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Self-Gravitating Relativistic Fluids: The Formation of a Free Phase Boundary in the Phase Transition from Soft to Hard

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1. The Existence Theorem

In [I] I introduced a relativistic fluid model with two phases, a soft phase, which holds when the density of mass-energy ρ is below a certain constant ρ_0 and in which the sound speed is zero, and a hard phase, which holds when ρ is above ρ_0 and in which the sound speed is equal to the speed of light. The model is of relevance in the study of the gravitational collapse of the degenerate cores of massive stars, the associated supernova explosions and the formation of neutron stars and black holes. The soft phase corresponds to degenerate stellar matter below nuclear density while the hard phase corresponds to homogeneous nuclear matter at supranuclear densities. The constant ρ_0, which by an appropriate choice of units we set equal to unity, corresponds to the nuclear saturation density (see Section 2 of [I]).

In [I] I began the study of the dynamics of the model in the spherically symmetric case. The problem then reduces to one on a 2-dimensional quotient space-time manifold Q. Starting with initial conditions which correspond entirely to the soft phase, we showed that predictions based solely on the soft phase break down beyond an achronal boundary $\partial\mathscr{I}^+(\mathscr{K})$, consisting of smooth spacelike segments Σ_i along which $\rho = 1$ and which form the spacelike part of the phase boundary, joined by pairs C_i^+, C_{i+1}^- of outgoing and incoming null segments (see Section 5 of [I]). The end points N_i^+ and N_i^- of the spacelike segment Σ_i at which Σ_i turns null, and which are, at the same time, the past end points of the null segments C_i^+ and C_{i+1}^- respectively, we called *boundary null points*. The data induced by the soft phase along the spacelike segments Σ_i, provide the initial conditions for a subsequent hard phase, determined in $\mathscr{D}^+(\Sigma_i)$, the future domain of dependence of Σ_i. In Section 6 of [I], we studied the associated hard-phase Cauchy problem in the large and analyzed the behavior near the null points, which, as we showed, are analogous to the *branch points* of minimal surface theory.

In [II] I began the investigation of the problem of extending the solution into the causal future of the boundary null points N_i^+, N_i^-. I formulated the problems of the formation and the continuation a free phase boundary and, after considering first the case of a vacuum free boundary I solved the continuation problem in general. I then turned to more global aspects of the free-boundary problem, in particular to the study of how a free phase boundary terminates. This led to the consideration of four cases, the last of which was proved to be non-generic, while the first corresponds to a spacelike transition from hard to soft with a null end point lying on the boundary of the causal past of the point of termination, the second case corresponds to a point of intersection with a spacelike transition from soft to hard, from which point the discontinuity propagates in the hard phase as a null shock, and the third case which corresponds to a point of intersection with a spacelike transition from hard to soft, from which point the discontinuity propagates as a contact discontinuity in the soft phase.

In the present paper we shall solve the problem of formation of a free phase boundary in the phase transition from soft to hard, formulated in Section 1 of [II]. We recall the domain $\mathscr{V}$ defined by the soft phase solution (4.7a) of [II]):

$$\mathscr{V} = \{(\tau,\chi):\chi \geqq 0,\ \tau_+(\chi) \leqq \tau \leqq \hat{\tau}(\chi)\}. \tag{1.1a}$$

The point $\tau = \chi = 0$ is now the outgoing boundary null point N^+ (see Section 1 of [II]), $\tau = \tau_+(\chi)$ is the equation of C^+, the outgoing null curve issuing from N^+, while $\tau = \hat{\tau}(\chi)$ is the equation of $\partial\mathscr{K}$, where, corresponding to the given soft-phase initial data, $\rho(\tau,\chi)$ along each flow line first becomes equal to 1 (see Section 5 of [I]). We have

$$\tau_+(0) = \hat{\tau}(0) = 0, \tag{1.1b}$$

$$e^{-\omega_0}\frac{d\tau_+}{d\chi}(0) = e^{-\omega_0}\frac{d\hat{\tau}}{d\chi}(0) = 1 \tag{1.1c}$$

where $\omega_0 = \omega(0,0)$. We cannot apply Theorem 3.1 of [II] to conclude the local existence of a solution to the formation problem. For, according to equation (6.48) of [I],

$$\zeta|_{N^+} = \frac{1}{a_{-0}} \tag{1.2a}$$

where $a_{-0} = a_-(0,0)$; therefore,

$$\dot{R}(0) = -\frac{1}{\zeta|_{N^+}} = -a_{-0}, \tag{1.2b}$$

$$\gamma(0) = -\frac{\dot{R}(0)}{a_{-0}} = 1. \tag{1.2c}$$

In view of (1.1b) and (1.2c), the hypotheses of Theorem 3.1 of [II] are not fulfilled.

We shall construct a solution to the formation problem as a limit of a sequence, each element of which is the solution of a regularized problem of the continuation type. To define the sequence of regularized problems we begin by selecting a sequence

$$((\tau_{0,n},\chi_{0,n}):n = 1,2,\ldots)$$

of points in the interior of $\mathscr{V}$ tending to $(0,0)$ as $n \to \infty$. Joining each point $(\tau_{0,n}, \chi_{0,n})$ to $(0,0)$ by a timelike geodesic, we define $\beta_{0,n}$ to be the velocity at $(0,0)$ of the corresponding geodesic relative to the flow lines. We then have

$$0 < \beta_{0,n} < 1, \quad \beta_{0,n} \to 1 \quad \text{as } n \to \infty.$$

Let $\tau = \tau_{+n}(\chi)$ be the equation of C_n^+, the outgoing null curve issuing from the point $(\tau_{0,n}, \chi_{0,n})$. We define the domains

$$\mathscr{V}_n = \{(\tau,\chi): \chi \geqq \chi_{0,n}, \tau_{+n}(\chi) \leqq \tau \leqq \hat{\tau}(\chi)\}. \tag{1.3a}$$

The domains $\mathscr{V}_n \subset \mathscr{V}$ are the images under the translations

$$\tau \mapsto \tau + \tau_{0,n}, \quad \chi \mapsto \chi + \chi_{0,n} \tag{1.3b}$$

of the domains

$$\tilde{\mathscr{V}}_n = \{(\tau,\chi): (\tau + \tau_{0,n}, \chi + \chi_{0,n}) \in \mathscr{V}_n\}. \tag{1.3c}$$

On the domains $\tilde{\mathscr{V}}_n$ we define the soft-phase solutions (r_n, ω_n, ρ_n) to be the corresponding pullbacks of the restrictions of the soft-phase solution (r, ω, ρ) to the domains $\mathscr{V}_n$:

$$\begin{aligned} r_n(\tau,\chi) &= r(\tau + \tau_{0,n}, \chi + \chi_{0,n}), \\ \omega_n(\tau,\chi) &= \omega(\tau + \tau_{0,n}, \chi + \chi_{0,n}), \\ \rho_n(\tau,\chi) &= \rho(\tau + \tau_{0,n}, \chi + \chi_{0,n}). \end{aligned} \tag{1.3d}$$

Let us denote

$$r_0 = r(0,0), \quad r_{0,n} = r_n(0,0) = r(\tau_{0,n}, \chi_{0,n}). \tag{1.4}$$

The initial data for the formation problem consist of the function R which defines r as a function of ϕ along C^{*+}, the incoming null curve issuing from N^+. This function, which, for $\phi|_{N^+} = 0$, is defined on some closed interval starting at 0, was constructed in Section 6 of [I]. We have

$$\dot{R}\zeta|_{C^{*+}} = -1, \tag{1.5}$$

and r, ζ along C^{*+} are given by equations (6.48), (6.49) of [I] in a neighborhood of N^+. To define initial data for the regularized problems, we set (see (4.5a) of [II])

$$v_{0,n} = \frac{1}{2} a_{-0}(1 - \beta_{0,n}), \quad \kappa_{0,n} = \frac{1}{2a_{-0}}(1 + \beta_{0,n}). \tag{1.6}$$

For each $n = 1, 2, \ldots$, we define the positive constant c_n to be the solution of

$$R(c_n) = r_0 - \tau_{0,n} v_{0,n}. \tag{1.7}$$

We then define R_n, the initial data for the nth regularized problem, by

$$R_n(t) = R(t + c_n) + r_{0,n} - r_0 + \tau_{0,n} v_{0,n} + \tau_{0,n}\kappa_{0,n} \frac{a_{-0}}{r_0}(a_{-0}^2 - 1 + 4\pi r_0^2)t. \tag{1.8a}$$

Note that according to this definition,

$$R_n(0) = r_{0,n}. \tag{1.8b}$$

We shall show that the nth regularized problem satisfies the hypotheses of Theorem 3.1 of [II], i.e.,

$$\gamma_n(0) := -\frac{\dot{R}_n(0)}{a_{-0,n}} < 1 \quad \forall n, \tag{1.9}$$

at least if n is large enough. Here

$$a_{-0,n} = a_{-n}(0,0) = a_-(\tau_{0,n}, \chi_{0,n}). \tag{1.10a}$$

We have (see (4.29b), (4.34b) of [II]):

$$\left(\frac{\partial a_-}{\partial \tau}\right)(0,0) = \frac{\mu_0}{2r_0},$$
$$\left(e^{-\omega}\frac{\partial a_-}{\partial \chi}\right)(0,0) = a_{-0}b_0 + \frac{1}{r_0}\left(1 - \frac{\mu_0}{2} - a_{-0}^2 - 4\pi r_0^2\right) \tag{1.10b}$$

where

$$\mu_0 = \frac{2m_0}{r_0}, \quad m_0 = m(0,0),$$
$$b_0 = \left(\frac{\partial \rho}{\partial \tau}\right)(0,0) > 0 \tag{1.10c}$$

(see Proposition 5.1 of [I]). It follows that

$$a_{-0,n} - a_{-0} = \tau_{0,n}\left[\left(\frac{\partial a_-}{\partial \tau}\right)(0,0) + \beta_{0,n}\left(e^{-\omega}\frac{\partial a_-}{\partial \chi}\right)(0,0)\right] + O(\tau_{0,n}^2)$$
$$= \tau_{0,n}\left\{\frac{\mu_0}{2r_0} + \beta_{0,n}\left[a_{-0}b_0 + \frac{1}{r_0}\left(1 - \frac{\mu_0}{2} - a_{-0}^2 - 4\pi r_0^2\right)\right]\right\} + O(\tau_{0,n}^2). \tag{1.10d}$$

According to (1.8a),

$$\dot{R}_n(t) = \dot{R}(t + c_n) + \tau_{0,n}\kappa_{0,n}\frac{a_{-0}}{r_0}(a_{-0}^2 - 1 + 4\pi r_0^2). \tag{1.11a}$$

Let Z be the function which defines ζ as a function of ϕ along C^{*+}. We have

$$\dot{R}Z = -1. \tag{1.11b}$$

Let us set

$$\xi_{0,n} = \frac{Z(c_n) - Z(0)}{\tau_{0,n}\nu_{0,n}}. \tag{1.11c}$$

Then, since $Z(0) = 1/a_{-0}$, we can write

$$\dot{R}(c_n) = -\frac{a_{-0}}{1 + \tau_{0,n}\nu_{0,n}a_{-0}\xi_{0,n}}. \tag{1.11d}$$

Therefore,

$$\dot{R}(c_n) = -a_{-0} + \tau_{0,n}\nu_{0,n}a_{-0}^2\xi_{0,n} + O((\tau_{0,n}\nu_{0,n}\xi_{0,n})^2). \tag{1.11e}$$

Setting

$$\triangle_n = 1 - \gamma_n(0), \tag{1.12a}$$

we obtain, by virtue of (1.10d) and (1.11e), that

$$\triangle_n = [(1 - \beta_{0,n})e_{0,n} + \beta_{0,n}b_0]\tau_{0,n} + E_n \tag{1.12b}$$

where

$$e_{0,n} = \frac{1}{2}a_{-0}^2\xi_{0,n} + \frac{1}{2r_0}\left[a_{-0} - \frac{(1 - 4\pi r_0^2)}{a_{-0}}\right] + \frac{\mu_0}{2r_0 a_{-0}} \tag{1.12c}$$

(see (4.26c) of [II]) and

$$E_n = O(\tau_{0,n}^2) + O((\tau_{0,n}\nu_{0,n}\xi_{0,n})^2). \tag{1.12d}$$

Now, r and ζ along C^{*+} are given, in a neighborhood of N^+, by equations (6.49) and (6.48) of [I] respectively, in terms of the parameter $\tilde{u}$:

$$\begin{aligned}\zeta(\tilde{u}, 0) &= \frac{1}{a_{-0}} + \frac{e^{-\omega_0/2}}{a_{-0}}\frac{b_0}{q^{+1/2}}\tilde{u}^{1/2} + O(\tilde{u}),\\ r(\tilde{u}, 0) &= r_0 - \tfrac{1}{4}e^{-\omega_0}a_{-0}\tilde{u} + O(\tilde{u}^{3/2})\end{aligned} \tag{1.13a}$$

(see (6.39), (6.40) of [I] for the definition of q^+). In view of (1.7), it follows that the parameter value $\tilde{u}_n$ corresponding to c_n according to

$$\phi(\tilde{u}_n, 0) = c_n \tag{1.13b}$$

is given by

$$\tfrac{1}{4}e^{-\omega_0}a_{-0}\tilde{u}_n = \tau_{0,n}\nu_{0,n} + O((\tau_{0,n}\nu_{0,n})^{3/2}). \tag{1.13c}$$

Thus,

$$Z(c_n) = \frac{1}{a_{-0}} + \frac{2b_0}{q^{+1/2}a_{-0}^{3/2}}(\tau_{0,n}\nu_{0,n})^{1/2} + O((\tau_{0,n}\nu_{0,n})^{3/2}). \tag{1.13d}$$

Consequently, by (11.11c),

$$\xi_{0,n} = \frac{2b_0}{q^{+1/2}a_{-0}^{3/2}} \cdot \frac{1}{(\tau_{0,n}\nu_{0,n})^{1/2}} + O(1) \to \infty \quad \text{as } n \to \infty. \tag{1.13e}$$

It follows that

$$e_{0,n} \to \infty \quad \text{as } n \to \infty. \tag{1.13f}$$

Hence

$$e_{0,n} > 0 \tag{1.13g}$$

and, by (1.12b),

$$\triangle_n > 0 \tag{1.13h}$$

for all sufficiently large n. Consequently (1.9) is satisfied if n is large enough.

We can thus apply Theorem 3.1 of [II] to conclude that there is a positive integer n_0 such that for each $n \geqq n_0$ the nth regularized problem has a solution with a maximal existence interval $\bar{\tau}_n > 0$. The corresponding free boundary $\mathscr{B}_n$ is on $[0, \bar{\tau}_n)$ a timelike curve of positive velocity β_n relative to the soft-phase flow lines issuing from the origin and contained in the interior of $\tilde{\mathscr{V}}_n, \gamma_n < 1$ on $[0, \bar{\tau}_n)$. On the domain $\mathscr{U}(\bar{\tau}_n)\backslash I^+(\bar{\tau}_n)$, where

$$\mathscr{U}(a) = \{(u,v): u \in [0,a], v \in [0,u]\}, \quad I^+(a) = \{(a,v): v \in [0,a]\}, \quad a > 0,$$

we have a genuine hard phase: $\sigma_n > 1$ in $\mathscr{U}(\bar{\tau}_n)\backslash I^+(\bar{\tau}_n)$. The proof of the existence of a limiting solution depends crucially on the following lemma.

Lemma 1.1.

$$\liminf_{n\to\infty} \bar{\tau}_n := \bar{\tau} > 0.$$

Proof. Suppose on the contrary that there is a sequence (n_i: $i = 1, 2, \ldots$), $n_i \to \infty$ as $i \to \infty$, such that

$$\bar{\tau}_{n_i} \to 0 \quad \text{as } i \to \infty. \tag{1.14}$$

Given any $a > 0$ let us define the domains

$$\tilde{\mathscr{V}}_n^a = \{(\tau, \chi) \in \tilde{\mathscr{V}}_n : \tau \leqq a\} \tag{1.15a}$$

and their translates

$$\mathscr{V}_n^{a+\tau_{0,n}} = \{(\tau, \chi) \in \mathscr{V}_n : \tau \leqq a + \tau_{0,n}\}. \tag{1.15b}$$

According to (4.15a, b), of [II], we have

$$\inf_{\mathscr{U}(\bar{\tau}_n)} \phi_n = \phi_n(0,0) = 0, \quad \sup_{\mathscr{U}(\bar{\tau}_n)} \phi_n = \phi_n(\bar{\tau}_n, \bar{\tau}_n) < \bar{\tau}_n. \tag{1.16}$$

If $\chi = \chi_{*n}(\tau)$ is the equation of $\mathscr{B}_n$, we have $\chi_{*n}(\tau) \to \bar{\chi}_n$ as $\tau \to \bar{\tau}_n$. According to the proof of Lemma 4.1 of [II], the supremum of the function r_n on $\mathscr{U}(\bar{\tau}_n)$ is attained on $\mathscr{B}_n$ and is therefore controlled by soft-phase solution (r_n, ω_n, ρ_n) corresponding to the nth regularized problem on $\tilde{\mathscr{V}}_n^{\bar{\tau}_n}$, or, equivalently, the original soft-phase solution (r, ω, ρ) on $\mathscr{V}_n^{\bar{\tau}_n + \tau_{0,n}}$:

$$\sup_{\mathscr{U}(\bar{\tau}_n)} r_n \leqq \sup_{\mathscr{V}_n^{\bar{\tau}_n+\tau_{0,n}}} r. \tag{1.17a}$$

Also, the infimum of r_n on $\mathscr{U}(\bar{\tau}_n)$ is attained on $I^+(\bar{\tau}_n)$, and

$$\inf_{I^+(\bar{\tau}_n)} r_n = \min\{r_n(\bar{\tau}_n, 0), r_n(\bar{\tau}_n, \bar{\tau}_n)\}. \tag{1.17b}$$

We have

$$r_n(\bar{\tau}_n, \bar{\tau}_n) \geqq \inf_{\mathscr{V}_n^{\bar{\tau}_n+\tau_{0,n}}} r \tag{1.17c}$$

and, in view of (1.16),

$$r_n(\bar{\tau}_n, 0) \geqq \inf_{[0,\bar{\tau}_n]} R_n. \tag{1.17d}$$

Therefore,

$$\inf_{\mathscr{U}(\bar{\tau}_n)} r_n \geqq \min \left\{ \inf_{[0,\bar{\tau}_n]} R_n, \inf_{\mathscr{V}_n^{\bar{\tau}_n+\tau_{0,n}}} r \right\}. \tag{1.17e}$$

By virtue of (1.14), as $i \to \infty$ the right-hand sides of (1.17a, e) with $n = n_i$ approach r_0; consequently for all large enough i,

$$\inf_{\mathscr{U}(\bar{\tau}_{n_i})} r_{n_i} \geqq \frac{r_0}{2}, \quad \sup_{\mathscr{U}(\bar{\tau}_{n_i})} r_{n_i} \leqq 2r_0. \tag{1.17f}$$

The proof of Lemma 4.1 of [II] then yields

$$\sup_{\mathscr{U}(\bar{\tau}_{n_i})} \int_0^v \left| \frac{\partial N_{n_i}}{\partial v} \right| dv \leqq C. \tag{1.18a}$$

Here N is the function

$$N(u,v) = \int_0^v \left((\mu - 4\pi r^2) \frac{\kappa}{r} \right) (u, v') \, dv' \tag{1.18b}$$

and C is a constant independent of n_i. Since, in view of (1.16) and (1.11a), for all large enough n

$$\sup_{u \in [0,\bar{\tau}_n]} \zeta_n(u, 0) \leqq \sup_{t \in [0,\bar{\tau}_n]} \left[-\frac{1}{\dot{R}_n(t)} \right] \leqq C \tag{1.19a}$$

where C is a constant independent of n, using (1.18a) and following the proof of Lemma 4.2 of [II] we then obtain, for all sufficiently large i that

$$\sup_{\mathscr{U}(\bar{\tau}_{n_i})} \zeta_{n_i} \leqq C \tag{1.19b}$$

where C is a constant independent of n_i. The proof of Lemma 4.3 of [II] then yields in turn, for all sufficiently large i that

$$\sup_{\mathscr{U}(\bar{\tau}_{n_i})} (\nu_{n_i}, \kappa_{n_i}) \leqq C, \tag{1.20}$$

$$\sup_{\mathscr{U}(\bar{\tau}_{n_i})} m_{n_i} \leqq C, \tag{1.21}$$

$$\sup_{\mathscr{U}(\bar{\tau}_{n_i})} \eta_{n_i} \leqq C \tag{1.22}$$

where C denotes constants independent of n_i.

According to Lemma 4.3 of [II], for each $n \geqq n_0$, $\gamma_n(\tau) \to \bar{\gamma}_n \in (0, 1]$ as well as $\rho_{*n}(\tau) \to \bar{\rho}_n \in [0, 1]$ as $\tau \to \bar{\tau}_n$, and we have the following four cases to consider:

Case 1. $\bar{\rho}_n < 1$, $\bar{\gamma}_n < 1$.

Case 2. $\bar{\rho}_n = 1$, $\bar{\gamma}_n < 1$.

Case 3. $\bar{\rho}_n < 1$, $\bar{\gamma}_n = 1$.

Case 4. $\bar{\rho}_n = 1$, $\bar{\gamma}_n = 1$.

