilon_0\,|(\nabla f)^* n|\,{}^{\mathrm{s}}\mathrm{div}\, h. \tag{5.35}$$

With (5.35), (2.27) implies that

$$\int_{\partial\Omega} M \cdot \nabla h = \varepsilon_0 \int_{\partial f(\Omega)} {}^{\mathrm{s}}\mathrm{div}\, h = \varepsilon_0 \int_{\partial f(\Omega)} ({}^{\mathrm{s}}\mathrm{div}\, m)\, m \cdot h; \tag{5.36}$$

---

[17] Notice that, again by $(5.19)$ and just as the pressure potential density, it follows from $(5.29)_2$ that $\hat{\tau}_m(f, \nabla f)$ depends on the displacement gradient through the first tangential derivatives only.

(5.36), with (5.26)–(5.28) and (5.33), is enough to prove that $T_m$ is a surface interaction potential for $s_m$:

$$\delta T_m\{f\}[h] = -\int_{\partial\Omega} s_m \cdot h. \tag{5.37}$$

We observe that, in view of (3.5), the membrane stress $M$ as given by (5.31) satisfies (2.28), *i.e.*, is tangential. We can then apply (2.30) and obtain

$$\int_{\partial\Omega} M \cdot \nabla h = \int_{\partial\Omega} M \cdot {}^{s}\nabla h = -\int_{\partial\Omega} {}^{s}\mathrm{Div}\, M \cdot h,$$

which, together, with (5.37), establishes $(5.30)_1$. □

Define now the *Kirchhoff membrane stress* $K$ to be

$$K := MF^{\mathrm{T}} = \tau_m(I - m \otimes m). \tag{5.38}$$

For all $x \in \Omega$ and $f \in \mathscr{D}$, and for all $t \in \mathscr{V}$ such that $t \cdot m(f(x)) = 0$, we have from (5.38) that

$$Kt = \tau_m t;$$

thus $K$ is a positive-definite multiple of the identity transformation of the tangent plane to $\partial f(\Omega)$ at $f(x)$ into itself. In addition, using among other things the symmetry of the Weingarten tensor (Proposition 2.7) one can show that

$${}^{s}\mathrm{div}\, K = s_m = {}^{s}\mathrm{Div}\, M.^{18} \tag{5.39}$$

Thus, as remarked in [3], on the one hand membrane loading cannot account for situations where compressive surface stresses are to be expected (*vid. e.g.* [27] and [28]): more general constitutive prescriptions than $(5.28)_2$ are needed to cover those cases. On the other hand, $s_m$ involves the second derivatives of the displacement field $f$ at the boundary, and therefore points to live surface loads more general than the *simple* loads examined by SPECTOR [10], [11], which by definition depend at most on first tangential derivatives.[19]

## 6. Properties of Extremals

The examples of dead, pressure and membrane interactions together suggest that we consider general conservative interactions as those described by Definition 4.2, with stored energy per unit volume $\hat{\sigma}$ obeying (4.9), (4.10) and stored energy per unit area $\hat{\tau}$ defined by assigning two class $C^2$ mappings

$$\hat{\tau}_\alpha : (\Sigma_\alpha \cup \Gamma) \times \mathscr{S}(1) \times \mathscr{V} \times \mathrm{Lin} \to \mathbb{R}, \quad \tau = \hat{\tau}_\alpha(x, u, v, A), \quad \alpha = 1, 2, \tag{6.1}$$

[18] For a general treatment of continuous bodies having the form of elastic membranes *vid.* [26].

[19] Notice that, as (5.4) shows, pressure loading $s_p$ does depend only on first tangential derivatives of $f$, and is therefore simple in the above sense.

and requiring that

$$\hat{\tau}|_{\Sigma_\alpha} := \hat{\tau}_\alpha|_{\Sigma_\alpha}, \quad \alpha = 1, 2.^{20} \tag{6.2}$$

In view of (6.1) and (6.2), for $x \in \Sigma_1 \cup \Sigma_2$ and $f \in \mathscr{D}$, the surface interaction potentials we consider have densities

$$x \mapsto \hat{\tau}(x, n(x), f(x), F(x)). \tag{6.3}$$

It would be desirable to have at our disposal an *a priori* selection criterion for $\hat{\tau}$ playing for (6.1), (6.2) the complementing role played by (4.10) for (4.9). Unfortunately, for live surface interactions the spatial symmetry issue has not received yet a treatment as thorough as it deserves; no generally accepted form such as (4.10) is presently available for invariance requirements under change in observer applicable to surface potentials.[21]

We then address ourselves to deduce other selection criteria for admissible classes of constitutive functionals $T$ from the properties that the extreme points of the variational problem (4.8), with $E$ chosen as just explained, must have: in other words, we shall regard $S$ as given, and require that $T$ be such as to obey the necessary conditions for an extremum identically on the domain $\mathscr{D}$ of $E$. Our next proposition lists all those conditions.

**6.1. Proposition.** *Let the total energy functional $E$ be defined by* (4.2), (4.3), (4.9), (4.10) *and* (6.2), *and let $f^{(0)}$ be an extreme point of $E$ on its domain $\mathscr{D}$. Then, $f^{(0)}$ satisfies the following conditions*:

(i) (*equilibrium in the interior*)

$$\mathrm{Div}\,(\partial_F \hat{\sigma}^{(0)}) = 0 \ \ in \ \ \Omega; \tag{6.4}$$

(ii) (*equilibrium on the regular part of the boundary*)

$$(\partial_F \hat{\sigma}^{(0)})\, n = -\partial_f \hat{\tau}^{(0)} + {}^{s}\mathrm{Div}\,(\partial_F \hat{\tau}^{(0)}) \quad on \ \Sigma_1 \cup \Sigma_2; \tag{6.5}$$

---

[20] Thus $\hat{\tau}$ is defined only over the regular part $\Sigma_1 \cup \Sigma_2$ of the boundary of $\Omega$. We remark that, as the normal field and the deformation gradient are in general discontinuous through the singular part $\Gamma$ of $\partial\Omega$, for all $f \in \mathscr{D}$ the scalar fields

$$x \mapsto \hat{\tau}_\alpha(x, n(x), f(x), \nabla_\alpha f(x)), \quad \alpha = 1, 2,$$

are pointwise different on $\Gamma$ even if $\hat{\tau}_1|_\Gamma$ and $\hat{\tau}_2|_\Gamma$ and are one and the same function from $\Gamma \times \mathscr{S}(1) \times \mathscr{V} \times \mathrm{Lin}$ into $\mathbb{R}$: it would be difficult to imagine a situation of practical interest where a single-valued $\hat{\tau}$ could be defined over the whole of $\partial\Omega$ (of course, this is not the case if $\Gamma$ is empty; *vid.* Remark 6.2 below).

[21] As an example of the invariance requirements that one may think of imposing, we recall that in [7] $\hat{\tau}$ was assumed to be *spatially symmetric* in the following reduced sense: for all fixed $x \in \Sigma_1 \cup \Sigma_2$ and $f \in \mathscr{D}$, let $\mathrm{Rot}\,(x, f)$ be the subgroup of Rot consisting of all rotations about the current normal $m$ at $f(x)$; then $\hat{\tau}$ is such that, for all $(x, u, v, A) \in (\Sigma_1 \cup \Sigma_2) \times \mathscr{S}(1) \times \mathscr{V} \times \mathrm{Lin}$ and for all $f \in \mathscr{D}$,

$$\hat{\tau}(x, u, Rv, RA) = \hat{\tau}(x, u, v, A) \quad \text{for all } R \in \mathrm{Rot}\,(x, f). \tag{\#}$$

(iii) (*tangency on the regular part of the boundary*)

$$(\partial_F \hat{\tau}^{(0)})\, \boldsymbol{n} = \boldsymbol{0} \qquad on\ \Sigma_1 \cup \Sigma_2; \tag{6.6}$$

*Moreover, if the region* $\Omega$ *is such that*

$$\boldsymbol{n}_1(\boldsymbol{x}) \neq -\boldsymbol{n}_2(\boldsymbol{x}) \qquad for\ all\ \boldsymbol{x} \in \Gamma, \tag{6.7}$$

*then* $\boldsymbol{f}^{(0)}$ *satisfies the additional condition*

(iv) (*equilibrium on the singular part of the boundary*)

$$(\partial_F \hat{\tau}_1^{(0)})\, \boldsymbol{n}_2 + (\partial_F \hat{\tau}_2^{(0)})\, \boldsymbol{n}_1 = \boldsymbol{0} \qquad on\ \Gamma. \tag{6.8}$$

**Proof.** For $\boldsymbol{f}^{(0)} \in \mathscr{D}$ an extremum of $E$, we have

$$\delta E\{\boldsymbol{f}^{(0)}\}\,[\boldsymbol{h}] = \int_\Omega \partial_F \hat{\sigma}^{(0)} \cdot \nabla \boldsymbol{h} + \int_{\partial\Omega} (\partial_f \hat{\tau}^{(0)} \cdot \boldsymbol{h} + \partial_F \hat{\tau}^{(0)} \cdot \nabla \boldsymbol{h}) = 0 \tag{6.9}$$

for all variations $\boldsymbol{h} \in \mathscr{H}$. Taking firstly $\boldsymbol{h}|_{\partial\Omega} \equiv \boldsymbol{0}$ and applying (4.6) we have (6.4). In addition, in view also of definitions (2.6), (2.8) and (2.9) yielding

$$\nabla \boldsymbol{h} = {}^{\mathrm{s}}\nabla \boldsymbol{h} + {}^{\mathrm{n}}\partial \boldsymbol{h} \otimes \boldsymbol{n},$$

