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On a Nonlinear Hyperbolic Variational Equation: I. Global Existence of Weak Solutions

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Abstract

We study the nonlinear hyperbolic partial differential equation,

$$(u_t + uu_x)_x = \tfrac{1}{2}u_x^2.$$

This partial differential equation is the canonical asymptotic equation for weakly nonlinear solutions of a class of hyperbolic equations derived from variational principles. In particular, it describes waves in a massive director field of a nematic liquid crystal.

Global smooth solutions of the partial differential equation do not exist, since their derivatives blow up in finite time, while weak solutions are not unique. We therefore define two distinct classes of admissible weak solutions, which we call dissipative and conservative solutions. We prove the global existence of each type of admissible weak solution, provided that the derivative of the initial data has

bounded variation and compact support. These solutions remain continuous, despite the fact that their derivatives blow up.

There are no *a priori* estimates on the second derivatives in any L^p space, so the existence of weak solutions cannot be deduced by using Sobolev-type arguments. Instead, we prove existence by establishing detailed estimates on the blowup singularity for explicit approximate solutions of the partial differential equation.

We also describe the qualitative properties of the partial differential equation, including a comparison with the Burgers equation for inviscid fluids and a number of illustrative examples of explicit solutions. We show that conservative weak solutions are obtained as a limit of solutions obtained by the regularized method of characteristics, and we prove that the large-time asymptotic behavior of dissipative solutions is a special piecewise linear solution which we call a kink-wave.

1. Introduction

In this paper we prove the global existence of admissible weak solutions of the initial-value problem for the partial differential equation

$$(u_t + uu_x)_x = \tfrac{1}{2}u_x^2. \tag{1.1}$$

We also study the qualitative properties of (1.1), including blowup, singularity formation, admissibility conditions for weak solutions, and long-time asymptotics.

In two companion papers, we analyze the zero-dissipation and dispersion limits of regularizations using viscosity and dispersivity [HZ1] and we show that (1.1) is a completely integrable, bi-Hamiltonian system [HZ2].

Significance of the equation

Equation (1.1) is a formal asymptotic equation describing weakly nonlinear solutions of any hyperbolic Euler-Lagrange equation derived from a variational principle of the form

$$\delta \int \sum_{i,j,p,q} A_{pq}^{ij}(\boldsymbol{x}, \boldsymbol{u}) \frac{\partial u^p}{\partial x^i} \frac{\partial u^q}{\partial x^j}\, d\boldsymbol{x} = 0, \tag{1.2}$$

provided that a certain "genuine nonlinearity" condition is satisfied [HS]. More generally, (1.2) may be supplemented by a constraint.

Our analysis of (1.1) suggests that the Euler-Lagrange equations associated with (1.2) have very interesting and unusual properties which have hardly been studied. Equations (1.2) thus constitute a new class of nonlinear hyperbolic partial differential equations.

A particular physical example leading to (1.1) and (1.2) is the motion of a massive director field in a nematic liquid crystal [HS, S]. The director field is described by a unit-vector field $\boldsymbol{n}(x, t) \in \mathbb{S}^2$. When there is no fluid motion, the director field satisfies an equation of the form [L, E]

$$\rho \boldsymbol{n}_{tt} + \mu \boldsymbol{n}_t + \frac{\delta \mathcal{W}}{\delta \boldsymbol{n}} + \lambda \boldsymbol{n} = 0.$$

Here, ρ and μ are constants, the Lagrange multiplier λ is determined by the constraint $\boldsymbol{n} \cdot \boldsymbol{n} = 1$, and the Oseen-Frank potential-energy functional, $\mathcal{W}$, is defined by

$$\mathcal{W}[\boldsymbol{n}] = \int W(\boldsymbol{n}, \nabla \boldsymbol{n})\, dx,$$

$$W(\boldsymbol{n}, \nabla \boldsymbol{n}) = \tfrac{1}{2} k_1 (\nabla \cdot \boldsymbol{n})^2 + \tfrac{1}{2} k_2 (\boldsymbol{n} \cdot \nabla \times \boldsymbol{n})^2 + \tfrac{1}{2} k_3 |\boldsymbol{n} \times (\nabla \times \boldsymbol{n})|^2.$$

Here, k_1, k_2, and k_3 are positive elastic constants. A special case is the one-constant model, where $k_1 = k_2 = k_3 = k$, when the potential energy $W = \frac{1}{2} k |\nabla \boldsymbol{n}|^2$ is independent of $\boldsymbol{n}$.

The elliptic equations for time-independent director fields have been extensively studied. In particular, the one-constant model leads to the equation for harmonic maps taking values in the two-sphere [C, EK].

There are two extreme cases of the time-dependent equations. The first is when viscous effects dominate inertia. Then we can set $\rho = 0$, and the evolution of the director field is governed by a gradient-flow parabolic partial differential equation [C]. This is the most important physical regime.

Here, we are interested in the second extreme case, when inertia effects dominate viscosity. Then we can set $\mu = 0$, and the director field satisfies a hyperbolic partial differential equation which is derived from the constrained variational principle

$$\delta \int \{\tfrac{1}{2} \rho\, \boldsymbol{n}_t \cdot \boldsymbol{n}_t - W(\boldsymbol{n}, \nabla \boldsymbol{n})\}\, dx\, dt = 0, \quad \boldsymbol{n} \cdot \boldsymbol{n} = 1.$$

Since W is a quadratic function of $\nabla \boldsymbol{n}$, with coefficients depending on $\boldsymbol{n}$, this variational principle is of the form (1.2).

All the interesting new nonlinear effects studied here are caused by the explicit dependence of W on $\boldsymbol{n}$. In that case, formal weakly nonlinear asymptotics leads to (1.1). For the one-constant approximation (corresponding to the harmonic-map problem), W is independent of $\boldsymbol{n}$, so that the wave speeds of the associated Euler-Lagrange equations are independent of $\boldsymbol{n}$. In this case, the equations are "linearly degenerate" and (1.1) does not apply. Thus, at least for the hyperbolic equations, there is a dramatic difference in the effects of nonlinearity between one-constant and three-constant nematic liquid crystals.

Comparison with the Burgers equation

The Burgers equation without viscosity,

$$u_t + (\tfrac{1}{2} u^2)_x = 0, \tag{1.3}$$

Table 1. Comparison of (1.1) with the Burgers equation (1.3).

	Burgers Equation	Equation (1.1)
Primitive Equation	conservation law	variational principle
Blowup	$u_x \to -\infty$	$u_x \to -\infty$
Weak Solutions	discontinuous, in BV	Hölder continuous
Typical Singularity	shock	cusp
Large-Time Limit	N-wave	kink-wave
Characteristics	cross	focus but do not cross
Oscillations	are killed	persist
Admissibility Classes	entropy solutions	dissipative solutions, conservative solutions
Zero Dissipation Limit	strong convergence to weak solution	strong convergence to dissipative solution
Zero Dispersion Limit	weak convergence, limit not a solution	strong convergence to conservative solution (?)

describes weakly nonlinear solutions of a genuinely nonlinear, hyperbolic system of conservation law [HK]

$$\sum_i f^i(u)_{x_i} = 0.$$

Equation (1.1) plays a similar role for the variational equations (1.2).

The properties of (1.3) reflect those of general quasilinear, hyperbolic conservation laws. Similarly, we expect that the properties of (1.1) reflect those of the general variational equations (1.2).

As we shall see, the qualitative properties of (1.1) show remarkable analogies and contrasts with those of the Burgers equation without viscosity. Table 1 summarizes some of these properties, which are discussed in greater detail below.

Qualitative properties

The method of characteristics shows that the first derivative, u_x, of smooth solutions of (1.1) blows up in finite time [HS]. A smooth solution can be extended past the blowup time by a weak solution, but, in contrast with the Burgers equation (1.3), shocks do not form. Instead we prove the surprising fact that (1.1) has global Hölder-continuous weak solutions, with $u(t, x) \in C^\alpha(\mathbb{R}^+ \times \mathbb{R})$ for any $\alpha < \frac{1}{3}$. A typical example is the explicit steady solution $u = |x|^{2/3}$.

Weak solutions of (1.1) are not unique, so (1.1) must be supplemented by an admissibility condition. We define two different classes of admissible weak solutions, which we call dissipative and conservative solutions. These are not the only possible admissibility classes, but they seem to be the most natural ones to use.

Dissipative solutions are motivated by the condition that their energy

$$E[u](t) = \int_{\mathbb{R}} u_x^2(t, x)\, dx \tag{1.4}$$

decays at the fastest possible rate. This is analogous to the *entropy-rate* criterion for conservation laws [D1]. The analog of an entropy condition (which only requires E to decrease) is not sufficient to pick out a unique weak solution of (1.1). In [HZ1], we show that dissipative solutions also satisfy a viscosity condition. We consider a viscous regularization of (1.1) with the simplest initial data displaying blowup of the derivative. We prove that the zero viscosity limit of the viscous solutions exists and is a dissipative weak solution of (1.1).

Conservative solutions have constant energy, even after they lose smoothness, and are thus compatible with the Hamiltonian structure of (1.1) [HZ2]. Numerical results strongly suggest that the zero-dispersion limit of solutions of an associated Korteweg-de Vries-type dispersive equation is a conservative weak solution of (1.1) [HZ1]. Furthermore, as we show in Section 4, the limit of solutions obtained by the regularized method of characteristics is also a conservative solution.

For smooth initial data, the dissipative and conservative solutions agree while the solution remains smooth, but they are distinct after the blowup time. Dissipative solutions are irreversible and, in general, they cannot be extended backwards in time. Conservative solutions are reversible and can be extended backwards and forwards in time.

The large-time behavior of solutions of (1.1) shows a striking similarity to the approach to an N-wave for solutions of the Burgers equation (1.3). For any $a \geqq 0$, we define the function

$$u^a(t, x) = \begin{cases} 0, & x \leqq 0, \\ 2x/t, & 0 < x < a^2 t^2/4, \\ a^2 t/2, & x \geqq a^2 t^2/4. \end{cases}$$

It is straightforward to verify that $u = u^a$ is a weak solution of (1.1) with constant energy, $E[u^a] = a^2$. We call this solution a *kink-wave*.

In the existence proof, we construct dissipative solutions of (1.1) when u_x has compact support in x. We show that, for these solutions, $u(t, \cdot)$ approaches a kink-wave in $H^1(\mathbb{R})$ as $t \to +\infty$. The parameter a is determined from the initial data by the time-invariant

$$\int_{\mathbb{R}} [u_x^+(t, x)]^2\, dx = \text{constant}.$$

Here, u_x^+ is the positive part of u_x. It follows that

$$a^2 = \int_{\mathbb{R}} [u_x^+(0, x)^2]\, dx.$$

Conservative solutions also approach a kink-wave as $t \to \infty$. The only difference is that in the conservative case, the energy is constant, so that

$$a^2 = \int_{\mathbb{R}} u_x^2(0, x)\, dx.$$

We do not write out any of the details here.

Equation (1.1) is a completely integrable equation [HZ2]. It is therefore not surprising that it has a large class of explicit solutions. These include piecewise linear and cusped solutions, such as the special cases mentioned above, as well as solutions obtained by the method of characteristics. Smooth solutions of (1.1) satisfy an infinite family of conservation laws, although only the first three have local densities. We introduce a new dependent variable V defined by

$$V(t, x) = u_x(t, x).$$

Then the first four conservation laws are

$$(|V_x|^{1/2})_t + (u|V_x|^{1/2})_x = 0, \tag{1.5}$$

$$(V^2)_t + (uV^2)_x = 0, \tag{1.6}$$

$$(uV^2)_t - (2uVu_t + u_t^2)_x = 0, \tag{1.7}$$

$$(2u^2V^2 + [D^{-1}(V^2)]^2)_t + (2u^3V^2)_x = 0. \tag{1.8}$$

In (1.8), D^{-1} denotes a suitable anti-derivative with respect to x [HZ2].

Global existence

The existence proof is based on the fact that (1.1) has explicit solutions with u piecewise linear and u_x a step function [HS]. We approximate general initial data by piecewise linear functions, then pass to the limit in the corresponding family of piecewise linear solutions. Since the derivatives of the approximating solutions blow up in finite time, there are no a priori estimates on higher-order derivatives which ensure strong compactness of the family of approximate solutions. Instead we prove compactness by a detailed analysis which shows that, after "chopping off" solutions arbitrarily close to points where their derivatives blow up, the derivatives of the approximating chopped-off solutions are bounded in BV.

We formulate the problem more precisely. Using the variable $V = u_x$, we can write (1.1) as the system

$$\begin{aligned} V_t + uV_x &= -\tfrac{1}{2}V^2, \\ u_x &= V. \end{aligned} \tag{1.9}$$

The function u is determined from (1.9b) only up to an additive function of time. We therefore supplement (1.9) with the initial and boundary conditions

$$\begin{aligned} V(0, x) &= V_0(x), \quad && x \in \mathbb{R}^+, \\ u(t, 0) &= 0, && t \in \mathbb{R}^+. \end{aligned} \tag{1.10}$$

Remark. Throughout this paper, we assume that $V_0(\cdot)$ is compactly supported, in which case $V(t, \cdot)$ is also compactly supported for all t. Note that

$$u(t, +\infty) = \int_0^{+\infty} V(t, x)\, dx$$

is typically nonzero, so that $u(t, \cdot)$ need not have compact support. This fact corresponds to a nonlinear instability of the original variational equations (1.2) [HS]. Since $x = 0$ is a characteristic boundary for (1.9), (1.10), there is no loss of generality in restricting compactly supported solutions to the half space $\mathbb{R}^+ \equiv (0, +\infty)$. Extending $\{u, V\}$ by zero gives a solution defined for all $x \in \mathbb{R}$.

If $V_0(x)$ is smooth, then the method of characteristics shows that a smooth solution $V(t, x)$ exists for a short time. However, if $V_0(x)$ is negative at some point, then $V(t, x)$ blows up in finite time because V satisfies a Ricatti equation along a characteristic. This blowup creates significant difficulties in proving global existence for problem (1.9), (1.10), as we now explain.

When V is smooth, (1.6) implies that the energy

$$E(V)(t) \equiv \int_0^\infty V^2(t, x)\, dx \tag{1.11}$$

is constant in time. For weak solutions, E is not necessarily constant and can actually increase. However, for dissipative solutions, E decreases, while for conservative solutions, E is constant. In either case we have an a priori L^2 estimate on V.

For any sequence of admissible solutions V^n whose initial data are bounded in L^2, it follows that there is a uniform bound

$$E(V^n)(t) \leqq C. \tag{1.12}$$

This estimate implies weak L^2 compactness of $\{V^n\}$, so we can extract a subsequence $V^n \rightharpoonup V$. The main difficulty in proving existence is that the weak limit V need not be a solution, since (1.9) contains a quadratically nonlinear term, so that $(V^n)^2$ need not converge to V^2.

We would like to show that $\{V^n\}$ is strongly compact in L^2. To do this by Sobolev embedding would require an L^p estimate on V_x^n, with $p \geqq 1$. We know of only one conservation law involving V_x, namely (1.5), but the resulting estimate is not sufficient to give strong compactness.

An alternative estimate would be a BV estimate. However, the total variation of V^n in x blows up together with V^n, so this also fails. We overcome the lack of a priori estimates by introducing a family of regularized approximate solutions $V^{n,\varepsilon}$ for which strong compactness can be established.