We shall show that under the hypothesis (1.14) each of the four cases leads to a contradiction. This would then establish the lemma.

Case 2 is the easiest to dismiss for, according to the results of Section 4 of [II], in this case $\partial\mathscr{K}$ must be non-timelike at (τ_n, χ_n). But $\partial\mathscr{K}$ is timelike in a neighborhood of the null point N^+ in $\mathscr{J}^+(N^+)$, the causal future of N^+, and by (1.14), all but a finite part of the sequence of points $(\bar{\tau}_{n_i}, \bar{\chi}_{n_i})$ is contained in this neighborhood.

The remaining cases are treated with the help of the *barrier function e*. We define this function in general by

$$e = \frac{\xi}{2\zeta^2} + \frac{1}{2r}\left[\frac{1}{\zeta} - (1 - 4\pi r^2)\zeta\right] + \frac{\mu\zeta}{2r} \tag{1.23a}$$

where ξ is defined by (4.25c) of [II]:

$$\frac{\partial\zeta}{\partial u} = v\xi.$$

Then, in view of (4.25d) of [II],

$$\frac{\partial\sigma}{\partial u} = \frac{v}{r}[r\eta\xi + (1 + 4\pi r^2\zeta^2)\zeta\eta - (1 - \mu)\zeta^2];$$

and at a point where $\sigma = 1$,

$$\frac{\partial\sigma}{\partial u} = 2v\zeta e. \tag{1.23b}$$

In particular, at an end point of the free boundary $\mathscr{B}$ where $\gamma = 1$, the barrier function e reduces to the *boundary barrier* $\bar{e}$ defined by (4.26c) of [II]:

$$\bar{e} = \frac{1}{2}\bar{a}_-^2\bar{\xi} + \frac{1}{2\bar{r}}\left[\bar{a}_- - \frac{(1 - 4\pi\bar{r}^2)}{\bar{a}_-}\right] + \frac{\bar{\mu}}{2\bar{r}\bar{a}_-}. \tag{1.23c}$$

To proceed we must estimate the oscillation of ξ along outgoing null curves. Recall the function α defined by equation (4.14a) of [II]:

$$\alpha = \frac{1}{v}\frac{\partial(r\phi)}{\partial u}, \tag{1.24a}$$

i.e.,

$$\alpha = r\zeta - \phi. \tag{1.24b}$$

This function satisfies (see (4.14b) of [II])

$$\alpha(u, v)e^{N(u,v)} = \alpha(u, 0) - \int_0^v (\phi N_+)(u, v')e^{N(u,v')}\kappa(u, v')\,dv' \tag{1.24c}$$

where

$$N_+ = \frac{1}{\kappa}\frac{\partial N}{\partial v}, \tag{1.25a}$$

i.e., by (1.18b),

$$N_+ = \frac{1}{r}(\mu - 4\pi r^2). \tag{1.25b}$$

We now define the function α_- by

$$\alpha_- = \frac{1}{\nu}\frac{\partial \alpha}{\partial u}. \tag{1.26a}$$

We have

$$\alpha_- = r\xi - 2\zeta. \tag{1.26b}$$

From (1.24c), using the fact that

$$\nu(u,v) = \nu(u,0)e^{N(u,v)},$$

we deduce the following equation for α_-:

$$\alpha_-(u,v) = e^{-2N(u,v)}\alpha_-(u,0) - \alpha(u,v)N_-(u,v)e^{-N(u,v)}$$
$$- \int_0^v [\zeta N_+ + \phi(N_{+-} + N_+N_-)](u,v')e^{2N(u,v')-2N(u,v)}\kappa(u,v')\,dv'. \tag{1.26c}$$

Here

$$N_- = \frac{1}{\nu}\frac{\partial N}{\partial u}, \tag{1.27a}$$

$$N_{+-} = \frac{1}{\nu\kappa}\frac{\partial^2 N}{\partial u \partial v}. \tag{1.27b}$$

Since $N_-(u,0) = 0$, we can express

$$N_-(u,v) = \int_0^v \frac{\nu(u,v')}{\nu(u,v)}(N_{+-}\kappa)(u,v')\,dv'$$
$$= \int_0^v e^{N(u,v')-N(u,v)}(N_{+-}\kappa)(u,v')\,dv'. \tag{1.27c}$$

Also, from (1.18b), using equations (6.5a) and (6.6a) of [I] we find

$$N_{+-} = \frac{2\mu}{r^2} - 4\pi\zeta^2(1 - 4\pi r^2). \tag{1.27d}$$

Now, by (1.17f), (1.19b), (1.21),

$$\sup_{\mathscr{U}(\bar{\tau}_{n_i})} |N_{+n_i}| \leqq C, \quad \sup_{\mathscr{U}(\bar{\tau}_{n_i})} |N_{+-n_i}| \leqq C \tag{1.28a}$$

where C is a constant independent of n_i. In view of (1.27c) and (1.20) it follows that

$$\sup_{\mathscr{U}(\bar{\tau}_{n_i})} |N_{n_i}| \leqq C\bar{\tau}_{n_i}, \quad \sup_{\mathscr{U}(\bar{\tau}_{n_i})} |N_{-n_i}| \leqq C\bar{\tau}_{n_i}. \tag{1.28b}$$

Recalling (1.16) we then conclude from (1.26c) that

$$\sup_{(u,v)\in\mathscr{U}(\bar{\tau}_{n_i})} |\alpha_{-n_i}(u,v) - \exp(-2N_{n_i}(u,v))\alpha_{-n_i}(u,0)| \leqq C\bar{\tau}_{n_i}. \tag{1.29a}$$

In view of (1.26b) and the fact that

$$\sup_{(u,v)\in\mathscr{U}(\bar{\tau}_{n_i})} |\alpha_{n_i}(u,v) - \exp(-N_{n_i}(u,v))\alpha_{n_i}(u,0)| \leqq C\bar{\tau}_{n_i} \tag{1.29b}$$

by (1.24c), we obtain

$$\sup_{(u,v)\in\mathscr{U}(\bar{\tau}_{n_i})} |\zeta_{n_i}(u,v) - \exp(-2N_{n_i}(u,v))\zeta_{n_i}(u,0)| \leqq C\bar{\tau}_{n_i}. \tag{1.29c}$$

Letting Z_n be the function which defines ζ_n as a function of ϕ_n along $I_n^-(0) = \{(u,0): u\in[0,\bar{\tau}_n]\}$, we have

$$\zeta_n(\bar{\tau}_n,0) = -\left(\frac{\dot{Z}_n}{\dot{R}_n}\right)(t_n) \tag{1.30a}$$

where

$$t_n = \phi_n(\bar{\tau}_n,0) < \bar{\tau}_n \tag{1.30b}$$

(see (1.16)). Also,

$$\dot{R}_n Z_n = -1, \tag{1.30c}$$

which gives

$$\dot{Z}_n = \frac{\ddot{R}_n}{\dot{R}_n^2}, \tag{1.30d}$$

while by (1.11b),

$$\dot{Z} = \frac{\ddot{R}}{\dot{R}^2}. \tag{1.30e}$$

Since

$$\ddot{R}_n(t) = \ddot{R}(t + c_n) \tag{1.30f}$$

by (1.11a), we can write

$$\dot{Z}_n(t_n) = \dot{Z}(t_n + c_n)\left(\frac{\dot{R}(t_n + c_n)}{\dot{R}_n(t_n)}\right)^2. \tag{1.30g}$$

Hence,

$$\zeta_n(\bar{\tau}_n,0) = -\frac{\dot{Z}(t_n + c_n)}{\dot{R}(t_n + c_n)}\left(\frac{\dot{R}(t_n + c_n)}{\dot{R}_n(t_n)}\right)^3. \tag{1.30h}$$

Now, by (1.11a), the factor $\dot{R}(t_n + c_n)/\dot{R}_n(t_n)$ tends to unity for $n = n_i \to \infty$. Also, $\dot{R}(t_n + c_n) \to \dot{R}(0) = -a_{-0}$. On the other hand, according to the results of

Section 6 of [I], ϕ along C^{*+} is given, in a neighborhood of N^+, in terms of the parameter $\tilde{u}$ by

$$\phi(\tilde{u},0) = \tfrac{1}{4} e^{-\omega_0 \tilde{u}} + O(\tilde{u}^{3/2}), \tag{1.31a}$$

while r and ζ are given by (1.13a). Equation (1.31a) implies that

$$\tilde{u} = 4e^{-\omega_0 t} + O(t^{3/2}). \tag{1.31b}$$

Consequently, by (1.13a),

$$Z(t) = \frac{1}{a_{-0}} + \frac{2b_0}{a_{-0}q^{+1/2}} t^{1/2} + O(t), \tag{1.32a}$$

$$\dot{Z}(t) = \frac{b_0}{a_{-0}q^{+1/2}} t^{-1/2} + O(1). \tag{1.32b}$$

Equation (1.30h) then yields

$$\xi_{n_i}(\bar{\tau}_{n_i}, 0) \to \infty \quad \text{as } i \to \infty. \tag{1.33a}$$

Hence, by estimate (1.29c),

$$\bar{\xi}_{n_i} := \xi_{n_i}(\bar{\tau}_{n_i}, \bar{\tau}_{n_i}) \to \infty \quad \text{as } i \to \infty. \tag{1.33b}$$

Consequently, in view of (1.23c),

$$\bar{e}_{n_i} \to \infty \quad \text{as } i \to \infty. \tag{1.33c}$$

In particular,

$$\bar{e}_{n_i} > 0 \tag{1.34a}$$

for all sufficiently large i.

Since, $\gamma_n(\bar{\tau}_n) = 1$, while $\gamma_n(\tau) < 1$ for $\tau < \bar{\tau}_n$, in cases 3 and 4, in these cases we must have

$$\limsup_{\tau \to \bar{\tau}_n} \frac{d\gamma_n}{d\tau}(\tau) \geqq 0. \tag{1.34b}$$

However, according to (4.30) of [II] in Case 3 we have

$$\lim_{\tau \to \bar{\tau}_n} \frac{d\gamma_n}{d\tau}(\tau) = -\bar{e}_n, \tag{1.34c}$$

while in Case 4, according to Lemma 4.4 of [II],

$$\limsup_{\tau \to \bar{\tau}_n} \frac{d\gamma_n}{d\tau}(\tau) \leqq -\min\{\bar{e}_n, \bar{b}_n\}. \tag{1.34d}$$

In view of (1.34c, d), the inequalities (1.34a, b) are contradictory.

The only case remaining to be considered is Case 1. In this case there is a point $(\bar{\tau}_n, v_{*n})$, $v_{*n} \in (0, \bar{\tau}_n)$, in the interior of the outgoing null segment $I^+(\bar{\tau}_n)$ (the future boundary of $\mathscr{U}(\bar{\tau}_n)$), where $\sigma_n(\bar{\tau}_n, v_{*n}) = 1$, while $\sigma_n > 1$ along the incoming null segment $[v_{*n}, \bar{\tau}_n) \times \{v_{*n}\} \subset \mathscr{U}(\bar{\tau}_n)$. Thus,

$$\frac{\partial \sigma_n}{\partial u}(\bar{\tau}_n, v_{*n}) \leqq 0 \tag{1.35a}$$

must hold at $(\bar{\tau}_n, v_{*n})$. Now, by (1.23b) at the point $(\bar{\tau}_n, v_{*n})$ we have

$$\frac{\partial \sigma_n}{\partial u}(\bar{\tau}_n, v_{*n}) = 2(v_n \zeta_n e_n)(\bar{\tau}_n, v_{*n}). \tag{1.35b}$$

We have

$$\xi_n(\bar{\tau}_n, 0) = -\left(\frac{\dot{Z}_n}{\dot{R}_n}\right)(t_n) \tag{1.36a}$$

where

$$t_n = \phi_n(\bar{\tau}_n, 0) < \bar{\tau}_n \tag{1.36b}$$

(see 1.16). As in equations (1.30a–h) we deduce

$$\xi_n(\bar{\tau}_n, 0) = -\frac{\dot{Z}(t_n + c_n)}{\dot{R}(t_n + c_n)}\left(\frac{\dot{R}(t_n + c_n)}{\dot{R}_n(t_n)}\right)^3. \tag{1.36c}$$

Since $t_{n_i} \to 0$ as $i \to \infty$ by (1.36b), it follows in view of (1.32b) that

$$\xi_{n_i}(\tau_{n_i}, 0) \to \infty \quad \text{as } i \to \infty. \tag{1.36d}$$

Hence, by the estimate (1.29c),

$$\xi_{n_i}(\bar{\tau}_{n_i}, v_{*n_i}) \to \infty \quad \text{as } i \to \infty. \tag{1.36e}$$

Consequently, in view of (1.23a),

$$e_{n_i}(\bar{\tau}_{n_i}, v_{*n_i}) \to \infty \quad \text{as } i \to \infty. \tag{1.36f}$$

In particular,

$$e_{n_i}(\bar{\tau}_{n_i}, v_{*n_i}) > 0 \tag{1.36g}$$

for all sufficiently large i. In view of (1.35b), the inequalities (1.35a) and (1.36g) are contradictory. The proof of the lemma is therefore complete.

In the arguments to follow we shall make use of

Proposition 1.1. *Let* $\mathscr{B}$, $(r, m, v, \kappa, \zeta, \eta)$, *defined on* (τ_1, τ_2), $\{(u, v) : u \in (\tau_1, \tau_2),$ $v \in (\tau_1, u]\}$, *respectively, be a solution of the free-boundary problem corresponding to a soft phase solution* (r, ω, ρ), *such that* $0 < \beta < 1$, $\rho_* < 1$, *whence also* $0 < \gamma < 1$. *Then at each* $\tau \in (\tau_1, \tau_2)$,

$$-\frac{1}{\gamma}\frac{d\gamma}{d\tau} = (1 - \beta)E + \beta B$$

where E and B are the functions

$$E = \frac{1}{2}\gamma a_{-*}^2 \zeta_* + \frac{1}{2r_*}\left[\gamma a_{-*}\sqrt{q} - \frac{(1-4\pi r_*^2)}{a_{-*}}\right] + \frac{\mu_*}{2r_* a_{-*}},$$

$$B = \frac{b_*}{\rho_*} + \frac{1}{r_*}\left[a_{-*}(\gamma\sqrt{q}-1) + \frac{4\pi r_*^2}{a_{-*}}(1-\rho_*)\right],$$

with

$$q = \frac{\rho_* + \gamma^2 - 2\rho_*\gamma^2}{1-\rho_*\gamma^2}, \quad b_* = \left(\frac{\partial \rho}{\partial \tau}\right)_*.$$

Proof. Since

$$\gamma = \frac{1}{\zeta_* a_{-*}},$$

we have

$$-\frac{1}{\gamma}\frac{d\gamma}{d\tau} = a_{-*}\gamma\frac{d\zeta_*}{d\tau} + \frac{1}{a_{-*}}\frac{da_{-*}}{d\tau}$$

(see (4.27) of [II]). By (4.28b) of [II], in view of equations (4.5a, d, e) of [II], we get

$$\gamma a_{-*}\frac{d\zeta_*}{d\tau} = (1-\beta)\left\{\frac{1}{2}\gamma a_{-*}^2\zeta_* + \frac{1}{2r_*}\left[\gamma a_{-*}\sqrt{q} - \frac{(1-4\pi r_*^2)}{a_{-*}}\right]\right\}$$
$$+ \frac{\beta}{r_*}\left[\gamma a_{-*}\sqrt{q} - \frac{(1-4\pi r_*^2)}{a_{-*}}\right].$$

Also, according to equation (4.35a) of [II],

$$\frac{1}{a_{-*}}\frac{da_{-*}}{d\tau} = (1-\beta)\frac{\mu_*}{2r_* a_{-*}} + \beta\left[\frac{b_*}{\rho_*} + \frac{1}{r_* a_{-*}}(1 - a_{-*}^2 - 4\pi r_*^2\rho_*)\right].$$

The proposition follows by substitution.

Remark. In the limiting case $\gamma \to \bar{\gamma} = 1$ the function E reduces to the boundary barrier $\bar{e}$ (see (1.23c)). If also $\rho_* \to \bar{\rho} = 1$, the function B reduces to $\bar{b}$ (see Lemma 4.4 of [II]).

Now let $\bar{\tau}$ be as in the statement of Lemma 1.1. Take any $\hat{\tau} \in (0, \bar{\tau})$. Then $\bar{\tau}_n \geqq \hat{\tau}$ if n is large enough.

Lemma 1.2. *There is a subsequence $(n_i\colon i = 1, 2, \dots)$ with the following properties:*
1. *The corresponding sequence of curves $(\mathscr{B}_{n_i})$ converges uniformly on $[0, \hat{\tau}]$ to a non-spacelike curve $\mathscr{B}$ contained in $\mathscr{V}^{\hat{\tau}}$.*
2. *The sequence of functions $(r_{n_i}, \phi_{n_i}, m_{n_i})$ converges uniformly in $\mathscr{U}(\hat{\tau})$ to (r, ϕ, m), Lipschitz functions on $\mathscr{U}(\hat{\tau})$.*
3. *The sequence of functions (N_{n_i}, K_{n_i}) converges uniformly in $\mathscr{U}(\hat{\tau})$ to (N, K), Lipschitz functions on $\mathscr{U}(\hat{\tau})$.*

4. *The sequence of functions* $(\zeta_{n_i}, \eta_{n_i})$ *converges uniformly in* $\mathscr{U}(\hat{\tau})$ *to* (ζ, η), *continuous functions on* $\mathscr{U}(\hat{\tau})$.

Proof. For each n, $\mathscr{B}_n$ is a C^1 timelike curve:

$$\chi = \chi_{*n}(\tau),\ 0 < \beta_n := e^{\omega_{*n}} \frac{d\chi_{*n}}{d\tau} < 1,\ \omega_{*n}(\tau) := \omega_n(\tau, \chi_{*n}(\tau)) \tag{1.37}$$

contained in the domain $\tilde{\mathscr{V}}_n^{\hat{\tau}}$. By the Ascoli-Arzelà theorem we can select a subsequence (n_i) such that χ_{*n_i} converges uniformly on $[0, \hat{\tau}]$ to a Lipschitz function χ_*. Then, in view of (1.3d) and the fact that the domains $\tilde{\mathscr{V}}_n^{\hat{\tau}}$ converge as $n \to \infty$ to the domain $\mathscr{V}^{\hat{\tau}}$, we have $\omega_{*n_i} \to \omega_*$ as $i \to \infty$ uniformly on $[0, \hat{\tau}]$, where $\omega_*(\tau) := (\tau, \chi_*(\tau))$. It follows that $\mathscr{B}$, given by $\chi = \chi_*(\tau)$, is a non-spacelike curve in the original soft-phase solution (r, ω, ρ), contained in $\mathscr{V}^{\hat{\tau}}$. Moreover, the functions $r_{*n_i}, m_{*n_i}, \rho_{*n_i}, a_{-*n_i}$ converge uniformly on $[0, \hat{\tau}]$ to the functions r_*, m_*, ρ_*, a_{-*} respectively.

Consider next the sequence of hard phase solutions (r_n, ϕ_n, m_n). In $\mathscr{U}(\hat{\tau})$ we have bounds analogous to those of (1.17a, e):

$$\sup_{\mathscr{U}(\hat{\tau})} r_n \leqq \sup_{\mathscr{V}_n^{\hat{\tau}_n + \tau_{0,n}}} r, \tag{1.38a}$$

$$\inf_{\mathscr{U}(\hat{\tau})} r_n \geqq \min\left\{ \inf_{[0,\hat{\tau}]} R_n,\ \inf_{\mathscr{V}_n^{\hat{\tau}_n + \tau_{0,n}}} r \right\}. \tag{1.38b}$$

Consequently, there is a constant C independent of n such that

$$\inf_{\mathscr{U}(\hat{\tau})} r_n \geqq \frac{r_0}{C}, \quad \sup_{\mathscr{U}(\hat{\tau})} r_n \leqq C r_0. \tag{1.38c}$$

Now, each ϕ_n is a time function, i.e., if $u_2 \geqq u_1$, $v_2 \geqq v_1$, then $\phi_n(u_2, v_2) \geqq \phi_n(u_1, v_1)$. Also, for each n the function ϕ_{*n}, defined by

$$\phi_{*n}(\tau) = \phi_n(\tau, \tau) \quad \forall \tau \in [0, \hat{\tau}]$$

satisfies

$$\frac{d\phi_{*n}}{d\tau} \leqq 1$$

according to Proposition 4.1 of [II]. Consequently,

$$0 = \phi_n(0,0) = \inf_{\mathscr{U}(\hat{\tau})} \phi_n \leqq \sup_{\mathscr{U}(\hat{\tau})} \phi_n = \phi_n(\hat{\tau}, \hat{\tau}) = \phi_{*n}(\hat{\tau}) \leqq \hat{\tau}. \tag{1.39}$$

In view of (1.38c) and (1.39), it then follows, as in (1.19b), (1.20), (1.22), that

$$\begin{aligned} &\sup_{\mathscr{U}(\hat{\tau})} \zeta_n \leqq C, \quad \sup_{\mathscr{U}(\hat{\tau})} (\nu_n, \kappa_n) \leqq C, \\ &\sup_{\mathscr{U}(\hat{\tau})} m_n \leqq C, \quad \sup_{\mathscr{U}(\hat{\tau})} \eta_n \leqq C. \end{aligned} \tag{1.40}$$

Here C denotes constants independent of n. Consequently, in view of equations (6.3a, b), (6.4a, b), (6.5a, b) of [I], the functions (r_n, ϕ_n, m_n) and their partial derivatives are bounded on $\mathscr{U}(\hat{\tau})$ uniformly in n. By the Ascoli-Arzelà theorem we can extract a subsequence from the subsequence (n_i) considered above, which for convenience of notation we still denote by (n_i), converging uniformly in $\mathscr{U}(\hat{\tau})$, to (r, ϕ, m), Lipschitz functions on $\mathscr{U}(\hat{\tau})$. Since the restrictions of (r_n, m_n) to $\{(\tau, \tau): \tau \in [0, \hat{\tau}]\}$ coincide with the functions (r_{*n}, m_{*n}) considered above, the restrictions of (r, m) to $\{(\tau, \tau): \tau \in [0, \hat{\tau}]\}$ coincide with the functions (r_*, m_*). Also, the limit function ϕ is a time function and ϕ_*, the restriction of ϕ to $\{(\tau, \tau): \tau \in [0, \hat{\tau}]\}$, has a Lipschitz constant not exceeding unity.