$(6.9)_2$ reduces to

$$\int_{\partial\Omega} [(\partial_F \hat{\sigma}^{(0)} \boldsymbol{n} + \partial_f \hat{\tau}^{(0)}) \cdot \boldsymbol{h} + \partial_F \hat{\tau}^{(0)} \cdot {}^{\mathrm{s}}\nabla \boldsymbol{h} + (\partial_F \hat{\tau}^{(0)})\, \boldsymbol{n} \cdot {}^{\mathrm{n}}\partial \boldsymbol{h}] = 0. \tag{6.10}$$

But on the regular part of $\partial\Omega$ the variation fields $\boldsymbol{h}$ and ${}^{\mathrm{n}}\partial \boldsymbol{h}$ may be assigned independently. Thus we deduce from (6.10) and (2.28) that $\partial_F \hat{\tau}^{(0)}$ is a tangential field on $\Sigma_1 \cup \Sigma_2$, which is (iii). At the same time, (6.10) and the consequence (2.31) of the surface divergence theorem yield

$$\int_{\Sigma_1 \cup \Sigma_2} [(\partial_F \hat{\sigma}^{(0)})\, \boldsymbol{n} + \partial_f \hat{\tau}^{(0)} - {}^{\mathrm{s}}\mathrm{Div}\, \partial_F \hat{\tau}] \cdot \boldsymbol{h}$$
$$+ \int_\Gamma [(\partial_F \hat{\tau}_1^{(0)})\, (\boldsymbol{t} \times \boldsymbol{n}_1) - (\partial_F \hat{\tau}_2^{(0)})\, (\boldsymbol{t} \times \boldsymbol{n}_2)] \cdot \boldsymbol{h} = 0. \tag{6.11}$$

Now taking $\boldsymbol{h}|_\Gamma \equiv \boldsymbol{0}$ we have (ii), whereas (6.11) reduces to the line integral. In view of hypothesis (6.7) and (2.4), we have

$$|\boldsymbol{n}_1 \times \boldsymbol{n}_2|\, \boldsymbol{t} \times \boldsymbol{n}_1 = \boldsymbol{n}_2 - (\boldsymbol{n}_1 \cdot \boldsymbol{n}_2)\, \boldsymbol{n}_1, \ |\boldsymbol{n}_1 \times \boldsymbol{n}_2|\, \boldsymbol{t} \times \boldsymbol{n}_2 = (\boldsymbol{n}_1 \cdot \boldsymbol{n}_2)\, \boldsymbol{n}_2 - \boldsymbol{n}_1;$$

thus the integrand in the line integral can be written as

$$\{[(\partial_F \hat{\tau}_1^{(0)})\, \boldsymbol{n}_2 + (\partial_F \hat{\tau}_2^{(0)})\, \boldsymbol{n}_1] - (\boldsymbol{n}_1 \cdot \boldsymbol{n}_2)\, [(\partial_F \hat{\tau}_1^{(0)})\, \boldsymbol{n}_1 + (\partial_F \hat{\tau}_2^{(0)})\, \boldsymbol{n}_2]\} \cdot \boldsymbol{h} = 0. \tag{6.12}$$

We now observe that, passing to the limit in (6.6), we obtain

$$(\partial_F \hat{\tau}_1^{(0)})\, \boldsymbol{n}_1 = (\partial_F \hat{\tau}_2^{(0)})\, \boldsymbol{n}_2 = \boldsymbol{0} \tag{6.13}$$

identically on $\Gamma$. Finally, (6.11), (ii), (6.12), (6.13) and the residual arbitrariness in the choice of $\boldsymbol{h}$ allow us to establish (iv). □

**6.2. Remark.** As an example of the adjustments alluded to in Remark 2.1, we note that, for $\Omega$ a regular region, the normal field is of class $C^2$, and the dependence of $\hat{\tau}$ on $\boldsymbol{n}$ may then be safely absorbed into the dependence on $\boldsymbol{x}$. Formally, (6.1) and (6.2) together should be read as

$$\hat{\tau} : \partial\Omega \times \mathscr{V} \times \mathrm{Lin} \mapsto \mathbb{R}, \qquad \tau = \hat{\tau}(\boldsymbol{x}, \boldsymbol{v}, A),$$

and (6.3) as

$$\boldsymbol{x} \mapsto \hat{\tau}(\boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x}), \nabla \boldsymbol{f}(\boldsymbol{x}));$$

Proposition 6.1 would then consist of the first three items only, with $\Sigma_1 \cup \Sigma_2$ replaced by $\partial\Omega$. □

**6.3. Remark.** If $\boldsymbol{n}_1 \equiv -\boldsymbol{n}_2$ on $\Gamma$, the integrand of the line integral in (6.11) vanishes, provided that $\nabla_1 \boldsymbol{f}^{(0)} \equiv \nabla_2 \boldsymbol{f}^{(0)}$ and

$$\hat{\tau}_1(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{u}, A) = -\hat{\tau}_2(\boldsymbol{x}, -\boldsymbol{n}, \boldsymbol{u}, A) \qquad \text{for } \boldsymbol{x} \in \Gamma \tag{6.14}$$

(this is the case for a pressure interaction on the whole boundary but not for a membrane interaction); in general, however, (6.11) takes the form

$$[\partial_F \hat{\tau}_1, (\boldsymbol{x}, \boldsymbol{n}_1, \boldsymbol{f}^{(0)}, \nabla_1 \boldsymbol{f}^{(0)}) + \partial_F \hat{\tau}_2(\boldsymbol{x}, -\boldsymbol{n}_1, \boldsymbol{f}^{(0)}, \nabla_2 \boldsymbol{f}^{(0)})] (\boldsymbol{t} \times \boldsymbol{n}_1) = 0 \qquad \text{for } \boldsymbol{x} \in \Gamma. \tag{6.15}$$

a condition on minimizers replacing (6.8) in the cuspidal case. □

**6.4. Remark.** Neither for pressure interactions nor for membrane interactions does the surface potential depend explicitly on the place $\boldsymbol{x}$; when this is the case, we shall say that a surface interaction potential is *homogeneous*. □

In the light of $(3.7)_1$ and (4.11), the equilibrium condition in the interior of $\Omega$ comes as no surprise. Likewise, the equilibrium condition on the regular part of $\partial\Omega$ simply suggests that, given a surface potential density $\hat{\tau}$, for $\boldsymbol{x} \in \Sigma_1 \cup \Sigma_2$ and $\boldsymbol{f} \in \mathscr{D}$ we associate with $\hat{\tau}$ the live surface load

$$\boldsymbol{x} \mapsto \hat{\boldsymbol{s}}(\boldsymbol{x}, \boldsymbol{n}(\boldsymbol{x}), \boldsymbol{f}(\boldsymbol{x}), \boldsymbol{F}(\boldsymbol{x})) := (-\partial_f \hat{\tau} + {}^s\mathrm{Div}\, \partial_F \hat{\tau})\, (\boldsymbol{x}, \boldsymbol{n}(\boldsymbol{x}), \boldsymbol{f}(\boldsymbol{x}), \boldsymbol{F}(\boldsymbol{x})). \tag{6.16}$$

Remarkably, all conditions (6.6), (6.8) and (6.15) involve exclusively the "membrane stress" $\partial_F \hat{\tau}$.

As to the tangentiality condition (6.6), we observe that both pressure and membrane potentials satisfy it identically. In the next section we shall start from a scrutiny of (6.6), and move on to motivate further consideration of densities $\hat{\tau}$ depending only on $\boldsymbol{x}$, $\boldsymbol{n}$, $\boldsymbol{f}$ and the first tangential derivatives of the displacement field $\boldsymbol{f}$, and then to discuss conditions such as to guarantee that $\hat{\boldsymbol{s}}$, too, shall depend at most on the first tangential derivatives.

As to the equilibrium condition on the singular part $\Gamma$ of $\partial\Omega$, in either one of its forms (6.8) or (6.15), we repeat that pressure interactions satisfy it whatever the region $\Omega$, and therefore the curve $\Gamma$, may be (provided $\nabla_1 \boldsymbol{f}^{(0)} \equiv \nabla_2 \boldsymbol{f}^{(0)}$). By Proposition 7.16 we shall characterize a class of homogeneous surface interaction potentials which satisfy identically both (6.8) and (6.15).

## 7. Tangential Interactions

We begin by establishing a preparatory result yielding an equivalent version of the tangentiality condition (6.6) for a surface potential density of type (6.2).