A more detailed description of the existence proof for dissipative solutions is as follows. We assume that V_0 in (1.10) has compact support and that

$$V_0(x) \in BV(\mathbb{R}^+).$$

We approximate $V_0(x)$ by a sequence of step functions $\{V_0^n(x)\}_{n=1}^\infty$ which have uniformly bounded BV-norms and satisfy

$$V_0^n \to V_0 \quad \text{in } L^2(\mathbb{R}^+) \quad \text{as } n \to +\infty.$$

We then construct explicit dissipative step-function solutions $V^n(t, x)$ of (1.9) with initial data V_0^n.

We consider an arbitrary finite time interval $0 \leqq t \leqq T$. Given any $\varepsilon > 0$, we chop $V^n(t, x)$ off when it is less than $-1/\varepsilon$. The resulting function $V^{n,\varepsilon}(t, x)$ is in $BV((0, T) \times \mathbb{R}^+)$ uniformly in n. Hence $\{V^{n,\varepsilon}\}_{n=1}^{\infty}$ is compact in $L^2((0, T) \times \mathbb{R}^+)$ for any fixed $\varepsilon > 0$. Furthermore, we show that the $L^2((0, T) \times \mathbb{R}^+)$-norm of the difference between V^n and $V^{n,\varepsilon}$ is of the order ε, independent of n, because $V^n(t, x)$ is large only in a very small region of space-time. Together, these estimates imply that $\{V^n\}$ is strongly compact in $L^2((0, T) \times \mathbb{R}^+)$, and the global existence of a weak solution follows.

The restriction of the initial data to BV is somewhat unnatural because the solution does not stay in BV at later times. The proof can be extended to give existence of a dissipative solution for initial data V_0 in a larger space $W(\mathbb{R}^+)$ of "chopped-BV" functions,

$$W(\mathbb{R}^+) = \{f \in L^2(\mathbb{R}^+) \mid \max\{f, -M\} \in BV(\mathbb{R}^+)\ \forall M > 0\}.$$

Existence of a global weak solution for rougher initial data, for example $V_0 \in L^2(\mathbb{R}^+)$, is an open question.

Compensated compactness [T] does not seem to be applicable to this problem. To see why, suppose we have a solution sequence $\{u^\varepsilon, V^\varepsilon\}$ such that

$$V_t^\varepsilon + (u^\varepsilon V^\varepsilon)_x = \tfrac{1}{2}(V^\varepsilon)^2 \in \text{bounded subset of } L^2,$$

$$((V^\varepsilon)^2)_t + (u^\varepsilon (V^\varepsilon)^2)_x = 0.$$

If $u^\varepsilon \to u$, $V^\varepsilon \rightharpoonup V$, $(V^\varepsilon)^2 \rightharpoonup Z$, then the div-curl lemma implies the trivial result $0 \rightharpoonup 0$, because V and V^2 propagate at the *same* velocity, namely the characteristic velocity u, whereas the essence of the div-curl lemma is that the product of oscillations propagating at *different* velocities cancels out in the limit.

The main obstacle to proving existence for rough initial data is the possibility of oscillations. As explained at the end of Section 3.2, concentrations do not occur. However, the example in Section 3.3 shows that oscillations in the initial data can persist at later times, and we are unable to rule out the generation of oscillations even if they are not present initially. The persistence of oscillations for (1.1) is in sharp contrast with the Burgers equation (1.3), where nonlinear effects immediately kill any oscillations in the initial data.

Uniqueness of dissipative or conservative solutions is an open question. In particular, the example at the end of Section 4 shows that uniqueness of conservative solutions does not hold unless the initial conditions are formulated in a careful way.

Outline of the paper

In Section 2, we formulate a precise definition of weak solutions of (1.9), (1.10). We describe the piecewise linear solutions used in the existence proof, and introduce the admissibility classes of dissipative and conservative solutions. We also use cusped solutions to give an explicit example of singularity formation from C^1 initial data for V. This example illustrates the different singularity structures of dissipative and conservative weak solutions.

The key existence results are proved in Section 3. In Section 3.1, we prove that existence of global dissipative solutions with compactly supported initial data $V_0 \in BV$. In Section 3.2, we prove existence for initial data in "chopped-BV". In Section 3.4, we indicate the necessary modifications for proving the existence of conservative solutions. In Section 3.3, we give a simple example illustrating the persistence of oscillations, and we discuss the dynamic behavior of the associated Young measure.

In Section 4, we construct solutions of (1.1) by the regularized method of characteristics. These solutions are globally single-valued because the characteristic surface of (1.1) never folds over, even though it becomes vertical at points where V blows up. We prove that the limit of solutions obtained by the regularized method of characteristics is a conservative weak solution of (1.1). This result also establishes global existence of a conservative solution when the initial data V_0 is continuous and compactly supported. This includes some initial data which is not of bounded variation.

Finally, in Section 5, we show that any dissipative solution approaches a kink-wave as $t \to +\infty$.

2. Admissible Weak Solutions

We consider the problem

$$
\begin{aligned}
&\left.\begin{aligned} V_t + (uV)_x &= \tfrac{1}{2}V^2 \\ u_x &= V \end{aligned}\right\} x > 0,\ t > 0, \\
&V(t, x)\,|_{t=0} = V_0(x), \\
&u(t, x)\,|_{x=0} = 0, \quad t > 0,
\end{aligned}
\tag{2.1}
$$

where $V_0(x) \in L^2(\mathbb{R}^+)$ is given. As we show below, global classical solutions of (2.1) typically do not exist even for smooth initial data. We therefore consider weak solutions.

Definition 2.1 (Weak solutions). A pair of functions $\{u(t, x), V(t, x)\}$ is a *weak solution* of problem (2.1) if

(a) $V \in L^\infty_{\text{loc}}(\mathbb{R}^+, L^2(\mathbb{R}^+))$,

(b) $u \in C([0, +\infty) \times [0, +\infty))$, $u(t, 0) = 0$ for $t > 0$,

(c) $V \in \text{Lip}_{\text{loc}}([0, +\infty), H^{-1}_{\text{loc}}(\mathbb{R}^+))$, $V(t, 0) = V_0(x)$ in $H^{-1}_{\text{loc}}(\mathbb{R}^+)$,

(d) $u_x = V$ in the sense of distributions,

(e) $V_t + (uV)_x = \frac{1}{2}V^2$ in the sense of distributions,

Here $\text{Lip}_{\text{loc}}([0, +\infty), X)$ denotes the space of Lipschitz continuous functions from $[0, T)$ to X for any $T > 0$.

Weak solutions are not unique, and we therefore need to supplement (2.1) with an admissibility criterion. Several different criteria are possible. The first criterion is

Definition 2.2 (Weakly admissible weak solutions). A weak solution $\{u, V\}$ of (2.1) is *weakly admissible* if

$$(V^2)_t + (uV^2)_x \leqq 0 \quad \text{in the sense of distributions.} \tag{2.2}$$

This criterion is analogous to a convex entropy condition for conservation laws. However, imposing weak admissibility is not sufficient to ensure uniqueness for problem (2.1). We therefore define two special classes of weakly admissible weak solutions: dissipative solutions and conservative solutions.

Dissipative solutions

To define dissipative solutions, we first consider a simple step-function solution of (2.1).

Example 1. Given the initial data

$$V_0(x) = \begin{cases} V_0, & 0 < x < l, \\ 0, & \text{otherwise}, \end{cases} \tag{2.3}$$

where $V_0 > 0$ and $l > 0$ are some constants, then

$$\begin{aligned} V(t,x) &= \begin{cases} \dfrac{2}{2/V_0 + t}, & 0 < x < (1 + \frac{1}{2}V_0 t)^2 l, \\ 0, & \text{otherwise}, \end{cases} \\ u(t,x) &= \begin{cases} \dfrac{2}{2/V_0 + t}x, & 0 < x < (1 + \frac{1}{2}V_0 t)^2 l, \\ \frac{1}{2}(2/V_0 + t)V_0^2 l, & x > (1 + \frac{1}{2}V_0 t)^2 l, \end{cases} \end{aligned} \tag{2.4}$$

is a weakly admissible weak solution (see Fig. 2.1).

If V_0 is negative in Example 1, then we note from (2.4) that $V(t, x)$ blows up at time $t^* = -2/V_0 > 0$. We continue the solution beyond t^* by $V = u = 0$ for $t > t^*$, $x \in \mathbb{R}^+$. The resulting function is a weakly admissible weak global solution of (2.1) (see Fig. 2.2).

Now if $V_0(x)$ is given by

$$V_0(x) = \begin{cases} V_1, & 0 < x < l_1, \\ V_2, & l_1 < x < l_2, \\ 0, & l_2 < x < \infty, \end{cases} \tag{2.5}$$

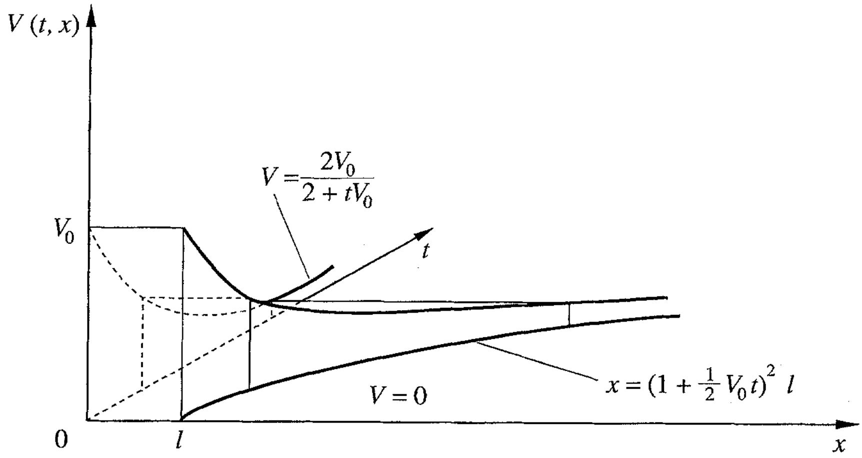

Fig. 2.1. An explicit solution: $V_0 > 0$.

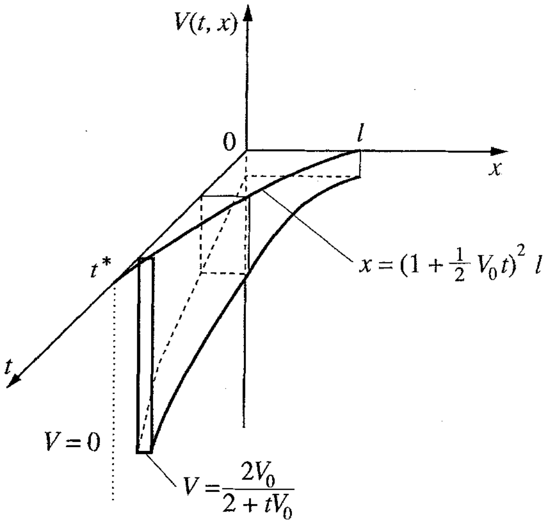

Fig. 2.2. An explicit solution: $V_0 < 0$.

we similarly obtain a weakly admissible weak solution

$$(2.6)\quad V = \begin{cases} \dfrac{2}{2/V_1 + t}, & 0 < x < (1 + \frac{1}{2}V_1 t)^2 l_1 m(V_1, t), \\ \dfrac{2}{2/V_2 + t}, & (1 + \frac{1}{2}V_1 t)^2 l_1 m(V_1, t) < x < (1 + \frac{1}{2}V_1 t)^2 l_1 m(V_1, t) \\ & \qquad + (1 + \frac{1}{2}V_2 t)^2 (l_2 - l_1) m(V_2, t), \\ 0, & \text{otherwise}, \end{cases}$$

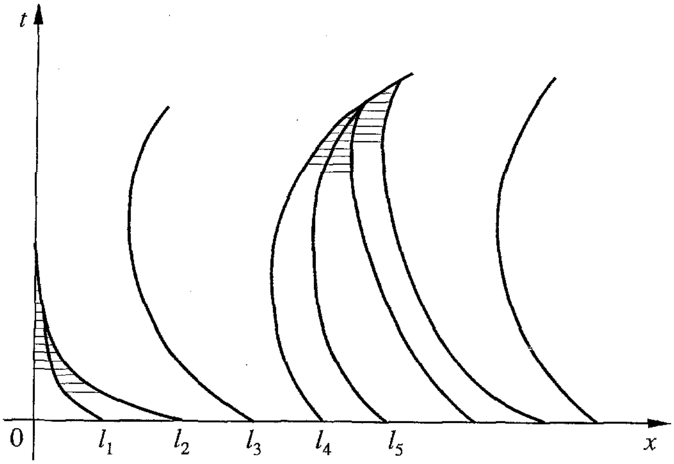

Fig. 2.3. A weakly admissible weak solution: $V(t, x)$ is a step function in x (the shading is for later use).

and u follows from V, where $m(V, t)$ is defined as

$$m(V, t) = \begin{cases} 1, & V \geqq 0, t > 0; \quad \text{or } V < 0, 0 < t < -2/V, \\ 0, & V < 0, t > -2/V. \end{cases}$$

More generally, if $V_0(x)$ is given as an n-step function

$$V_0(x) = V_i, \quad x \in (l_{i-1}, l_i), \ i = 1, 2, \ldots, n, n+1, \tag{2.7}$$

where

$$0 = l_0 < l_1 < \cdots < l_n < l_{n+1} = \infty, \quad V_{n+1} = 0,$$

then, extending by zero after blowup, we can similarly obtain a weakly admissible weak solution (see Fig. 2.3). It is straightforward to verify explicitly that these are weakly admissible weak solutions of (2.1) in the sense of Definitions 2.1 and 2.2. We call these solutions *piecewise linear dissipative solutions.* We also refer to them as step-function solutions, especially when we regard V rather than u as the main dependent variable.

We observe that the L^2-norm of V in a strip (with integration in the space variable x) is conserved in time if $V_i \geqq 0$ in that strip. If the solution blows up in a strip, then the L^2-norm jumps down to zero at the blowup time. It follows that, for all the solutions constructed above,

$$\int_0^\infty V^2(t, x)\, dx \leqq \int_0^\infty V_0^2(x)\, dx. \tag{2.8}$$

More generally, we define dissipative solutions as follows.

Definition 2.3 (Dissipative solutions). A weak solution $\{u, V\}$ of (2.1) is a *dissipatively admissible weak solution,* or *dissipative solution* for short, if it is the strong limit in $C_{\text{loc}} \otimes L^2_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^+)$ of a sequence of piecewise linear dissipative solutions.

This type of solution is called a strongly admissible solution in [HS]. In the next section, we prove that there exists a global dissipative solution of problem (2.1) for quite general initial data by using the piecewise linear dissipative solutions.

Proposition 2.1. *A dissipative solution is weakly admissible.*

Proof. A straightforward computation [HS] shows that every piecewise linear dissipative solution is weakly admissible. The proposition then follows from the continuity of the functional $(V^2)_t + (uV^2)_x$, in the sense of distributions, with respect to strong convergence in $C_{\text{loc}} \otimes L^2_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^+)$. We omit a detailed proof. □

Remark. Dissipative solutions are also closely connected with viscous regularizations of (2.1). In [HZ1], we prove that solutions of the equation with viscosity

$$
\begin{aligned}
V^\varepsilon_t + u^\varepsilon V^\varepsilon_x &= \varepsilon V^\varepsilon_{xx} - \tfrac{1}{2}(V^\varepsilon),\\
u^\varepsilon_x &= V^\varepsilon,\\
V^\varepsilon(t, x)|_{t=0} &= V_0(x),\\
u^\varepsilon(t, x)|_{x=0} &= 0
\end{aligned}
\tag{2.9}
$$

with initial data (2.3), where $V_0 < 0$, converge as $\varepsilon \to 0+$ to the dissipative solution in Fig. 2.2. For general initial data, it is straightforward to show that if the smooth solutions $\{u^\varepsilon, V^\varepsilon\}$ converge strongly in $C_{\text{loc}} \otimes L^2_{\text{loc}}$, then the limit is a weakly admissible weak solution. However, the convergence of the solutions of (2.9) to a dissipative solution of (2.1) has not been established except for the initial data in (2.3). We hope that dissipative solutions are unique because of their connection with solutions with vanishing viscosity.