Next, we consider the functions (N_n, K_n). As in (1.28a) we have

$$\sup_{\mathscr{U}(\hat{\tau})} |N_{+n}| \leqq C, \quad \sup_{\mathscr{U}(\hat{\tau})} |N_{+-n}| \leqq C \tag{1.41a}$$

where C is a constant independent of n. Since (see 1.27c)

$$N_n(u, v) = \int_0^v N_{+n} \kappa_n(u, v')\, dv', \tag{1.41b}$$

$$N_{-n}(u, v) = \int_0^v e^{N_n(u, v') - N_n(u, v)} (N_{+-n} \kappa_n)(u, v')\, dv', \tag{1.41c}$$

(1.41a) together with (1.40) implies that

$$\sup_{\mathscr{U}(\hat{\tau})} |N_n| \leqq C\hat{\tau}, \quad \sup_{\mathscr{U}(\hat{\tau})} |N_{-n}| \leqq C\hat{\tau}, \tag{1.41d}$$

where C is a constant independent of n.

The positive functions K_n are given by (see (4.17b) of [II])

$$K_n(u, v) = 4\pi \int_v^u (r_n \nu_n \zeta_n^2)(u', v)\, du'. \tag{1.42a}$$

Letting

$$\frac{\partial K_n}{\partial v} = \kappa_n K_{+n}, \quad \frac{\partial K_n}{\partial u} = \nu_n K_{-n}, \tag{1.42b}$$

$$\frac{\partial^2 K_n}{\partial u \partial v} = \nu_n \kappa_n K_{+-n}, \tag{1.42c}$$

we have

$$K_{-n} = 4\pi r_n \zeta_n^2, \tag{1.42d}$$

and using equations (6.3b), (6.6b), (6.7b) of [I], we find that

$$K_{+-n} = 4\pi \zeta_n [2\eta_n - (1 - 4\pi r_n^2)\zeta_n]. \tag{1.42e}$$

Now, by (1.40),

$$\sup_{\mathscr{U}(\hat{\tau})} |K_{-n}| \leqq C \tag{1.42f}$$

where C is a constant independent of n. Since

$$K_n(u,v) = \int_v^u (K_{-n}\nu_n)(u',v)\,du', \tag{1.42g}$$

(1.42f) yields

$$\sup_{\mathscr{U}(\hat{\tau})} K_n \leqq C\hat{\tau}. \tag{1.42h}$$

Differentiating the equation $K_n(v,v) = 0$, we obtain

$$\left(\frac{\partial K_n}{\partial v} + \frac{\partial K_n}{\partial u}\right)(v,v) = 0,$$

i.e.,

$$\kappa_{*n}K_{+*n} + \nu_n K_{-*n} = 0. \tag{1.42i}$$

Now, by equations (4.5b) of [II],

$$\frac{\nu_{*n}}{\kappa_{*n}} = a^2_{-*n}\left(\frac{1-\beta_n}{1+\beta_n}\right);$$

hence (since $\beta_n > 0$),

$$\frac{\nu_{*n}}{\kappa_{*n}} \leqq a^2_{-*n}. \tag{1.42j}$$

In view of (1.42f), equation (1.42i) together with (1.42j) implies that the functions K_{+*n} are bounded on $[0,\hat{\tau}]$ uniformly in n:

$$\sup_{[0,\hat{\tau}]} |K_{+*n}| \leqq C \tag{1.42k}$$

where C is a constant independent of n. Integrating (1.42c) with respect to u we obtain

$$(\kappa_n K_{+n})(u,v) = (\kappa_{*n}K_{+*n})(v) + \int_v^u (K_{+-n}\nu_n\kappa_n)(u',v)\,du'.$$

Dividing this equation by

$$\kappa_n(u,v) = \kappa_{*n}(v)e^{-K_n(u,v)}$$

yields

$$K_{+n}(u,v) = e^{K_n(u,v)}K_{+*n}(v) + \int_v^u e^{K_n(u,v)-K_n(u',v)}(K_{+-n}\nu_n)(u',v)\,du'. \tag{1.42l}$$

Substituting the estimate (1.42k) into (1.42l) we then obtain

$$\sup_{\mathscr{U}(\hat{\tau})} |K_{+n}| \leqq C \tag{1.42m}$$

where C is a constant independent of n.

By (1.41a, d), (1.42f, h, m), the functions (N_n, K_n) and their partial derivatives are bounded on $\mathscr{U}(\hat{\tau})$ uniformly in n. The Ascoli-Arzelà theorem then allows us to extract a subsequence out of the subsequence (n_i), which for convenience of notation we again denote by (n_i), converging uniformly in $\mathscr{U}(\hat{\tau})$, to (N, K), Lipschitz functions on $\mathscr{U}(\hat{\tau})$. Also, $N(u,0) = 0$, $K(v,v) = 0$.

Finally we consider the functions (ζ_n, η_n). According to (1.24c), the functions $\alpha_n = r_n\zeta_n - \phi_n$ satisfy

$$\alpha_n(u,v) = \alpha_n(u,0)\,e^{-N_n(u,v)} - A_n(u,v) \tag{1.43a}$$

where

$$A_n(u,v) = \int_0^v \left(\phi_n \frac{\partial N_n}{\partial v}\right)(u,v')\,e^{N_n(u,v')-N_n(u,v)}\,dv'. \tag{1.43b}$$

By virtue of the results above, the functions A_n and their partial derivatives are bounded on $\mathscr{U}(\hat{\tau})$ uniformly in n. We can therefore select a subsequence from the subsequence (n_i), which we again denote by (n_i), converging uniformly in $\mathscr{U}(\hat{\tau})$ to a Lipschitz function A on $\mathscr{U}(\hat{\tau})$. To prove the uniform convergence of $\alpha_{n_i}, \zeta_{n_i}$, we consider the functions $\zeta_n(\cdot,0)$ on $[0,\hat{\tau}]$. From (1.30c), (1.11a, b) we have

$$\zeta_n(u,0) = Z_n(\phi_n(u,0)) = \frac{Z(\phi_n(u,0)+c_n)}{1 - l_n Z(\phi_n(u,0)+c_n)} \tag{1.43c}$$

where l_n are the constants

$$l_n = \tau_{0,n}\kappa_{0,n}\frac{a_{-0}}{r_0}(a_{-0}^2 - 1 + 4\pi r_0^2). \tag{1.43d}$$

Since Z is continuous and the functions $\phi_{n_i}(\cdot,0)$ converge uniformly on $[0,\hat{\tau})$ to $\phi(\cdot,0)$ while the constants $c_n, l_n \to 0$, it follows that the functions $\zeta_{n_i}(\cdot,0)$ converge uniformly on $[0,\hat{\tau})$ to $\zeta(\cdot,0)$, a continuous function on $[0,\hat{\tau}]$. Setting

$$\alpha(\cdot,0) = r(\cdot,0)\zeta(\cdot,0) - \phi(\cdot,0),$$

we then obtain

$$\alpha_{n_i} \to \alpha := \alpha(\cdot,0)e^{-N} - A \quad \text{as } i \to \infty \tag{1.43e}$$

uniformly in $\mathscr{U}(\hat{\tau})$. Consequently the ζ_{n_i} also converge uniformly in $\mathscr{U}(\hat{\tau})$ to

$$\zeta := r^{-1}(\alpha + \phi).$$

It follows in particular that

$$\frac{1}{\zeta_{*n_i}a_{-*n_i}} := \gamma_n \to \gamma := \frac{1}{\zeta_* a_{-*}} \quad \text{as } i \to \infty \tag{1.44}$$

uniformly on $[0,\hat{\tau}]$.

According to equations (4.19a, b) of [II], the functions η_n satisfy

$$(r_n\eta_n)(u,v) = e^{K_n(u,v)}\left[(r_{*n}\eta_{*n})(v) - J_n(u,v)\right] \tag{1.45a}$$

where

$$J_n(u,v) = \int_u^v e^{-K_n(u',v)}((1-\mu_n)v_n\zeta_n)(u',v)\,du'. \tag{1.45b}$$

By virtue of the results above and in view of equations (6.6b), (6.7b), of [I], the functions J_n and their partial derivatives are bounded on $\mathscr{U}(\hat{\tau})$ uniformly in n. We can therefore select a subsequence from the subsequence (n_i), which we again denote by (n_i), converging uniformly in $\mathscr{U}(\hat{\tau})$ to a Lipschitz function J on $\mathscr{U}(\hat{\tau})$. Thus, to prove the uniform convergence to η_{n_i} we must show that the functions η_{*n_i} converge uniformly on $[0, \hat{\tau}]$. Now, according to equation (4.5e) of [II] we can express

$$\eta_{*n} = a_{-*n}\sqrt{\frac{\rho_{*n} + \gamma_n^2 - 2\rho_{*n}\gamma_n^2}{1 - \rho_{*n}\gamma_n^2}}. \tag{1.45c}$$

In terms of the variables

$$x_n = 2(1 - \rho_{*n}), \quad y_n = 1 - \gamma_n^2, \quad z_n = \frac{x_n y_n}{x_n + 2y_n}, \tag{1.45d}$$

we can write

$$\eta_{*n} = a_{-*n}\sqrt{q_n}, \quad q_n = \frac{1 - 2z_n}{1 - z_n}. \tag{1.45e}$$

By virtue of (1.44) we have

$$(x_{n_i}, y_{n_i}) \to (x, y) \quad \text{as } i \to \infty \tag{1.45f}$$

uniformly on $[0, \hat{\tau}]$. Since ζ_n is bounded on $\mathscr{U}(\hat{\tau})$ uniformly in n, by (1.40), it follows that

$$\gamma_n \geqq \varepsilon \tag{1.45g}$$

where ε is a positive constant independent of n. Also, $\gamma_n < 1$ and $0 \leqq \rho_{*n} < 1$. Hence

$$0 < x_n \leqq 2, \quad 0 < y_n \leqq 1 - \varepsilon^2, \tag{1.45h}$$

$$0 \leqq x \leqq 2, \quad 0 \leqq y \leqq 1 - \varepsilon^2. \tag{1.45i}$$

Now, the function

$$f(x, y) = \begin{cases} xy/(x + 2y) & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

is continuous on the closed positive quadrant. It follows that

$$z_{n_i} = f(x_{n_i}, y_{n_i}) \to f(x, y) := z \quad \text{as } i \to \infty \tag{1.45j}$$

uniformly on $[0, \hat{\tau}]$. Moreover,

$$0 \leqq z \leqq y \leqq 1 - \varepsilon^2. \tag{1.45k}$$

We conclude that

$$\frac{1 - 2z_{n_i}}{1 - z_{n_i}} := q_{n_i} \to q := \frac{1 - 2z}{1 - z} \quad \text{as } i \to \infty. \tag{1.45l}$$

Hence

$$\eta_{*n_i} \to \eta_* := a_{-*}\sqrt{q} \quad \text{as } i \to \infty \tag{1.45m}$$

uniformly on $[0, \hat{\tau}]$. The proof of the Lemma is therefore complete.

To proceed further we must show that we can select a subsequence (n_i) such that the corresponding functions β_{n_i} converge uniformly on $[0, \hat{\tau}]$. This is accomplished in two steps, the two lemmas which follow. According to equation (4.5c) of [II], β_n is given by

$$\beta_n = \frac{1 - \gamma_n^2}{1 + \gamma_n^2 - 2\gamma_n^2 \rho_{*n}}. \tag{1.46a}$$

In terms of the variables x_n, y_n we can write this in the form

$$\beta_n = \frac{y_n}{(1 - y_n)x_n + y_n}. \tag{1.46b}$$

Now the function

$$g(x, y) = \frac{y}{(1 - y)x + y} \tag{1.46c}$$

is continuous on $\bar{Q}_\varepsilon \backslash (0, 0)$ where

$$\bar{Q}_\varepsilon = \{(x, y) : x \geqq 0, 1 - \varepsilon^2 \geqq y \geqq 0\}, \tag{1.46d}$$

but is not continuous at the point $(0, 0)$. Therefore (1.45f) does not imply the uniform convergence of the β_{n_i}.

Let us first make the following general remarks. The functions β_n belong to $C^0[0, \hat{\tau}] \subset L^\infty[0, \hat{\tau}]$ and satisfy $0 < \beta_n < 1$. By Alaoglu's theorem, $\bar{B}_1(L^\infty[0, \hat{\tau}])$, the closed unit ball of $L^\infty[0, \hat{\tau}] = L^1[0, \hat{\tau}]^\star$, is weak-star compact. We can therefore select a subsequence from the subsequence (n_i), which we again denote by (n_i), such that the corresponding functions β_{n_i} converge weak-star to a function $\beta \in \bar{B}_1(L^\infty[0, \hat{\tau}])$, that is,

$$\int_0^{\hat{\tau}} \beta_{n_i} f \, d\tau \;\to\; \int_0^{\hat{\tau}} \beta f \, d\tau \quad \forall f \in L^1[0, \hat{\tau}]. \tag{1.47}$$

Since $\beta \in \bar{B}_1(L^\infty[0, \hat{\tau}])$, we have $\beta \leqq 1$ almost everywhere. We shall show that also $\beta \geqq 0$ almost everywhere. For, let

$$N = \{\tau \in [0, \hat{\tau}] : \beta(\hat{\tau}) < 0\}.$$

Then

$$N = \bigcup_{m=1}^{\infty} N_m$$

where

$$N_m = \left\{\tau \in [0, \hat{\tau}] : \beta(\tau) \leqq -\frac{1}{m}\right\}.$$

Let χ_{N_m} denote the characteristic function of N_m. Setting $f = \chi_{N_m}$ in (1.47), we obtain

$$0 \leqq \int_{N_m} \beta_{n_i} d\tau \;\to\; \int_{N_m} \beta \, dt \leqq -\frac{1}{m} \operatorname{meas} N_m.$$

Hence, meas $N_m = 0$ and N, being the countable union of sets of measure zero, is itself of measure zero. We conclude that $0 \leqq \beta \leqq 1$ and we can adjust β on a set of measure zero to achieve $0 \leqq \beta \leqq 1$ everywhere. Now, suppose that M is a measurable subset of $[0, \hat{\tau}]$ and suppose that

$$\beta_{n_i} \to \tilde{\beta} \quad \text{pointwise on } M. \tag{1.48a}$$

Then

$$\beta_{|M} = \tilde{\beta} \quad \text{a.e.} \tag{1.48b}$$

For, on one hand, by the dominated-convergence theorem, for any $f \in L^1[0, \hat{\tau}]$,

$$\int_M \beta_{n_i} f \, d\tau \to \int_M \tilde{\beta} f \, d\tau,$$

while on the other hand, replacing f in (1.47) by $f\chi_M$, where χ_M is the characteristic function of M, we obtain

$$\int_M \beta_{n_i} f \, d\tau \to \int_M \beta f \, d\tau.$$

Consequently,

$$\int_M (\beta - \tilde{\beta}) f \, d\tau = 0 \quad \forall f \in L^1[0, \hat{\tau}].$$

In particular, setting $f = \chi_M(\beta - \tilde{\beta})$, we obtain the conclusion (1.48b). Thus we can adjust β on a subset of measure zero to coincide with $\tilde{\beta}$ on M. In view of the fact that $0 \leqq \tilde{\beta} \leqq 1$, this does not affect our previous adjustment.

As a consequence of the above, the functions

$$\nu_* := \frac{1}{2} a_{-*}(1 - \beta), \quad \kappa_* := \frac{1}{2a_{-*}}(1 + \beta) \tag{1.49a}$$

belong to $L^\infty[0, \hat{\tau}]$ and we have

$$\nu_{*n_i} \to \nu_*, \quad \kappa_{*n_i} \to \kappa_* \tag{1.49b}$$

in the weak-star sense. We set

$$\nu(u, v) = \nu_*(u) e^{N(u,v) - N(u,u)}, \quad \kappa(u, v) = \kappa_*(v) e^{-K(u,v)} \tag{1.49c}$$

where N, K are the functions appearing in the statement of Lemma 1.2. We shall show that these functions are in fact given by

$$N(u, v) = \int_0^v \left((\mu - 4\pi r^2) \frac{\kappa}{r} \right)(u, v') \, dv',$$

$$K(u, v) = 4\pi \int_v^u (r\nu\zeta^2)(u', v) \, du'. \tag{1.49d}$$

The proof relies on the following lemma:

Lemma 1.3. *Let (f_{n_i}) be a sequence of functions in $C^0(\mathscr{U}(\hat{\tau}))$ converging uniformly to a function f and let*

$$g_{n_i}(u,v) = \int_0^v (f_{n_i}\kappa_{n_i})(u,v')\,dv',$$

$$g(u,v) = \int_0^v (f\kappa)(u,v')\,dv,$$

$$h_{n_i}(u,v) = \int_v^u (f_{n_i}\nu_{n_i})(u',v)\,du',$$

$$h(u,v) = \int_v^u (f\nu)(u',v)\,du',$$

Then

$$g_{n_i} \to g, \quad h_{n_i} \to h \quad \textit{uniformly on } \mathscr{U}(\hat{\tau}).$$

Proof. We demonstrate the conclusion for g_{n_i}, g. The conclusion for h_{n_i}, h is proved similarly. Let

$$\tilde{f}_{n_i} = f_{n_i}\exp(-K_{n_i}), \quad \tilde{f} = f\exp(-K). \tag{1.50a}$$

Since

$$\int_0^v |\tilde{f}_{n_i} - \tilde{f}|\,\kappa_{*n_i}\,dv \to 0 \quad \text{uniformly on } \mathscr{U}(\hat{\tau}), \tag{1.50b}$$

it suffices to show that

$$\tilde{g}_{n_i} \to g \quad \text{uniformly on } \mathscr{U}(\hat{\tau}) \tag{1.50c}$$

where

$$\tilde{g}_{n_i}(u,v) = \int_0^v \tilde{f}(u,v')\kappa_{*n_i}(v')\,dv'. \tag{1.50d}$$

Note that

$$g(u,v) = \int_0^v \tilde{f}(u,v')\kappa_*(v')\,dv'. \tag{1.50e}$$

To prove (1.50c), given $(u_0,v_0) \in \mathscr{U}(\hat{\tau})$ and $\delta > 0$, let

$$Q_\delta(u_0,v_0) = \{(u,v) \in \mathscr{U}(\hat{\tau}) : |u-u_0| \leqq \delta, |v-v_0| \leqq \delta\}. \tag{1.50f}$$

Also, let

$$C_1 := \sup_i \left(\sup_{[0,\hat{\tau}]} \kappa_{*n_i} \right), \quad C_2 := \sup_{\mathscr{U}(\hat{\tau})} |\tilde{f}| \tag{1.50g}$$

and, given $\delta > 0$,

$$\varepsilon(\delta) = \sup\{|\tilde{f}(u,v) - \tilde{f}(u_0,v)| : (u,v), (u_0,v) \in \mathscr{U}(\hat{\tau})\}. \tag{1.50h}$$

Then for any $(u,v)\in Q_\delta(u_0,v_0)$ we have

$$|\tilde{g}_{n_i}(u,v)-\tilde{g}_{n_i}(u_0,v_0)| = \left|\int_0^v \tilde{f}(u,v')\kappa_{*n_i}(v')\,dv - \int_0^{v_0}\tilde{f}(u_0,v')\kappa_{*n_i}(v')\,dv'\right|$$

$$\leqq \int_0^{v_0} |\tilde{f}(u,v')-\tilde{f}(u_0,v')|\kappa_{*n_i}(v')\,dv' + \int_{v_0}^{v} |\tilde{f}(u,v')|\kappa_{*n_i}(v')\,dv'$$

$$\leqq \hat{\tau}\varepsilon(\delta)C_1 + \delta C_2 C_1. \tag{1.50i}$$

Similarly,

$$|g(u,v)-g(u_0,v_0)| \leqq (\hat{\tau}\varepsilon(\delta)+\delta C_2)C_1. \tag{1.50j}$$

Now by virtue of the weak-star convergence of κ_{*n_i} to κ_*, given any $(u_0,v_0)\in\mathscr{U}(\hat{\tau})$ and any $\eta>0$ we can find a $N((u_0,v_0),\eta)$ sufficiently large so that

$$|\tilde{g}_{n_i}(u_0,v_0)-g(u_0,v_0)| \leqq \tfrac{1}{3}\eta \quad \forall n \leqq N((u_0,v_0),\eta). \tag{1.50k}$$

On the other hand, given any $\eta>0$ we can also choose $\delta(\eta)$ sufficiently small so that

$$(\hat{\tau}\varepsilon(\delta(\eta)) + \delta(\eta)C_2)C_1 \leqq \tfrac{1}{3}\eta. \tag{1.50l}$$

For, it follows from the uniform continuity of $\tilde{f}$ that $\varepsilon(\delta)\to 0$ as $\delta\to 0$. Thus for all $(u,v)\in Q_{\delta(\eta)}(u_0,v_0)$ and for all $n\geqq N((u_0,v_0),\eta)$ we have

$$|\tilde{g}_{n_i}(u,v)-g(u,v)| \leqq |\tilde{g}_{n_i}(u_0,v_0)-g(u_0,v_0)|$$

$$+ |\tilde{g}_{n_i}(u,v)-\tilde{g}_{n_i}(u_0,v_0)| + |g(u,v)-g(u_0,v_0)|$$

$$\leqq \tfrac{1}{3}\eta + \tfrac{1}{3}\eta + \tfrac{1}{3}\eta = \eta. \tag{1.50m}$$

Finally, for each $\delta>0$ a finite family

$$\{Q_\delta(u_j,v_j): j=1,\ldots,m(\delta)\}$$

of sets Q_δ suffices to cover $\mathscr{U}(\hat{\tau})$. Setting

$$M(\eta) = \max_{j=1,\ldots,m(\delta(\eta))} N((u_j,v_j),\eta)$$

we then obtain

$$\sup_{\mathscr{U}(\hat{\tau})} |\tilde{g}_{n_i}-g| \leqq \eta \quad \forall n \geqq M(\eta), \tag{1.50n}$$

and (1.50c) is proved.