**7.1. Proposition.** *Let* $\hat{\psi}$ *be a smooth mapping from* Lin *into* $\mathbb{R}$. *Then, for each fixed* $\boldsymbol{a} \in \mathscr{V}$,

$$(\partial_A \hat{\psi}(A))\, \boldsymbol{a} \equiv 0 \quad \Leftrightarrow \quad \hat{\psi}(A) \equiv \hat{\psi}(A(\boldsymbol{I} - \boldsymbol{a} \otimes \boldsymbol{a}))\,. \tag{7.1}$$

**Proof.** For fixed $\boldsymbol{a} \in \mathscr{V}$, let

$$\hat{\psi}(A) \equiv \hat{\psi}(A(\boldsymbol{I} - \boldsymbol{a} \otimes \boldsymbol{a}))\,.$$

Then differentiation yields

$$\partial_A \hat{\psi}(A) \equiv (\partial_A \hat{\psi}(A(\boldsymbol{I} - \boldsymbol{a} \otimes \boldsymbol{a})))\,(\boldsymbol{I} - \boldsymbol{a} \otimes \boldsymbol{a})$$

and the leftward implication follows. To prove the converse, for $\boldsymbol{a} \in \mathscr{V}$ again fixed we define the real-valued mapping on Lin$\times\mathscr{V}$

$$\overline{\psi}(Y, \boldsymbol{y}) := \hat{\psi}(Y + \boldsymbol{y} \otimes \boldsymbol{a})\,. \tag{7.2}$$

Now, differentiating (7.2) with respect to $\boldsymbol{y}$ we have

$$\partial_y \overline{\psi}(Y, \boldsymbol{y}) \equiv (\partial_A \hat{\psi}(A))\, \boldsymbol{a} \qquad \text{for } A = Y + \boldsymbol{y} \otimes \boldsymbol{a}\,.$$

If

$$(\partial_A \hat{\psi}(A))\, \boldsymbol{a} \equiv \mathbf{0}\,,$$

it follows that $\overline{\psi}$ actually depends on the first variable only. Writing now each $A \in$ Lin as

$$A = A(\boldsymbol{I} - \boldsymbol{a} \otimes \boldsymbol{a}) + (A\boldsymbol{a}) \otimes \boldsymbol{a}\,,$$

and taking

$$Y = A(\boldsymbol{I} - \boldsymbol{a} \otimes \boldsymbol{a}), \qquad \boldsymbol{y} = A\boldsymbol{a}$$

in (7.2), we have the desired conclusion. □

Now let $\hat{\tau}$ in (6.2) be tangential not only at an extreme point of the total energy functional, as required by (6.6), but rather on the whole of $\mathscr{D}$, as guaranteed by assuming that the membrane stress $\partial_F \hat{\tau}$ be tangential in the sense of (2.28), *i.e.*, that

$$(\partial_A \hat{\tau}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v}, A))\, \boldsymbol{u} = \mathbf{0} \tag{7.3}$$

for all $(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v}, A) \in (\Sigma_1 \cup \Sigma_2) \times \mathscr{S}(1) \times \mathscr{V} \times \text{Lin}$. Putting then, for each fixed triplet $(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v}) \in (\Sigma_1 \cup \Sigma_2) \times \mathscr{S}(1) \times \mathscr{V}$,

$$\hat{\tau}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v}, A) = \hat{\psi}(A) \qquad \text{for all } A \in \text{Lin}, \tag{7.4}$$

we conclude from (7.1) that (7.3) implies that

$$\hat{\tau}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v}, A) \equiv \hat{\tau}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v}, A(\boldsymbol{I} - \boldsymbol{u} \otimes \boldsymbol{u}))\,. \tag{7.5}$$

Thus, for the surface interaction potential densities of type (6.2), assuming tangency in the sense of (7.3) is equivalent to restricting attention to the subclass of *tangential* potentials, namely, those having surface densities of the following type:

$$\boldsymbol{x} \mapsto \hat{\tau}(\boldsymbol{x}, \boldsymbol{n}(\boldsymbol{x}), \boldsymbol{f}(\boldsymbol{x}), {}^{s}\boldsymbol{F}(\boldsymbol{x})). \tag{7.6}$$

### *7.1 Subpotentials*

Beside the hint provided by the necessary property (6.6) of extremals, we shall here offer another reason for considering only tangential potentiels, so putting that choice into better perspective.

We begin by recalling a notion of restricted convexity for functions defined on Lin (*vid.* [22], p. 43ff. and [23], Sect. 4.1.1).

A smooth mapping $\hat{\psi}$ from Lin into $\mathbb{R}$ is said to be *rank-one convex* if, for all $A \in \text{Lin}$ and $\boldsymbol{a}, \boldsymbol{b} \in \mathscr{V}$,

$$\hat{\psi}(A + \alpha \boldsymbol{a} \otimes \boldsymbol{b}) \leqq (1 - \alpha)\, \hat{\psi}(A) + \alpha \hat{\psi}(A + \boldsymbol{a} \otimes \boldsymbol{b}) \quad \text{for all } \alpha \in [0, 1]. \tag{7.7}$$

It is easily seen that $\hat{\psi}$ is rank-one convex if and only if the function on [0, 1] defined by

$$\alpha \mapsto \hat{\psi}(A + \alpha \boldsymbol{a} \otimes \boldsymbol{b}) \tag{7.8}$$

is convex for all choices of $A$, $\boldsymbol{a}$ and $\boldsymbol{b}$. If $\hat{\psi}$ is of class $C^2$, another well known characterization of rank-one convexity is that $\hat{\psi}$ obeys (7.7) if and only if, for each $A \in \text{Lin}$ fixed, the linear mapping $\partial_A^2 \hat{\psi}(A)$ from Lin into itself obeys the *Legendre-Hadamard Condition*:

$$\boldsymbol{a} \otimes \boldsymbol{b} \cdot (\partial_A^2 \hat{\psi}(A))\, [\boldsymbol{a} \otimes \boldsymbol{b}] \geqq 0 \tag{7.9}$$

for all choices of $\boldsymbol{a}$ and $\boldsymbol{b}$ (*vid.* again [22] or [23], *loc. cit.*).

If both $\hat{\psi}$ and $-\hat{\psi}$ are rank-one convex, $\hat{\psi}$ is said to be *rank-one affine*. For $\hat{\psi}$ a rank-one affine mapping there are two real numbers $\gamma_1, \gamma_2$ and two tensors $C_1$, $C_2$ such that the representation formula holds:

$$\hat{\psi}(A) = \gamma_1 + \gamma_2 \det A + C_1 \cdot A^* + C_2 \cdot A. \tag{7.10}$$

The following characterization of rank-one affine mappings of class $C^2$ will be of use later on.

**7.2. Proposition.** *A $C^2$-mapping $\hat{\psi}$ from* Lin *into* $\mathbb{R}$ *is rank-one affine if and only if, for all* $A \in \text{Lin}$, *the mapping* $\partial_A^2 \hat{\psi}(A)$ *obeys the skew-symmetry conditions*

$$\begin{aligned} \boldsymbol{c} \otimes \boldsymbol{d} \cdot (\partial_A^2 \hat{\psi}(A))\, [\boldsymbol{a} \otimes \boldsymbol{b}] &= -\boldsymbol{c} \otimes \boldsymbol{b} \cdot (\partial_A^2 \hat{\psi}(A))\, [\boldsymbol{a} \otimes \boldsymbol{d}] \\ &= -\boldsymbol{a} \otimes \boldsymbol{d} \cdot (\partial_A^2 \hat{\psi}(A))\, [\boldsymbol{c} \otimes \boldsymbol{b}] \end{aligned} \tag{7.11}$$

*for all* $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \in \mathscr{V}$.

**Proof.** In view of (7.7) and (7.8), we conclude that $\hat{\psi}$ is rank-one affine if and only if

$$\alpha \mapsto \hat{\psi}(A + \alpha \boldsymbol{a} \otimes \boldsymbol{b}) = (1 - \alpha)\, \hat{\psi}(A) + \alpha \hat{\psi}(A + \boldsymbol{a} \otimes \boldsymbol{b});$$

differentiating this function twice, we conclude, under the hypotheses, that $\hat{\psi}$ is rank-one affine if and only if

$$\boldsymbol{a} \otimes \boldsymbol{b} \cdot (\partial_A^2 \hat{\psi}(A))\, [\boldsymbol{a} \otimes \boldsymbol{b}] = 0 \tag{7.12}$$

for all $a, b \in \mathscr{V}$ (*cf.* (7.9)); in turn, condition (7.12) is known to be equivalent to $(7.11)_1$.[22] Finally, as for all fixed $A \in \mathrm{Lin}$ the linear mapping $\partial_A^2 \hat{\varphi}(A)$ of Lin into itself is symmetric in the sense of Footnote 8, $(7.11)_2$ also follows. □

Consider now the mapping $\overline{\varphi}$ defined by (7.2), and put

$$\overline{\varphi}_{\mathrm{sub}}(Y) := \inf\{\overline{\varphi}(Y, y) \mid y \in \mathscr{V}\}. \tag{7.13}$$