For completeness, we mention one other admissibility condition for selecting a dissipative type of weak solution. This condition resembles the Oleinik entropy condition [SM] for hyperbolic conservation laws.

Definition 2.4 (Upper-bounded solutions). A weak solution $\{u, V\}$ of (2.1) is an *upper-bounded solution* if for every $t_0 > 0$, there is a constant K, possibly depending on t_0, such that

$$V(t, x) \leqq K$$

for almost all $(t, x) \in (t_0, +\infty) \times \mathbb{R}^+$.

Remark. If the initial data are bounded above, that is, if $V_0(x) \leqq K$, then we can require instead that $V(t, x) \leqq K$ for almost all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^+$.

A piecewise linear solution is upper-bounded if and only if it is dissipative. Thus, the condition of upper-boundedness is sufficiently strong to ensure uniqueness in the class of piecewise linear solutions. We hope that this condition is sufficient to ensure uniqueness for general initial data, and we plan to study this question in future work.

The following proposition gives the simplest relations between upper-bounded solutions and the other types of admissible solutions introduced above.

Proposition 2.2. (a) *Any dissipative solution of* (2.1) *is upper-bounded. More precisely,*

$$V(t,x) \leqq \frac{2}{t} \quad \text{for almost all } (t,x) \in \mathbb{R}^+ \times \mathbb{R}^+.$$

(b) *Any solution of* (2.1) *which is the limit in* $C_{\mathrm{loc}} \otimes L^2_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^+)$ *of smooth solutions* $\{u^\varepsilon, V^\varepsilon\}$ *of the viscous regularization* (2.9) *is upper-bounded.*

Proof. First we prove (a). For a dissipative piecewise linear solution, $V(t,\cdot)$ is a step function. The values of V are given by

$$V = \begin{cases} \dfrac{2V_n}{2+tV_n} & \text{if } 0 < t < -2/V_n, \\ 0, & \text{otherwise} \end{cases}$$

for some finite number of constants $\{V_n\}$. If $V_n < 0$, then $V < 0$ for $t < -2/V_n$ and $V = 0$ for $t \geqq -2/V_n$. If $V_0 \geqq 0$, then $V \leqq 2/t$ for all $t \in \mathbb{R}^+$. In either case, we always have $V \leqq 2/t$. Part (a) then follows from the fact that any dissipative solution is the pointwise a.e. limit of piecewise linear dissipative solutions.

Part (b) follows from maximum principle arguments for the first equation in (2.9) [HZ1]. □

Conservative solutions

The dissipative solution is not the only weakly admissible weak solution of problem (2.1) with data (2.3) when $V_0 < 0$. We do not have to continue the solution beyond t^* by zero; instead, there is a one-parameter family of weakly admissible weak solutions to the same problem [HS]. A second natural choice for the continuation, apart from zero, is the step-function solution which preserves the L^2-norm. In that case,

$$(2.10)\qquad \begin{aligned} V(t,x) &= \begin{cases} \dfrac{2}{2/V_0 + t}, & 0 < x < (1 + \tfrac{1}{2}V_0 t)^2 l, \quad 0 < t < \infty, \\ 0, & \text{otherwise,} \end{cases} \\ u(t,x) &= \int_0^x V(t,y)\,dy. \end{aligned}$$

This solution is illustrated in Fig. 2.4.

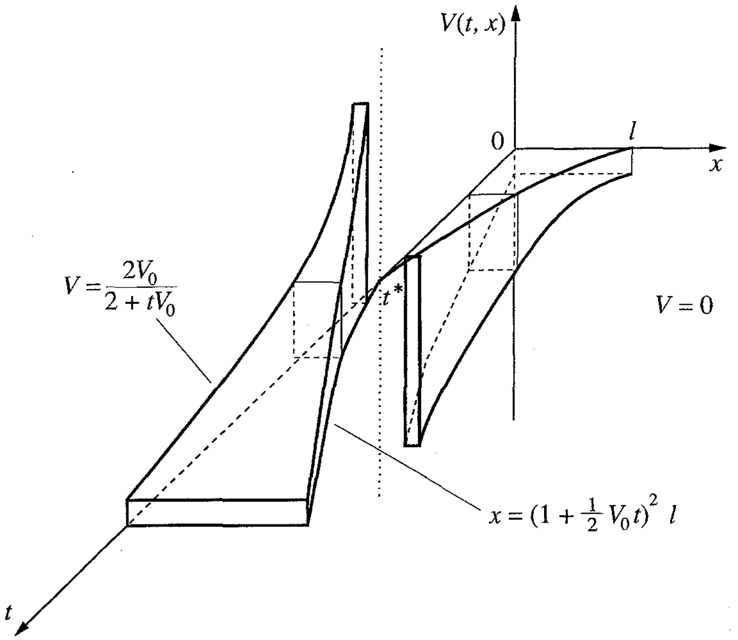

Fig. 2.4. L^2-conservation solution.

Equation (2.10) defines a weakly admissible weak solution with the property that

$$\int_0^\infty V^2(t,x)\,dx = \int_0^\infty V_0^2(x)\,dx \quad \text{for almost every } t \in \mathbb{R}^+.$$

General piecewise linear solutions which conserve energy can be constructed in a way similar to that for the dissipative solutions described above. Each of these solutions satisfies the energy equation (1.6) even after blowup occurs. This leads to a second class of weakly admissible weak solutions.

Definition 2.4 (Conservative solution). A weakly admissible weak solution $\{u, V\}$ of (2.1) is a *conservatively admissible weak solution*, or *conservative solution* for short, if

$$(V^2)_t + (uV^2)_x = 0 \quad \text{in the sense of distributions.}$$

Conservative solutions are the natural weak solutions compatible with the method of characteristics solution (see Section 4) and the Hamiltonian structure of (1.1) (see [HZ2]).

Remark. Another motivation for conservative solutions concerns the zero-dispersion limit of

$$\begin{aligned} V_t^\varepsilon + u^\varepsilon V_x^\varepsilon + \varepsilon V_{xxx}^\varepsilon &= -\tfrac{1}{2}(V^\varepsilon)^2, \\ u_x^\varepsilon &= V^\varepsilon. \end{aligned} \tag{2.11}$$

Numerical calculations in [HZ1] suggest that $\{u^\varepsilon, V^\varepsilon\}$ converges strongly as $\varepsilon \to 0$ and that the limit is a conservative solution.

Summary of admissibility classes

We have introduced several different kinds of admissible solutions of (2.1). The following diagram summarizes the known relations between the various admissibility classes:

$$\begin{array}{ccccccccc}
\text{zero-} & & & & \text{weakly} & (\supset) & \text{upper-bounded} & \supset & \text{zero-} \\
\text{dispersion} & (\subset) & \text{conservative} & \subset & \text{admissible} & & \cup & & \text{viscosity} \\
\text{limit} & & & & \text{solution} & \supset & \text{dissipative} & (\supset) & \text{limit}
\end{array}$$

The inclusions contained in parentheses are conjectured but not completely proved.

Cusped solutions

Equation (1.1) has explicit cusped weak solutions. We do not use these solutions in our existence proof, but they do illustrate energy considerations for weak solutions and reveal some interesting differences between dissipative and conservative solutions.

It is straightforward to check that the time-independent function

$$u = \begin{cases} a|x|^{2/3}, & x \leqq 0, \\ b|x|^{2/3}, & x \geqq 0, \end{cases} \tag{2.12}$$

is a distributional solution of (1.1) for any choice of constants a and b. The energy equation for (2.12) is

$$(V^2)_t + (uV^2)_x = \tfrac{4}{9}(b^3 - a^3)\delta(x).$$

This solution is therefore weakly admissible when $b \leqq a$ and conservative when $b = a$. For example, the monotone increasing function $u = \operatorname{sgn} x\,|x|^{2/3}$ is not a weakly admissible weak solution. The singularity at $x = 0$ is a source of energy, since the energy flux $uV^2 = (\frac{4}{9})\operatorname{sgn} x$ is directed away from the singularity. The monotone decreasing solution $u = -\operatorname{sgn} x\,|x|^{2/3}$ dissipates energy since the energy flux is directed inwards on either side. Finally, the two cusped solutions $u = \pm|x|^{2/3}$ are conservative, and their energy fluxes are continuous across $x = 0$.

We can use this type of stationary solution to give an explicit solution illustrating dynamic singularity formation in monotone decreasing initial data. We consider piecewise C^2-initial data

$$u(0, x) = \begin{cases} 3 - 3(x+1)^{2/3} & \text{when } x \geqq 0, \\ -u(0, -x) & \text{when } x \leqq 0. \end{cases} \tag{2.13}$$

Although this function is unbounded, we can patch it to constants at large $|x|$; we do not do this, since it does not affect the local behavior near $x = 0$.

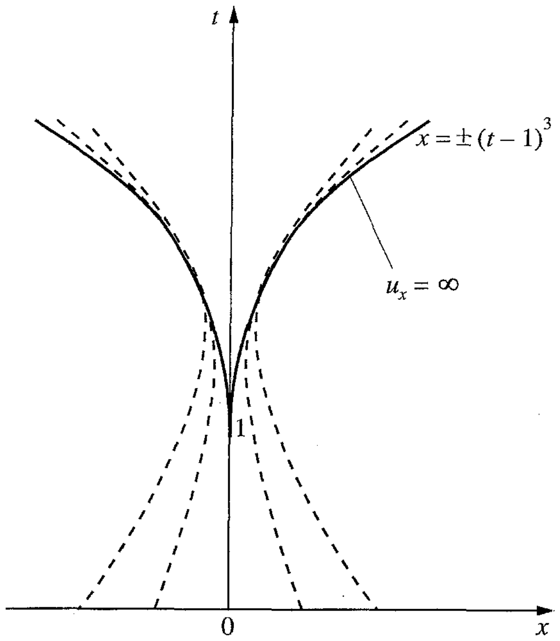

Fig. 2.5. Cusp singularity of a conservative solution.

The minimum value of the derivative is $u_x(0, 0) = -2$, so the blowup time is $t^* = 1$. We choose solutions $u(t, x)$ which are odd functions of x. A conservative solution (which can be found by using the regularized method of characteristics described in Section 4) is

$$u(t, x) = 3(t-1)^2 - 3\,|x - (t-1)^3|^{2/3}, \quad x \geqq 0.$$

This is a piecewise C^2-solution for $t < 1$. At $t = 1$, two cusp singularities emerge from the point $(t, x) = (1, 0)$ and then propagate away from each other along $x = \pm(t-1)^3$; see Fig. 2.5. A dissipative solution is given by the same formula for $t < 1$, but for $t \geqq 1$ it is continued by

$$u(t, x) = -3 \operatorname{sgn} x |x|^{2/3}.$$

For the dissipative solution, there is a jump discontinuity in u_{tt} across the line $t = 1$: $u_{tt}(1-, x) = 6$, but $u_{tt}(1+, x) = 0$ for all x; see Fig. 2.6.

Finally, we note that Theorems 3.1, 3.3, and 4.1 all apply to the initial function (2.13) (if it is patched to constants at large x) and yield the existence of dissipative and conservative weak solutions. However, if we take as initial data the solution after the blowup time, none of the above theorems applies, since the derivative of $x^{2/3}$ does not have bounded variation and is not continuous. Theorem 3.2 is applicable in both cases, since the derivative of $x^{2/3}$ is in "chopped" BV.

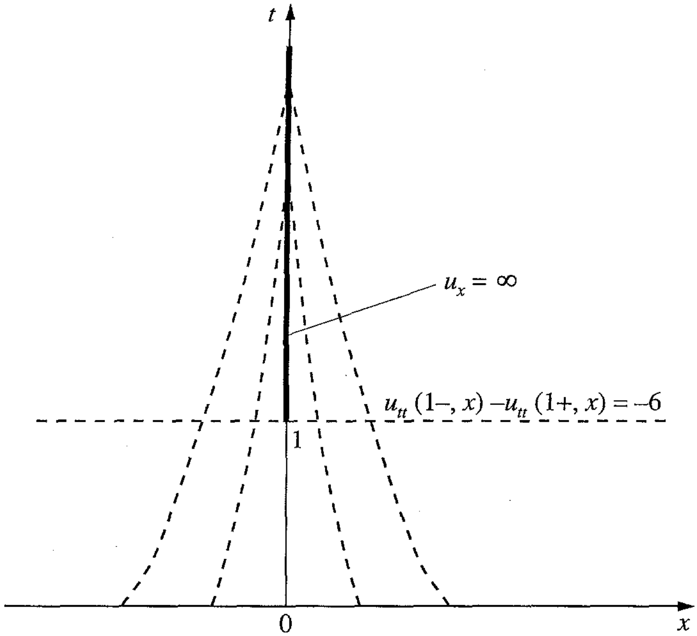

Fig. 2.6. Cusp singularity of a dissipative solution.

3. Global Existence of Admissible Weak Solutions

3.1. Existence of dissipative solutions

We first prove our key result.

Theorem 3.1. *There exists a global dissipatively admissible weak solution* $\{u, V\}$ *of problem* (2.1) *provided that* $V_0(x)$ *has bounded total variation and compact support in* $[0, +\infty)$. *This solution satisfies the regularity conditions*

(i) u *is Hölder continuous in* $[0, +\infty) \times [0, +\infty)$ *with exponent* $\alpha \in (0, \frac{1}{3})$,

(ii) $V \in L^p((0, T) \times \mathbb{R}^+)$ *for any* $p \in [2, 3)$, $T < \infty$.

Proof. By rescaling (t, x), we can assume that $V_0(x)$ is supported in $[0, 1]$. Since $V_0(x) \in BV[0, 1]$, it is bounded:

$$|V_0(x)| \leqq M < \infty, \quad x \in [0, 1].$$

We approximate $V_0(x)$ with step functions $\{V_0^n(x)\}_{n=1}^{\infty}$ defined by

$$V_0^n(x) = V_i^n, \quad x \in \left(\frac{i-1}{n}, \frac{i}{n}\right), \quad i = 1, 2, \ldots, n,$$

where the V_i^n are constants. These step functions satisfy

$$V_0^n(x) \to V_0(x) \quad \text{a.e. and in } L^p[0,1], \quad 1 \leqq p < \infty, \tag{3.1}$$

$$\| V_0^n(\cdot) \|_{BV[0,1]} \leqq C, \tag{3.2}$$

$$\| V_0^n \|_{L^\infty[0,1]} \leqq M. \tag{3.3}$$

We construct the dissipatively admissible weak solutions $V^n(t,x)$ with data $V_0^n(x)$ as in Section 3 (see Fig. 2.3). We then have the estimate

$$\int_0^\infty |V^n(t,x)|^2\,dx \leqq \int_0^\infty |V_0^n(x)|^2\,dx \leqq C < \infty, \quad n = 1, 2, 3, \ldots. \tag{3.4}$$

In general, solutions blow up at times $t^*_{n,i} = -2/V_i^n$. The earliest time is

$$t^*_{n,\min} = \frac{2}{\max_{1 \leqq i \leqq n}(-V_i^n)} > \frac{2}{M}.$$

Thus the $BV(\mathbb{R}^+)$ norm of the approximate solution $V^n(t,\cdot)$ generally blows up. We chop $V^n(t,x)$ off to form a new, bounded sequence $\{V^{n,\varepsilon}(t,x)\}_{n=1}^\infty$, so that $\{V^{n,\varepsilon}\}_{n=1}^\infty$ is compact in $L^2((0,T)\times\mathbb{R}^+)$ for any $T > 0$ and so that

$$\int_0^T \int_0^\infty |V^{n,\varepsilon} - V^n|^2\,dx\,dt \leqq \varepsilon C$$

uniformly in n for all $\varepsilon > 0$ small.