Applying Lemma 1.3 to the cases

$$f_{n_i} = \frac{1}{r_{n_i}}(\mu_{n_i} - 4\pi r_{n_i}^2), \quad f = \frac{1}{r}(\mu - 4\pi r^2)$$

and

$$f_{n_i} = 4\pi r_{n_i} \nu_{n_i} \zeta_{n_i}^2, \quad f = 4\pi r \nu \zeta^2$$

we obtain (the conclusion for g_{n_i}, g):

$$N_{n_i} \to \int_0^v (\mu - 4\pi r^2)\frac{\kappa}{r}\,dv \quad \text{uniformly on } \mathscr{U}(\hat{\tau})$$

and (the conclusion for h_{n_i}, h):

$$K_{n_i} \to \int_v^u r\nu\zeta^2\,du \quad \text{uniformly on } \mathscr{U}(\hat{\tau}).$$

Since $N_{n_i} \to N$ and $K_{n_i} \to K$ by Lemma 1.2, the functions N and K are thus given by (1.49d). It follows that ν and κ satisfy equations (6.6b) and (6.6a) of [I] respectively.

We may also apply Lemma 1.3 to the cases

$$f_{n_i} = 1 - \mu_{n_i}, \quad f = 1 - \mu$$

and

$$f_{n_i} = 1, \quad f = 1$$

to obtain (the conclusion for g_{n_i}, g):

$$r_{n_i}(u,v) - r_{n_i}(u,0) \to \int_0^v ((1-\mu)\kappa)(u,v')\,dv',$$

$$r_{n_i}(u,v) - r_{*n_i}(v) \to -\int_v^u \nu(u',v)\,du'.$$

Since $r_{n_i}(u,v) - r_{n_i}(u,0) \to r(u,v) - r(u,0)$ and $r_{n_i}(u,v) - r_{*n_i}(v) \to r(u,v) - r_*(v)$ by Lemma 1.2, we have

$$r(u,v) - r(u,0) = \int_0^v ((1-\mu)\kappa)(u,v')\,dv',$$

$$r(u,v) - r_*(v) = -\int_v^u \nu(u',v)\,du'.$$

Thus r satisfies equations (6.3a, b) of [I].

Applying Lemma 1.3 to the cases

$$f_{n_i} = \eta_{n_i}, \quad f = \eta$$

and

$$f_{n_i} = \zeta_{n_i}, \quad f = \zeta,$$

we obtain (the conclusion for g_{n_i}, g):

$$\phi_{n_i}(u,v) - \phi_{n_i}(u,0) \to \int_0^v (\eta\kappa)(u,v')\,dv',$$

and

$$\phi_{n_i}(u,v) - \phi_{*n_i}(v) \to -\int_v^u (\zeta\nu)(u',v)\,du'.$$

Since $\phi_{n_i}(u,v) - \phi_{n_i}(u,0) \to \phi(u,v) - \phi(u,0)$ and $\phi_{n_i}(u,v) - \phi_{*n_i}(v) \to \phi(u,v) - \phi_*(v)$ by Lemma 1.2, we have

$$\phi(u,v) - \phi(u,0) = \int_0^v (\eta\kappa)(u,v')\,dv',$$

and

$$\phi(u,v) - \phi_*(v) = \int_v^u (\zeta\nu)(u',v)\,du'.$$

Thus ϕ satisfies equations (6.4a, b) of [I].

In addition, we apply Lemma 1.3 to the cases

$$f_{n_i} = 2\pi r_{n_i}^2[\eta_{n_i}^2 + (1-\mu_{n_i})], \quad f = 2\pi r^2[\eta^2 + (1-\mu)]$$

and

$$f_{n_i} = -2\pi r_{n_i}^2[(1-\mu_{n_i})\zeta_{n_i}^2 + 1], \quad f = -2\pi r^2[(1-\mu)\zeta^2 + 1]$$

to obtain (the conclusion for g_{n_i}, g):

$$m_{n_i}(u,v) - m_{n_i}(u,0) \to 2\pi \int_0^v (r^2[\eta^2 + (1-\mu)]\kappa)(u,v')\,dv',$$

and

$$m_{n_i}(u,v) - m_{*n_i}(v) \to -2\pi \int_v^u (r^2[(1-\mu)\zeta^2 + 1]\nu)(u',v)\,du'.$$

Since $m_{n_i}(u,v) - m_{n_i}(u,0) \to m(u,v) - m(u,0)$ and $m_{n_i}(u,v) - m_{*n_i}(v) \to m(u,v) - m_*(v)$ by Lemma 1.2, we have

$$m(u,v) - m(u,0) = 2\pi \int_0^v (r^2[\eta^2 + (1-\mu)]\kappa)(u,v')\,dv',$$

and

$$m(u,v) - m_*(v) = -2\pi \int_v^u (r^2[(1-\mu)\zeta^2 + 1]\nu)(u',v)\,du'.$$

Thus m satisfies equations (6.5a, b) of [I].

Finally the application of Lemma 1.3 to the cases

$$f_{n_i} = \frac{1}{r_{n_i}}[\eta_{n_i} - (1 - 4\pi r_{n_i}^2)\zeta_{n_i}], \quad f = \frac{1}{r}[\eta - (1 - 4\pi r^2)\zeta]$$

and

$$f_{n_i} = \frac{1}{r_{n_i}}[(1 + 4\pi r_{n_i}^2\zeta_{n_i}^2)\eta_{n_i} - (1-\mu_{n_i})\zeta_{n_i}], \quad f = \frac{1}{r}[(1 + 4\pi r^2\zeta^2)\eta - (1-\mu)\zeta]$$

yields

$$\zeta_{n_i}(u,v) - \zeta_{n_i}(u,0) = \int_0^v \left(\frac{1}{r_{n_i}}[\eta_{n_i} - (1 - 4\pi r_{n_i}^2)\zeta_{n_i}]\kappa_{n_i}\right)(u,v')\,dv'$$

$$\to \int_0^v \left(\frac{1}{r}[\eta - (1 - 4\pi r^2)\zeta]\kappa\right)(u,v')\,dv'$$

and

$$\eta_{n_i}(u,v) - \eta_{*n_i}(v) = \int_v^u \left(\frac{1}{r_{n_i}}[(1 + 4\pi r_{n_i}^2 \zeta_{n_i}^2)\eta_{n_i} - (1-\mu_{n_i})\zeta_{n_i}]\nu_{n_i}\right)(u',v)\,du'$$

$$\to \int_v^u \left(\frac{1}{r}[(1 + 4\pi r^2 \zeta^2)\eta - (1-\mu)\zeta]\nu\right)(u',v)\,du'.$$

Consequently, since $\eta_{n_i} \to \eta$ and $\zeta_{n_i} \to \zeta$ by Lemma 1.2,

$$\zeta(u,v) - \zeta(u,0) = \int_0^t \left(\frac{1}{r}[\eta - (1 - 4\pi r^2)\zeta]\kappa\right)(u,v')\,dv',$$

$$\eta(u,v) - \eta(v) = \int_v^u \left(\frac{1}{r}[(1 + 4\pi r^2 \zeta^2)\eta - (1-\mu)\zeta]\nu\right)(u',v)\,du';$$

hence the functions ζ and η satisfy equations (6.7b) and (6.7a) of [I] respectively. We conclude that $(r, \phi, m, \nu, \kappa, \zeta, \eta)$ is a solution of the hard-phase equations.

Let us now define the closed subsets

$$X = \{\tau \in [0,\hat{\tau}] : x(\tau) = 0\}, \tag{1.51a}$$

$$Y = \{\tau \in [0,\hat{\tau}] : y(\tau) = 0\}. \tag{1.51b}$$

We then define the subsets:

$$\begin{aligned} C_1 &= X^c \cap Y^c, \quad C_2 = X \cap Y^c, \\ C_3 &= X^c \cap Y, \quad C_4 = X \cap Y. \end{aligned} \tag{1.51c}$$

Since

$$x = 2(1-\rho_*), \quad y = 1 - \gamma^2, \tag{1.51d}$$

the subsets C_1, C_2, C_3, C_4 correspond to Case 1, Case 2, Case 3, Case 4, respectively (see Section 1 and the proof of Lemma 1.1). The subset C_1 is open and, by virtue of (1.45f) and (1.46c), $\beta_{n_i} \to \beta$ pointwise on C_1 and

$$0 < \beta_{|C_1} < 1. \tag{1.51e}$$

The convergence is uniform in closed subsets of C_1. Thus β is continuous on C_1. By (1.45f) and (1.46c), $\beta_{n_i} \to \beta$ pointwise on C_2 and C_3, and

$$\beta_{|C_2} = 1, \quad \beta_{|C_3} = 0. \tag{1.51f}$$

Thus only on the closed subset

$$C_4 = \{\tau \in [0,\hat{\tau}] : (x(\tau), y(\tau)) = (0,0)\},$$

which corresponds to the point of non-continuity of the function g on $\bar{Q}_\varepsilon$, does (1.45f) fail to yield pointwise convergence. The argument instead rests on

Lemma 1.4. *For every* $\tau_1 \in (0, \hat{\tau}]$, *there exists a* $\tau_0 \in (0, \tau_1)$ *such that* $\tau_0 \in C_1$.

Proof. Suppose on the contrary that there is a $\tau_1 \in (0, \hat{\tau}]$ such that for all $\tau \in (0, \tau_1)$ we have $\tau \notin C_1$, that is,

$$\tau \in \bigcup_{i=2}^{4} C_i.$$

We may assume that τ_1 is as small as we wish.

Consider first the case that there is a $\check{\tau} \in (0, \tau_1)$ such that $\check{\tau} \in C_2$, that is, $\check{\tau} \in X$, $\check{\tau} \in Y^c$. Since Y^c is open, there is an $\varepsilon > 0$ and an interval

$$I_\varepsilon := [\check{\tau} - \varepsilon, \check{\tau} + \varepsilon] \tag{1.52a}$$

such that $I_\varepsilon \subset (0, \tau_1) \cap Y^c$. Thus $I_\varepsilon \cap X^c = \emptyset$ because

$$I_\varepsilon \cap X^c \subset (0, \tau_1) \cap Y^c \cap X^c = (0, \tau_1) \cap C_1,$$

and according to our hypothesis the last set is empty. We conclude that $I_\varepsilon \subset X$; hence

$$I_\varepsilon \subset (0, \tau_1) \cap C_2$$

also. It follows that

$$\rho_* = 1 \quad \text{on } I_\varepsilon, \tag{1.52b}$$

and also that

$$\beta_{n_i} \to 1 \quad \text{uniformly on } I_\varepsilon, \tag{1.52c}$$

which implies that for any $\tau', \tau'' \in I_\varepsilon$,

$$\chi_{*n_i}(\tau'') - \chi_{*n_i}(\tau') = \int_{\tau'}^{\tau''} \exp(-\omega_{*n_i}) \beta_{n_i} \, d\tau \to \int_{\tau'}^{\tau''} \exp(-\omega_*) \beta \, d\tau.$$

Since

$$\chi_{*n_i}(\tau'') - \chi_{*n_i}(\tau') \to \chi_*(\tau'') - \chi_*(\tau'),$$

χ_* is continuously differentiable in I_ε and

$$e^{\omega_*} \frac{d\chi_*}{d\tau} = 1. \tag{1.52d}$$

This means that the segment of the curve $\mathscr{B}$ corresponding to the interval I_ε is null outgoing. On the other hand, according to (1.52b), this segment must coincide with the corresponding segment of $\partial\mathscr{K}$. However the latter is strictly timelike.

Consider next the case that there is a $\check{\tau} \in (0, \tau_1)$ such that $\check{\tau} \in C_3$, that is, $\check{\tau} \in Y$, $\check{\tau} \in X^c$. Since X^c is open, there is an $\varepsilon > 0$ and an interval I_ε as in (1.52a) such that $I_\varepsilon \subset (0, \tau_1) \cap X^c$. We have $I_\varepsilon \cap Y^c = \emptyset$, because

$$I_\varepsilon \cap Y^c \subset (0, \tau_1) \cap X^c \cap Y^c = (0, \tau_1) \cap C_1,$$

and according to our hypothesis the last set is empty. We conclude that $I_\varepsilon \subset Y$; hence

$$I_\varepsilon \subset (0, \tau_1) \cap C_3.$$

It follows that

$$\gamma = 1 \quad \text{on } I_\varepsilon \tag{1.53a}$$

and also that

$$\beta_{n_i} \to 0 \quad \text{uniformly on } I_\varepsilon. \tag{1.53b}$$

According to Proposition 1.1 we can express

$$\gamma_{n_i}(\check{\tau} + \varepsilon) - \gamma_{n_i}(\check{\tau} - \varepsilon) = -\int_{\check{\tau}-\varepsilon}^{\check{\tau}+\varepsilon} \gamma_{n_i}[(1 - \beta_{n_i})E_{n_i} + \beta_{n_i}B_{n_i}]\, d\tau. \tag{1.53c}$$

Now, by (1.53a) the left-hand side tends to zero as $i \to \infty$. Consequently, in conjunction with (1.53b),

$$\lim_{i\to\infty} \int_{\check{\tau}-\varepsilon}^{\check{\tau}+\varepsilon} e_{*n_i}\, d\tau = 0. \tag{1.53d}$$

Here e_{*n_i} is the restriction to $\mathscr{B}_{n_i}$ of the barrier function (1.23a) corresponding to the n_i solution:

$$e_{*n_i} = \frac{1}{2} a^2_{-*n_i} \zeta_{*n_i} + \frac{1}{2r_{*n_i}}\left[a_{-*n_i} - \frac{(1 - 4\pi r^2_{*n_i})}{a_{-*n_i}} \right] + \frac{\mu_{*n_i}}{2r_{*n_i} a_{-*n_i}} \quad \text{on } I_\varepsilon.$$

We shall show that, if τ_1 is sufficiently small, which, as we have remarked, can in the present context be assumed without loss of generality, then, in fact

$$\lim_{i\to\infty} \left(\inf_{I_\varepsilon} e_{-*n_i} \right) > 0, \tag{1.53e}$$

contradicting (1.53d). For, it can be shown that, as in (1.29c),

$$\sup_{(u,v)\in\mathscr{U}(\tau_1)} |\xi_{-n_i}(u, v) - \exp(-2N_{n_i})(u, v)\xi_{n_i}(u, 0)| \leqq C\tau_1, \tag{1.53f}$$

while (see (1.30a–h) and (1.43d))

$$\begin{aligned} \inf_{u\in[0,\tau_1]} \xi_{n_i}(u, 0) &= \inf_{t\in[0,\phi_{n_i}(\tau_1)]} \left(-\frac{\dot{Z}_{n_i}(t)}{\dot{R}_{n_i}(t)} \right) \\ &= \inf_{t\in[c_n,\phi_{n_i}(\tau_1)+c_n]} \left[-\frac{\dot{Z}(t)}{\dot{R}(t)} \left(\frac{\dot{R}(t)}{\dot{R}(t) + l_{n_i}} \right)^3 \right]. \end{aligned} \tag{1.53g}$$

Hence

$$\liminf_{i\to\infty} \left(\inf_{u\in[0,\tau_1]} \xi_{n_i}(u, 0) \right) = \inf_{t\in[0,\phi(\tau_1)]} \left(-\frac{\dot{Z}(t)}{\dot{R}(t)} \right). \tag{1.53h}$$

Since $\phi(\tau_1, 0) \to 0$ as $\tau_1 \to 0$, it follows, in view of (1.32b) that

$$\liminf_{i\to\infty} \left(\inf_{u\in[0,\tau_1]} \xi_{n_i}(u, 0) \right) \to \infty \quad \text{as } \tau_1 \to 0. \tag{1.53i}$$

In conjunction with (1.53f) this yields

$$\liminf_{i\to\infty}\left(\inf_{[0,\tau_1]} e_{-*n_i}\right)\to\infty \quad \text{as } \tau_1\to 0, \tag{1.53j}$$

which, if τ_1 is small enough, implies (1.53e).

The preceding development leads to the conclusion that $(0,\tau_1)\subset C_4$. Then for any closed interval $I_\varepsilon\subset(0,\tau_1)$ we have

$$\rho_* = \gamma = 1 \quad \text{on } I_\varepsilon \tag{1.54a}$$

and (1.53c) holds. Therefore, considering the remark at the end of Proposition 1.1, since the left-hand side of (1.53c) tends to zero as before, we obtain

$$\lim_{i\to\infty}\int_{\check{\tau}-\varepsilon}^{\check{\tau}+\varepsilon}[(1-\beta_{n_i})e_{*n_i}+\beta_{n_i}b_{*n_i}]\,d\tau = 0. \tag{1.54b}$$

Now, the left-hand side of (1.54b) is not less than

$$2\varepsilon\liminf_{i\to\infty}\left(\inf_{I_\varepsilon}\min\{e_{-*n_i}, b_{*n_i}\}\right),$$

and, as we have shown, (1.53e) holds, while by Proposition 5.1 of [I], in view of (1.54a), $b_{*n_i}\to b_*$ uniformly and

$$\inf_{I_\varepsilon} b_* > 0.$$

We have therefore again reached a contradiction and the proof of the lemma is complete.

Next we show

Lemma 1.5. *If $\hat{\tau}$ is sufficiently small, then C_1 coincides with* $(0,\hat{\tau}]$.

Proof. Let I be a component of the open set C_1. It suffices to show that the right end point of I is included in I. The proof of this depends on showing that γ is continuously differentiable in I and applying Proposition 1.1. The continuous differentiability of γ in I in turn reduces to that of ζ_*.

For any interval $J\subset[0,\hat{\tau}]$, we define the domains $\mathscr{U}_J^+,\mathscr{U}_J^-\subset\mathscr{U}$, by

$$\mathscr{U}_J^+ = \{(u,v): u\in J, v\in[0,u]\}, \tag{1.55a}$$

$$\mathscr{U}_J^- = \{(u,v): v\in J, u\in[v,\hat{\tau}]\}. \tag{1.55b}$$

That β is continuous and takes values in $(0,1)$ on I and that the convergence of β_{n_i} to β is uniform in any closed subinterval $J\subset I$, imply that ν_* and κ_* are continuous on I and the convergence of ν_{*n_i} and κ_{*n_i} to ν_* and κ_* respectively is uniform in any closed sub-interval $J\subset I$. It follows that ν is continuous in $\mathscr{U}_I^+$ and κ is continuous in $\mathscr{U}_I^-$ and the convergence of ν_{n_i} to ν is uniform in $\mathscr{U}_J^+$, while the convergence of κ_{n_i} to κ is uniform in $\mathscr{U}_J^-$, for any closed subinterval $J\subset I$. It follows that in $\mathcal{U}_I^+ \cap \mathcal{U}_I^-$, v, ζ are continuously differentiable with respect to v while κ, η are continuously differentiable with respect to u. Also, r, ϕ, m are C^1 functions in $\mathcal{U}_I^+ \cap \mathcal{U}_I^-$.