Using the notation (7.4) again, we see that, for each fixed triplet $(x, u, v) \in (\Sigma_1 \cup \Sigma_2) \times \mathscr{S}(1) \times \mathscr{V}$, the mapping

$$\hat{\tau}_{\mathrm{sub}}(x, u, v, A) := \overline{\varphi}_{\mathrm{sub}}(A) \quad \text{for all } A \in \mathrm{Lin}, \tag{7.14}$$

is well defined. We then have grounds for the following

**7.3. Definition.** *Given a surface interaction potential $T$ with density $\hat{\tau}$ of type* (6.1)–(6.3), *the associated* subpotential *is the tangential functional*

$$T_{\mathrm{sub}}\{f\} := \int_{\partial\Omega} \hat{\tau}_{\mathrm{sub}}(x, n, f, {}^{s}F)\, \mathrm{d}\,(\mathrm{Srf}) \tag{7.15}$$

*with density*

$$\hat{\tau}_{\mathrm{sub}}(x, n, r, {}^{s}F) := \inf\{\hat{\tau}(x, n, f, {}^{s}\nabla f + y \otimes n) \mid y \in \mathscr{V}\}. \tag{7.16}$$

*Likewise,*

$$E_{\mathrm{sub}} := S + T_{\mathrm{sub}} \tag{7.17}$$

*is the subpotential associated with E.*

**7.4. Remark.** The terminology we use is inspired by the similar but different concept of "subenergy" that ERICKSEN [29] has taken from FLORY [30] in order to model certain types of phase transitions within the framework of thermoelasticity. □

**7.5. Proposition.** *Let $T$ be a surface interaction potential, with rank-one convex density $\hat{\tau}$, and let the associated subpotential $T_{\mathrm{sub}}$ be Gateaux-differentiable over $\mathscr{D}$. Then, if $f^{(0)}$ is an extreme point of $E$ jn $\mathscr{D}$, it also renders $E_{\mathrm{sub}}$ stationary.*

**Proof.** Recall that, on confining attention to conservative interactions by means of Definition 4.2, we already stipulated that $E$ is G-differentiable over $\mathscr{D}$. Assuming now, in addition, that it makes sense to write

$$\lim_{\varepsilon \to 0} \frac{E_{\mathrm{sub}}\{f + \varepsilon h\} - E_{\mathrm{sub}}\{f\}}{\varepsilon} =: \delta E_{\mathrm{sub}}\{f\}\,[h] \tag{7.18}$$

for all $f \in \mathscr{D}$ and $h \in \mathscr{H}$, we wish to show that

$$\delta E_{\mathrm{sub}}\{f^{(0)}\}\,[h] = 0 \quad \text{for all } h \in \mathscr{H} \tag{7.19}$$

at each extreme point $f^{(0)}$ of $E$.

[22] Note that $(7.11)_1$ can be written equivalently as

$$((\partial_A^2 \hat{\varphi}(A))\,[a \otimes b])\, c = -((\partial_A^2 \hat{\varphi}(A))\,[a \otimes c])\, b \quad \text{for all } a, b, c \in \mathscr{V}. \tag{\#}$$

First, it follows from the very definitions of $E$ and $E_{\text{sub}}$ that

$$E\{f\} \geqq E_{\text{sub}}\{f\} \quad \text{for all } f \in \mathscr{D}. \tag{7.20}$$

A less obvious fact is that, under the hypotheses,

$$E\{\boldsymbol{f}^{(0)}\} = E_{\text{sub}}\{\boldsymbol{f}^{(0)}\} \tag{7.21}$$

at each extreme point $\boldsymbol{f}^{(0)}$ of $E$. To see this, recall that, at $\boldsymbol{f}^{(0)}$ (6.6) requires that $\hat{\tau}$ is such that

$$(\partial_{\boldsymbol{F}}\hat{\tau}^{(0)})\,\boldsymbol{n} = \boldsymbol{0}. \tag{7.22}$$

Moreover, if $\hat{\tau}$ is rank-one convex, *i.e.*, if the mapping $\hat{\psi}$ defined by (7.4) is rank-one convex for each fixed triplet $(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v}) \in (\Sigma_1 \cup \Sigma_1) \times \mathscr{S}(1) \times \mathscr{V}$, then

$$\boldsymbol{b} \mapsto \hat{\psi}({}^{\text{s}}\nabla \boldsymbol{f}^{(0)} + \boldsymbol{b} \otimes \boldsymbol{n})$$

is a convex mapping whose derivative at $\boldsymbol{b} = {}^{\text{n}}\partial \boldsymbol{f}^{(0)}$ equals $(\partial_{\boldsymbol{F}}\hat{\tau}^{(0)})\,\boldsymbol{n}$ and therefore vanishes because of (7.22); but then this mapping has at ${}^{\text{n}}\partial \boldsymbol{f}^{(0)}$ an absolute minimum, and thus, in view also of definition (7.16),

$$\hat{\tau}_{\text{sub}}^{(0)} = \hat{\tau}^{(0)}, \tag{7.23}$$

which implies (7.21). With this and (7.20), we have

$$E_{\text{sub}}\{\boldsymbol{f}^{(0)} + \varepsilon \boldsymbol{h}\} - E_{\text{sub}}\{\boldsymbol{f}^{(0)}\} \leqq E\{\boldsymbol{f}^{(0)} + \varepsilon \boldsymbol{h}\} - E\{\boldsymbol{f}^{(0)}\},$$

so that

$$\lim_{\varepsilon \to 0^+} \frac{E_{\text{sub}}\{\boldsymbol{f}^{(0)} + \varepsilon \boldsymbol{h}\} - E_{\text{sub}}\{\boldsymbol{f}^{(0)}\}}{\varepsilon} \leqq 0 \tag{7.24$_1$}$$

and

$$\lim_{\varepsilon \to 0^-} \frac{E_{\text{sub}}\{\boldsymbol{f}^{(0)} + \varepsilon \boldsymbol{h}\} - E_{\text{sub}}\{\boldsymbol{f}^{(0)}\}}{\varepsilon} \geqq 0. \tag{7.24$_2$}$$

(7.19) then follows from (7.24) and the assumed differentiability of $E_{\text{sub}}$. □

The last proposition shows that, for reasonably general assignments of surface interaction potentials depending in principle on the entire displacement gradient, the extreme points of the resulting total potentials integrate a subset of the extremal set of an associated functional so constructed as to give to normal derivatives the least weight possible in the minimization process.

### *7.2 Simple Potentials*

Here and henceforth we restrict attention to tangential potentials

$$T\{\boldsymbol{f}\} = \int_{\partial\Omega} \hat{\tau}(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{f}, {}^{\text{s}}\boldsymbol{F})\, \mathrm{d}\,(\text{Srf}). \tag{7.25}$$

As is clear from (6.16), these potentials correspond in general to surface loads depending on first and second tangential derivatives. Our goal in this section is to characterize a class of surface potentials such that the corresponding surface loads are *sinmple, i.e.*, depend at most on first derivatives. Interestingly, the total potentials featuring simple surface interactions may be thought of as defined over a domain larger than $\mathscr{D}$, as second derivatives are not assumed to exist at points of $\Sigma_1 \cup \Sigma_2$.

**7.6. Definition.** *A tangential potential $T$ with density $\hat{\tau}$ is simple if it gives rise to a simple surface load $\hat{s}$*, i.e., *if*

$$s = -\partial_f \hat{\tau}(x, n, f, {}^sF) + {}^s\mathrm{Div}\, \partial_F \hat{\tau}(x, n, f, {}^sF) = \hat{s}(x, n, f, {}^sF). \qquad (7.26)$$

**7.7. Proposition.** *A tangential potential $T$ is simple if and only if its density $\tau$ admits the following representation formula:*

$$\hat{\tau}(x, u, v, A) = \hat{\gamma}(x, u, v) + (\hat{c}(x, u, v) \otimes u) \cdot A^* + \hat{C}(x, u, v)\, U \cdot A. \quad (7.27)$$

**Proof.** Observe first that, by $(7.26)_1$, the part of $s$ containing second derivatives of $f$ vanishes identically on $\Sigma_1 \cup \Sigma_2$ if and only if $\partial_A^2 \hat{\tau}(x, u, v, A)$ has the skew-symmetry property $(7.11)_1$ for all $(x, u, v)) \in (\Sigma_1 \cup \Sigma_2) \times \mathscr{S}(1) \times \mathscr{V}$, *i.e.*, in view of (7.4) and Proposition 7.3, if and only if the mapping

$$A \mapsto \hat{\psi}(A) = \hat{\tau}(x, u, v, A) \qquad (7.28)$$

is rank-one affine. Secondly, by Proposition 7.1, if $T$ is tangential we may replace $A$ with $A(I - u \otimes u)$ in the representation formula (7.10); the result is

$$\hat{\psi}(A) = \gamma_1 + C_1 \cdot (A(I - u \otimes u))^* + C_2 \cdot A(I - u \otimes u)$$

or rather, by (1.28),

$$\hat{\psi}(A) = \gamma_1 + (C_1 u \otimes u) \cdot A^* + C_2(I - u \otimes u) \cdot A;$$

from this, with the notations

$$\gamma_1 =: \gamma_1, \quad C_1 u =: c \quad \text{and} \quad -C_2 U =: C \quad (\textit{cf.}\ (1.11)),$$

(7.27) follows. □

Resuming now the developments at the end of Subsection 5.1 in the present more general context, we turn to characterize those simple surface loadings that are alternatively described by surface and volume interaction potentials. As our attention is restricted, for the rest of this subsection, to bodies comprising regular regions $\Omega$, we shall drop the explicit dependence on the normal $n$ to $\partial\Omega$ in the expression of surface interaction potentials (*cf.* Remark 6.2).