Let $0 < \varepsilon < 2/M$. Let $V^{n,\varepsilon}(t,x)$ be defined as follows: replace the solution $V^n(t,x)$ in the shaded region from $t^*_{n,i} - \varepsilon$ to $t^*_{n,i}$ by the constant at $t = t^*_{n,i} - \varepsilon$ (see Figures 2.3 and 3.1). The regularized step-function solution, $V^{n,\varepsilon}(t,x)$, is well defined in $\mathbb{R}^+ \times \mathbb{R}^+$. It is equal to $-2/\varepsilon$ in the shaded regions (see Figure 2.3) where $V^n(t,x)$ exceeds $-2/\varepsilon$, and is otherwise equal to $V^n(t,x)$. Thus

$$\| V^{n,\varepsilon}(t,x) \|_{L^\infty} \leqq \frac{2}{\varepsilon}. \tag{3.5}$$

Given $T > 0$, we next establish the following estimates which imply strong compactness of $\{V^n\}_{n=1}^\infty$:

$$\int_0^T \int_0^\infty |V^{n,\varepsilon} - V^n|^2\,dx\,dt \leqq \varepsilon C, \tag{3.6}$$

$$\sup_{0 \leqq t \leqq T} \| V^{n,\varepsilon}(t,\cdot) \|_{BV(\mathbb{R}^+)} \leqq \frac{C}{\varepsilon^2}, \tag{3.7}$$

$$\sup_{x \in \mathbb{R}^+} \| V^{n,\varepsilon}(\cdot,x) \|_{BV[0,T]} \leqq \frac{C}{\varepsilon^2}. \tag{3.8}$$

Here, $C = C(T, M, V_0)$ is a constant independent of n, and $\varepsilon \in (0,1]$.

Proof of (3.6). For convenience, let us introduce some notation. Fix n. For each $i = 1, 2, \ldots, n$, let R_i denote either the i-th shaded region, or an empty set when

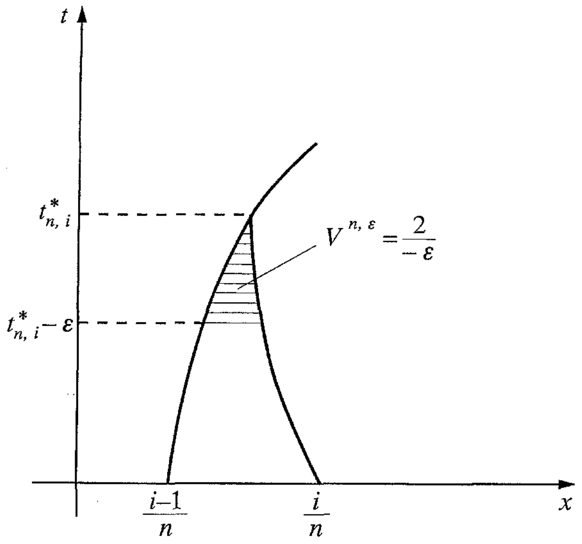

Fig. 3.1. Definition of $V^{n,\varepsilon}$.

there is no shaded region in the strip. Also, let $x_i(t)$ denote the right boundary of a strip (the i-th strip) which has a bottom side $(\frac{i-1}{n}, \frac{i}{n})$ on the x-axis. Now we have

$$\int_0^T \int_0^\infty |V^{n,\varepsilon} - V^n|^2 \, dx \, dt = \sum_{i=1}^n \iint_{R_i} |V^{n,\varepsilon} - V^n|^2 \, dx \, dt$$

$$\leqq \sum_{i=1}^n \iint_{R_i} |V^n|^2 \, dx \, dt.$$

Using the energy estimate (2.8) in each strip with a nonempty R_i, we find that

$$\iint_{R_i} |V^n|^2 \, dx \, dt = \int_{t^*_{n,i}-\varepsilon}^{t^*_{n,i}} \int_{x_{i-1}(t)}^{x_i(t)} |V^n|^2 \, dx \, dt = \varepsilon \int_{(i-1)/n}^{i/n} |V_0^n|^2 \, dx.$$

Therefore

$$\int_0^T \int_0^\infty |V^{n,\varepsilon} - V^n|^2 \, dx \, dt \leqq \varepsilon \int_0^\infty |V_0^n|^2 \, dx \leqq C\varepsilon.$$

Proof of (3.7). Fix a t in $(0, T)$. The BV norm of $V^{n,\varepsilon}(t, \cdot)$ is the sum of jumps of $V^{n,\varepsilon}$ across all the boundaries $x_i(t)$ of the strips. There are three cases to consider, depending on the signs of V_i^n and V_{i+1}^n. The first case is
Case (3.7a), *when V_i^n and V_{i+1}^n are both positive.* On each i-th strip, we have

$$V^n(t, x) = \frac{2}{2/V_i^n + t}.$$

So the jump in $V^{n,\varepsilon} = V^n(t, x)$ across the i-th boundary $x_i(t)$ is

$$(3.9) \qquad |V^{n,\varepsilon}(t, x_i(t)+) - V^{n,\varepsilon}(t, x_i(t)-)|$$

$$= \frac{4\left|\dfrac{1}{V_i^n} - \dfrac{1}{V_{i+1}^n}\right|}{\left(\dfrac{2}{V_i^n}+t\right)\left(\dfrac{2}{V_{i+1}^n}+t\right)} \leqq \frac{4\left|\dfrac{1}{V_i^n} - \dfrac{1}{V_{i+1}^n}\right|}{\dfrac{2}{V_i^n}\cdot\dfrac{2}{V_{i+1}^n}} = |V_{i+1}^n - V_i^n|.$$

The second case is

Case (3.7b), *when V_i^n and V_{i+1}^n have opposite signs.* We need to estimate the rate at which the equation *amplifies* the initial data. For simplicity, we assume that $V_i^n < 0 < V_{i+1}^n$.

From the general formula

$$V^n(t, x) = \frac{2}{2/V_i^n + t} = \frac{2}{2 + V_i^n t} V_i^n \quad \text{in the } i\text{-th strip},$$

we have for our present case

$$(3.10) \qquad 0 \leqq V^{n,\varepsilon}(t, x) \leqq V_{i+1}^n \quad \text{in the } (i+1)\text{-st strip},$$

$$(3.11) \qquad 0 > V^{n,\varepsilon}(t, x) \geqq \left(\frac{\varepsilon + T}{\varepsilon}\right) V_i^n \quad \text{in the } i\text{-th strip}.$$

We prove the second inequality in (3.11); the rest is immediate. In fact, we split the initial data V_i^n into two categories (see Figure 3.2); $V_i^n \leqq -2/(\varepsilon + T)$ and $0 > V_i^n > -2/(\varepsilon + T)$. In the first category, we use

$$V^{n,\varepsilon}(t, x) \geqq \frac{2}{-\varepsilon} = \frac{2}{\varepsilon(-V_i^n)} V_i^n \geqq \left(\frac{\varepsilon + T}{\varepsilon}\right) V_i^n.$$

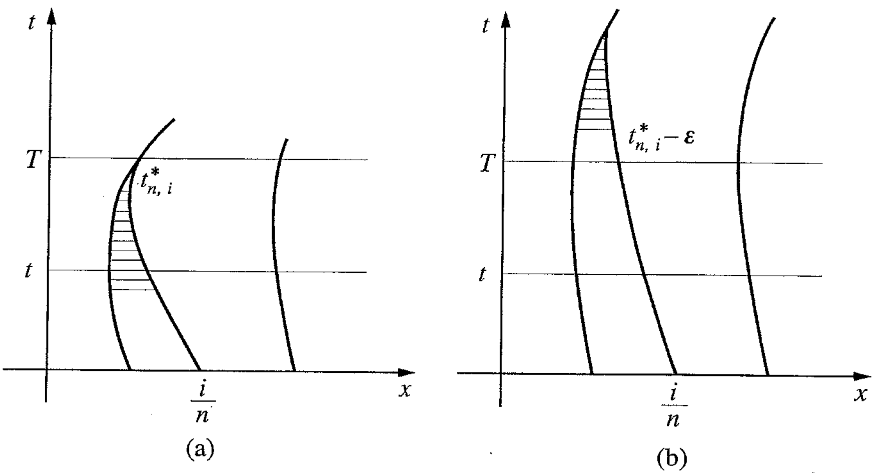

Fig. 3.2. Calculation of amplification of data. (a) Case $V_i^n \leqq \frac{-2}{\varepsilon+T}$. (b) Case $0 > V_i^n > \frac{-2}{\varepsilon+T}$.

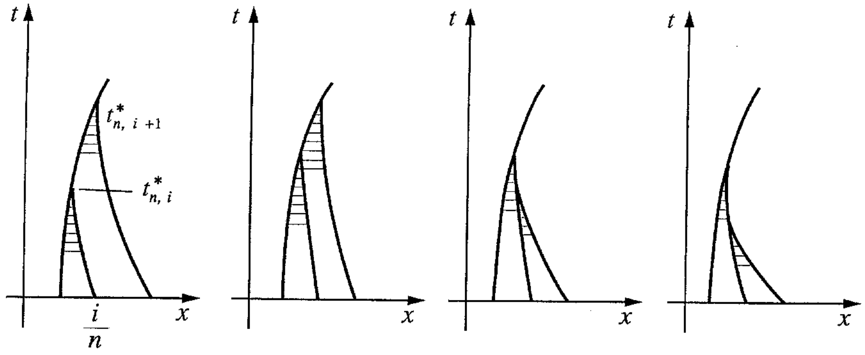

Fig. 3.3. Four possibilities of case (3.7c).

In the second category, we note that the solution blows up at time

$$t^*_{n,i} = \frac{2}{-V_i^n} > T + \varepsilon.$$

Therefore

$$V^{n,\varepsilon}(t, x) \geqq \frac{2}{2 + V_i^n T} V_i^n > \left(\frac{\varepsilon + T}{\varepsilon}\right) V_i^n.$$

So (3.10) and (3.11) are proved, and the amplifying factor is bounded by a constant independent of n. Thus

$$|V^{n,\varepsilon}(t, x_i +) - V^{n,\varepsilon}(t, x_i -)| \leqq \frac{\varepsilon + T}{\varepsilon} |V_{i+1}^n - V_i^n|. \tag{3.12}$$

The last case is the one leading to the largest amplification of the variation. An initial variation of the order ε produces a variation of the order ε^{-1} at later times.

Case (3.7c), *when V_i^n and V_{i+1}^n are both negative.* According to the relative location of the two neighboring shaded regions, we have four different subcases depicted in Figure 3.3.

The first subcase occurs when $t^*_{n,i+1} - \varepsilon > t^*_{n,i}$.

The second subcase occurs when $t^*_{n,i+1} > t^*_{n,i} > t^*_{n,i+1} - \varepsilon$.

The third subcase occurs when $t^*_{n,i+1} + \varepsilon > t^*_{n,i} > t^*_{n,i+1}$.

The last subcase occurs when $t^*_{n,i+1} + \varepsilon < t^*_{n,i}$.

We deal with the first subcase first. For $t \in [0, t^*_{n,i} - \varepsilon]$, we find that the same argument for (3.7a) works here also because

$$(1 + \tfrac{1}{2}V_{i+1}^n t)(1 + \tfrac{1}{2}V_i^n t) \geqq \left(\frac{\varepsilon}{\varepsilon + T}\right)^2.$$

Hence

$$(3.13)\qquad |V^{n,\varepsilon}(t, x_i(t)+) - V^{n,\varepsilon}(t, x_i(t)-)| = \left|\frac{V_{i+1}^n}{1+\frac{1}{2}V_{i+1}^n t} - \frac{V_i^n}{1+\frac{1}{2}V_i^n t}\right|$$

$$= \frac{|V_{i+1}^n - V_i^n|}{(1+\frac{1}{2}V_{i+1}^n t)(1+\frac{1}{2}V_i^n t)}$$

$$\leqq \left(\frac{\varepsilon + T}{\varepsilon}\right)^2 |V_{i+1}^n - V_i^n|.$$

For $t \in [t_{n,i}^* - \varepsilon, t_{n,i}^*]$, we find that

(3.14)

$$|V^{n,\varepsilon}(t, x_i(t)+) - V^{n,\varepsilon}(t, x_i(t)-)| \leqq |V^{n,\varepsilon}(t_{n,i}^* - \varepsilon, x_i+) - V^{n,\varepsilon}(t_{n,i}^* - \varepsilon, x_i-)|,$$

because $V^{n,\varepsilon}(t, x_i(t)+)$ is a decreasing function of t and because $V^{n,\varepsilon}(t, x_i(t)+) > V^{n,\varepsilon}(t, x_i(t)-) = -2/\varepsilon$ in $[t_{n,i}^* - \varepsilon, t_{n,i}^*]$. Combining (3.13) and (3.14), we have

$$(3.15)\quad |V^{n,\varepsilon}(t, x_i(t)+) - V^{n,\varepsilon}(t, x_i(t)-)| \leqq |V_{i+1}^n - V_i^n| \left(\frac{\varepsilon+T}{\varepsilon}\right)^2, \quad t \in [0, t_{n,i}^*],$$

for the first subcase. We note that t cannot be larger than $t_{n,i}^*$ in the first subcase, since our boundary $x_i(t)$ terminates at $t_{n,i}^*$.

For the second and third subcases, the calculation is similar. The only new part is when $t \in [t_{n,i+1}^* - \varepsilon, t_i^*]$ in the second subcase, or $t \in [t_{n,i}^* - \varepsilon, t_{n,i+1}^*]$ in the third subcase. We note that in these cases the jump is simply zero. For the fourth subcase, the calculation is exactly the same as in the first subcase.

We have therefore proved (3.7).

Proof of (3.8). For any fixed $x_0 \in (0, +\infty)$, we estimate the $BV[0, T]$ norm of $V^{n,\varepsilon}(\cdot, x_0)$. Note that all the strip boundary curves $x = x_i(t)$, $1 \leqq i \leqq n$, are quadratic. So a vertical line $x = x_0$ intersects any of these curves $x = x_i(t)$ at most twice. A typical case is depicted in Figure 3.4.

The $BV[0, T]$ norm of $V^{n,\varepsilon}(\cdot, x_0)$ is the sum of two parts:

(I) Simple jumps across the boundaries $x = x_i(t)$. The result is at most equal to twice the $BV(0, +\infty)$ norm of $V^{n,\varepsilon}(t, \cdot)$ as is given in (3.7).