To prove that ζ is also continuously differentiable with respect to u in $\mathcal{U}_I^+$, we consider the functions α_{-n_i}. We can write (see (1.26c))

$$\alpha_{-n_i}(u,v) = \exp(-2N_{n_i}(u,v))\,\alpha_{-n_i}(u,0) - A_{-n_i}(u,v) \tag{1.56a}$$

where

$$A_{-n_i}(u,v) = \alpha(u,v)N_{-n_i}(u,v)\exp(-N_{n_i}(u,v)) + \int_0^v [\zeta_{n_i}N_{+n_i} + \phi_{n_i}(N_{+-n_i} + N_{+n_i}N_{-n_i})](u,v') \times \exp(2N_{n_i}(u,v') - 2N_{n_i}(u,v))\,\kappa_{n_i}(u,v')\,dv'. \tag{1.56b}$$

We apply Lemma 1.3 to the case

$$f_{n_i} = \exp(N_{n_i})\,N_{+-n_i}, \quad f = \exp(N)\,N_{+-}$$

where (see 1.27d)

$$N_{+-n_i} := \frac{2\mu_{n_i}}{r_{n_i}^2} - 4\pi\zeta_{n_i}^2(1-4\pi r_{n_i}^2), \quad N_{+-} := \frac{2\mu}{r^2} - 4\pi\zeta^2(1-4\pi r^2)$$

to obtain (the conclusion for g_{n_i}, g):

$$N_{-n_i} \to N_- \quad \text{uniformly on } \mathcal{U}(\hat{\tau}) \tag{1.56c}$$

where (see 1.43c)

$$N_-(u,v) := \int_0^v e^{N(u,v')-N(u,v)}(N_{+-}\kappa)(u,v')\,dv'.$$

We then apply Lemma 1.3 once again, this time to the case

$$f_{n_i} = [\zeta_{n_i}N_{+n_i} + \phi_{n_i}(N_{+-n_i} + N_{+n_i}N_{-n_i})]\exp(2N_{n_i}),$$
$$f = [\zeta N_+ + \phi(N_{+-} + N_+N_-)]\exp(2N)$$

where (see 1.25b)

$$N_{+n_i} := \frac{1}{r_{n_i}}(\mu_{n_i} - 4\pi r_{n_i}^2), \quad N_+ := \frac{1}{r}(\mu - 4\pi r^2)$$

to obtain (the conclusion for g_{n_i}, g):

$$A_{-n_i} \to A_- \quad \text{uniformly on } \mathcal{U}(\hat{\tau}) \tag{1.56d}$$

where

$$A_-(u,v) = \alpha(u,v)N_-(u,v)e^{-N(u,v)} + \int_0^v [\zeta N_+ + \phi(N_{+-} + N_+N_-)](u,v')\,e^{2N(u,v')-2N(u,v)}\kappa(u,v')\,dv'. \tag{1.56e}$$

Moreover, as in (1.29a) we deduce that

$$\sup_{\mathscr{U}(\hat{\tau})} |A_{-n_i}| \leqq C\hat{\tau}.$$

Hence

$$\sup_{\mathscr{U}(\hat{\tau})} |A_-| \leqq C\hat{\tau}. \tag{1.56f}$$

Next we show that the sequence of functions $\xi_{n_i}(\cdot, 0)$ is uniformly convergent in any closed subinterval $J \subset I$. We have (see (1.30a–h) and (1.43d))

$$\xi_{n_i}(u,0) = -\left(\frac{\dot{Z}_{n_i}}{\dot{R}_{n_i}}\right)(\phi_{n_i}(u,0)) \\ -\frac{\dot{Z}(\phi_{n_i}(u,0)+c_{n_i})}{\dot{R}(\phi_{n_i}(u,0)+c_{n_i})}\left(\frac{\dot{R}(\phi_{n_i}(u,0)+c_{n_i})}{\dot{R}(\phi_{n_i}(u,0)+c_{n_i})+l_{n_i}}\right)^3. \tag{1.57a}$$

Proposition 4.1 of [II] implies (see (1.39)) that

$$0 = \phi_{n_i}(0,0) \leqq \phi_{n_i}(u,0) \leqq \phi_{n_i}(u,u) \leqq u. \tag{1.57b}$$

For any closed subinterval $J \subset I$, $\inf_{u \in J} \phi_{n_i}(u, 0)$ is in fact bounded from below by a positive constant independent of i. For, there is a closed interval J_0 on the left of J such that $J_0 \subset C_1$. Since $\sup_{J_0} \beta < 1$ and hence $\inf_{J_0} v_* > 0$ (see (1.51a)), it follows that on $J_0 \times \{0\}$ the functions v_{n_i} are uniformly bounded from below by a positive constant. Since the same is true for the functions ζ_{n_i} and we have $\partial\phi_{n_i}/\partial u = v_{n_i}\zeta_{n_i}$, it follows that

$$\inf_i \left(\inf_{u \in J_0} \frac{\partial \phi_{n_i}}{\partial u}\right) := \varepsilon_0 > 0. \tag{1.57c}$$

Consequently

$$\inf_{u \in J} \phi_{n_i}(u,0),\ \inf_{u \in J} \phi(u,0) \geqq \varepsilon_0 |J_0|. \tag{1.57d}$$

Since $\dot{Z}(t)$ is continuous away from $t = 0$ (see (1.32b)), we conclude that the sequence $(\xi_{n_i}(\cdot, 0))$ converges uniformly in J to $\xi(\cdot, 0)$, where

$$\xi(u,0) = -\frac{\dot{Z}(\phi(u,0))}{\dot{R}(\phi(u,0))} \tag{1.57e}$$

is a continuous function in I, extending continuously to the right end point τ_1 of I. Hence the sequence $(\alpha_{-n_i}(\cdot, 0))$ converges uniformly in J to $\alpha_-(\cdot, 0) = r(\cdot, 0)\xi(\cdot, 0) - 2\zeta(\cdot, 0)$. In conjunction with (1.56d) this allows us to conclude that

$$\alpha_{-n_i} \to \alpha_- \quad \text{uniformly in } \mathscr{U}_J^+ \tag{1.57f}$$

where

$$\alpha_-(u,v) = e^{-2N(u,v)}\alpha_-(u,0) - A_-(u,v) \tag{1.57g}$$

is a continuous function in $\mathscr{U}_I^+$, extending continuously to its future boundary $\{\tau_1\} \times [0, \tau_1]$. Consequently,

$$\xi_{n_i} \to \xi := r^{-1-}(\alpha_- + 2\zeta) \quad \text{uniformly in } \mathscr{U}_J^+ \tag{1.57h}$$

for any closed subinterval $J \subset I$. Thus, for any $u_1, u_2 \in I$ and any $v \in [0, \min\{u_1, u_2\}]$,

$$\zeta(u_2, v) - \zeta(u_1, v) = \lim_{i\to\infty} (\zeta_{n_i}(u_2, v) - \zeta_{n_i}(u_1, v))$$

$$= \lim_{i\to\infty} \left(\int_{u_1}^{u_2} (v_{n_i}\xi_{n_i})(u, v)\, du \right) = \int_{u_1}^{u_2} (v\xi)(u, v)\, du.$$

Hence ζ is continuously differentiable with respect to u in $\mathscr{U}_I^+$ and

$$\frac{\partial \zeta}{\partial u} = v\xi.$$

In view of the continuous differentiability of ζ with respect to v in $\mathscr{U}_I^+ \cap \mathscr{U}_I^-$ established previously, we conclude that ζ is a C^1 function in $\mathscr{U}_I^+ \cap \mathscr{U}_I^-$, and hence ζ_* and γ are C^1 functions in I and Proposition 1.1 applies.

We are now ready to prove that the end point τ_1 of I is included in I. For, suppose on the contrary that $\tau_1 \notin I$. Then

$$\tau_1 \in \bigcup_{i=2}^{4} C_i.$$

Now $\tau_1 \in C_2$ is impossible, for according to the results of Section 4 of [I], this would imply that $\partial\mathscr{K}$ is non-timelike at the point $(\tau_1, \chi_1 := \chi_*(\tau_1))$, contradicting, when $\hat{\tau}$ is small enough, the fact that $\partial\mathscr{K}$ is timelike in a neighborhood of the null point N^+ in $\mathscr{J}^+(N^+)$. On the other hand, if $\tau_1 \in C_3 \cup C_4$, then $\gamma(\tau_1) = 1$ while $\gamma(\tau) < 1$ for all $\tau \in I$. Hence,

$$\limsup_{\tau\to\tau_1} \frac{d\gamma}{d\tau}(\tau) \geqq 0. \tag{1.58a}$$

However, by virtue of Proposition 1.1 we have

$$\lim_{\tau\to\tau_1} \frac{d\gamma}{d\tau}(\tau) = -e_1 \tag{1.58b}$$

if $\tau \in C_3$, while

$$\limsup_{\tau\to\tau_1} \frac{d\gamma}{d\tau}(\tau) \leqq -\min\{e_1, b_1\} \tag{1.58c}$$

if $\tau_1 \in C_4$. Here,

$$e_1 = \frac{1}{2} a_{-*1}^2 \xi_{*1} + \frac{1}{2r_{*1}} \left[a_{-*1} - \frac{(1 - 4\pi r_{*1}^2)}{a_{-*1}} \right] + \frac{\mu_{*1}}{2r_{*1} a_{-*1}}$$

is the boundary barrier function at the point τ_1 and

$$b_1 = \left(\frac{\partial \rho}{\partial \tau} \right)(\tau_1, \chi_1).$$

Now the proof of Lemma 1.4 shows that in fact $e_1 > 0$ if $\hat{\tau}$ is small enough. Consequently, (1.58b, c) and the fact that $b_1 > 0$ when $\tau_1 \in C_4$ (Proposition 5.1 of [I]) imply that

$$\lim_{\tau \to \tau_1} \frac{d\gamma}{d\tau}(\tau) < 0,$$

in contradiction to 1.58a. The proof of the lemma is therefore complete.

We are now ready to prove

Theorem 1.1. *Let (r, ω, ρ) be a soft-phase solution corresponding to smooth initial data and let $\mathscr{V}$ be the domain*

$$\mathscr{V} = \{(\tau, \chi): \chi \geqq 0, \tau_+(\chi) \leqq \tau \leqq \hat{\tau}(\chi)\}$$

where $(0,0) = N^+$ is an outgoing boundary null point, $\tau = \tau_+(\chi)$ is the equation of C^+, the outgoing null curve issuing from N^+, while $\tau = \hat{\tau}(\chi)$ is the equation of $\partial\mathscr{K}$, where, corresponding to the given soft-phase initial data, $\rho(\tau, \chi)$ along each flow line first becomes equal to 1. Let R be a function defined on an interval $[0, \hat{\tau}]$ and representing r as a function of ϕ along C^{*+}, the incoming null curve issuing from N^+. Then there is a $\hat{\tau} > 0$ and a solution to the problem of formation of a free phase boundary such that $\mathscr{B}$ is a C^1 curve: $[0, \hat{\tau}] \mapsto \mathscr{V}$, $\tau \mapsto \chi_*(\tau)$, issuing from N^+, which has positive velocity β relative to the soft-phase flow lines, is strictly timelike ($\beta < 1$) and contained in the interior of $\mathscr{V}$ in $(0, \hat{\tau}]$, becoming null outgoing at N^+ ($\beta(0) = 1$). Also, r, m and ϕ are C^1 functions while ν, κ, ζ, η are C^0 functions on $\mathscr{U}(\hat{\tau})$, defining a solution corresponding to genuine hard phase, except at N^+ where $\sigma = 1$.

Proof. By virtue of Lemma 1.5, ν is continuous in

$$\mathscr{U}^+_{(0,\hat{\tau}]} = \mathscr{U}(\hat{\tau}) \backslash (0,0)$$

and κ is continuous in

$$\mathscr{U}^-_{(0,\hat{\tau}]} = \mathscr{U}(\hat{\tau}) \backslash ([0, \hat{\tau}] \times \{0\}).$$

Since

$$\mathscr{U}^+_{(0,\hat{\tau}]} \cap \mathscr{U}^-_{(0,\hat{\tau}]} = \mathscr{U}^-_{(0,\hat{\tau}]},$$

the functions ν, ζ are thus continuously differentiable with respect to v while the functions κ, η are continuously differentiable with respect to u in $\mathscr{U}^-_{(0,\hat{\tau}]}$. Also, the functions r, ϕ, m are C^1 functions in $\mathscr{U}^-_{(0,\hat{\tau}]}$. We now show that $\beta(\tau) \to 1$ as $\tau \to 0$. This would imply that ν_* and κ_* (see (1.51a)) extend continuously to $\tau = 0$ and

$$\nu_*(0) = 0, \quad \kappa_*(0) = \frac{1}{a_{-0}}, \tag{1.59a}$$

which would in turn imply that ν extends continuously to the point $(0,0)$ where it vanishes, κ extends continuously to $[0, \hat{\tau}] \times \{0\}$, the past boundary of $\mathscr{U}(\hat{\tau})$, the partial derivatives of the functions ν, ζ with respect to v and of the functions κ, η

with respect to u extend continuously to $[0,\hat{\tau}] \times \{0\}$, while the functions r, m and ϕ are C^1 on the whole of $\mathscr{U}(\hat{\tau})$.

From the expression (see (1.45d), (1.46b))

$$\beta = \frac{y}{(1-y)x + y}, \quad x = 2(1-\rho_*), \ y = 1 - \gamma^2 \tag{1.59b}$$

we see that $\beta \to 1$ as $\tau \to 0$ is equivalent to

$$\frac{x}{y} \to 0 \quad \text{as } \tau \to 0. \tag{1.59c}$$

For, since ρ_*, ζ_* are continuous and $\rho_*(0) = 1$, $\zeta_*(0) = 1/a_{-0}$, we have $x, y \to 0$ as $\tau \to 0$. Now (1.59c) would follow if we show that

$$\frac{1-\rho_*}{\tau} \to 0 \quad \text{as } \tau \to 0, \tag{1.59d}$$

$$\liminf_{\tau \to 0} \left(-\frac{1}{\gamma}\frac{d\gamma}{d\tau} \right) > 0. \tag{1.59e}$$

By Proposition 5.1 of [I] we have $\partial\rho/\partial\tau > 0$ along $\partial\mathscr{K}$, the future boundary of $\mathscr{V}$. Consequently, if χ_1 is sufficiently small, the minimum of ρ along the segment of the flow line $\chi = \chi_1$ lying in $\mathscr{V}$ is attained on C^+, the past boundary of $\mathscr{V}$. Letting ρ_+ be ρ as function of τ along C^+, i.e., letting

$$\rho_+(\tau) = \rho(\tau, \chi_+(\tau)) \tag{1.59f}$$

where $\chi = \chi_+(\tau)$ is the equation of C^+, this implies that

$$1 - \rho(\tau) \leqq 1 - \rho_+(\tau). \tag{1.59g}$$

Now

$$\left(\frac{d\rho_+}{d\tau} \right)(0) = 0. \tag{1.59h}$$

For, C^+ is tangent at N^+ to $\partial\mathscr{K}$, the level set of ρ corresponding to the value 1. Therefore,

$$1 - \rho_+(\tau) = o(\tau) \tag{1.59i}$$

and, in view of (1.59g), the limit (1.59d) is proved. To prove (1.59e) we apply Proposition 1.1. Since $\gamma, \rho_* \to 1$ as $\tau \to 0$, we obtain

$$\liminf_{\tau \to 0} \left(-\frac{1}{\gamma}\frac{d\gamma}{d\tau} \right) \geqq \liminf_{\tau \to 0} \min\{e_*, b_*\}. \tag{1.59j}$$

Here e_* is the boundary barrier function:

$$e_* = \frac{1}{2} a_{-*}^2 \xi_* + \frac{1}{2r_*}\left[a_{-*} - \frac{(1 - 4\pi r_*^2)}{a_{-*}} \right] + \frac{\mu_*}{2r_* a_{-*}}. \tag{1.59k}$$

The proof of Lemma 1.5 shows that ξ is a continuous function in $\mathscr{U}(\hat{\tau})\backslash(0,0)$ and (see (1.56f))

$$|\xi(u,v) - e^{-2N}(u,v)\xi(u,0)| \leqq Cu. \tag{1.59l}$$

Since $\xi(u,0)$ is given by (1.57e), $\phi(u,0) \to 0$ as $u \to 0$, while according to (1.32b), $\dot{Z}(t) \to \infty$ as $t \to 0$ (and $\dot{R}(t) \to \dot{R}(0) = -a_{-0}$ by (1.2b)), we have

$$\xi(u,0) \to \infty \quad \text{as } u \to 0. \tag{1.59m}$$

Hence, in view of (1.59l) and (1.59k),

$$e_*(\tau) \to \infty \quad \text{as } \tau \to 0. \tag{1.59n}$$

In conjunction with the fact that $\lim_{\tau\to 0} b_*(\tau) = b_0 > 0$, this yields, through (1.59j), the result (1.59e). The conclusion $\beta(\tau) \to 1$ as $\tau \to 0$ follows.

It remains to be shown that the solution obtained in $\mathscr{U}(\hat{\tau})$ corresponds to a genuine hard phase. More precisely, it suffices to show that we can find a $\hat{\tau}' \in (0,\hat{\tau}]$ such that the restriction of the solution to $\mathscr{U}(\hat{\tau}')$ satisfies $\sigma > 1$ except at $(0,0)$. We shall do this with the help of the (interior) barrier function e of (1.23a). Now, on $(0,\hat{\tau}]$ we have $\sigma_* > 1$ (see (4.20a, b) of [II]). Thus if, on the contrary, such a $\hat{\tau}'$ cannot be found, then for every $\tau_1 \in (0,\hat{\tau}]$, the set

$$\mathscr{E}(\tau_1) := \{(u,v) \in \mathscr{U}(\tau_1)\backslash(0,0) : \sigma(u,v) \leqq 1\} \tag{1.60a}$$

is not empty; consequently, there is a $\check{v} \in [0,\tau_1)$ such that $\mathscr{E}(\tau_1)$ has a non-empty intersection with the incoming null segment

$$I^-_{\tau_1}(\check{v}) := [\check{v},\tau_1] \times \{\check{v}\}. \tag{1.60b}$$

Suppose that $\check{v} > 0$. Then since $\sigma_*(\check{v}) > 1$, there is a first point, $(\check{u},\check{v})$ along $I^-_{\tau_1}(\check{v})$ at which $\sigma = 1$:

$$\check{u} = \sup\{u : \sigma(u,\check{v}) > 1\}. \tag{1.60c}$$

Thus,

$$\frac{\partial\sigma}{\partial u}(\check{u},\check{v}) \leqq 0 \tag{1.60d}$$

must hold. However, by (1.23b),

$$\frac{\partial\sigma}{\partial u}(\check{u},\check{v}) = 2(v\zeta e)(\check{u},\check{v}) \tag{1.60e}$$

and (1.59l, m) imply that

$$\inf_{\mathscr{U}(\tau_1)} e > 0 \tag{1.60f}$$

if τ_1 is sufficiently small. It follows that $\check{v} = 0$; hence $\mathscr{E}(\tau_1) \subset I^-_{\tau_1}(0)\backslash(0,0)$ and, by continuity, $\sigma = 1$ on $\mathscr{E}(\tau_1)$. Consequently, either there is an open segment $(u_0,u_1) \times \{0\} \subset I^-_{\tau_1}(0)$ not intersecting $\mathscr{E}(\tau_1)$ such that its future end point $(u_1,0)$ belongs to $\mathscr{E}(\tau_1)$ or there is a $\tau_0 \in (0,\tau_1]$ such that $\mathscr{E}(\tau_1)$ coincides with $I^-_{\tau_0}(0)\backslash(0,0)$. However, according to the first alternative, $\sigma > 1$ on $(u_0,u_1) \times \{0\}$ and $\sigma(u_1,0) = 1$, whence $(\partial\sigma/\partial u)(u_1,0) \leqq 0$, while according to the second alternative $\sigma = 1$, whence

$\partial\sigma/\partial u = 0$ on $I_{\tau_0}^-(0)\backslash(0,0)$. Consequently both alternatives contradict (1.23b) in conjunction with (1.60f). The proof of the theorem is therefore complete.

2. The Local Form of the Free Phase Boundary in a Neighborhood of the Null point

In the present section we shall derive the local form of the free phase boundary $\mathscr{B}$ in a neighborhood of the null point, for any solution of the problem of formation of a free phase boundary in the phase transition from soft to hard, as formulated in Section 1 of [II]. Specifically, we shall assume that $\mathscr{B}$ is a C^1 curve: $[0,\bar{\tau}) \mapsto \mathscr{V}$, $\tau \mapsto \chi_*(\tau)$, issuing from N^+, which has positive velocity β relative to the soft-phase flow lines, is strictly timelike ($\beta < 1$) and contained in the interior of $\mathscr{V}$ in $(0,\bar{\tau})$. It then follows that γ is continuous and $0 < \gamma < 1$ on $(0,\bar{\tau})$, and also that $\mathscr{B}$ becomes null outgoing at N^+, that is, $\beta(0) = 1$. Moreover, we shall assume that $\gamma \to 1$ as $\tau \to 0$, which follows from the continuity of the hard-phase solution, in particular, the function ζ, at $N^+ = (0,0)$.

The uniqueness of the solution of the formation problem will be established in the next section; the proof relies on the results of the present section.

We shall show that in the limit $\tau \to 0$, the variables

$$x := 2(1-\rho_*), \quad y := 1-\gamma^2 \tag{2.1}$$

satisfy a closed system of ordinary differential equations.

We introduce the following notation. If f and g are two continuous functions on $(0,\bar{\tau})$ and g is positive, we write $f \sim g$ if $f(\tau)/g(\tau) \to 1$ as $\tau \to 0$.