We state without proof a proposition where some results of [15] and [16] are rephrased.

**7.8. Proposition.** *A functional*

$$F\{f\} = \int_\Omega \hat{\varphi}(x, f, \nabla f)\, \mathrm{d}\,(\mathrm{Vol})$$

*on* $C^2(\bar{\Omega})$ *is a null Lagrangian if and only if there is a smooth vector-valued mapping*

$$(\boldsymbol{x}, \boldsymbol{u}, A) \mapsto \hat{\boldsymbol{v}}(\boldsymbol{x}, \boldsymbol{u}, A), \tag{7.29}$$

*obeying the skew-symmetry condition*

$$(\partial_A(\hat{\boldsymbol{v}}(\boldsymbol{x}, \boldsymbol{v}, A) \cdot \boldsymbol{b}))\, \boldsymbol{a} = -(\partial_A(\hat{\boldsymbol{v}}(\boldsymbol{x}, \boldsymbol{v}, A) \cdot \boldsymbol{a}))\, \boldsymbol{b} \tag{7.30}$$

*for all* $\boldsymbol{a}, \boldsymbol{b} \in \mathscr{V}$ *and for all fixed* $(\boldsymbol{x}, \boldsymbol{u}, A) \in \bar{\Omega} \times \mathscr{V} \times \mathrm{Lin}$, *such that*

$$\hat{\varphi}(\boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x}), \nabla \boldsymbol{f}(\boldsymbol{x})) = \mathrm{Div}\, \hat{\boldsymbol{v}}(\boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x}), \nabla \boldsymbol{f}(\boldsymbol{x})) \tag{7.31}$$

*for all* $\boldsymbol{x} \in \bar{\Omega}$ *and* $\boldsymbol{f} \in C^2(\bar{\Omega})$.

We are now in a position to state our announced characterization.

**7.9. Proposition.** *Given a simple tangential potential*

$$T\{\boldsymbol{f}\} = \int_{\partial\Omega} \hat{\tau}(\boldsymbol{x}, \boldsymbol{f}, {}^{s}\boldsymbol{F})\, \mathrm{d}\,(\mathrm{Srf}), \tag{7.32}$$

*there is an associated volume potential*

$$F\{\boldsymbol{f}\} = \int_{\Omega} \hat{\varphi}(\boldsymbol{x}, \boldsymbol{f}, \boldsymbol{F})\, \mathrm{d}\,(\mathrm{Vol}) \tag{7.33}$$

*such that*

$$T\{\boldsymbol{f}\} = F\{\boldsymbol{f}\} \quad \text{for all } \boldsymbol{f} \in C^2(\bar{\Omega}); \tag{7.34}$$

*F is a null Lagrangian. Conversely, if a volume potential F as in* (7.33) *is a null Lagrangian, then there is a simple tangential potential T as in* (7.32) *such that* (7.34) *holds.*

**Proof.** Given $\hat{\tau}$, for $(\boldsymbol{x}, \boldsymbol{u}, A) \in \partial\Omega \times \mathscr{V} \times \mathrm{Lin}$, we consider the vector-valued mapping defined by

$$\hat{\boldsymbol{v}}(\boldsymbol{x}, \boldsymbol{u}, A) := \hat{\tau}(\boldsymbol{x}, \boldsymbol{u}, A)\, \boldsymbol{n}(\boldsymbol{x}) - (\partial_A \hat{\tau}(\boldsymbol{x}, \boldsymbol{u}, A))^{\mathrm{T}} A\boldsymbol{n}(\boldsymbol{x}). \tag{7.35}$$

We observe first that, because $T$ is tangential,

$$(\partial_A \hat{\tau}(\boldsymbol{x}, \boldsymbol{u}, A))\, \boldsymbol{n}(\boldsymbol{x}) \equiv \boldsymbol{0},$$

so that

$$\hat{\boldsymbol{v}}(\boldsymbol{x}, \boldsymbol{u}, A) \cdot \boldsymbol{n}(\boldsymbol{x}) = \hat{\tau}(\boldsymbol{x}, \boldsymbol{u}, A). \tag{7.36}$$

Secondly, differentiating $\hat{\boldsymbol{v}}$ with respect to the third argument, we see that

$$(\partial_A(\hat{\boldsymbol{v}} \cdot \boldsymbol{b}))\, \boldsymbol{a} = -[(\boldsymbol{a} \cdot \boldsymbol{n})\,(\partial_A \hat{\tau})\, \boldsymbol{b} - (\boldsymbol{b} \cdot \boldsymbol{n})\,(\partial_A \hat{\tau})\, \boldsymbol{a}] - ((\partial_A^2 \hat{\tau})\,[\boldsymbol{F}\boldsymbol{n} \otimes \boldsymbol{b}])\, \boldsymbol{a}. \tag{7.37}$$

On the right-hand side of (7.37), the addend within square brackets is obviously skew-symmetric with respect to the interchange of $\boldsymbol{a}$ and $\boldsymbol{b}$, whereas the other addend is likewise skew-symmetric because $T$ is simple and hence, by Proposition 7.7, $\hat{\tau}$ obeys $(7.11)_1$; consequently, $\hat{\boldsymbol{v}}$ satisfies (7.30).[23] As $\Omega$ is by assumption

[23] It can be shown that the most general $\hat{\boldsymbol{v}}$ obeying (7.30) and (7.36) has the form $\hat{\boldsymbol{v}}(\boldsymbol{x}, \boldsymbol{u}, A) = \hat{\tau}(\boldsymbol{x}, \boldsymbol{u}, A)\, \boldsymbol{n}(\boldsymbol{x}) - (\partial_A \hat{\tau}(\boldsymbol{x}, \boldsymbol{u}, A))^{\mathrm{T}} A\boldsymbol{n}(\boldsymbol{x}) + (\boldsymbol{c}_1(\boldsymbol{x}, \boldsymbol{u}) + A^{\mathrm{T}} \boldsymbol{c}_2(\boldsymbol{x}, \boldsymbol{u})) \times \boldsymbol{n}(\boldsymbol{x})$, with $\boldsymbol{c}_1(\boldsymbol{x}, \boldsymbol{u})$, $\boldsymbol{c}_2(\boldsymbol{x}, \boldsymbol{u})$ two arbitrary vector-valued mappings.

of class $C^3$, $\hat{v}$ as in (7.35) can be extended to a class $C^2$ mapping over $\bar{\Omega}\times\mathcal{V}\times\mathrm{Lin}$; the direct assertion then follows by defining $\hat{\varphi}$ as in (7.31).

A proof of the converse begins by constructing $\hat{\tau}$ from the vector-valued mapping $\hat{v}$ associated with a given null Lagrangian density $\hat{\varphi}$ by Proposition 7.8:

$$\hat{\tau}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{A}) := \hat{\boldsymbol{v}}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{A}) \cdot \boldsymbol{n}(\boldsymbol{x}). \tag{7.38}$$

It remains to verify that such a $\hat{\tau}$ is both tangential and simple. But, $\hat{\tau}$ is tangential because, by (7.38) and (7.30),

$$(\partial_A \hat{\tau})\, \boldsymbol{n} = (\partial_A (\hat{\boldsymbol{v}} \cdot \boldsymbol{n}))\, \boldsymbol{n} = \boldsymbol{0},$$

so that (7.6) follows from appealing to Proposition 7.1; $\hat{\tau}$ is simple because, by (7.33) and (7.31),

$$\delta F\{f\}\,[\boldsymbol{h}] = \int_{\partial\Omega} (\partial_F \hat{\varphi}(\boldsymbol{x}, f, F))\, \boldsymbol{n} \cdot \boldsymbol{h}\, \mathrm{d}\,(\mathrm{Srf})$$

and thus the corresponding surface loading

$$\hat{\boldsymbol{s}} = -(\partial_F \hat{\varphi})\, \boldsymbol{n}$$

depends only on first derivatives. □

### *7.3 Regular Potentials*

Let $\mathcal{O}$ denote the collection of all domains $\Omega$ described under (ii) in the beginning of Section 2; moreover, for $\Omega$ chosen in $\mathcal{O}$, let the mappings $\hat{\tau}_\alpha$ ($\alpha = 1, 2$) define through use of (6.1)–(6.3) a surface interaction potential density $\hat{\tau}$.