(II) The monotone gain in the continuous regions. This region can be split into two parts. The first part is the shaded regions, in which $V^{n,\varepsilon} = 2/\varepsilon$ is constant, so that

$$V_t^{n,\varepsilon} = 0.$$

Hence, there is no contribution to the variation from the shaded regions. The second part is the unshaded regions. To estimate the variation there, we use the equation

$$V_t^n + u^n V_x^n = -\tfrac{1}{2}(V^n)^2$$

and $V_x^n = 0$ to find that

$$V_t^n = -\tfrac{1}{2}(V^n)^2.$$

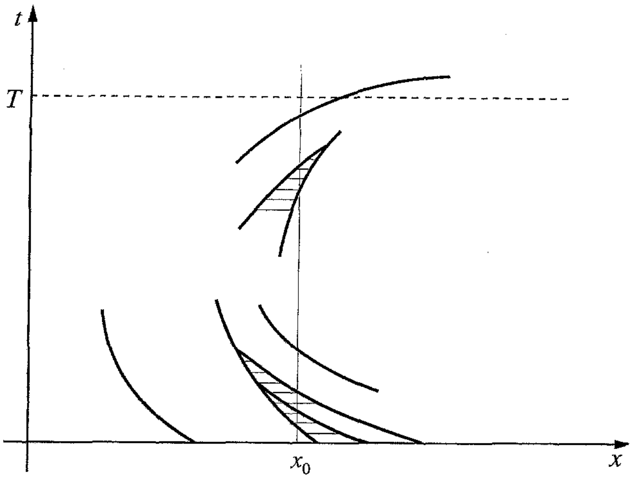

Fig. 3.4. Proof of (3.8).

Therefore

$$|V_t^{n,\varepsilon}| = |V_t^n| \leqq \frac{1}{2}\cdot\left(\frac{2}{-\varepsilon}\right)^2 = \frac{2}{\varepsilon^2},$$

and the total variation in the continuous regions is

$$\text{Variation in continuous regions of } \; |V^{n,\varepsilon}(\cdot, x_0)| \leqq \frac{2T}{\varepsilon^2}.$$

Thus, the total variation in the whole interval $[0, T]$ is

$$\begin{aligned} \|V^{n,\varepsilon}(\cdot, x_0)\|_{BV[0,T]} &= \sum_{i=1}^{n} \text{jumps across } x_i(t) \\ &\qquad + \text{Variation in continuous regions} \\ &\leqq 2 \sup_{0\leqq t\leqq T} \|V^{n,\varepsilon}(t,\cdot)\|_{BV} + \frac{2T}{\varepsilon^2} \\ &\leqq \frac{C(T, M)}{\varepsilon^2}. \end{aligned} \tag{3.16}$$

Taking the supremum over $x_0 \in \mathbb{R}^+$ of (3.16), we obtain (3.8).

From (3.7) and (3.8) it follows that $\{V^{n,\varepsilon}\}_{n=1}^\infty$ is a bounded subset of

$$L^\infty \cap BV([0, T]\times\mathbb{R}^+).$$

Therefore, $\{V^{n,\varepsilon}\}_{n=1}^\infty$ is compact in $L^2([0, T]\times\mathbb{R}^+)$ [EG]. Since (3.6) holds, $\{V^n\}_{n=1}^\infty$ is compact in $L^2([0, T]\times\mathbb{R}^+)$ [LS]. Thus there exists a subsequence $\{V^{n_j}\}_{j=1}^\infty$ which converges strongly in $L^2([0, T]\times\mathbb{R}^+)$ to a limit V. One also obtains that the corresponding sequence $\{u^{n_j}\}_{j=1}^{\infty}$ is compact in $L^2([0, T] \times \mathbb{R}^+_{\text{loc}})$ since $u_t^n + u^n V^n = \frac{1}{2}\int_0^x (V^n)^2\,dx$. Passing to a further subsequence of $\{u^{n_j}\}_{j=1}^{\infty}$, we obtain a pair $\{u, V\}$ which satisfies conditions (a), (d), and (e) of Definition 2.1.

To establish the regularity for $\{u, V\}$, we assert that for all $T > 0$,

$$\int_0^T \int_0^\infty |V^n(t, x)|^p\,dx\,dt \leqq C_{T,p} \int_0^\infty |V_0(x)|^2\,dx, \quad 2 \leqq p < 3.$$

In fact, we compute

$$\begin{aligned}\int_0^T \int_0^\infty |V^n(t, x)|^p\,dx\,dt &= \sum_{i=1}^n \int_0^T \int_{x_{i-1}(t)}^{x_i(t)} |V^n(t, x)|^p\,dx\,dt \\ &= \sum_{i=1}^n \int_0^T \left(\int_{x_{i-1}}^{x_i} |V^n|^2\,dx \right) |V^n|^{p-2}\,dt \\ &= \sum_{i=1}^n \left(\frac{(V_i^n)^2}{n} \int_0^T |V^n|^{p-2}\,dt \right).\end{aligned}$$

In positive strips where $V_i^n \geqq 0$, we have $V^n \leqq 2/t$. Therefore

$$\int_0^T |V^n|^{p-2}\,dt \leqq \int_0^T \left(\frac{2}{t}\right)^{p-2} dt = \frac{2^{p-2}}{3-p}\,T^{3-p}.$$

In strips where $-\frac{1}{T} \leqq V_i^n < 0$, we have $t_{n,i}^* = \dfrac{2}{-V_i^n} \geqq 2T$. Hence

$$\int_0^T |V^n|^{p-2}\,dt = \int_0^T \left(\frac{2}{t_{n,i}^* - t}\right)^{p-2} dt \leqq 2^{p-2} T^{3-p}.$$

And in strips where $V_i^n \leqq -\frac{1}{T}$, we have $0 < t_{n,i}^* \leqq 2T$. Therefore

$$\int_0^T |V^n|^{p-2}\,dt \leqq \int_0^T \frac{2^{p-2}}{|t_{n,i}^* - t|^{p-2}}\,dt \leqq 2^{p-2} \int_{-2T}^T \frac{dt}{|t|^{p-2}} < \infty.$$

Hence

$$\int_0^T \int_0^\infty |V^n(t, x)|^p\,dx\,dt \leqq C_{T,p} \sum_{i=1}^n \frac{(V_i^n)^2}{n} \leqq C_{T,p} \int_0^\infty |V_0(x)|^2\,dx.$$

Therefore the assertion is true, so that condition (ii) holds. Furthermore, from the assertion and the equation

$$u_t^n + u^n V^n = \tfrac{1}{2} \int_0^x (V^n(t, y))^2\,dy,$$

we have

$$\int_0^T \int_0^R |u_t^n|^p\,dx\,dt \leqq C_{T,p,R} \int_0^\infty |V_0(x)|^2\,dx.$$

Therefore $\{u^n\}_{n=1}^{\infty}$ is a bounded subset of $W^{1,p}([0,T]\times[0,R])$ for all $p\in[2,3)$, $T>0, R>0$. By the Sobolev embedding theorem, $W^{1,p}([0,T]\times[0,R])\subset C^{1-2/p}$. We obtain that $\{u^n\}_{n=1}^{\infty}$ is uniformly Hölder continuous for any exponent $\alpha<\frac{1}{3}$. Therefore the limit u is also Hölder continuous for any exponent $\alpha<\frac{1}{3}$, so that (i) holds. Since $u^n(t,0)=0$, it follows from the uniform convergence of $u^n\to u$ that $u(t,0)=0$ also, so that condition (b) of Definition 2.1 holds.

It remains to check that V satisfies condition (c) in Definition 2.1. This follows from equation (e) of Definition 2.1, which implies that

$$V_t=-(uV)_x+\tfrac{1}{2}V^2\in L^{\infty}_{\mathrm{loc}}([0,\infty),H^{-1}_{\mathrm{loc}}(\mathbb{R}^+)+L^1(\mathbb{R}^+))\subset L^{\infty}_{\mathrm{loc}}([0,\infty),H^{-1}_{\mathrm{loc}}(\mathbb{R}^+)).$$

We give a detailed proof in Appendix B.

This finishes the proof of Theorem 3.1.

3.2. *More general initial data*

The solution V constructed in Theorem 3.1 is only in $L^2((0,T)\times\mathbb{R}^+)$ even though the initial function is in $BV(\mathbb{R}^+)$. In this section, we prove existence of global solutions under a weaker assumption on the initial data. We define a space W of "chopped" BV functions which lies between L^2 and BV. We then prove that if $V_0(x)$ is in $W(\mathbb{R}^+)$, there exists a solution $V(t,x)$ in $W((0,T)\times\mathbb{R}^+)$.

Given $\Omega\subset\mathbb{R}^n$, we define the space $W(\Omega)$ to be

$$W(\Omega)=\{f\in L^2(\Omega)\mid \|f_M\|_{BV(\Omega)}<\infty\ \forall M>0\}$$

where

$$f_M(x)=\begin{cases} f(x) & \text{if } f(x)\geqq -M,\\ -M & \text{if } f(x)<-M.\end{cases}$$

Corollary 3.1. *The solution obtained in Theorem 3.1 is in* $W((0,T)\times\mathbb{R}^+)$ *for all* $T>0$.

The proof of Corollary 3.1 is implicit in the proof of Theorem 3.1 and is contained in the proof of the next more general theorem. Therefore we omit it.

We need a lemma whose proof is given in Appendix A.

Lemma 3.1. *For any* $V_0(x)\in W([0,1])$, *there exists a sequence of step functions* $\{V_0^n\}_{n=1}^{\infty}$ *defined on* $[0,1]$ *with the properties*:

(1) $V_0^n(x)\to V_0(x)$ *in* $L^2([0,1])$,

(2) $\|(V_0^n)_M(x)\|_{BV[0,1]}\leqq C_M<\infty\quad\forall n,\forall M>0.$

We now state our existence result.

Theorem 3.2. *There exists a dissipatively admissible weak solution* $\{u,V\}$ *to problem* (2.1) *provided that* $V_0(x)$ *is supported in* $[0,1]$ *and belongs to* $W[0,1]$. *For any*

$T > 0$, the solution V is in $W((0, T) \times \mathbb{R}^+)$ and $L^p((0, T) \times \mathbb{R}^+)$ for $p \in [2, 3)$, and u is Hölder continuous with exponent $\alpha < \frac{1}{3}$ in both space and time.

Proof of Theorem 3.2. For $V_0(x) \in W[0, 1]$, we find a sequence $\{V_0^n(x)\}$ which satisfies the properties of Lemma 3.1. For each n, $V_0^n(x)$ is a step function. Using the construction of Section 2, we obtain a dissipative step-function solution $V^n(t, x)$ with initial data $V_0^n(x)$. This solution satisfies

$$\int_0^\infty |V^n(t, x)|^2 \, dx \leqq \int_0^\infty |V_0^n(x)|^2 \, dx \leqq C.$$

For any $0 < \varepsilon \leqq 1$, we define (see Figure 3.5)

$$V^{n,\varepsilon}(t, x) = \begin{cases} V^n(t, x) & \text{if } V^n(t, x) \geqq -2/\varepsilon, \\ -2/\varepsilon & \text{if } V^n(t, x) < -2/\varepsilon. \end{cases}$$

$V^{n,\varepsilon}(t, x)$ is defined in the whole $\mathbb{R}^+ \times \mathbb{R}^+$. In particular,

$$V^{n,\varepsilon}(0, x) = (V_0^n)_{2/\varepsilon}(x) \equiv \text{the cut-off of } V_0^n \text{ at } -2/\varepsilon,$$

$$\| V^{n,\varepsilon}(0, x) \|_{BV(\mathbb{R}^+)} \leqq C_\varepsilon \quad \forall n = 1, 2, \ldots . \tag{3.17}$$

This $V^{n,\varepsilon}(t, x)$ is the same as the $V^{n,\varepsilon}$ defined in the proof of Theorem 3.1 for $t > \varepsilon$. As a matter of fact, it satisfies all of the estimates (3.6)–(3.8) in exactly the same way. Thus there exists a subsequence $\{u^{n_j}, V^{n_j}\}_{j=1}^\infty$ which converges in $C([0, T] \times \mathbb{R}^+_{\text{loc}}) \otimes L^2([0, T] \times \mathbb{R}^+)$ to a limit $\{u, V\}$ which is a solution, and $\{(V^{n_j})_M\}_{j=1}^\infty$ converges in L^2 to V_M, which is in BV for all $M > 0$. The proof of other regularity conditions is the same as in Theorem 3.1. This completes the proof of Theorem 3.2.

We remark that the space L^2 is a more natural space for problem (2.1) than the space W. It would be interesting to start with $L^2(\mathbb{R}^+)$ data and obtain an $L^\infty((0, T), L^2(\mathbb{R}^+))$ solution. The problem is that we do not have strong enough

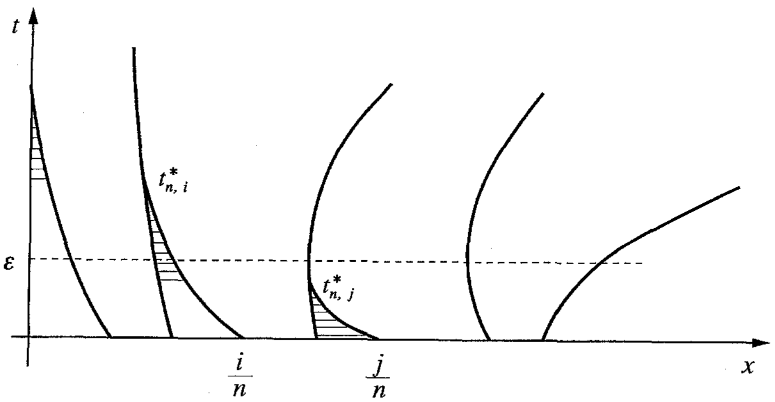

Fig. 3.5. Definition of $V^{n,\varepsilon}$.

estimates to obtain compactness in L^2. We note that, by interpolation, an $L^p((0, T) \times \mathbb{R}^+)$, $p < 3$, estimate would imply compactness in L^2 if one can establish compactness in L^1 (i.e., no oscillations). In this sense concentration is eliminated. In the next section, however, we give an example which shows that oscillations can persist in solution sequences.

3.3. Persistence of oscillations

Consider problem (2.1) with data $V_0(x)$ given by

$$V_0^n(x) = \begin{cases} 1, & x \in \left(\dfrac{2k}{n}, \dfrac{2k+1}{n}\right),\ k = 0, 1, \ldots, \left[\dfrac{n-1}{2}\right], \\ -1, & x \in \left(\dfrac{2k+1}{n}, \dfrac{2k+2}{n}\right),\ k = 0, 1, \ldots, \left[\dfrac{n}{2}\right] - 1 \end{cases}$$

where n is a natural number (see Figure 3.6). The step-function solution $V^n(t, x)$ is given explicitly for $0 < t < 2$ by

$$V^n(t, x) = \begin{cases} \dfrac{2}{t+2} & \text{for } x_{2k} < x < x_{2k+1},\ k = 0, 1, \ldots, \left[\dfrac{n-1}{2}\right], \\ \dfrac{2}{t-2} & \text{for } x_{2k+1} < x < x_{2k+2},\ k = 0, 1, \ldots, \left[\dfrac{n}{2}\right] - 1, \end{cases}$$

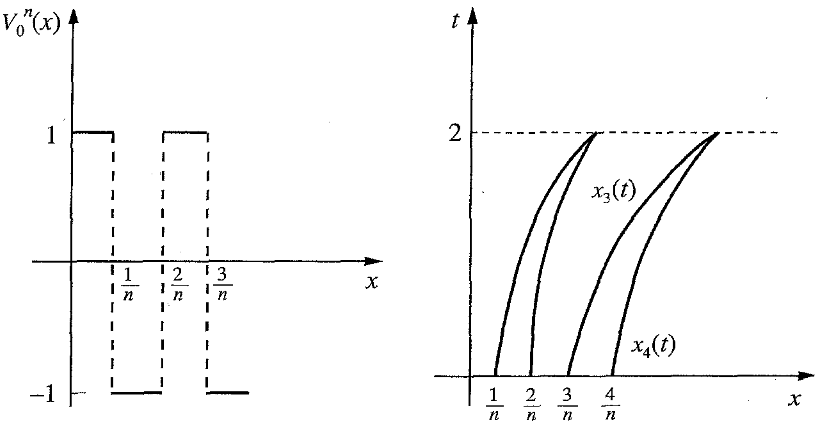

Fig. 3.6. Oscillation.

where

$$x_{2k}(t) = \frac{2k}{4n}(t+2)^2 - \frac{2k}{n}t, \qquad k = 0, 1, 2, \ldots, \left[\frac{n-1}{2}\right],$$

$$x_{2k+1} = \frac{2k+1}{4n}(t+2)^2 - \frac{2k}{n}t, \quad k = 0, 1, 2, \ldots, \left[\frac{n}{2}\right] - 1.$$

The two lines $x = x_{2k+1}(t)$ and $x = x_{2k+2}(t)$ meet at $t = 2$. Therefore in the region $0 \leqq t \leqq 2$, the solution $V^n(t, x)$ changes sign from region (x_{2k}, x_{2k+1}) to (x_{2k+1}, x_{2k+2}), and oscillations occur when n approaches infinity.