The variables x, y are continuous and positive on $(0,\bar{\tau})$ and $x, y \to 0$ as $\tau \to 0$. We have

$$\beta = \frac{y}{(1-y)x+y} \to 1 \quad \text{as } \tau \to 0. \tag{2.2a}$$

Hence,

$$\frac{x}{y} \to 0 \quad \text{as } \tau \to 0, \tag{2.2b}$$

$$1-\beta = \frac{(1-y)x}{(1-y)x+y} \sim \frac{x}{y}. \tag{2.2c}$$

We first derive an equation for x. We have

$$\frac{dx}{d\tau} = -2\frac{d\rho_*}{d\tau} = -2\left(\frac{\partial\rho}{\partial\tau} + \beta e^{-\omega}\frac{\partial\rho}{\partial\chi}\right)_*.$$

In terms of the functions

$$b := \frac{\partial\rho}{\partial\tau}, \quad c := -\frac{\partial\rho}{\partial\tau} - e^{-\omega}\frac{\partial\rho}{\partial\chi} \tag{2.3a, b}$$

defined by the soft phase solution, we can write

$$\frac{dx}{d\tau} = 2\beta c_* - 2(1-\beta)b_*. \tag{2.4}$$

As usual, the subscript $*$ denotes the restriction to the phase boundary. That C^+ is tangent at N^+ to $\partial\mathscr{K}$, the level set of ρ corresponding to the value 1, is expressed by the condition

$$c_*(0) = 0. \tag{2.5a}$$

We have

$$\frac{dc_*}{d\tau} = \left(\frac{\partial c}{\partial \tau} + \beta e^{-\omega}\frac{\partial c}{\partial \chi}\right)_*.$$

In particular (in view of 2.2a),

$$\frac{dc_*}{d\tau}(0) = \left(\frac{\partial c}{\partial \tau} + e^{-\omega}\frac{\partial c}{\partial \chi}\right)(N^+). \tag{2.5b}$$

Now, since c is a continuously differentiable function defined on the domain of definition of the soft phase solution, the right-hand side of (2.5b) can be evaluated by taking the limit at N^+ along the fixed phase boundary $\Sigma \subset \partial\mathscr{K}$ (see Section 5 of [I]). Letting $\phi = \phi_*(\chi)$ be the equation of Σ, we have

$$\rho(\phi_*(\chi), \chi) = 1.$$

Hence, differentiating implicitly with respect to χ, we obtain

$$e^{-\omega_*}\frac{d\phi_*}{d\chi} := \delta = -\left(\frac{e^{-\omega}\partial\rho/\partial\chi}{\partial\rho/\partial\tau}\right)_*$$

(see equation 5.13 of [I]). Thus, in terms of the functions b, c we can express

$$1 - \delta = -\frac{c_*}{b_*}. \tag{2.5c}$$

The function q along Σ defined by equation (6.39) of [I]:

$$q = e^{-\omega_*}\frac{d\delta}{d\chi}, \tag{2.5d}$$

is given by

$$q = \delta\left(\frac{\partial(c/b)}{\partial\tau}\right)_* + e^{-\omega_*}\left(\frac{\partial(c/b)}{\partial\chi}\right)_*. \tag{2.5e}$$

In particular, at N_+, since $c(N^+) = 0$ and $\delta(N^+) = 1$, the quantity $q^+ := q(N^+)$ is given by

$$q^+ = \frac{1}{b_0}\left(\frac{\partial c}{\partial\tau} + e^{-\omega}\frac{\partial c}{\partial\chi}\right)(N^+). \tag{2.5f}$$

Therefore, in view of 2.5b we have

$$\frac{dc_*}{d\tau}(0) = b_0 q^+. \tag{2.5g}$$

From (2.5a, g) it follows that

$$c_* \sim b_0 q^+ \tau. \tag{2.5h}$$

By virtue of (2.5h) and (2.2c), we conclude from (2.4) that

$$\frac{dx}{d\tau} = f_1 - f_2 \tag{2.6a}$$

where

$$f_1 := 2\beta c_* \sim 2b_0 q^+ \tau, \quad f_2 := 2(1-\beta)b_* \sim 2b_0 \frac{x}{y}. \tag{2.6b}$$

Next, we derive a limiting equation for y. By Proposition 1.1 and the fact that $x, y \to 0$ as $\tau \to 0$, we obtain

$$\frac{dy}{d\tau} \sim 2[(1-\beta)e_* + b_0]. \tag{2.7}$$

To proceed we must investigate the limiting form of the product $(1-\beta)e_*$ as $\tau \to 0$. First, we note that (see (1.59k))

$$(1-\beta)e_* \sim \tfrac{1}{2} a_{-0}^2 (1-\beta)\xi_*. \tag{2.8a}$$

Now according to (1.57e),

$$\xi(u,0) = -\frac{\dot{Z}(\phi(u,0))}{\dot{R}(\phi(u,0))}. \tag{2.8b}$$

From (1.13a) and (1.31a) we obtain

$$r_0 - r(u,0) = a_{-0}\phi(u,0) + O((\phi(u,0))^{3/2}). \tag{2.8c}$$

Also

$$\dot{R}(\phi(u,0)) = -a_{-0} + O((\phi(u,0))^{1/2}). \tag{2.8d}$$

Moreover, according to (1.32b),

$$\dot{Z}(\phi(u,0)) = \frac{b_0}{a_{-0} q^{+1/2}} (\phi(u,0))^{-1/2} + O(1). \tag{2.8e}$$

Substituting yields

$$\sqrt{r_0 - r(u,0)}\,\xi(u,0) = \frac{b_0}{a_{-0}^{3/2} q^{+1/2}} + O((\phi(u,0))^{1/2}). \tag{2.8f}$$

In view of (1.59l), this implies that

$$\sqrt{r_0 - r(\tau,0)}\,\xi_*(\tau) \sim \frac{b_0}{a_{-0}^{3/2} q^{+1/2}}. \tag{2.8g}$$

Hence, by (2.8a) and (2.2c),.

$$(1-\beta(\tau))e_*(\tau) \sim \frac{b_0 a_{-0}^{1/2}}{2q^{+1/2}}\left(\frac{x}{y}\right)(\tau)\frac{1}{\sqrt{r_0 - r(\tau,0)}}. \tag{2.8h}$$

Let us define a new variable z by

$$z(\tau) = \left(\frac{2}{a_{-0}}\right)^{1/2}\sqrt{r_0 - r(\tau,0)}. \tag{2.8i}$$

Then (2.8h) becomes

$$(1-\beta)e_* \sim \frac{b_0}{(2q^+)^{1/2}}\frac{x}{yz}. \tag{2.8j}$$

Substituting this into (2.7) yields

$$\frac{dy}{d\tau} \sim 2b_0\left[1 + \frac{1}{(2q^+)^{1/2}}\frac{x}{yz}\right]. \tag{2.8k}$$

Since

$$\frac{\partial r}{\partial u}(\tau,0) = -\nu(\tau,0) \tag{2.9a}$$

and since (1.49c), (1.49a) and (2.2c) imply that

$$\nu(\tau,0) = \frac{1}{2}a_{-*}(\tau)(1-\beta(\tau))e^{-N_*(\tau)} \sim \frac{1}{2}a_{-0}\left(\frac{x}{y}\right)(\tau), \tag{2.9b}$$

the variable z satisfies

$$\frac{dz}{d\tau} \sim \frac{x}{2yz} \quad (z(0)=0). \tag{2.9c}$$

The limiting equations (2.6a, b), (2.8k), (2.9c) constitute a complete system of limiting equations for the variables x, y, z. By virtue of (2.9c), the limiting equation (2.8k) reduces to

$$\frac{dy}{d\tau} \sim \frac{d}{d\tau}\{2b_0[\tau + (2/q^+)^{1/2}z]\}. \tag{2.10}$$

Now,

$$\frac{df}{d\tau} \sim \frac{dg}{d\tau} \tag{2.11a}$$

together with

$$f(0) = g(0) = 0 \tag{2.11b}$$

implies that

$$f \sim g. \tag{2.11c}$$

For, (2.11a) means that there is a positive continuous function h such that

$$\frac{df}{d\tau} = h\frac{dg}{d\tau} \tag{2.11d}$$

and $h(\tau) \to 1$ as $\tau \to 0$. Integrating yields

$$f = \tilde{h} g \tag{2.11e}$$

where

$$\tilde{h} = \frac{1}{g} \int_0^g h \, dg. \tag{2.11f}$$

Thus g is a C^1 strictly increasing function and, according to (2.11f), $\tilde{h}$ is the mean-value function of h relative to the measure dg. Consequently, $\tilde{h}$ is continuous and positive and $\tilde{h}(\tau) \to h(0) = 1$ as $\tau \to 0$. Condition (2.11c) follows from (2.11e).

Applying these considerations to the case

$$f = y, \quad g = 2b_0[\tau + (2/q^+)^{1/2} z],$$

we conclude that

$$y \sim 2b_0[\tau + (2/q^+)^{1/2} z]. \tag{2.12}$$

Therefore, we can eliminate the variable y and reduce the system (2.6a, b), (2.9c), to the following system for the variables x, z:

$$\frac{dx}{d\tau} = f_1 - f_2, \quad f_1 \sim 2b_0 q^+ \tau, \quad f_2 \sim \frac{x}{\tau + (2/q^+)^{1/2} z}, \tag{2.13a}$$

$$\frac{dz}{d\tau} \sim \frac{x}{4b_0 z[\tau + (2/q^+)^{1/2} z]}. \tag{2.13b}$$

This means that there is a $\tau_1 > 0$ and positive continuous functions h_1, h_2, h_3 on $(0, \tau_1)$, tending to 1 as $\tau \to 0$, such that the variables x, z satisfy on $(0, \tau_1)$ the system of ordinary differential equations

$$\frac{dx}{d\tau} = 2b_0 q^+ \tau h_1 - \frac{h_2 x}{\tau + (2/q^+)^{1/2} z}, \tag{2.14a}$$

$$\frac{dz}{d\tau} = \frac{h_3 x}{4b_0 z[\tau + (2/q^+)^{1/2} z]}. \tag{2.14b}$$

Also,

$$x, z > 0 \quad \text{on } (0, \tau_1), \quad \text{and} \quad x(\tau), z(\tau) \to 0 \quad \text{as } \tau \to 0. \tag{2.14c}$$

We shall derive upper and lower bounds for x. Equation (2.14a) implies that

$$\frac{dx}{d\tau} \leqq 2b_0 q^+ \tau h_1. \tag{2.15a}$$

Integrating this, we obtain the upper bound

$$x \leqq b_0 q^+ \tau^2 h_4 \tag{2.15b}$$

where

$$h_4 = \tau^{-2} \int_0^\tau 2\tau' h_1(\tau') \, d\tau' \tag{2.15c}$$

is a function which is positive and continuous on $(0, \tau_1)$, tending to 1 as $\tau \to 0$. To obtain a lower bound, we remark that since $z \geqq 0$, equation (2.14a) also implies that

$$\frac{dx}{d\tau} \geqq 2b_0 q^+ \tau h_1 - h_2 \frac{x}{\tau},$$

i.e.,

$$\frac{d(\tau x)}{d\tau} \geqq 2b_0 q^+ \tau^2 h_1 + (1 - h_2)x. \tag{2.15d}$$

Now, by the upper bound (2.15b),

$$(1 - h_2)x = o(\tau^2).$$

Hence (for τ_1 suitably small) (2.15d) may be written in the form

$$\frac{d(\tau x)}{d\tau} \geqq 2b_0 q^+ \tau^2 h_5 \tag{2.15e}$$

where h_5 is a positive continuous function on $(0, \tau_1)$, tending to 1 as $\tau \to 0$. Integrating (2.15e) we obtain the lower bound

$$x \geqq \frac{2b_0}{3} q^+ \tau^2 h_6 \tag{2.15f}$$

where

$$h_6 = \tau^{-3} \int_0^\tau 3\tau'^2 h_5(\tau')\, d\tau' \tag{2.15g}$$

is again a positive continuous function on $(0, \tau_1)$, tending to 1 as $\tau \to 0$.

Equation (2.14b) can be written in the form

$$\frac{d(2b_0 z^2)}{d\tau} = \frac{h_3 x}{\tau + (2/q^+)^{1/2} z}.$$

The sum of this equation and (2.14a) is

$$\frac{d(x + 2b_0 z^2)}{d\tau} = 2b_0 q^+ \tau h_1 + \frac{(h_3 - h_2)x}{\tau + (2/q^+)^{1/2} z}. \tag{2.16a}$$

Since $h_3 - h_2 \to 0$ as $\tau \to 0$, by virtue of the upper bound (2.15b) we have

$$\frac{|h_3 - h_2| x}{\tau + (2/q^+)^{1/2} z} \leqq \frac{|h_3 - h_2| x}{\tau}$$

$$\leqq h_4 |h_3 - h_2| b_0 q^+ \tau^2 = o(\tau).$$

Hence (2.16a) may be written as

$$\frac{d(x + 2b_0 z^2)}{d\tau} = 2b_0 q^+ \tau h_7 \tag{2.16b}$$

where h_7 is a positive continuous function on $(0, \tau_1)$, tending to 1 as $\tau \to 0$. Integrating this equation yields

$$x + 2b_0 z^2 = b_0 q^+ \tau^2 h_8 \tag{2.16c}$$

where

$$h_8 = \tau^{-2} \int_0^\tau 2\tau' h_7(\tau')\, d\tau' \tag{2.16d}$$

is again a positive continuous function on $(0, \tau_1)$, tending to 1 as $\tau \to 0$. Substituting in (2.16c) the lower bound (2.15f), we deduce that

$$z^2 \leqq \frac{q^+}{6} \tau^2 h_9 \tag{2.16e}$$

where

$$h_9 = 3h_8 - 2h_6 \tag{2.16f}$$

is, for τ_1 suitably small, a positive continuous function on $(0, \tau_1)$, tending to 1 as $\tau \to 0$. By virtue of (2.16e), the function $q^+\tau^2 - 2z^2$ is positive on $(0, \tau_1)$, provided that τ_1 is taken suitably small; consequently, equation (2.16c) can be written in the form

$$x = h_{10} b_0 (q^+ \tau^2 - 2z^2) \tag{2.16g}$$

where

$$h_{10} = 1 + \frac{q^+ \tau^2 (h_8 - 1)}{q^+ \tau^2 - 2z^2}. \tag{2.16h}$$

By (2.16e),

$$|h_{10} - 1| \leqq \frac{|h_8 - 1|}{1 - (h_9/3)}. \tag{2.16i}$$

Therefore, if τ_1 is taken suitably small, h_{10} is a positive continuous function on $(0, \tau_1)$, tending to 1 as $\tau \to 0$. Finally substituting (2.16g) into (2.14b), we reduce the limiting system to a single limiting equation for the variable z:

$$\frac{dz}{d\tau} = \frac{hq^+(\tau - (2/q^+)^{1/2} z)}{4z} \tag{2.17a}$$

where

$$h = h_3 h_{10} \tag{2.17b}$$

is a positive continuous function on $(0, \tau_1)$, tending to 1 as $\tau \to 0$.

We set

$$z = \left(\frac{q^+}{2}\right)^{1/2} \lambda\tau. \tag{2.18a}$$

Then (2.17a) assumes the form

$$\tau \frac{d\lambda}{d\tau} = -\frac{[2\lambda^2 + h(\lambda - 1)]}{2\lambda}. \tag{2.18b}$$

The roots of the quadratic in the numerator are

$$\lambda_+ = \frac{-h+\sqrt{h^2+8h}}{4}, \quad \lambda_- = \frac{-h-\sqrt{h^2+8h}}{4}. \tag{2.18c}$$

We have

$$\lambda_+(\tau) \to \lambda_+(0) = \tfrac{1}{2}, \quad \lambda_-(\tau) \to \lambda_-(0) = -1 \quad \text{as } \tau \to 0, \tag{2.18d}$$

and we can write (2.18c) in the form

$$\tau \frac{d\lambda}{d\tau} = -\frac{(\lambda-\lambda_+(\tau))(\lambda-\lambda_-(\tau))}{\lambda}. \tag{2.18e}$$

We assert that

$$\lambda(\tau) \to \lambda_+(0) = \tfrac{1}{2} \quad \text{as } \tau \to 0. \tag{2.18f}$$

For, if the contrary holds, then there is an $\varepsilon > 0$ and sequence $(\tau_n\colon n = 2, 3, \ldots)$ $\subset (0, \tau_1)$ decreasing to 0 as $n \to \infty$, such that the corresponding sequence $(\lambda(\tau_n)\colon n = 2, 3, \ldots)$ is contained in $(0, \lambda_+(0) - \varepsilon] \cup [\lambda_+(0) + \varepsilon, \infty)$; therefore, at least one of the intervals $(0, \lambda_+(0) - \varepsilon]$, $[\lambda_+(0) + \varepsilon, \infty)$ contains infinitely many $\lambda(\tau_n)$.

Suppose first that the interval $(0, \lambda_+(0) - \varepsilon]$ contains a subsequence $(\lambda(\tau_{n_i})\colon i = 1, 2, \ldots)$. Now, there is a $\delta > 0$ such that

$$\sup_{\tau \in (0,\delta)} \lambda_-(\tau) < 0, \tag{2.19a}$$

$$\inf_{\tau \in (0,\delta)} \lambda_+(\tau) \geqq \lambda_+(0) - \frac{\varepsilon}{2}. \tag{2.19b}$$

Also, there is an I such that $\tau_{n_i} < \delta$ for all $i > I$. Take any $i > I$. Then, in view of (2.19a, b), equation (2.18e) implies that for all $\tau \in (0, \tau_{n_i})$

$$\tau \frac{d\lambda}{d\tau} > \lambda_+ - \lambda \geqq \frac{\varepsilon}{2} \tag{2.19c}$$

For, (2.19c) is true at τ_{n_i} and if it is true for all $\tau \in (\check{\tau}, \tau_{n_i}]$, then we have

$$\lambda(\check{\tau}) < \lambda(\tau_{n_i}) \leqq \lambda_+(0) - \varepsilon \tag{2.19d}$$

which by (2.19b) and (2.18e) implies that (2.19c) is true at $\check{\tau}$. Thus (2.19c) follows for all $\tau \in (0, \tau_{n_i}]$ by continuous induction. However, integration of (2.19c) on $[\tau, \tau_{n_i}]$ yields

$$\lambda(\tau) \leqq \lambda(\tau_{n_i}) - \frac{\varepsilon}{2} \log\left(\frac{\tau_{n_i}}{\tau}\right), \tag{2.19e}$$

according to which as τ is decreased from the value τ_{n_i}, $\lambda(\tau)$ and therefore $z(\tau)$ become 0 at a positive value τ_0 of τ. By (2.9c) we conclude that x vanishes identically on $(0, \tau_0)$: a contradiction.

Suppose next that the interval $[\lambda_+(0)+\varepsilon, \infty)$ contains a subsequence $(\lambda(\tau_{n_i})$: $i = 1, 2, \ldots)$. There is a $\delta > 0$ such that

$$\sup_{\tau \in (0,\delta)} \lambda_+(\tau) \geqq \lambda_+(0) + \frac{\varepsilon}{2} \tag{2.20a}$$

and (2.19a) holds, and again there is an I such that $\tau_{n_i} < \delta$ for all $i > I$. Take any $i > I$. Then in view of (2.19a) and (2.20a), (2.18e) implies that for all $\tau \in (0, \tau_{n_i})$

$$\tau \frac{d\lambda}{d\tau} < -(\lambda - \lambda_+) \leqq -\frac{\varepsilon}{2}. \tag{2.20b}$$

For, this is true at τ_{n_i} and if it is true for all $\tau \in (\check{\tau}, \tau_{n_i}]$, then we have

$$\lambda(\check{\tau}) > \lambda(\tau_{n_i}) \geqq \lambda_+(0) + \varepsilon \tag{2.20c}$$

which by (2.20a) and (2.18e) implies that (2.19c) is true at $\check{\tau}$. Thus (2.20b) follows for all $\tau \in (0, \tau_{n_i}]$ by continuous induction. However, integration of (2.20b) on $[\tau, \tau_{n_i}]$ yields

$$\lambda(\tau) \geqq \lambda(\tau_{n_i}) + \frac{\varepsilon}{2} \log\left(\frac{\tau_{n_i}}{\tau}\right). \tag{2.20d}$$

Thus $\lambda(\tau) \to \infty$ as $\tau \to 0$, in contradiction to the estimate (2.16d), according to which (see (2.18a))

$$\lambda \leqq \left(\frac{h_9}{3}\right)^{1/2}. \tag{2.20e}$$

Therefore, for each $\varepsilon > 0$, the set

$$(0, \lambda_+(0) - \varepsilon] \cup [\lambda_+(0) + \varepsilon, \infty) = (0, \infty) \setminus (\lambda_+(0) - \varepsilon, \lambda_+(0) + \varepsilon)$$

contains only finitely many of the $\lambda(\tau_n)$ and the assertion (2.18f) is established.