On the singular part $\Gamma$ of $\partial\Omega$ the equilibrium condition (6.8) reads:

$$(\partial_F \hat{\tau}_1(\boldsymbol{x}, \boldsymbol{n}_1(\boldsymbol{x}), f^{(0)}(\boldsymbol{x}), \nabla_1 f^{(0)}(\boldsymbol{x}))\, \boldsymbol{n}_2(\boldsymbol{x}) + (\partial_F \hat{\tau}_2(\boldsymbol{x}, \boldsymbol{n}_2(\boldsymbol{x}), f^{(0)}(\boldsymbol{x}), \nabla_2 f^{(0)}(\boldsymbol{x})), \boldsymbol{n}_1(\boldsymbol{x}) \equiv \boldsymbol{0} \tag*{(7.39)$_1$}$$

at each extremum point $f^{(0)} \in \mathcal{D}$ of the underlying total energy functional, when (6.7) applies; otherwise, if $\Gamma$ is everywhere cuspidal, *i.e.*, if $\boldsymbol{n}_1 = -\boldsymbol{n}_2$ on $\Gamma$, (7.39)$_1$ is replaced by

$$[\partial_F \hat{\tau}_1(\boldsymbol{x}, \boldsymbol{n}_1(\boldsymbol{x}), f^{(0)}(\boldsymbol{x}), \nabla_1 f^{(0)}(\boldsymbol{x})) + \partial_F \hat{\tau}_2(\boldsymbol{x}, -\boldsymbol{n}_1(\boldsymbol{x}), f^{(0)}(\boldsymbol{x}), \nabla_2 f^{(0)}(\boldsymbol{x}))]\, (\boldsymbol{t}(\boldsymbol{x}) \times \boldsymbol{n}_1(\boldsymbol{x})) \equiv \boldsymbol{0} \tag*{(7.39)$_2$}$$

(*cf.* Remark 6.3). On the other hand, the companion equilibrium condition (6.6) prevailing on the regular part of $\partial\Omega$ can be split into separate tangentiality conditions for the individual $\hat{\tau}_\alpha$:

$$(\partial_F \hat{\tau}_\alpha(\boldsymbol{x}, \boldsymbol{n}(\boldsymbol{x}), f^{(0)}, \nabla f^{(0)})\, \boldsymbol{n}(\boldsymbol{x}) \equiv \boldsymbol{0} \quad \text{on } \Sigma_\alpha, \quad \alpha = 1, 2. \tag*{(7.39)$_3$}$$

Comparison with (7.39)$_3$ suggests that we regard (7.39)$_{1,2}$ as conditions of joint tangentiality on the pair $(\hat{\tau}_1, \hat{\tau}_2)$, prevailing at extrema. We may ask whether there are surface interactions such as to satisfy all conditions (7.39) identically in $\mathcal{O}$ and $\mathcal{D}$ with $\nabla_1 f \equiv \nabla_2 f$.

Just as the analogous question which led us to consider potentials with density of type (7.6) obeying condition $(7.39)_3$ identically in $\mathscr{D}$, this question is constitutive in nature.

In order to answer it, we begin by noting, that, as far as condition (7.39) is concerned, the choice of an element $\Omega \in \mathcal{O}$ amounts to choice of a class $C^3$ closed simple curve $\Gamma$ and two class $C^2$ normal fields $\boldsymbol{n}_\alpha = \boldsymbol{n}_\alpha(\boldsymbol{x})$ over $\Gamma$. If they have to satisfy (7.39) whatever $\Gamma$ may be, the mappings $\hat{\tau}_\alpha$ cannot depend explicitly on the place $\boldsymbol{x}$. We then introduce the following definition.

**7.10. Definition.** *A pair* $(\hat{\tau}_1, \hat{\tau}_2)$ *of class* $C^2$ *mappings*

$$\hat{\tau}_\alpha : \mathscr{S}(1) \times \mathscr{V} \times \mathrm{Lin} \to \mathbb{R}, \qquad \tau = \hat{\tau}_\alpha(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{A}), \qquad \alpha = 1, 2,$$

*is* jointly tangential *if, for all* $(\boldsymbol{v}, \boldsymbol{A}) \in \mathscr{V} \times \mathrm{Lin}$ *fixed, the following two conditions hold*:

$$\partial_A \hat{\tau}_1(\boldsymbol{u}_1, \boldsymbol{v}, \boldsymbol{A})\, \boldsymbol{u}_2 + \partial_A \hat{\tau}_2(\boldsymbol{u}_2, \boldsymbol{v}, \boldsymbol{A})\, \boldsymbol{u}_1 = \boldsymbol{0} \quad \textit{for all } \boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathscr{S}(1); \qquad (7.40)_1$$

$$\partial_A \hat{\tau}_\alpha(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{A})\, \boldsymbol{u} = \boldsymbol{0}, \alpha = 1, 2, \quad \textit{for all } \boldsymbol{u} \in \mathscr{S}(1). \qquad (7.40)_3$$

**7.11. Proposition.** *If* $(\hat{\tau}_1, \hat{\tau}_2)$ *is a jointly tangential pair, then for both the associated membrane stresses* $\partial_A \hat{\tau}_\alpha$ *one and the same representation formula holds*:

$$\partial_A \hat{\tau}_1(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{A}) = \partial_A \hat{\tau}_2(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{A}) = \hat{\boldsymbol{B}}(\boldsymbol{v}, \boldsymbol{A})\, \boldsymbol{U} \qquad (7.41)$$

*for all* $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{A}) \in \mathscr{S}(1) \times \mathscr{V} \times \mathrm{Lin}$, *with* $\hat{\boldsymbol{B}}$ *a class* $C^1$ *mapping from* $\mathscr{V} \times \mathrm{Lin}$ *into* Lin, *and* $\boldsymbol{U}$ *the skew tensor associated to* $\boldsymbol{u}$ *by* (1.5).

**7.12. Remark.** A straightforward, but important, consequence of (7.41) is that

$$\partial_A \hat{\tau}_1(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{A}) + \partial_A \hat{\tau}_2(-\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{A}) = \boldsymbol{0} \quad \text{for all } \boldsymbol{u} \in \mathscr{S}(1). \qquad (7.40)_2$$

Thus, if $(\hat{\tau}_1, \hat{\tau}_2)$ is jointly tangential and if, for each $\Omega$ in $\mathcal{O}$ and $f \in C^2(\bar{\Omega})$, we define the class $C^2$ tensor fields

$$\boldsymbol{x} \mapsto \hat{\boldsymbol{B}}(f(\boldsymbol{x}), \nabla f(\boldsymbol{x}))\, \boldsymbol{N}_\alpha(\boldsymbol{x})|_{\Sigma_\alpha \cup \Gamma}, \qquad \alpha = 1, 2, \qquad (7.42)$$

with $\boldsymbol{N}_\alpha(\boldsymbol{x})$ the skew tensor associated to $\boldsymbol{n}_\alpha(\boldsymbol{x})$, then, as $(7.40)_i$ reflects the constitutive significance of $(7.39)_i$ for $i = 1, 2, 3$ in the order, the fields (7.42) satisfy all conditions (7.39) identically in $\mathcal{O}$ and $\mathscr{D}$. □

**Proof of Proposition 7.11.** Let the variables $(\boldsymbol{v}, \boldsymbol{A})$ be fixed in $\mathscr{V} \times \mathrm{Lin}$ (for convenience, the arguments $\boldsymbol{v}, \boldsymbol{A}$ will not be displayed in the following developments). As $\hat{\tau}_\alpha$ satisfies $(7.40)_3$, we conclude from (1.11) in Proposition 1.1 that, for all $\boldsymbol{u} \in \mathscr{S}(1)$,

$$\partial_A \hat{\tau}_\alpha(\boldsymbol{u}) = \hat{\boldsymbol{B}}_\alpha(\boldsymbol{u})\, \boldsymbol{U}, \qquad (7.43)$$

with $\boldsymbol{U}$ the skew tensor associated to $\boldsymbol{u}$ by (1.5) and

$$\hat{\boldsymbol{B}}_\alpha(\boldsymbol{u}) := -\partial_A \hat{\tau}_\alpha(\boldsymbol{u})\, \boldsymbol{U}. \qquad (7.44)$$

We then rewrite $(7.40)_1$ as

$$\hat{B}_1(|u_1|^{-1} u_1)\, U_1 u_2 + \hat{B}_2(|u_2|^{-1} u_2)\, U_1 u_2 = 0, \tag{7.45}$$

this time for all $u_1, u_2 \in \mathscr{V}$. Differentiating (7.45) with respect to $u_1$ and $u_2$ in the direction of $h_1$ and $h_2$, respectively, we have

$$(\partial_{u_1}(\hat{B}_1(|u_1|^{-1} u_1)\, U_1)\,[h_1])\, h_2 + (\partial_{u_2}(\hat{B}_2(|u_2|^{-1} u_2)\, U_2)\,[h_2])\, h_1 = 0$$

identically in $u_1$, $u_2$ and $h_1$, $h_2$. It follows that there exists a linear mapping $\mathbb{B}$ from $\mathscr{V}$ into Lin, with $\mathbb{B}$ depending at most on $(v, A)$, such that

$$\begin{aligned}(\mathbb{B}[h_1])\, h_2 &:= (\partial_{u_1}(\hat{B}_1(|u_1|^{-1} u_1)\, U_1)\,[h_1])\, h_2 \\ &= -(\partial_{u_2}(\hat{B}_2(|u_2|^{-1} u_2)\, U_2)\,[h_2])\, h_1 .\end{aligned} \tag{7.46}$$

Integrating $(7.46)_1$ we obtain

$$\hat{B}_1(|u_1|^{-1} u_1)\, U_1 = \mathbb{B}[u_1] + C, \tag{7.47}$$

where $C$ also depends at most on $(v, A)$; but, by multiplication of (7.47) on the right by $u_1$, we see that

$$0 = (\mathbb{B}[u_1])\, u_1 + C u_1 ,$$

so that, as $u_1$ is here arbitrary,

$$(\mathbb{B}[h_1])\, h_2 = -(\mathbb{B}([h_2])\, h_1 \quad \text{for all } h_1, h_2 \in \mathscr{V}, \quad C = 0. \tag{7.48}$$

(7.43), (7.47) and (7.48) yield

$$\partial_A \hat{\tau}_1(u) = \partial_A \hat{\tau}_2(u) = \mathbb{B}[u]$$

for all $u \in \mathscr{S}(1)$; moreover, due to $(7.48)_1$, the condition

$$\mathbb{B}[u] = BU \quad \text{for all } u \in \mathscr{V}$$

uniquely determines the second-order tensor $B$ in terms of $\mathbb{B}$.[24] □

Proposition 7.11 indicates that the mappings $\hat{\tau}_1, \hat{\tau}_2$ cannot differ much from each other if $(\hat{\tau}_1, \hat{\tau}_2)$ has to be a jointly tangential pair. Actually, an even more stringent result than (7.41) is valid.