The weak limits of V^n and $(V^n)^2$ are simple to compute explicitly. We find that

$$V^n(t, x) \rightharpoonup V(t),$$

$$(V^n)^2(t, x) \rightharpoonup V^2(t) + \frac{16}{(t^2+4)^2}$$

in $L^2((0, 2) \times \mathbb{R}^+)$ and in the sense of distributions, respectively. Here

$$V(t) = \frac{2}{t+2}\left(\frac{x_{2k+1} - x_{2k}}{x_{2k+2} - x_{2k}}\right) + \frac{2}{t-2}\left(\frac{x_{2k+2} - x_{2k+1}}{x_{2k+2} - x_{2k}}\right) = \frac{2t}{t^2+4}.$$

That $\lim (V^n)^2 \neq (\lim V^n)^2$ means that the weak limit V is not a solution of (2.1).

The dynamic behavior of the Young measure $\nu_{t,x}$ associated with this sequence of solutions is interesting. For $0 \leqq t < 2$ the Young measure is a sum of two delta measures:

$$\nu_{t,x} = \frac{(2+t)^2}{2(4+t^2)}\delta_{2/(2+t)} + \frac{(2-t)^2}{2(4+t^2)}\delta_{2/(t-2)}. \tag{3.18}$$

As $t \uparrow 2$, the coefficient of the second delta measure tends to zero, while the point on which it is supported tends to $-\infty$. For dissipative solutions, the Young measure reduces to the single delta-measure

$$\nu_{t,x} = \delta_{2/(2+t)}$$

when $t > 2$. Thus, for $t > 2$ the sequence converges strongly to a solution. For conservative solutions, on the other hand, the Young measure is given by (3.18) for all $t > 0$.

3.4. Existence of conservative solutions

In this subsection we prove the existence of conservative solutions of problem (2.1). With different conditions on the initial data the existence of these solutions also follows from the method of characteristics (see Section 4).

Theorem 3.3. *Suppose that the initial function $V_0(x)$ is compactly supported and is in $BV(\mathbb{R}^+)$. Then there exists a conservative weak solution $\{u, V\}$ of problem* (2.1) *which satisfies*

(α) $\int_0^\infty V^2(t,x)\,dx = \int_0^\infty V_0^2(x)\,dx$ *for almost every* $t \in \mathbb{R}^+$,
(β) $(V^2)_t + (uV^2)_x = 0$ *in the sense of distributions,*
(γ) u *is Hölder continuous in* $[0,\infty)\times[0,\infty)$ *with exponent* $\alpha < \frac{1}{3}$,
(δ) $V \in L^p((0,T)\times\mathbb{R}^+)$ *for any* $p \in [2,3)$ *and any* $T > 0$.

Proof of Theorem 3.3. For simplicity we assume that the initial function $V_0(x)$ is supported in $[0,1]$ and is in $BV[0,1]$. We use the sequence $\{V_0^n(x)\}$ from the proof of Theorem 3.1 to approximate $V_0(x)$. Using the construction of conservative solutions described in Section 2, we obtain step-function solutions $\{V^n(t,x)\}$ to problem (2.1) with data $\{V_0^n(x)\}$. The solutions $\{V^n(t,x)\}$ are weakly admissible and preserve the L^2 norm:

$$\int_0^\infty |V^n(t,x)|^2\,dx = \int_0^\infty |V_0^n(x)|^2\,dx \quad \text{for } t \in \mathbb{R}^+, \text{ a.e.} \tag{3.19}$$

We show that $\{V^n\}_{n=1}^\infty$ is strongly compact in $L^2((0,T)\times\mathbb{R}^+)$. Condition ($\alpha$) on the L^2 conservation of V follows from (3.19) and this strong convergence. For

$$0 < \varepsilon < 2\left(\sup_{[0,1]} |V_0(x)|\right)^{-1},$$

we set

$$V^{n,\varepsilon}(t,x) = \begin{cases} 2/\varepsilon, & V^n(t,x) \geqq 2/\varepsilon, \\ V^n(t,x), & |V^n(t,x)| < 2/\varepsilon, \\ -2/\varepsilon, & V^n(t,x) \leqq -2/\varepsilon. \end{cases}$$

We note that we need to have $V^{n,\varepsilon}(0,x) = V^n(0,x) = V_0^n(x)$ for later estimates (i.e., V_0^n should not be chopped off with $2/\varepsilon$). As in the proof of (3.6) we obtain that

$$\int_0^T \int_0^\infty |V^{n,\varepsilon}(t,x) - V^n(t,x)|^2\,dx\,dt \leqq \varepsilon C \quad \forall n = 1,2,\ldots. \tag{3.20}$$

The estimates (3.7), (3.8) do not hold in the present case. Instead, we assert that the following estimates hold:

$$\int_0^T \int_0^\infty |V_x^{n,\varepsilon}|\,dx\,dt \leqq \frac{C}{\varepsilon^2}, \tag{3.21}$$

$$\int_0^T \int_0^\infty |V_t^{n,\varepsilon}|\,dx\,dt \leqq \frac{C}{\varepsilon^2}. \tag{3.22}$$

Proof of (3.21). We note that $V_x^{n,\varepsilon} = 0$ in continuous regions. The only contribution to (3.21) are jumps of $V^{n,\varepsilon}$ across strip boundaries. We try to follow the proof of (3.7). In the first case, (3.7a), we have no new difficulty: the estimate (3.9) is still valid. In the second case, (3.7b), we have to estimate the new wave emerging after $t_{n,i}^*$. For

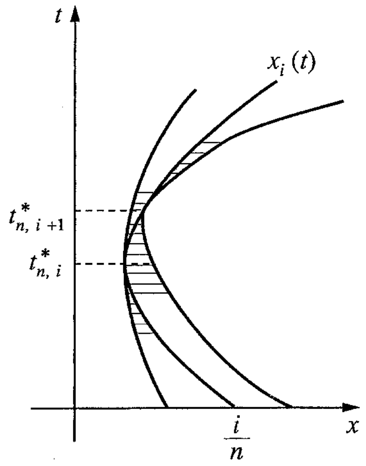

Fig. 3.7. For proof of (3.21).

$t \in [0, t^*_{n,i}]$, the estimate (3.12) is still valid. For $t \in [t^*_{n,i}, T]$, we note that our initial value V_i^n satisfies $V_i^n \leqq -2/(\varepsilon + T)$. Thus for $t > t^*_{n,i}$,

$$
\begin{aligned}
(3.23) \qquad |V^{n,\varepsilon}(t, x_i +) - V^{n,\varepsilon}(t, x_i -)| &\leqq \frac{2}{\varepsilon} \leqq \frac{\varepsilon + T}{\varepsilon}\left(\frac{2}{\varepsilon + T}\right) \\
&\leqq \frac{\varepsilon + T}{\varepsilon}\left(V_{i+1}^n + \frac{2}{\varepsilon + T}\right) \\
&\leqq \frac{\varepsilon + T}{\varepsilon}|V_{i+1}^n - V_i^n|.
\end{aligned}
$$

Thus (3.23) (or (3.12)) holds for all $t \in [0, T]$. In the last case, (3.7c), we find that estimate (3.15) is no longer valid for all $t \in [0, T]$. Without loss of generality, let us assume that $V_i^n \leqq V_{i+1}^n < 0$. Therefore we have the order $0 < t^*_{n,i} \leqq t^*_{n,i+1}$ for the blowup times. Now if $t^*_{n,i} \geqq T$, the estimate (3.15) is valid for all $t \in [0, T]$ because the new positive waves appear after $t^*_{n,i}$. Suppose that $0 < t^*_{n,i} < T$, and that $t^*_{n,i+1} \leqq t^*_{n,i} + \varepsilon$ (see Fig. 3.7). It follows that

$$
(3.24) \qquad -V_i^n \geqq \frac{2}{T}, \quad -V_{i+1}^n \geqq \frac{2}{T + \varepsilon}.
$$

If we denote $\mathrm{Jump}(x_i(t)) = |V^{n,\varepsilon}(t, x_i(t) +) - V^{n,\varepsilon}(t, x_i(t) -)|$, we find that

$$
(3.25) \quad \mathrm{Jump}(x_i(t)) \leqq \begin{cases}
(1 + T/\varepsilon)^2 |V_{i+1}^n - V_i^n|, & t \in (0, t^*_{n,i}), \\
4/\varepsilon, & t \in (t^*_{n,i}, t^*_{n,i+1}), \\
0, & t \in (t^*_{n,i+1}, t^*_{n,i} + \varepsilon), \\
4/\varepsilon, & t \in (t^*_{n,i} + \varepsilon, t^*_{n,i+1} + \varepsilon), \\
(1 + T/\varepsilon)^2 |V_{i+1}^n - V_i^n|, & t \in (t^*_{n,i+1} + \varepsilon, +\infty).
\end{cases}
$$

The last estimate of (3.25) holds because, for $t > t^*_{n,i+1} + \varepsilon > t^*_{n,i} + \varepsilon$,

$$\begin{aligned}\text{Jump}(x_i(t)) &= \frac{|V^n_{i+1} - V^n_i|}{|(1 + \frac{1}{2}V^n_{i+1}t)(1 + \frac{1}{2}V^n_i t)|} \\ &\leqq \frac{|V^n_{i+1} - V^n_i|}{\left(\frac{-V^n_{i+1}}{2}\right)\varepsilon \cdot \left(\frac{-V^n_i}{2}\right)\varepsilon} \\ &\leqq \frac{T(T+\varepsilon)}{\varepsilon^2}|V^n_{i+1} - V^n_i|,\end{aligned}$$

where we have used (3.24). The critical estimate is

$$|t^*_{n,i+1} - t^*_{n,i}| \leqq \frac{2|V^n_{i+1} - V^n_i|}{|(-V^n_{i+1})(-V^n_i)|} \leqq \frac{(T+\varepsilon)^2}{2}|V^n_{i+1} - V^n_i|. \tag{3.26}$$

This estimate implies that the time interval in which $\text{Jump}(x_i(t)) \geqq 4/\varepsilon$ is big is controlled by the initial jump. The remaining case occurs when $0 < t^*_{n,i} < T$ and $t^*_{n,i+1} > t^*_{n,i} + \varepsilon$. In this case, $\text{Jump}(x_i(t))$ can still be as big as $4/\varepsilon$, but we find that the initial jump is big too. As a matter of fact, we have from the assumption $t^*_{n,i+1} > t^*_{n,i} + \varepsilon$ that

$$\frac{2}{-V^n_{i+1}} - \frac{2}{-V^n_i} \geqq \varepsilon.$$

Hence,

$$0 \leqq -V^n_{i+1} \leqq \frac{2V^n_i}{\varepsilon V^n_i - 2}.$$

Therefore,

$$\begin{aligned}V^n_{i+1} - V^n_i &\geqq -\frac{2V^n_i}{\varepsilon V^n_i - 2} - V^n_i \\ &\geqq -\frac{4/T}{2\varepsilon/T - 2} - \frac{2}{T} = \frac{2\varepsilon}{T(T+\varepsilon)}\end{aligned}$$

where we have used $t^*_{n,i} < T$, so that $-V^n_i \geqq 2/T$. Thus

$$\begin{aligned}\text{Jump}(x_i(t)) &\leqq \frac{4}{\varepsilon} = \frac{2(T+\varepsilon)T}{\varepsilon^2} \cdot \frac{2\varepsilon}{T(T+\varepsilon)} \\ &\leqq \frac{2T(T+\varepsilon)}{\varepsilon^2}|V^n_{i+1} - V^n_i|, \quad t \in [0, T].\end{aligned} \tag{3.27}$$

So, for the case (3.7c), $\mathrm{Jump}(x_i(t))$ is given by one of the three estimates: (3.15) if $t^*_{n,i} \geqq T$, (3.25) if $t^*_{n,i} < T$ and $t^*_{n,i+1} \leqq t^*_{n,i} + \varepsilon$, and (3.27) if $t^*_{n,i} < T$ and $t^*_{n,i+1} > t^*_{n,i} + \varepsilon$. Thus

$$\int_0^T \int_0^\infty |V_x^{n,\varepsilon}|\,dx\,dt = \int_0^T \int_0^\infty \sum_{i=1}^n \mathrm{Jump}(x_i(t))\delta(x - x_i(t)) \frac{dx\,dt}{\sqrt{1 + \left(\dfrac{dx_i(t)}{dt}\right)^2}}$$

$$\leqq \sum_{i=1}^n \int_0^T \mathrm{Jump}(x_i(t))\,dt$$

$$\leqq \sum_{i=1}^n \left(\frac{2(T+\varepsilon)^2}{\varepsilon^2} |V_{i+1}^n - V_i^n| T + \frac{4}{\varepsilon} \cdot 2 \cdot \frac{(T+\varepsilon)^2}{2} |V_{i+1}^n - V_i^n|\right)$$

$$\leqq \frac{6T(T+\varepsilon)^2}{\varepsilon^2} \| V_0 \|_{BV}$$

where we define $\delta(x - x_i(t))$ by

$$\langle \varphi(t, x), \delta(x - x_i(t)) \rangle = \int_0^\infty \varphi(t, x_i(t)) \sqrt{1 + \left(\frac{d}{dt} x_i(t)\right)^2}\, dt \quad \forall \varphi \in C_c^\infty.$$

Proof of (3.22). $V_t^{n,\varepsilon}$ can be split into two parts. $V_t^{n,\varepsilon} = V_{1,t}^{n,\varepsilon} + V_{2,t}^{n,\varepsilon}$. The first part is the simple jump across $x_i(t)$ and takes the form

$$V_{1,t}^{n,\varepsilon} = \sum_{i=1}^n \pm \mathrm{Jump}(x_i(t))\delta(x - x_i(t)) \frac{x_i'(t)}{\sqrt{1 + (x_i'(t))^2}}. \tag{3.28}$$

The second part occurs in each strip and satisfies

$$|V_{2,t}^{n,\varepsilon}| \leqq \frac{2}{\varepsilon^2}. \tag{3.29}$$

The estimate (3.22) follows from (3.28) and (3.29).

Now that the estimates (3.20), (3.21), and (3.22) are established, the rest of the existence proof is similar to the proof of Theorem 3.1, and we omit the details.