By (2.18a), condition (2.18f) implies that

$$z \sim \frac{1}{2}\left(\frac{q^+}{2}\right)^{1/2} \tau. \tag{2.21a}$$

This in turn implies, by (2.12), that

$$y \sim 3b_0\tau, \tag{2.21b}$$

and by (2.16f), that

$$x \sim \tfrac{3}{4} b_0 q^+ \tau^2. \tag{2.21c}$$

It is instructive to compare the local form of ρ_* as given by (2.21c) with the local form of ρ_+, that is, ρ as a function of τ along C^+. We have

$$\frac{d\rho_+}{d\tau} = -c_+ \tag{2.22a}$$

where

$$c_+ = c(\tau, \chi_+(\tau)) \tag{2.22b}$$

(see (2.3b)),

$$\rho_+(0) = 1, \quad \frac{d\rho_+}{d\tau}(0) = -c(N^+) = 0,$$

and by (2.5b, g),

$$\frac{d^2\rho_+}{d\tau^2}(0) = -\left(\frac{\partial c}{\partial \tau} + e^{-\omega}\frac{\partial c}{\partial \chi}\right)(N^+) = -b_0 q^+. \tag{2.22c}$$

Consequently,

$$\rho_+(\tau) = 1 - k\tau^2 + o(\tau^2) \tag{2.22d}$$

where k is the positive constant

$$k = \tfrac{1}{2} b_0 q^+. \tag{2.22e}$$

In terms of the constant k the result (2.21c) is expressed by

$$\rho_*(\tau) = 1 - \tfrac{3}{4} k\tau^2 + o(\tau^2). \tag{2.22f}$$

Let us recall that according to (4.8a–c) of [II],

$$\frac{d\phi_*}{d\tau} = \frac{1}{2}(f_+ + f_-) \tag{2.23a}$$

where

$$f_+ = (1+\beta)\sqrt{1 - \frac{\beta x}{1+\beta}}, \tag{2.23b}$$

$$f_- = (1-\beta)\sqrt{1 + \frac{\beta x}{1-\beta}}. \tag{2.23c}$$

Let

$$g(s) = \sqrt{1+s} - (1 + \tfrac{1}{2}s - \tfrac{1}{8}s^2). \tag{2.23d}$$

Then the Taylor expansion of g begins with cubic terms and we can express

$$\frac{1}{2}(f_+ + f_-) = -\frac{1}{8}\frac{\beta^2 x^2}{(1-\beta^2)} + f_0 \tag{2.23e}$$

where

$$f_0 = \frac{1}{2}(1+\beta) g\left(-\frac{\beta x}{1+\beta}\right) + \frac{1}{2}(1-\beta) g\left(\frac{\beta x}{1-\beta}\right). \tag{2.23f}$$

The results (2.21b, c) imply that

$$\frac{x^2}{1-\beta^2} \sim \frac{9}{4} b_0 k\tau^3, \tag{2.23g}$$

while $f_0 = O(\tau^4)$. It follows that

$$\frac{d\phi_*}{d\tau} = 1 - \frac{9}{32} b_0 k\tau^3 + o(\tau^3). \tag{2.23h}$$

We summarize the results of this section:

Theorem 2.1. *Consider any solution of the problem of formation of a free phase boundary in the phase transition from soft to hard, as formulated in Section 1 of* [II]. *Let ρ as a function of τ along C^+ be given, in a neighborhood of N^+, by*

$$\rho_+(\tau) = 1 - k\tau^2 + o(\tau^2)$$

where k is a positive constant. Then, in a neighborhood of N^+, ρ as a function of τ along $\mathscr{B}$ satisfies

$$\rho_*(\tau) = 1 - \tfrac{3}{4} k\tau^2 + o(\tau^2),$$

and γ is given by

$$\gamma(\tau) = 1 - \tfrac{3}{2} b_0\tau + o(\tau)$$

where b_0 is the positive constant:

$$b_0 = \left(\frac{\partial\rho}{\partial\tau}\right)(N^+).$$

Moreover,

$$1 - \beta(\tau) = \frac{k}{2b_0}\tau + o(\tau),$$

$$r(\tau, 0) = r_0 - \frac{a_{-0}}{8b_0} k\tau^2 + o(\tau^2),$$

$$\phi_*(\tau) = \tau - \tfrac{9}{128} b_0 k\tau^4 + o(\tau^4)$$

(*see Figure 1*).

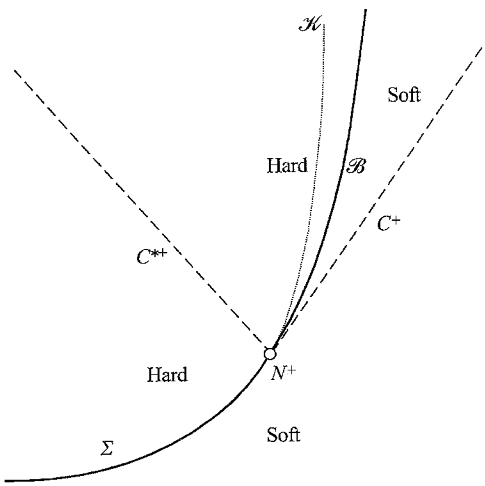

Figure 1

3. The Uniqueness Theorem

The aim of this section is the proof of

Theorem 3.1. *Let* $\mathscr{B}'$, $(r', m', v', \kappa', \zeta', \eta')$, *and* $\mathscr{B}''$, $(r'', m'', v'', \kappa'', \zeta'', \eta'')$, *both defined on* $[0, \tau_1]$, $\mathscr{U}(\tau_1)$, *be two solutions of the problem of formation of a free phase boundary in the phase transition from soft to hard, as formulated in Section 1 of* [II]. *Suppose that the two solutions have the same hard-phase initial data along* C^{*+}, *that is, the same function* R, *and also the same soft-phase initial data along* C^+, *and hence correspond to the same soft-phase solution* (r, ω, ρ). *Then the two solutions coincide.*

The proof of the theorem relies on the results (2.21) on the local form of the variables x, y and z in a neighborhood of the null point. Let us recall that the variable

$$x = 2(1 - \rho_*) \tag{3.1a}$$

satisfies equation (2.4):

$$\frac{dx}{d\tau} = 2\beta c_* - 2(1 - \beta) b_* \tag{3.1b}$$

and that β is given by

$$\beta = \frac{y}{(1 - y)x + y}. \tag{3.1c}$$

Let us also recall that

$$y = 1 - \gamma^2, \quad \gamma = \frac{1}{a_{-*}\zeta_*} \tag{3.2a}$$

and that ζ_* is given by (see (1.43e)):

$$\zeta_* = r_*^{-1}(\alpha_* + \phi_*) \tag{3.2b}$$

where

$$\alpha_*(\tau) = e^{-N_*(\tau)}\alpha(\tau, 0) - A_*(\tau). \tag{3.2c}$$

Let us further recall from (1.31a) and (1.13a) that ϕ, r, ζ along C^{*+} are given in terms of the parameter $\tilde{u}$ by

$$\phi_{|C^{*+}} = \tfrac{1}{4} e^{-\omega_0}\tilde{u} f(\tilde{u}^{1/2}), \tag{3.3a}$$

$$r_0 - r_{|C^{*+}} = \tfrac{1}{4} e^{-\omega_0} a_{-0}\tilde{u} g(\tilde{u}^{1/2}), \tag{3.3b}$$

$$\zeta_{|C^{*+}} = \frac{h(\tilde{u}^{1/2})}{a_{-0}}. \tag{3.3c}$$

Here f, g, h are smooth positive functions and

$$f(0) = g(0) = h(0) = 1, \quad \frac{dh}{ds}(0) = e^{-\omega_0/2}\frac{b_0}{q^{+1/2}}. \tag{3.3d}$$

Consider the function

$$\tilde{g}(s) = s\sqrt{g(s)}. \tag{3.3e}$$

The function $\tilde{g}$ is smooth, with $d\tilde{g}/ds$ is positive, and

$$\tilde{g}(0) = 0, \quad \frac{d\tilde{g}}{ds}(0) = 1. \tag{3.3f}$$

Consequently, $\tilde{g}$ posesses a smooth inverse function $\tilde{g}^{-1}$, with $d\tilde{g}^{-1}/dt$ positive, and

$$\tilde{g}^{-1}(0) = 0, \quad \frac{d\tilde{g}^{-1}}{dt}(0) = 1. \tag{3.3g}$$

Since, in view of 3.3b,

$$\sqrt{r_0 - r_{|C^{*+}}} = \tfrac{1}{2}\, e^{-\omega_0/2} a_{-0}^{1/2} \tilde{g}(\tilde{u}^{1/2}),$$

recalling the definition of the variable z from (2.8i):

$$z(\tau) = \left(\frac{2}{a_{-0}}\right)^{1/2} \sqrt{r_0 - r(\tau, 0)}, \tag{3.4a}$$

we can then express

$$\tilde{u}^{1/2} = \tilde{g}^{-1}(2^{1/2} e^{\omega_0/2} z) \tag{3.4b}$$

and, by (3.3c),

$$\zeta(\tau, 0) = \frac{P(z(\tau))}{a_{-0}} \tag{3.4c}$$

where

$$P(z) = (h \circ \tilde{g}^{-1})(2^{1/2} e^{\omega_0/2} z) \tag{3.4d}$$

is a smooth positive function with

$$\frac{dP}{dz}(0) = b_0 \left(\frac{2}{q^+}\right)^{1/2}. \tag{3.4e}$$

Moreover, using (3.3a) we also express

$$\phi(\tau, 0) = S(z(\tau)) \tag{3.4f}$$

where

$$S(z) = \tfrac{1}{4}\, e^{-\omega_0} ((\tilde{f} \circ \tilde{g}^{-1})(2^{1/2} e^{\omega_0/2} z))^2, \tag{3.4g}$$

$$\tilde{f}(s) = s\sqrt{f(s)}. \tag{3.4h}$$

The function $\tilde{f}$ is smooth, with $d\tilde{f}/ds$ positive, and

$$\tilde{f}(0) = 0, \quad \frac{d\tilde{f}}{ds}(0) = 1. \tag{3.4i}$$

Consequently, S is a smooth strictly increasing function with

$$S(0) = \frac{dS}{dz}(0) = 0, \quad \frac{d^2S}{dz^2}(0) = 1. \tag{3.4j}$$

In view of (3.4a, c, f), we can express

$$\alpha(\tau, 0) = Q(z(\tau)) \tag{3.4k}$$

where

$$Q(z) = \frac{\sqrt{r_0 - \frac{1}{2}a_{-0}z^2}}{a_{-0}} P(z) - S(z) \tag{3.4l}$$

is a smooth function with

$$\frac{dQ}{dz}(0) = \frac{r_0^{1/2}}{a_{-0}} \frac{dP}{dz}(0) = \frac{b_0}{a_{-0}} \left(\frac{2r_0}{q^+}\right)^{1/2}. \tag{3.4m}$$

Substituting (3.4k) into (3.2c) and substituting in turn the result into (3.2b), we obtain

$$\zeta_* = r_*^{-1}(e^{-N_*}Q(z) + \phi_* - A_*). \tag{3.5}$$

Finally, by (2.9b), the variable z satisfies the equation

$$\frac{dz}{d\tau} = \frac{1}{2} \frac{a_{-*}}{a_{-0}} \frac{(1-\beta)}{z} e^{-N_*}. \tag{3.6}$$

Consider now two solutions as in the statement of Theorem 3.1. From (3.1c) we have

$$\beta' - \beta'' = p_1(y' - y'') - p_2(x' - x'') \tag{3.7a}$$

where

$$p_1 = \frac{x'}{p_0}, \quad p_2 = \frac{y'(1-y'')}{p_0}, \tag{3.7b}$$

$$p_0 = [(1-y')x' + y'][(1-y'')x'' + y'']. \tag{3.7c}$$

Note that, according to the results (2.21b, c),

$$p_1 \sim \frac{x'}{y'y''} \sim \frac{q^+}{12b_0}, \quad p_2 \sim \frac{1}{y''} \sim \frac{1}{3b_0\tau}. \tag{3.7d}$$

From (3.2a), we have

$$y' - y'' = \gamma'\gamma''(\gamma' + \gamma'')(a'_{-*}\zeta'_* - a''_{-*}\zeta''_*) \tag{3.8a}$$

and from (3.6),

$$\begin{aligned} \zeta'_* - \zeta''_* &= (r'_*)^{-1}e^{-N'_*}(Q(z') - Q(z'')) \\ &\quad - (r'_*r''_*)^{-1}(Q(z'')e^{-N''_*} + \phi''_* - A''_*)(r'_* - r''_*) \\ &\quad + (r'_*)^{-1}[Q(z'')(e^{-N'_*} - e^{-N''_*}) + (\phi'_* - \phi''_*) - (A'_* - A''_*)]. \end{aligned} \tag{3.8b}$$

Let us denote

$$L = |\phi'_* - \phi''_*| + |N'_* - N''_*| + |A'_* - A''_*|, \tag{3.9a}$$

$$l = \sup_{\tau \in (0, \tau_1]} \left(\frac{L(\tau)}{\tau}\right). \tag{3.9b}$$

The proof of Theorem 3.1 relies on the following lemma:

Lemma 3.1. *Consider two solutions as in the statement of Theorem 3.1. If τ_1 is sufficiently small, then there is a constant C such that*

$$|x'(\tau) - x''(\tau)| \leqq Cl\tau^2, \quad |y'(\tau) - y''(\tau)|, |z'(\tau) - z''(\tau)| \leqq Cl\tau \quad \text{in } [0, \tau_1].$$

Proof. We introduce the variable (see (2.16c))

$$w = x + 2b_0 z^2. \tag{3.10a}$$

Then by (3.1b) and (3.6), w satisfies the equation

$$\frac{dw}{d\tau} = 2\beta c_* - 2(1-\beta) f_* \tag{3.10b}$$

where

$$f_* = b_* - (a_{-*}/a_{-0}) e^{-N_*} b_0. \tag{3.10c}$$

We shall base our estimates on the variable w instead of x. The advantage of doing this is the following. Whereas the coefficient c_* of β in equations (3.1b), (3.10b) is the same and vanishes at $\tau = 0$ (see (2.5a)), the coefficient b_* of $1-\beta$ in the former equation does not vanish at $\tau = 0$, while the coefficient f_* of $1-\beta$ in the latter equation does. Consequently, in a neighborhood of the null point, the difference $w' - w''$ depends less sensitively on the difference $\beta' - \beta''$ than does the difference $x' - x''$. We can express

$$x' - x'' = w' - w'' - 2b_0(z' + z'')(z' - z''). \tag{3.10d}$$

From (3.10a) we obtain

$$\frac{d}{d\tau}(w' - w'') = 2(\beta' - \beta'')c'_* + 2\beta''(c'_* - c''_*) + 2(\beta' - \beta'')f'_* - 2(1 - \beta'')(f'_* - f''_*). \tag{3.10e}$$

Now let g be a smooth function defined by the soft-phase solution, such as r, ρ, a_-, b, c, and let g'_* and g''_* be the restrictions of g to $\mathscr{B}'$ and $\mathscr{B}''$ respectively. Then we have

$$|g'_* - g''_*|(\tau) = \left| \int_{\chi''_*(\tau)}^{\chi'_*(\tau)} \left(\frac{\partial g}{\partial \chi}\right)(\tau, \chi)\, d\chi \right| \leqq C|\chi'_* - \chi''_*|(\tau). \tag{3.11a}$$

In the case $g = \rho$, since

$$\left(\frac{\partial \rho}{\partial \tau}\right)_0 + e^{-\omega_0}\left(\frac{\partial \rho}{\partial \chi}\right)_0 = 0, \quad \text{while} \quad \left(\frac{\partial \rho}{\partial \tau}\right)_0 > 0,$$

we have

$$\frac{\partial \rho}{\partial \chi} < 0 \quad \text{in a neighborhood of } N^+.$$

Hence,

$$|\rho'_* - \rho''_*|(\tau) = \left| \int_{\chi''_*(\tau)}^{\chi'_*(\tau)} \left(\frac{\partial \rho}{\partial \chi}\right)(\tau, \chi)\, d\chi \right| \geqq C^{-1}|\chi'_* - \chi''_*|(\tau). \tag{3.11b}$$

It follows that

$$|\chi'_* - \chi''_*|(\tau) \leqq C|x' - x''|(\tau). \tag{3.11c}$$

Therefore, substituting in (3.11a), we obtain

$$|g'_* - g''_*|(\tau) \leqq C|x' - x''|(\tau). \tag{3.11d}$$

From (3.7a) and from the relations $p_1 = O(1)$ and $p_2 = O(\tau^{-1})$ coming from (3.7d), we obtain

$$|\beta' - \beta''| \leqq C(\tau^{-1}|x' - x''| + |y' - y''|). \tag{3.12a}$$

Also, from (3.8a, b), and from (3.11d) in the cases $g = r, a_-$, we get

$$|y' - y''| \leqq C(|x' - x''| + |z' - z''| + L) \tag{3.12b}$$

(see 3.9a); hence, substitution yields

$$|\beta' - \beta''| \leqq C(\tau^{-1}|x' - x''| + |z' - z''| + L). \tag{3.12c}$$

Going back to (3.11a) and taking into account the fact that $1 - \beta$, c_*, f_* are all $O(\tau)$, as well as (3.11d) in the cases $g = a_-, b, c$, we obtain

$$\frac{d}{d\tau}|w' - w''| \leqq C(\tau|\beta' - \beta''| + |x' - x''| + \tau L). \tag{3.13a}$$

Hence, substituting (3.12c) into (3.13a) we obtain

$$\frac{d}{d\tau}|w' - w''| \leqq C(|x' - x''| + \tau|z' - z''| + \tau L). \tag{3.13b}$$

By (3.10d) and the fact that $z = O(\tau)$ by (2.21a), we find

$$|x' - x''| \leqq C(|w' - w''| + \tau|z' - z''|). \tag{3.13c}$$

Substituting (3.13c) into (3.13b) then yields

$$\frac{d}{d\tau}|w' - w''| \leqq C(|w' - w''| + \tau|z' - z''| + \tau L). \tag{3.13d}$$

Next we consider the difference $z' - z''$. By (3.6) this difference satisfies the equation

$$\frac{d}{d\tau}(z' - z'') = \frac{1}{2}\frac{a'_{-*}}{a_{-0}}\, e^{-N'_*}\Delta + E; \tag{3.14a}$$

here Δ is the difference

$$\Delta = \frac{1 - \beta'}{z'} - \frac{1 - \beta''}{z''} = -\frac{(\beta' - \beta'')}{z'} - \frac{(1 - \beta'')}{z'z''}(z' - z''), \tag{3.14b}$$

and

$$E = \frac{1}{2a_{-0}} \frac{(1-\beta'')}{z''}(a'_{-*}e^{-N'_*} - a''_{-*}e^{-N''_*}). \tag{3.14c}$$

Since

$$\frac{d}{d\tau}|z' - z''| = \operatorname{sgn}(z' - z'')\frac{d}{d\tau}(z' - z''),$$

we need only estimate the product

$$\operatorname{sgn}(z' - z'')\frac{d}{d\tau}(z' - z'') \tag{3.14d}$$

from above. In particular, the last term on the right in $\triangle$ contributes negatively to this product, and hence requires no further consideration. To evaluate the contribution of the first term in $\triangle$ to the product (3.14d), we substitute (3.10d) into (3.7a) to obtain

$$\beta' - \beta'' = p_1(y' - y'') + 2p_2b_0(z' + z'')(z' - z'') - p_2(w' - w''). \tag{3.14e}$$

The difference $z' - z''$ in the second term on the right enters the first term $-(\beta' - \beta'')/z'$ of $\triangle$ with a negative coefficient. Hence this also contributes negatively to the product (3.14d) and need not be considered further. Consider now the difference $Q(z') - Q(z'')$ which enters the difference $\zeta'_* - \zeta''_*$ (see (3.8b)), and hence also the difference $y' - y''$ (see (3.8a)), with a positive coefficient, and the first term of $\triangle$ with a negative coefficient. We can write

$$Q(z') - Q(z'') = Q_1(z', z'')(z' - z''). \tag{3.14f}$$

According to (3.4m) we have $(dQ/dz)(0) > 0$; thus, if τ_1 is sufficiently small, then $Q_1 > 0$. Consequently, the term in $Q(z') - Q(z'')$ contributes negatively to the product (3.14d) and need not be considered further. In view of (2.21a), (3.7d), by using (3.11d) in the cases $g = r, a_-$, the contribution of the remaining terms can be estimated by

$$C\tau^{-1}(|x' - x''| + \tau^{-1}|w' - w''| + L) \leqq C(|z' - z''| + \tau^{-2}|w' - w''| + \tau^{-1}L).$$

We can also estimate

$$|E| \leqq C(|x' - x''| + L) \leqq C(\tau|z' - z''| + |w' - w''| + L), \tag{3.14g}$$

for, according to the results of Section 2, $(1 - \beta)/z = O(1)$. We conclude that

$$\frac{d}{d\tau}|z' - z''| \leqq C\tau^{-1}(|z' - z''| + \tau^{-1}|w' - w''| + L). \tag{3.14h}$$

Setting

$$\vartheta = \sup_{\tau \in (0, \tau_1]} \left(\frac{|z' - z''|(\tau)}{\tau} \right) \tag{3.15a}$$

and recalling (3.9b), we obtain from the estimate (3.13d) that

$$\frac{d}{d\tau}|w' - w''| \leqq C(|w' - w''| + \tau^2(\vartheta + l)). \tag{3.15b}$$

Integrating and using the fact that $w(0) = 0$ yields

$$|w' - w''| \leqq C(\vartheta + l)\tau^3. \tag{3.15c}$$

Substituting this into (3.14h), integrating and using the fact that $z(0) = 0$ then yields

$$|z' - z''| \leqq C(\tau^2\vartheta + \tau l). \tag{3.15d}$$

This implies that

$$\vartheta \leqq C(\tau_1\vartheta + l), \tag{3.15e}$$

which, when τ_1 is small enough, in turn implies that

$$\vartheta \leqq Cl. \tag{3.15f}$$

In view of (3.15c), (3.13c) and (3.12b), the lemma then follows.