**7.13. Proposition.** *If $(\hat{\tau}_1, \hat{\tau}_2)$ is a jointly tangential pair, then there are two class $C^2$ mappings*

$$\hat{\gamma}_\alpha : \mathscr{S}(1) \times \mathscr{V} \to \mathbb{R}, \quad \gamma_\alpha = \hat{\gamma}_\alpha(u, v) \tag{7.49}$$

*such that the following representation formulae hold*:

$$\hat{\tau}_\alpha(u, v, A) - \hat{\gamma}_\alpha(u, v) = (\hat{c}(v) \otimes u) \cdot A^* + \hat{C}(v)\, U \cdot A, \quad \alpha = 1, 2. \tag{7.50}$$

[24] To detail the algebra, we note that, in view of $(7.46)_1$, $\mathbb{B}$ can be repesented as $\mathbb{B} = b \otimes W$, with $b \in \mathscr{V}$ and $W \in \mathrm{Skw}$, so that in turn $B$ has the representation $B = b \otimes w$, where $w$ is of course the vector associated with $W$.

**Proof.** Recall that, in view of (7.41), the membrane stress $\partial_A \hat{\tau}_\alpha(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{A})$ depends linearly on $\boldsymbol{u}$:

$$\partial_A \hat{\tau}_\alpha(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{A}) = \hat{\boldsymbol{B}}(\boldsymbol{v}, \boldsymbol{A})\, \boldsymbol{U} \tag{7.51}$$

for $\alpha = 1$ or 2.

Our first step is to prove that $\hat{\tau}_\alpha$ is rank-one affine, *i.e.*, that, for all $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{A}) \in \mathscr{S}(1) \times \mathscr{V} \times \mathrm{Lin}$,

$$\boldsymbol{c} \otimes \boldsymbol{d} \cdot (\partial_A^2 \hat{\tau}_\alpha(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{A}))\, [\boldsymbol{c} \otimes \boldsymbol{d}] = 0 \tag{7.52}$$

for all $\boldsymbol{c}, \boldsymbol{d} \in \mathscr{V}$ (*cf.* (7.12)). We begin by noticing a consequence of (7.51):

$$(\partial_A^2 \hat{\tau}_\alpha(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{A}))\, [\boldsymbol{H}] = \partial_A \hat{\boldsymbol{B}}(\boldsymbol{v}, \boldsymbol{A})\, [\boldsymbol{H}]\, \boldsymbol{U} \quad \text{for all } \boldsymbol{H} \in \mathrm{Lin}, \tag{7.53}$$

and by writing

$$\begin{aligned} \boldsymbol{c} \otimes \boldsymbol{d} \cdot (\partial_A^2 \hat{\tau}_\alpha(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{A}))\, [\boldsymbol{c} \otimes \boldsymbol{d}] &= \boldsymbol{c} \cdot (\partial_A \boldsymbol{B}(\boldsymbol{v}, \boldsymbol{A})\, [\boldsymbol{c} \otimes \boldsymbol{d}]\, \boldsymbol{U})\, \boldsymbol{d} \\ &= -\boldsymbol{c} \otimes \boldsymbol{u} \cdot (\partial_A \hat{\boldsymbol{B}}(\boldsymbol{v}, \boldsymbol{A})\, [\boldsymbol{c} \otimes \boldsymbol{d}]\, \boldsymbol{D}) = -\boldsymbol{c} \otimes \boldsymbol{u} \cdot \partial_A^2 \hat{\tau}_\alpha(\boldsymbol{d}, \boldsymbol{v}, \boldsymbol{A})\, [\boldsymbol{c} \otimes \boldsymbol{d}], \end{aligned} \tag{7.54}$$

where repeated use has been made of (7.53) and $\boldsymbol{D}$ is the skew tensor uniquely associated with $\boldsymbol{d}$. We notice next that, as $\partial_A^2 \hat{\tau}_\alpha(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{A})$ is symmetric,

$$\partial_A^2 \hat{\tau}_\alpha(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{A})\, [\boldsymbol{H}] = \partial_A (\partial_A \hat{\tau}_\alpha(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{A}) \cdot \boldsymbol{H}) \quad \text{for all } \boldsymbol{H} \in \mathrm{Lin}, \tag{7.55}$$

so that in particular, for $\boldsymbol{H} = \boldsymbol{c} \otimes \boldsymbol{d}$,

$$\partial_A^2 \hat{\tau}_\alpha(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{A})\, [\boldsymbol{c} \otimes \boldsymbol{d}] = \partial_A (\boldsymbol{c} \cdot \partial_A \hat{\tau}_\alpha(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{A})\, \boldsymbol{d}). \tag{7.56}$$

Combining then (7.54) and (7.56), and recalling that $\hat{\tau}_\alpha$ is tangential:

$$\partial_A \hat{\tau}_\alpha(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{A})\, \boldsymbol{u} = \boldsymbol{0} \quad \text{for all } (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{A}) \in \mathscr{S}(1) \times \mathscr{V} \times \mathrm{Lin}, \tag{7.57}$$

we establish (7.52), as

$$-\boldsymbol{c} \otimes \boldsymbol{u} \cdot \partial_A^2 \hat{\tau}_\alpha(\boldsymbol{d}, \boldsymbol{v}, \boldsymbol{A})\, [\boldsymbol{c} \otimes \boldsymbol{d}] = -\boldsymbol{c} \otimes \boldsymbol{u} \cdot \partial_A (\boldsymbol{c} \cdot \partial_A \hat{\tau}_\alpha(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{A})\, \boldsymbol{d}) = 0.$$

Our second step consists in adapting in the obvious way the argument in the proof of Proposition 7.7 to conclude that $\hat{\tau}_\alpha$, being rank-one affine and tangential (and being, moreover, homogeneous with associated membrane stress linear in $\boldsymbol{u}$ as (7.51) requires), must have the form

$$\hat{\tau}_\alpha(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{A}) = \hat{\gamma}_\alpha(\boldsymbol{u}, \boldsymbol{v}) + (\hat{\boldsymbol{c}}_\alpha(\boldsymbol{v}) \otimes \boldsymbol{u}) \cdot \boldsymbol{A}^* + \hat{\boldsymbol{C}}_\alpha(\boldsymbol{v})\, \boldsymbol{U} \cdot \boldsymbol{A}, \tag{7.58}$$

with

$$\hat{\boldsymbol{C}}_\alpha(\boldsymbol{v})\, \boldsymbol{U} + \partial_A \boldsymbol{A}^*(\boldsymbol{A})\, [\hat{\boldsymbol{c}}_\alpha(\boldsymbol{v}) \otimes \boldsymbol{u}] = \hat{\boldsymbol{B}}(\boldsymbol{v}, \boldsymbol{A})\, \boldsymbol{U}, \quad \alpha = 1, 2. \tag{7.59}$$

Finally, from (7.59) we deduce that, for each fixed $\boldsymbol{v} \in \mathscr{V}$,

$$[\hat{\boldsymbol{C}}_1(\boldsymbol{v}) - \hat{\boldsymbol{C}}_2(\boldsymbol{v})]\, \boldsymbol{U} \cdot \boldsymbol{A} + 2(\hat{\boldsymbol{c}}_1(\boldsymbol{v}) - \hat{\boldsymbol{c}}_2(\boldsymbol{v})) \otimes \boldsymbol{u} \cdot \boldsymbol{A}^* = 0, \tag{7.60}$$

identically in $\boldsymbol{u}$ and $\boldsymbol{A}$.[25] Exploiting the arbitrariness in the choice of $\boldsymbol{A}$, we see