Finally, we note that the conservative piecewise linear solutions satisfy

$$((V^n)^2)_t + (u^n (V^n)^2)_x = 0,$$

in the sense of distributions [HS]. Passing to the limit in a strongly convergent subsequence implies that the solution also satisfies this conservation law.

This completes the proof of Theorem 3.3. $\square$

4. Method of Characteristics

The main purpose of this section is to show that there is a close connection between conservative weak solutions and the method of characteristics. We begin with a heuristic discussion.

For smooth solutions, equation (1.1) can be solved exactly by using the method of characteristics [HS]. An implicit solution of (1.1), with initial data

$$u(0, x) = F(x),$$

is given by

$$u = U(t, \xi) := F(\xi) + tG(\xi),$$

$$x = X(t, \xi) := \xi + tF(\xi) + \tfrac{1}{2}t^2 G(\xi).$$

Here G is a function such that

$$G'(\xi) = \tfrac{1}{2} f^2(\xi), \quad f(\xi) = F'(\xi),$$

and $'$ denotes the derivative with respect to ξ.

The derivative X_ξ is given by

$$X_\xi = Y^2, \quad Y = 1 + \tfrac{1}{2}tf.$$

It follows that

$$V = u_x = \frac{U_\xi}{X_\xi} = \frac{f}{Y}$$

blows up when $X_\xi = 0$. However, even after the blowup time, the transformation from characteristic to spatial coordinates is typically one-to-one since X_ξ is non-negative. The only exception is when $\{\xi : f(\xi) = c\}$ has non-zero measure, in which case an interval of characteristics converge at the same point. (This is exactly what happens for the piecewise linear solutions used previously.)

Geometrically, the surface $\xi = Z(t, x)$ formed by the characteristics becomes vertical when the derivative of u blows up, but the surface never folds over (see Fig. 4.1). We can therefore use the method of characteristics to construct global single-valued solutions of (1.1). This again contrasts with the behavior of characteristics for conservation laws such as the Burgers equation (1.3).

To make these arguments rigorous, we introduce a solution u^ε by the regularized method of characteristics. We assume that $f(x) = F'(x)$ is continuous with compact support. Given any $\varepsilon > 0$, we define

$$U^\varepsilon(t, \xi) = \frac{F(\xi) + tG(\xi)}{1 + \varepsilon^2},$$

$$X^\varepsilon(t, \xi) = \frac{(1 + \varepsilon^2)\xi + tF(\xi) + \tfrac{1}{2}t^2 G(\xi)}{1 + \varepsilon^2}, \tag{4.1}$$

$$G(\xi) = \tfrac{1}{2} \int_{-\infty}^{\xi} f^2(z)\, dz, \quad Y(t, \xi) = 1 + \tfrac{1}{2}tf(\xi).$$

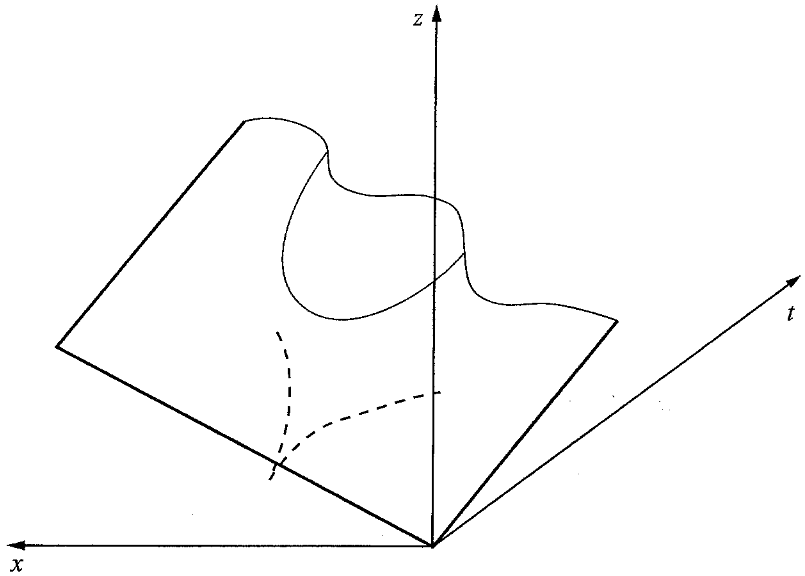

Fig. 4.1. A typical characteristic surface for conservative solutions.

It follows that

$$X_\xi^\varepsilon = \frac{Y^2 + \varepsilon^2}{1 + \varepsilon^2} \geqq \frac{\varepsilon^2}{1 + \varepsilon^2} > 0. \tag{4.2}$$

The implicit function theorem implies that the change of variables $(t, \xi) \mapsto (t, x)$ is a C^1-diffeomorphism of $\mathbb{R}^2$. We denote the inverse map by

$$\xi = Z^\varepsilon(t, x). \tag{4.3}$$

We let

$$u^\varepsilon(t, x) = U^\varepsilon(t, Z^\varepsilon(t, x)) \in C^1(\mathbb{R}^2). \tag{4.4}$$

From (4.1)–(4.4), the derivative $V^\varepsilon = u_x^\varepsilon$ is given by

$$V^\varepsilon(t, x) = \frac{f(Z^\varepsilon(t, x))\, Y(t, Z^\varepsilon(t, x))}{Y^2(t, Z^\varepsilon(t, x)) + \varepsilon^2}. \tag{4.5}$$

We prove that $\{u^\varepsilon, V^\varepsilon\} \to \{u, V\}$ as $\varepsilon \to 0$ and that the limit $\{u, V\}$ is a conservative weak solution of (1.1).

Before stating and proving a general theorem, we illustrate how the regularized method of characteristics picks out a conservative solution for the basic step-function initial data

$$f(x) = \begin{cases} -1, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Using (4.1), and eliminating ξ in terms of x (assuming that $F(x) = 0$ for $x < 0$), we find that the solution by the regularized method of characteristics for this initial function is given by

$$V^\varepsilon(t, x) = \begin{cases} \dfrac{2(t-2)}{(t-2)^2 + 4\varepsilon^2}, & 0 < x < \xi^\varepsilon(t), \\ 0, & \text{otherwise}, \end{cases}$$

where

$$\xi^\varepsilon(t) = \frac{(t-2)^2 + 4\varepsilon^2}{4(1+\varepsilon^2)}.$$

After the blowup time, $t^* = 2$, this solution approaches the *conservative* step-function solution

$$V(t, x) = \begin{cases} \dfrac{2}{t-2}, & 0 < x < \frac{1}{4}(t-2)^2, \\ 0 & \text{otherwise}, \end{cases}$$

shown in Figure 2.4, rather than the dissipative one shown in Figure 2.2. More precisely, it is straightforward to check that

$$\int_K |V^\varepsilon - V|^2 \, dt \, dx = O(\varepsilon)$$

for any compact set K, so that $V^\varepsilon \to V$ in $L^2_{\text{loc}}(\mathbb{R}^2)$.

Theorem 4.1. *Suppose that* $f(x) = F'(x) \in C_c(\mathbb{R})$. *There exists a global conservative solution of* (2.1) *with initial data* $u(0, x) = F(x)$ *and* $V(0, x) = f(x)$. *This solution is the limit in* $C_{\text{loc}} \otimes L^2_{\text{loc}}(\mathbb{R}^2)$ *of solutions* $\{u^\varepsilon, V^\varepsilon\}$ *by the regularized method of characteristics defined in* (4.1)–(4.5).

Proof. We show that $V^\varepsilon(t, x)$ has a finite pointwise limit $V(t, x)$ almost everywhere. The limit exists on a set Ω consisting of those points (t, x) where $Y(t, Z^\varepsilon(t, x))$ is bounded uniformly away from zero as $\varepsilon \to 0$.

Throughout the proof, we restrict t to an arbitrary finite interval $[-T, T]$.

Given any $0 < \varepsilon < 1$ and $0 < \eta < 1$, we define

$$\Omega^\varepsilon_\eta = \{(t, x) \in [-T, T] \times \mathbb{R} : |Y(t, Z^\varepsilon(t, x))| \geqq \eta\}. \tag{4.6}$$

The convergence of the regularized characteristic variables Z^ε follows from the following nesting property of the Ω^ε_η.

Lemma 4.1. *For any* $\eta > 0$, *there exists* $\varepsilon_0(\eta) > 0$ *such that*

$$\Omega^\varepsilon_{3\eta/2} \subset \Omega^{\varepsilon_0(\eta)}_\eta \subset \Omega^\varepsilon_{\eta/2} \quad \forall 0 < \varepsilon \leqq \varepsilon_0. \tag{4.7}$$

Proof. From (4.1) and the assumption that f is continuous with compact support, there is a modulus of continuity $\rho: \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|\xi - \zeta| \leqq \rho(\eta) \;\Rightarrow\; |Y(t, \xi) - Y(t, \zeta)| < n/2. \tag{4.8}$$

We define M and ε_0 by

$$M = \sup_{t, \xi} |tF(\xi) + \tfrac{1}{2}t^2 G(\xi)|, \quad \varepsilon_0^2(\eta) = \rho(\eta)\eta^2/4M. \tag{4.9}$$

First we prove that $\Omega_\eta^{\varepsilon_0} \subset \Omega_{\eta/2}^{\varepsilon}$. Differentiating $x = X^\varepsilon(t, Z^\varepsilon(t, x))$ in (4.1) with respect to ε^2, with (t, x) fixed shows that

$$\frac{\partial Z^\varepsilon}{\partial \varepsilon^2} = \frac{1}{1 + \varepsilon^2} \, \frac{tF(Z^\varepsilon) + \frac{1}{2}t^2 G(Z^\varepsilon)}{Y^2(t, Z^\varepsilon) + \varepsilon^2}. \tag{4.10}$$

To explain the idea of the proof, assume that $|Y(t, Z^\varepsilon)| \geqq \eta/2$. Then estimating the right-hand side of (4.10) gives

$$\left| \frac{\partial Z^\varepsilon}{\partial \varepsilon^2} \right| \leqq \frac{4M}{\eta^2}.$$

Integrating this result and using (4.9) implies that $|Z^{\varepsilon_0} - Z^\varepsilon| \leqq \rho(\eta)$. It follows from (4.8) that $|Y(t, Z^\varepsilon)|$ differs from $|Y(t, Z^{\varepsilon_0})|$ by at most $\eta/2$, and is therefore greater than $\eta/2$, in consonance with our initial assumption.

We give a detailed proof by contradiction. Suppose that $(t, x) \in \Omega_\eta^{\varepsilon_0(\eta)}$. Let

$$\begin{aligned} \varepsilon_* &= \inf\{\varepsilon_1 > 0 \mid |Y(t, Z^\varepsilon(t, x))| \geqq \eta/2 \quad \forall \varepsilon \in (\varepsilon_1, \varepsilon_0(\eta))\} \\ &= \inf\{\varepsilon_1 > 0 \mid (t, x) \in \Omega_{\eta/2}^{\varepsilon} \quad \forall \varepsilon \in (\varepsilon_1, \varepsilon_0(\eta))\}. \end{aligned}$$

We want to show that $\varepsilon_* = 0$. Suppose that $\varepsilon_* > 0$. Integrating (4.10) and then using (4.9) and the fact that $Y(t, Z^\varepsilon) \geqq \eta/2$ for $\varepsilon \geqq \varepsilon_*$ implies that

$$\begin{aligned} |Z^{\varepsilon_0} - Z^{\varepsilon_*}| &= \left| \int_{\varepsilon_*^2}^{\varepsilon_0^2} \frac{1}{1 + \varepsilon^2} \, \frac{tF(Z^\varepsilon) + \frac{1}{2}t^2 G(Z^\varepsilon)}{Y^2(t, Z^\varepsilon) + \varepsilon^2} \, d\varepsilon^2 \right| \\ &\leqq \frac{4M(\varepsilon_0^2 - \varepsilon_*^2)}{\eta^2} < \rho(\eta). \end{aligned}$$

It follows from this inequality and (4.8) that

$$|Y(t, Z^{\varepsilon_*})| \geqq |Y(t, Z^{\varepsilon_0})| - |Y(t, Z^{\varepsilon_0}) - Y(t, Z^{\varepsilon_*})| > \eta/2.$$

Since $Y(t, Z^\varepsilon)$ is a continuous function of ε, this inequality implies that $|Y(t, Z^\varepsilon)| \geqq \eta/2$ for some interval $\varepsilon_1 < \varepsilon < \varepsilon_*$, contradicting the definition of ε_*. Therefore, $\varepsilon_* = 0$ and $\Omega_\eta^{\varepsilon_0} \subset \Omega_{\eta/2}^{\varepsilon}$ for all $0 < \varepsilon < \varepsilon_0$.

An almost identical argument shows that if $|Y(t, Z^\varepsilon(t, x))| \geqq 3\eta/2$ for some $\varepsilon \in (0, \varepsilon_0)$, then $|Y(t, Z^{\varepsilon_0(\eta)}(t, x))| \geqq \eta$. We do not write out the details. It follows that

$$\Omega_{3\eta/2}^{\varepsilon} \subset \Omega_\eta^{\varepsilon_0(\eta)}.$$

This completes the proof of Lemma 4.1. $\square$

We abbreviate

$$\Omega_\eta = \Omega_\eta^{\varepsilon_0(\eta)}$$

and define the "good" set

$$\Omega = \bigcup_\eta \Omega_\eta.$$

The next Lemma shows that $m(\Omega^c) = 0$, where m is the Lebesgue measure and c denotes the complement in $[-T, T] \times \mathbb{R}$.

Lemma 4.2. *For $0 < \eta \leqq \frac{1}{2}$, there is a constant C independent of η such that*

$$m(\Omega_\eta^c) \leqq C\eta^3.$$

Proof. We consider the change of variables $(t, x) \mapsto (y, \xi)$ defined by

$$y = Y(t, Z^\varepsilon(t, x)), \quad \xi = Z^\varepsilon(t, x). \tag{4.11}$$

By (4.1)–(4.3), this transformation is one-to-one on any set where $f(\xi) \neq 0$, and its Jacobian is given by

$$dt\, dx = \frac{2}{1 + \varepsilon^2} \frac{y^2 + \varepsilon^2}{|f(\xi)|} dy\, d\xi. \tag{4.12}$$

The definition of Ω_η^ε implies that $|f(\xi)|$ is bounded uniformly away from zero on $\Omega_\eta^{\varepsilon c}$ when $\eta \leqq \frac{1}{2}$. Using (4.11)–(4.12), with $\varepsilon = \varepsilon_0(\eta)$, and (4.9), we compute that

$$\begin{aligned} m(\Omega_\eta^c) &= \int_{\Omega_\eta^c} dt\, dx \\ &= \int_{\Omega_\eta^c} \frac{2}{1 + \varepsilon_0^2} \frac{y^2 + \varepsilon_0^2}{|f(\xi)|} dy\, d\xi \\ &\leqq C \int_{|y| \leqq \eta} (y^2 + \varepsilon_0^2)\, dy \leqq C(\eta^3 + \eta\varepsilon_0^2) \leqq C\eta^3. \quad \square \end{aligned}$$

Next we prove that the characteristic variables Z^ε have a pointwise limit on Ω.