Lemma 3.2. *Consider two solutions as in the statement of Theorem 3.1. If τ_1 is sufficiently small, then there is a constant C such that*

$$|\phi'_*(\tau) - \phi''_*(\tau)| \leqq Cl\tau^4 \quad \textit{in } [0, \tau_1].$$

Proof. From (2.23a, e), we have

$$\frac{d}{d\tau}(\phi'_* - \phi''_*) = -\frac{1}{8}\frac{\beta'^2}{1-\beta'^2}x'^2 + \frac{1}{8}\frac{\beta''^2}{1-\beta''^2}x''^2 + f'_0 - f''_0. \tag{3.16a}$$

From (3.12c), by virtue of Lemma 3.1, we get

$$|\beta' - \beta''| \leqq Cl\tau. \tag{3.16b}$$

In view of Theorem 2.1 and Lemma 3.1, it follows that

$$\left|\frac{\beta'^2}{1-\beta'^2}x'^2 - \frac{\beta''^2}{1-\beta''^2}x''^2\right| \leqq \frac{\beta'^2}{1-\beta'^2}|x'^2 - x''^2| + x''^2\frac{|\beta'^2 - \beta''^2|}{(1-\beta'^2)(1-\beta''^2)}$$
$$\leqq C\tau|x' - x''| + C\tau^2|\beta' - \beta''| \leqq Cl\tau^3. \tag{3.16c}$$

By (2.23f) we can express

$$f'_0 - f''_0 = \tfrac{1}{2}(h'_+ - h''_+) + \tfrac{1}{2}(h'_- - h''_-) \tag{3.16d}$$

where

$$h_+ = (1+\beta)g\left(-\frac{\beta x}{1+\beta}\right), \quad h_- = (1-\beta)g\left(\frac{\beta x}{1-\beta}\right). \tag{3.16e}$$

Since the Taylor expansion of g begins with cubic terms (see (2.23d)), the results of Theorem 2.1 imply that

$$g\left(-\frac{\beta x}{1+\beta}\right) = O(\tau^6), \quad g\left(\frac{\beta x}{1-\beta}\right) = O(\tau^3). \tag{3.16f}$$

Also (note that g is analytic on $(-1, \infty)$),

$$|g(s') - g(s'')| \leqq \sup_{[s'', s']} \left|\frac{dg}{ds}\right| \cdot |s' - s''|.$$

Hence, setting

$$s' = -\frac{\beta' x'}{1+\beta'}, \quad s'' = -\frac{\beta'' x''}{1+\beta''},$$

we can estimate

$$\left|g\left(-\frac{\beta' x'}{1+\beta'}\right) - g\left(-\frac{\beta'' x''}{1+\beta''}\right)\right| \leqq C\tau^4 \left|\frac{\beta' x'}{1+\beta'} - \frac{\beta'' x''}{1+\beta''}\right| \leqq Cl\tau^6 \quad (3.16g)$$

by (3.16b), while setting

$$s' = \frac{\beta' x'}{1-\beta'}, \quad s'' = \frac{\beta'' x''}{1-\beta''}$$

we can estimate

$$\left|g\left(\frac{\beta' x'}{1-\beta'}\right) - g\left(\frac{\beta'' x''}{1-\beta''}\right)\right| \leqq C\tau^2 \left|\frac{\beta' x'}{1-\beta'} - \frac{\beta'' x''}{1-\beta''}\right| \leqq Cl\tau^3 \quad (3.16h)$$

by Theorem 3.1. The estimates (3.16b, f, g, h) imply that

$$|h'_+ - h''_+| \leqq Cl\tau^6, \quad |h'_- - h''_-| \leqq Cl\tau^4. \quad (3.16i)$$

Hence

$$|f'_0 - f''_0| \leqq Cl\tau^4, \quad (3.16j)$$

which together with (3.16a, e) yields

$$\frac{d}{d\tau}|\phi'_* - \phi''_*| \leqq Cl\tau^3. \quad (3.16k)$$

The lemma then follows by integration.

Proof of Theorem 3.1. Let us denote

$$\mathscr{U}(\tau) = \{(u, v) : 0 \leqq v \leqq u \leqq \tau\}.$$

From (1.49c) we have

$$v'(u,v) - v''(u,v) = (v'_*(u) - v''_*(u))e^{N'(u,v) - N'_*(u)} + v''_*(u)(e^{N'(u,v) - N'_*(u)} - e^{N''(u,v) - N''_*(u)}), \quad (3.17a)$$

$$\kappa'(u,v) - \kappa''(u,v) = (\kappa'_*(v) - \kappa''_*(v))e^{-K'(u,v)} + \kappa''_*(v)(e^{-K'(u,v)} - e^{-K''(u,v)}), \quad (3.17b)$$

and, from (1.49a),

$$\nu'_* - \nu''_* = \tfrac{1}{2}(a'_{-*} - a''_{-*})(1-\beta') - \tfrac{1}{2}a''_{-*}(\beta' - \beta''), \tag{3.17c}$$

$$\kappa'_* - \kappa''_* = -\frac{(a'_{-*} - a''_{-*})}{2a'_{-*}a''_{-*}}(1+\beta') + \frac{1}{2a''_{-*}}(\beta' - \beta''). \tag{3.17d}$$

Using (3.16b) and the fact that

$$|a'_{-*} - a''_{-*}| \leqq C|x' - x''| \leqq Cl\tau^2, \tag{3.17e}$$

by (3.11d) and Lemma 3.1 we deduce that

$$|\nu'_* - \nu''_*|,\ |\kappa'_* - \kappa''_*| \leqq Cl\tau. \tag{3.17f}$$

Taking also into account that

$$\nu_*(\tau) = O(\tau), \quad \sup_{\mathscr{U}(\tau)} \nu = O(\tau) \tag{3.17g}$$

(see Theorem 1.1), we then obtain

$$\sup_{\mathscr{U}(\tau)} |\nu' - \nu''| \leqq Cl\tau + C\tau \sup_{\mathscr{U}(\tau)} |N' - N''|, \tag{3.17h}$$

$$\sup_{\mathscr{U}(\tau)} |\kappa' - \kappa''| \leqq Cl\tau + C \sup_{\mathscr{U}(\tau)} |K' - K''|. \tag{3.17i}$$

From (1.49d) we deduce that

$$\sup_{\mathscr{U}(\tau)} |N' - N''| \leqq C\tau \left[\sup_{\mathscr{U}(\tau)} (|r' - r''| + |m' - m''|) + \sup_{\mathscr{U}(\tau)} |\kappa' - \kappa''| \right], \tag{3.18a}$$

$$\sup_{\mathscr{U}(\tau)} |K' - K''| \leqq C\tau \left[\tau \sup_{\mathscr{U}(\tau)} (|r' - r''| + |\zeta' - \zeta''|) + \sup_{\mathscr{U}(\tau)} |\nu' - \nu''| \right]. \tag{3.18b}$$

Substituting into (3.17h, i) we obtain

$$\sup_{\mathscr{U}(\tau)} |\nu' - \nu''| \leqq Cl\tau + C\tau^2 \left[\sup_{\mathscr{U}(\tau)} (|r' - r''| + |m' - m''|) + \sup_{\mathscr{U}(\tau)} |\kappa' - \kappa''| \right], \tag{3.18c}$$

$$\sup_{\mathscr{U}(\tau)} |\kappa' - \kappa''| \leqq Cl\tau + C\tau \left[\tau \sup_{\mathscr{U}(\tau)} (|r' - r''| + |\zeta' - \zeta''|) + \sup_{\mathscr{U}(\tau)} |\nu' - \nu''| \right]. \tag{3.18d}$$

When τ is suitably small these inequalities imply that

$$\sup_{\mathscr{U}(\tau)} |\nu' - \nu''| \leqq Cl\tau + C\tau^2 \sup_{\mathscr{U}(\tau)} (|r' - r''| + |m' - m''|) + C\tau^4 \sup_{\mathscr{U}(\tau)} |\zeta' - \zeta''|, \tag{3.18e}$$

$$\sup_{\mathscr{U}(\tau)} |\kappa' - \kappa''| \leqq Cl\tau + C\tau^2 \sup_{\mathscr{U}(\tau)} (|r' - r''| + |\zeta' - \zeta''|) + C\tau^3 \sup_{\mathscr{U}(\tau)} |m' - m''|. \tag{3.18f}$$

Substituting into (3.18a, b) then yields

$$\sup_{\mathscr{U}(\tau)} |N' - N''| \leqq Cl\tau^2 + C\tau \sup_{\mathscr{U}(\tau)} (|r' - r''| + |m' - m''|) + C\tau^3 \sup_{\mathscr{U}(\tau)} |\zeta' - \zeta''|, \tag{3.18g}$$

$$\sup_{\mathscr{U}(\tau)} |K' - K''| \leqq Cl\tau^2 + C\tau^2 \sup_{\mathscr{U}(\tau)} (|r' - r''| + |\zeta' - \zeta''|) + C\tau^3 \sup_{\mathscr{U}(\tau)} |m' - m''|. \tag{3.18h}$$

We have

$$r(u, v) = r_*(v) - \int_v^u \nu(u, v')\, dv'. \tag{3.19a}$$

Now, by (3.11d) and Lemma 3.1,

$$|r'_* - r''_*| \leqq C|x' - x''| \leqq Cl\tau^2. \tag{3.19b}$$

It follows that

$$\sup_{\mathscr{U}(\tau)} |r' - r''| \leqq Cl\tau^2 + \tau \sup_{\mathscr{U}(\tau)} |\nu' - \nu''|. \tag{3.19c}$$

According to equation (3.4a) of [II] we have

$$m(u, v) = m_*(v) e^{K(u,v)} - F(u, v) \tag{3.19d}$$

where (see (2.3c) of [II])

$$F(u, v) = 2\pi \int_v^u e^{K(u,v) - K(u',v)} (r^2 \nu(\zeta^2 + 1))(u', v)\, du'. \tag{3.19e}$$

By (3.11d) and Lemma 3.1,

$$|m'_* - m''_*| \leqq C|x' - x''| \leqq Cl\tau^2. \tag{3.19f}$$

We can also estimate (taking into account (3.17g)):

$$\sup_{\mathscr{U}(\tau)} |F' - F''| \leqq C\tau \left[\tau \sup_{\mathscr{U}(\tau)} (|K' - K''| + |r' - r''| + |\zeta' - \zeta''|) + \sup_{\mathscr{U}(\tau)} |\nu' - \nu''| \right]. \tag{3.19g}$$

Hence,

$$\sup_{\mathscr{U}(\tau)} |m' - m''| \leqq Cl\tau^2 + C \sup_{\mathscr{U}(\tau)} |K' - K''| + C\tau^2 \sup_{\mathscr{U}(\tau)} (|r' - r''| + |\zeta' - \zeta''|)$$
$$+ C\tau \sup_{\mathscr{U}(\tau)} |\nu' - \nu''|. \tag{3.19h}$$

Substituting (3.18e, h) into (3.19c, h), we deduce, when τ is small enough, the inequalities

$$\sup_{\mathscr{U}(\tau)} |r' - r''| \leqq Cl\tau^2 + C\tau^3 \sup_{\mathscr{U}(\tau)} |m' - m''| + C\tau^5 \sup_{\mathscr{U}(\tau)} |\zeta' - \zeta''|, \tag{3.20a}$$

$$\sup_{\mathscr{U}(\tau)} |m' - m''| \leqq Cl\tau^2 + C\tau^2 \sup_{\mathscr{U}(\tau)} |r' - r''| + C\tau^2 \sup_{\mathscr{U}(\tau)} |\zeta' - \zeta''|. \tag{3.20b}$$

When τ is suitably small these imply that

$$\sup_{\mathscr{U}(\tau)} |r' - r''| \leqq Cl\tau^2 + C\tau^5 \sup_{\mathscr{U}(\tau)} |\zeta' - \zeta''|, \tag{3.20c}$$

$$\sup_{\mathscr{U}(\tau)} |m' - m''| \leqq Cl\tau^2 + C\tau^2 \sup_{\mathscr{U}(\tau)} |\zeta' - \zeta''|. \tag{3.20d}$$

Substituting these into (3.18e, f, g, h) yields

$$\sup_{\mathscr{U}(\tau)} |v' - v''| \leqq Cl\tau + C\tau^4 \sup_{\mathscr{U}(\tau)} |\zeta' - \zeta''|, \tag{3.20e}$$

$$\sup_{\mathscr{U}(\tau)} |\kappa' - \kappa''| \leqq Cl\tau + C\tau^2 \sup_{\mathscr{U}(\tau)} |\zeta' - \zeta''|, \tag{3.20f}$$

$$\sup_{\mathscr{U}(\tau)} |N' - N''| \leqq Cl\tau^2 + C\tau^3 \sup_{\mathscr{U}(\tau)} |\zeta' - \zeta''|, \tag{3.20g}$$

$$\sup_{\mathscr{U}(\tau)} |K' - K''| \leqq Cl\tau^2 + C\tau^2 \sup_{\mathscr{U}(\tau)} |\zeta' - \zeta''|. \tag{3.20h}$$

We have

$$\phi(u,v) = \phi_*(v) + \int_v^u (v\zeta)(u,v')\, dv'. \tag{3.21a}$$

Consequently, we can estimate

$$\sup_{\mathscr{U}(\tau)} |\phi' - \phi''| \leqq \sup_{[0,\tau]} |\phi'_* - \phi''_*| + C\tau \sup_{\mathscr{U}(\tau)} |v' - v''| + C\tau^2 \sup_{\mathscr{U}(\tau)} |\zeta' - \zeta''|. \tag{3.21b}$$

Substituting the result of Lemma 3.2 and (3.20e) into (3.21b) yields

$$\sup_{\mathscr{U}(\tau)} |\phi' - \phi''| \leqq Cl\tau^2 + C\tau^2 \sup_{\mathscr{U}(\tau)} |\zeta' - \zeta''|. \tag{3.21c}$$

We now estimate the difference $\zeta' - \zeta''$ in $\mathscr{U}(\tau)$. According to (1.43e) we have

$$\zeta = r^{-1}(\alpha + \phi), \tag{3.22a}$$

$$\alpha(u,v) = \alpha(u,0)e^{-N(u,v)} - A(u,v). \tag{3.22b}$$

In particular,

$$\alpha_*(u) = \alpha(u,0)e^{-N_*(u)} - A_*(u),$$

and we can express

$$\alpha(u,0) = (\alpha_*(u) + A_*(u))e^{N_*(u)}.$$

Substituting this in (3.22b), we obtain

$$\alpha(u,v) = (\alpha_*(u) + A_*(u))e^{N_*(u) - N(u,v)} - A(u,v). \tag{3.22c}$$

Now,

$$\zeta_* = \frac{1}{a_{-*}\gamma} = \frac{1}{a_{-*}} \frac{1}{\sqrt{1-y}}.$$

Hence, by virtue of Lemma 3.1 and (3.17e),

$$|\zeta'_* - \zeta''_*| \leqq C|a'_{-*} - a''_{-*}| + C|y' - y''| \leqq Cl\tau. \tag{3.22d}$$

Since

$$\alpha_* = r_* \zeta_* - \phi_*,$$

we then obtain, by virtue of Lemma 3.2 and (3.19b) that

$$|\alpha'_* - \alpha''_*| \leqq C|r'_* - r''_*| + C|\zeta'_* - \zeta''_*| + |\phi'_* - \phi''_*| \leqq Cl\tau. \tag{3.22e}$$

Using (3.22e) we deduce from (2.22c) that

$$\sup_{\mathscr{U}(\tau)} |\alpha' - \alpha''| \leqq Cl\tau + C \sup_{\mathscr{U}(\tau)} |N' - N''| + C \sup_{\mathscr{U}(\tau)} |A' - A''|. \tag{3.22f}$$

In view of (3.22a), this implies that

$$\sup_{\mathscr{U}(\tau)} |\zeta' - \zeta''| \leqq Cl\tau + C \sup_{\mathscr{U}(\tau)} |r' - r''| + C \sup_{\mathscr{U}(\tau)} |\phi' - \phi''|$$
$$+ \sup_{\mathscr{U}(\tau)} |N' - N''| + C \sup_{\mathscr{U}(\tau)} |A' - A''|. \tag{3.22g}$$

Substituting the estimates (3.20c, g), (3.21c) into (3.22g) yields

$$\sup_{\mathscr{U}(\tau)} |\zeta' - \zeta''| \leqq Cl\tau + C\tau^2 \sup_{\mathscr{U}(\tau)} |\zeta' - \zeta''| + C \sup_{\mathscr{U}(\tau)} |A' - A''|. \tag{3.22h}$$

Consider finally the integral A (see (1.43b), (1.49c)):

$$A(u,v) = \int_0^v \left(\phi(\mu - 4\pi r^2) \frac{\kappa}{r} \right)(u,v')\, e^{N(u,v') - N(u,v)}\, dv'. \tag{3.23a}$$

Since $\sup_{\mathscr{U}(\tau)} \phi \leqq \tau$, we deduce that

$$\sup_{\mathscr{U}(\tau)} |A' - A''| \leqq C\tau^2 \left(\sup_{\mathscr{U}(\tau)} |r' - r''| + \sup_{\mathscr{U}(\tau)} |m' - m''| \right)$$
$$+ C\tau^2 \left(\sup_{\mathscr{U}(\tau)} |\kappa' - \kappa''| + \sup_{\mathscr{U}(\tau)} |N' - N''| \right)$$
$$+ C\tau \sup_{\mathscr{U}(\tau)} |\phi' - \phi''|. \tag{3.23b}$$

Substituting the estimates (3.20c, d, f, g), (3.21c) into (3.23b) we then obtain

$$\sup_{\mathscr{U}(\tau)} |A' - A''| \leqq Cl\tau^3 + C\tau^3 \sup_{\mathscr{U}(\tau)} |\zeta' - \zeta''|. \tag{3.23c}$$

Substituting (3.23c) in turn into (3.22h) yields the inequality

$$\sup_{\mathscr{U}(\tau)} |\zeta' - \zeta''| \leqq Cl\tau + C\tau^2 \sup_{\mathscr{U}(\tau)} |\zeta' - \zeta''|,$$

which when τ is suitably small implies that

$$\sup_{\mathscr{U}(\tau)} |\zeta' - \zeta''| \leqq Cl\tau. \tag{3.24a}$$

In view of this estimate, the inequalities (3.20c, d, e, f, g, h), (3.21c) and (3.23c) reduce to

$$\sup_{\mathscr{U}(\tau)} |r' - r''| \leqq Cl\tau^2, \quad \sup_{\mathscr{U}(\tau)} |m' - m''| \leqq Cl\tau^2, \tag{3.24b}$$

$$\sup_{\mathscr{U}(\tau)} |\nu' - \nu''| \leqq Cl\tau, \quad \sup_{\mathscr{U}(\tau)} |\kappa' - \kappa''| \leqq Cl\tau, \tag{3.24c}$$

$$\sup_{\mathscr{U}(\tau)} |N' - N''| \leqq Cl\tau^2, \quad \sup_{\mathscr{U}(\tau)} |K' - K''| \leqq Cl\tau^2, \tag{3.24d}$$

$$\sup_{\mathscr{U}(\tau)} |\phi' - \phi''| \leqq Cl\tau^2, \tag{3.24e}$$

$$\sup_{\mathscr{U}(\tau)} |A' - A''| \leqq Cl\tau^3. \tag{3.24f}$$

In particular, we obtain

$$\sup_{(0,\tau]} |N'_* - N''_*| \leqq Cl\tau^2, \quad \sup_{(0,\tau]} |A'_* - A''_*| \leqq Cl\tau^3, \tag{3.25a}$$

which, combined with the result of Lemma 3.2, implies (see (3.9a)) that

$$L(\tau) \leqq Cl\tau^2. \tag{3.25b}$$

Therefore,

$$\sup_{\tau\in(0,\tau_1]} \left(\frac{L(\tau)}{\tau}\right) := l \leqq C l\tau_1, \tag{3.25c}$$

which when $C\tau_1 < 1$ implies that $l = 0$. In view of Lemma 3.1, the estimates (3.24a–f) and the equation

$$(r\eta)(u,v) = e^{K(u,v)}[(r_*\eta_*)(v) - J(u,v)] \tag{3.26a}$$

where

$$\eta_* = a_{-*}\sqrt{\frac{x + 2y - 2xy}{x + 2y - xy}}, \tag{3.26b}$$

$$J(u,v) = \int_u^v e^{-K(u',v)}((1-\mu)\nu\zeta)(u',v)\,du' \tag{3.26c}$$

(see (1.45a–e)), we conclude that for τ_1 suitably small, the solutions $\mathscr{B}'$, $(r', m', \nu', \kappa', \zeta', \eta')$ and $\mathscr{B}''$, $(r'', m'', \nu'', \kappa'', \zeta'', \eta'')$, coincide on $[0, \tau_1]$, $\mathscr{U}(\tau_1)$. The uniqueness without a smallness condition on τ_1 then follows immediately from the uniqueness in the large of the solution of the continuation problem, Theorem 3.2 of [II].

References

[I] D. CHRISTODOULOU: Self-gravitating relativistic fluids: a two-phase model, *Arch. Rational Mech. Anal.* **130** (1995), 343–400.

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