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[25] To arrive at (7.60) we have made use of the fact, following from the symmetry of $\partial_A \boldsymbol{A}^*$, (1.18), (1.21) and (1.30), that

$$\partial_A \boldsymbol{A}^*[\boldsymbol{A}] = 2\boldsymbol{A}^*.$$

with the use of (1.15) that (7.60) is equivalent to

$$[\hat{C}_1(v) - \hat{C}_2(v)]\, U = \mathbf{0} \quad \text{and} \quad (\hat{c}_1(v) - \hat{c}_2(v)) \otimes u = \mathbf{0}$$

identically in $u$, or rather, to

$$\hat{C}_1(v) = \hat{C}_2(v) \quad \text{and} \quad \hat{c}_1(v) = \hat{c}_2(v) \quad \text{for all } v \in \mathscr{V}. \tag{7.61}$$

(7.58) and (7.61) together then yield the desired conclusion. □

In the light of the above proposition we see that, given a jointly tangential pair $(\hat{\tau}_1, \hat{\tau}_2)$ and a domain $\Omega \in \mathcal{O}$, one can construct an associated homogeneous and tangential surface interaction potential $T$ with density

$$\hat{\tau} : \mathscr{S}(1) \times \mathscr{V} \times \mathrm{Lin} \to \mathbb{R}, \quad \tau = \hat{\tau}(u, v, A) = \hat{\gamma}(u, v) + (\hat{c}(v) \otimes u) \cdot A^* + \hat{C}(v)\, U \cdot A,$$

by requiring that, for all $f \in C^2(\bar{\Omega})$,

$$\hat{\gamma}(n(x), f(x))|_{\Sigma_\alpha} := \hat{\gamma}_\alpha(n(x), f(x))|_{\Sigma_\alpha}, \quad \alpha = 1, 2. \tag{7.62}$$

As such a $T$ is simple, it is obvious from Proposition 7.9 that one can also construct a null Lagrangian

$$F\{f\} = \int_\Omega \hat{\varphi}(f, F)\, d\,(\mathrm{Vol}) \tag{7.63}$$

such that

$$T\{f\} = F\{f\} + \int_{\partial\Omega} \hat{\gamma}(n, f)\, d\,(\mathrm{Srf}) \quad \text{for all } f \in C^2(\bar{\Omega}). \tag{7.64}$$

Conversely, given a null Lagrangian density $\hat{\varphi}$ as in (7.63) and two mappings

$$\hat{\gamma}_\alpha : \mathscr{S}(1) \times \mathscr{V} \to \mathbb{R}, \quad \gamma_\alpha = \hat{\gamma}_\alpha(u, v), \quad \alpha = 1, 2, \tag{7.65}$$

it is not difficult to construct a jointly tangential pair of mappings $(\hat{\tau}_1, \hat{\tau}_2)$ such that the homogeneous tangential potential $T$ defined by (7.64) be regular with respect to $(\hat{\tau}_1, \hat{\tau}_2)$.

In this way, surface interactions obeying all conditions (7.39) identically in $\mathcal{O}$ and $\mathscr{D}$ are characterized as those whose surface densities are specified by a jointly tangential pair of mappings, and an answer to our original question is thereby provided. Without great loss of generality, as Proposition 7.13 makes clear, we finally turn to consider interactions specified by one density mapping, rather than two. In this connection, we introduce a definition which parallels strictly Definition 7.10.

**7.14. Definition.** *A class* $C^2$ *mapping*

$$\hat{\tau} : \mathscr{S}(1) \times \mathscr{V} \times \mathrm{Lin} \to \mathbb{R}, \quad \tau = \hat{\tau}(u, v, A),$$

*is* regular *if, for all* $(v, A) \in \mathscr{V} \times \mathrm{Lin}$ *fixed, the following condition holds:*

$$\partial_A \hat{\tau}(u_1, v, A)\, u_2 + \partial_A \hat{\tau}(u_2, v, A)\, u_1 = \mathbf{0} \quad \textit{for all } u_1, u_2 \in \mathscr{S}(1). \tag{7.66}$$

**7.15. Remark.** Clearly, for $\hat{\tau}_1 \equiv \hat{\tau}_2$ $(7.40)_1$ reduces to (7.66); moreover, $(7.40)_3$ reduces to

$$(\partial_A \hat{\tau}(u, v, A))\, u = \mathbf{0} \quad \text{for all } u \in \mathscr{S}(1), \tag{7.67}$$

and taking $u_1 = u_2 = u$ in (7.66) we see that a regular $\hat{\tau}$ satisfies (7.67); finally, again as a consequence of the linearity of $\partial_A \hat{\tau}$ (*cf.* Remark 7.12), we see that a regular $\hat{\tau}$ also satisfies

$$\partial_A \hat{\tau}(\boldsymbol{u}, \boldsymbol{v}, A) + \partial_A \hat{\tau}(-\boldsymbol{u}, \boldsymbol{v}, A) = \boldsymbol{0} \quad \text{for all } \boldsymbol{u} \in \mathcal{S}(1), \tag{7.68}$$

which is the form presently taken by $(7.40)_2$. □

In view of Proposition 7.13, a regular $\hat{\tau}$ admits the representation

$$\hat{\tau}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{A}) = \hat{\gamma}(\boldsymbol{u}, \boldsymbol{v}) + (\hat{\boldsymbol{c}}(\boldsymbol{v}) \otimes \boldsymbol{u}) \cdot \boldsymbol{A}^* + \hat{\boldsymbol{C}}(\boldsymbol{v})\, \boldsymbol{U} \cdot \boldsymbol{A}. \tag{7.69}$$

For each $\Omega \in \mathcal{O}$ fixed, we say that the homogeneous tangential potential

$$T\{f\} = \int_{\partial\Omega} \hat{\tau}(\boldsymbol{n}, \boldsymbol{f}, {}^{s}\boldsymbol{F})\, \mathrm{d}\,(\mathrm{Srf}), \tag{7.70}$$

with density $\hat{\tau}$ given by (7.69), is a *regular* surface interaction potential. Regular interactions can be regarded as a generalization of a pressure interaction on the entire boundary, whose density is (*cf.* $(5.5)_1$)

$$\hat{\tau}_p(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{A}) = \hat{\pi}(\boldsymbol{v})\, (\boldsymbol{v} \otimes \boldsymbol{u}) \cdot \boldsymbol{A}^*; \tag{7.71}$$

they are characterized by our last proposition.

**7.16 Proposition.** *A tangential potential $T$ as in* (7.70) *is regular if and only if there is a null Lagrangian $F$ as in* (7.63) *and a mapping*

$$\hat{\gamma} : \mathcal{S}(1) \times \mathcal{V} \to \mathbb{R}, \qquad \gamma = \hat{\gamma}(\boldsymbol{u}, \boldsymbol{v}),$$

*such that* (7.64) *holds.*

**Proof.** At this stage, it suffices to show how to construct the density $\hat{\varphi}$ of $F$ from the density $\hat{\tau}$ of $T$ (and conversely). To this end, given $\hat{\tau}$ as in (7.69), for each $\boldsymbol{f} \in \mathcal{D}$ we write

$$\hat{\tau}(\boldsymbol{n}, \boldsymbol{f}, \boldsymbol{F}) - \hat{\gamma}(\boldsymbol{n}, \boldsymbol{f}) = \hat{\boldsymbol{v}}(\boldsymbol{f}, \boldsymbol{F}) \cdot \boldsymbol{n}, \tag{7.72$_1$}$$

with

$$\hat{\boldsymbol{v}}(\boldsymbol{f}, \boldsymbol{F}) \cdot \boldsymbol{w} := \boldsymbol{F}^{*\mathrm{T}} \hat{\boldsymbol{c}}(\boldsymbol{f}) \cdot \boldsymbol{w} - \boldsymbol{F}^{\mathrm{T}} \hat{\boldsymbol{C}}(\boldsymbol{f}) \cdot \boldsymbol{W} \tag{7.72$_2$}$$

for all $\boldsymbol{w} \in \mathcal{V}$ and $\boldsymbol{W}$ the skew tensor associated with $\boldsymbol{w}$. By virtue of divergence theorem, (7.72) yields

$$\hat{\varphi}(\boldsymbol{f}, \boldsymbol{F}) = \mathrm{Div}\, \hat{\boldsymbol{v}}(\boldsymbol{f}, \boldsymbol{F}). \tag{7.73}$$

Moreover, it is not difficult to give $\hat{\varphi}$ the canonical form of a null Lagrangian density [15], [16]: performing the differentiation indicated in (7.73) we obtain, with the use of (4.6) and (5.9),

$$\hat{\varphi}(\boldsymbol{f}, \boldsymbol{F}) = \hat{\delta}(\boldsymbol{f}) \det \boldsymbol{F} + \hat{\boldsymbol{D}}(\boldsymbol{f}) \cdot \boldsymbol{F}^*, \tag{7.74$_1$}$$

with

$$\hat{\delta}(\boldsymbol{f}) := \mathrm{div}\, \hat{\boldsymbol{c}}(\boldsymbol{f}), \quad 3\hat{\boldsymbol{D}}(\boldsymbol{f}) \cdot \boldsymbol{a} \otimes \boldsymbol{b} := -\boldsymbol{b} \cdot \mathrm{curl}\,(\hat{\boldsymbol{C}}(\boldsymbol{f})\, \boldsymbol{b}) \quad \text{for all } \boldsymbol{a}, \boldsymbol{b} \in \mathcal{V}. \ \square \tag{7.74$_2$}$$

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