Lemma 4.3. *For any $(t, x) \in \Omega$, there is a pointwise limit, $Z^\varepsilon(t, x) \to Z(t, x)$ as $\varepsilon \to 0$. Moreover, $|Y(t, Z(t, x))| \geqq \eta/2$ on Ω_η and $Y(t, Z(t, x)) \neq 0$ on Ω.*

Proof. If $(t, x) \in \Omega$, then $(t, x) \in \Omega_\eta$ for some $\eta > 0$. Lemma 4.1 implies that

$$|Y(t, Z^\varepsilon(t, x))| \geqq \eta/2 \tag{4.13}$$

for all $\varepsilon \leqq \varepsilon_0(\eta)$. Therefore, from (4.9), we have

$$\frac{1}{1+\varepsilon^2}\,\frac{tF(Z^\varepsilon)+\frac{1}{2}t^2G(Z^\varepsilon)}{Y^2(t,Z^\varepsilon)+\varepsilon^2} \leqq \frac{4M}{\eta^2} = \frac{\rho}{\varepsilon_0^2}.$$

Integrating (4.10) with respect to ε^2, and using this inequality to estimate the result, gives

$$|Z^\varepsilon - Z^\delta| \leqq \frac{\rho}{\varepsilon_0^2}(\varepsilon^2-\delta^2)$$

for all $\varepsilon_0 \geqq \varepsilon \geqq \delta > 0$. It follows that Z^ε is a uniform Cauchy sequence on Ω_η and therefore has a pointwise limit Z.

Taking the limit of (4.13) as $\varepsilon \to 0$, and using the continuity of Y, implies that $|Y(t, Z(t,x))| \geqq \eta/2 \neq 0$. This completes the proof of the lemma. □

From Lemma 4.3, equation (4.5), and the continuity of f, we can define the limiting function

$$V(t,x) = \begin{cases} \lim\limits_{\varepsilon\to 0} V^\varepsilon(t,x) = \dfrac{f(Z(t,x))}{Y(t,Z(t,x))} & \text{if } (t,x)\in\Omega, \\ 0, & \text{if } (t,x)\in\Omega^c. \end{cases} \tag{4.14}$$

We now prove the main convergence result.

Lemma 4.4. *$V^\varepsilon \to V$ strongly in $L^p([-T,T]\times\mathbb{R})$ for any $1 \leqq p < 3$.*

Proof. For any $\eta > 0$ and $\varepsilon < \varepsilon_0(\eta)$, we have

$$\|V^\varepsilon - V\|^p = \int_{\Omega_\eta} |V^\varepsilon - V|^p\,dt\,dx + \int_{\Omega_\eta^c} |V^\varepsilon - V|^p\,dt\,dx, \tag{4.15}$$

where $\|\cdot\|$ is the $L^p([-T,T]\times\mathbb{R})$ norm. From (4.5) and Lemma 4.3,

$$\sup_{(t,x)\in\Omega_\eta} |V^\varepsilon - V| \leqq \frac{C}{\eta},$$

so it follows from (4.14) and the Lebesgue dominated convergence theorem that

$$\lim_{\varepsilon\to 0}\int_{\Omega_\eta} |V^\varepsilon - V|^p\,dt\,dx = 0. \tag{4.16}$$

To estimate the second term on the right-hand side of (4.15), we use

$$\left(\int_{\Omega_\eta^c} |V^\varepsilon - V|^p\,dt\,dx\right)^{1/p} \leqq \left(\int_{\Omega_\eta^c} |V^\varepsilon|^p\,dt\,dx\right)^{1/p} + \left(\int_{\Omega_\eta^c} |V|^p\,dt\,dx\right)^{1/p}. \tag{4.17}$$

By Lemma 4.1, $\Omega_\eta^c \subset \Omega_{3\eta/2}^{\varepsilon c}$. Consequently, using (4.11), (4.12) to change integration variables from (t, x) to (y, ξ) and then using (4.5), we obtain

$$
\begin{aligned}
(4.18) \qquad \int_{\Omega_\eta^c} |V^\varepsilon|^p \, dt\, dx &\leqq \int_{\Omega_{3\eta/2}^{\varepsilon c}} |V^\varepsilon|^p \, dt\, dx \\
&= \int_{\Omega_{3\eta/2}^{\varepsilon c}} |V^\varepsilon|^p \frac{2}{1+\varepsilon^2} \frac{y^2+\varepsilon^2}{|f(\xi)|} \, dy\, d\xi \\
&= \frac{2}{1+\varepsilon^2} \int_{\Omega_{3\eta/2}^{\varepsilon c}} |f(\xi)|^{p-1} \frac{|y|^p}{(y^2+\varepsilon^2)^{p-1}} \, dy\, d\xi \\
&\leqq C \int |f(\xi)|^{p-1} \, d\xi \cdot \int_{|y| \leqq 3\eta/2} |y|^{2-p} \, dy \\
&\leqq C\eta^{3-p}.
\end{aligned}
$$

Using Fatou's lemma and the fact that $V^\varepsilon \to V$ a.e., we also have

$$
(4.19) \qquad \int_{\Omega_\eta^c} |V|^p \, dt\, dx = \int_{\Omega_\eta^c} \liminf_{\varepsilon \to 0} |V^\varepsilon|^p \, dt\, dx \leqq \liminf_{\varepsilon \to 0} \int_{\Omega_\eta^c} |V^\varepsilon|^p \, dt\, dx \leqq C\eta^{3-p}.
$$

Combining (4.15)–(4.19) implies that

$$
\limsup_{\varepsilon \to 0} \| V^\varepsilon - V \| \leqq C\eta^{(3-p)/p},
$$

and taking the limit as $\eta \to 0$ proves the result. □

By the Sobolev embedding theorem, it further follows that $u^\varepsilon \to u$ uniformly, with $V = u_x$. This establishes the convergence of $\{u^\varepsilon, V^\varepsilon\} \to \{u, V\}$ in $C \otimes L^2$ $([-T, T] \times \mathbb{R})$ and shows that (d) of Definition 2.1 is satisfied.

To show that the limit is a weak solution, we need to prove that $\{u, V\}$ satisfies (e) of Definition 2.1. From (4.1)–(4.5), we compute that

$$
(4.20) \qquad V_t^\varepsilon + u^\varepsilon V_x^\varepsilon + \frac{1}{2}(V^\varepsilon)^2 = \frac{1}{2}\varepsilon^2 \frac{f^2}{(Y^2+\varepsilon^2)^2},
$$

where the right-hand side is evaluated at $\xi = Z^\varepsilon(t, x)$. If $f \in C^1$, this equation is valid in the classical sense; otherwise it holds in the sense of distributions.

Using (4.11), (4.12), we find that the $L^1(\mathbb{R}^2)$ norm of the right-hand side of (4.20) is given by

$$
\int_{\mathbb{R}^2} \frac{1}{2}\varepsilon^2 \frac{f^2}{(Y^2+\varepsilon^2)^2} \, dt\, dx = \frac{\varepsilon^2}{1+\varepsilon^2} \int_{\mathbb{R}^2} \frac{|f(\xi)|}{y^2+\varepsilon^2} \, dy\, d\xi = \frac{\varepsilon\pi}{1+\varepsilon^2} \int_{\mathbb{R}} |f(\xi)| \, d\xi.
$$

The right-hand side therefore tends to zero in the sense of distributions as $\varepsilon \to 0$. It follows that $\{u, V\}$ satisfies (e) of Definition 2.1 and is a weak solution. (We omit the detailed verification of conditions (a)–(c), with $V_0 = f$.)

To show that this solution is conservative, we compute from (4.1)–(4.5) that the energy equation for the regularized solution is

$$(V^{\varepsilon 2})_t + (u^\varepsilon V^{\varepsilon 2})_x = \varepsilon^2 \frac{Yf^3}{(Y^2+\varepsilon^2)^3}.$$

We shall prove that the right-hand side tends to zero in the sense of distributions as $\varepsilon \to 0$. It follows that the limiting solution satisfies the energy equation exactly.

The right-hand side of the regularized energy equation does not tend to zero in L^1_{loc}, but some cancellation occurs and it does tend to zero in the sense of distributions. We consider

$$I^\varepsilon[\varphi] = \varepsilon^2 \int \frac{\varphi Y f^3}{(Y^2+\varepsilon^2)^3}\, dt\, dx,$$

where $\varphi(t, x)$ is an arbitrary test function. Using (4.2) to change variables from x to $\xi = Z^\varepsilon(t, x)$ gives

$$I^\varepsilon[\varphi] = \frac{\varepsilon^2}{1+\varepsilon^2} \int_\Gamma \frac{\varphi Y f^3}{(Y^{12}+\varepsilon^2)^2}\, dt\, d\xi.$$

Here, we restrict the region of integration to $\Gamma = \mathbb{R} \times A$ where $A = \{\xi : f(\xi) \neq 0\}$. Using Fubini's theorem and integrating by parts with respect to t gives

$$I^\varepsilon[\varphi] = -\frac{\varepsilon^2}{1+\varepsilon^2} \int_\Gamma [\varphi_t(t, X^\varepsilon(t,\xi)) + X_t^\varepsilon(t,\xi)\varphi_x(t, X^\varepsilon(t,\xi))]$$

$$\times \left\{ \int_{-\infty}^{t} \frac{Y(t',\xi) f^3(\xi)}{(Y^2(t',\xi)+\varepsilon^2)^2}\, dt' \right\} dt\, d\xi.$$

Using (4.1), we have that $X_t^\varepsilon = U^\varepsilon$ and

$$\int_{-\infty}^{t} \frac{Y(t',\xi) f(\xi)}{(Y^2(t',\xi)+\varepsilon^2)^2}\, dt' = -\frac{1}{Y^2(t,\xi)+\varepsilon^2}.$$

Using these in the expression for $I^\varepsilon[\varphi]$ gives

$$I^\varepsilon[\varphi] = \frac{\varepsilon^2}{1+\varepsilon^2} \int_\Gamma \frac{(\varphi_t + U^\varepsilon \varphi_x) f^2}{Y^2+\varepsilon^2}\, dt\, d\xi.$$

Since U^ε is uniformly bounded on compact sets,

$$|I^\varepsilon[\varphi]| \leqq C\varepsilon^2 \int_\Gamma \frac{f^2}{Y^2+\varepsilon^2}\, dt\, d\xi \leqq C\varepsilon^2 \int_\Gamma \frac{|f|}{y^2+\varepsilon^2}\, dy\, d\xi \leqq C\varepsilon \int_{\mathbb{R}} |f(\xi)|\, d\xi,$$

where we use (4.1) to change integration variables from t to $y = Y(t, \xi)$ and where C is a constant independent of ε. It follows that $I^\varepsilon[\varphi] \to 0$ as $\varepsilon \to 0$.

This completes the proof of Theorem 4.1. $\square$

The proof gives an explicit expression for the conservative solution, namely, (4.14). This expression is the same as the one obtained by the method of characteristics for smooth solutions, but with the characteristic variable $Z(t, x)$ defined a.e. as a pointwise limit of regularized characteristic variables. Characteristics are not defined uniquely from $dx/dt = u(t, x)$, since u is not Lipschitz continuous in x. It is interesting to compare this construction of characteristics for weak solutions of (1.1) with the generalized characteristics introduced by DAFERMOS [D2] for conservation laws (where u is discontinuous).

Finally, we mention a problem concerning lack of uniqueness. There are many conservative solutions which take on initial values $u(0, x) = V(0, x) = 0$. The one constructed by Theorem 4.1 is the zero solution. However, another solution is the conservative step-function solution

$$V(t, x) = \begin{cases} 2/t, & 0 < x - \xi < a^2t^2/4, \\ 0, & \text{otherwise.} \end{cases}$$

More generally, we can place arbitrary amounts of energy at a finite number of points, and V still takes on zero initial values in the sense of Definition 2.1. Thus, in addition to initial values for V it is also necessary to give compatible initial values for V^2 (in the space of bounded measures, for example). This difficulty does not arise in the dissipative case, where the only admissible piecewise linear continuation of $V(0, x) = 0$ is $V(t, x) = 0$ for all $t > 0$.

5. Large-Time Asymptotics

In this section, we prove that the dissipative solutions whose existence we have established in Theorems 3.1 and 3.2 all have the same asymptotic form as $t \to +\infty$. The asymptotic state is completely determined by a single invariant of the solution, namely, the energy of the positive part of V. We use the natural space $L^2(\mathbb{R}_x^+)$ for solutions $V(t, x)$. The $L^p(\mathbb{R}_x^+)$ spaces with $p \neq 2$ are not suitable for $V(t, x)$, because the L^p norm of $V(t, x)$ blows up in time if $p \neq 2$.

Theorem 5.1. *As time approaches infinity, all solutions $V(t, x)$ of the problem* (2.1) *established in Theorem 3.2 converge in $L^2(\mathbb{R}_x^+)$ to $U(t, x)$ given by*

$$U(t, x) = \begin{cases} 2/t, & 0 < x < \frac{1}{4}E(V^+(0, \cdot))t^2, \\ 0, & \textit{otherwise} \end{cases} \tag{5.1}$$

where

$$E(V^+(0, \cdot)) = \int_0^\infty (V^+(0, x))^2\, dx,$$

$$V^+(t, x) = \max\{V(t, x), 0\}.$$

We remark that this conclusion was conjectured in [HS]. The function $U(t, x)$ satisfies the first two equations in (2.1), but it takes on very singular initial data.

The asymptotics of $u(t, x)$ follows immediately from Theorem 5.1:

$$u(t, +\infty) \sim \int_0^\infty U(t, x)\,dx = \frac{t}{2} E(V^+(0, \cdot)).$$

It follows that the constant state $u = 0$ is nonlinearly unstable. If $u = 0$ is perturbed initially by a compactly supported pulse $u_0(x)$ with $E(u_{0x}^+(x)) > 0$, then asymptotically $u(t, +\infty)$ grows linearly in time.

Sketch of the Proof of Theorem 5.1. We need to prove that

$$\lim_{t\to\infty} \int_0^\infty |V(t, x) - U(t, x)|^2\,dx = 0. \tag{5.2}$$

We use the sequence of approximate solutions $\{V^n\}$ from Theorem 3.2, a sequence which converges to V in $L^2((0, T)\times \mathbb{R}^+)$ for all $T > 0$. Consider the simplest case:
(i) $V(0, x) \leqq 0$.
It is clear that any initial value $V^n(0, x)$ less than $-2/t$, where $t > 0$, blows up before time t. From the construction of V^n, it follows that the energy at $t > 0$ comes only from initial values larger than $-2/t$. Thus

$$\int_0^\infty |V^n(t, x)|^2\,dx \leqq \frac{4}{t^2}.$$

Passing to the limit $n \to \infty$, we find that

$$\int_0^\infty |V(t, x)|^2\,dx \leqq \frac{4}{t^2}. \tag{5.3}$$

But $E(V^+(0, x)) = 0$; thus $U \equiv 0$ in this case. So (5.2) is established for this case with the explicit rate given in (5.3).

The next simplest case is
(ii) $V(0, x) \geqq 0$.
The solution $V(t, x)$ is bounded on the entire domain $\mathbb{R}^+ \times \mathbb{R}^+$, so $u(t, x)$ is Lipschitz continuous with respect to x. Therefore the characteristics for (2.1) can be uniquely defined everywhere:

$$\frac{dX}{ds} = u(s, X), \quad 0 < s < \infty,$$
$$X(t) = x, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^+. \tag{5.4}$$

Let x_0 be the intersection with the x-axis of the characteristics $X(s; t, x)$ starting at (t, x):

$$x_0 = x_0(t, x) \equiv X(0; t, x).$$

Then

$$V(t,x)=\frac{2V(x_0(t,x),0)}{2+tV(x_0(t,x),0)},$$

$$\int_0^\infty V^2(t,x)\,dx=\int_0^\infty V^2(0,x)\,dx.$$

Integrating the first equation in (2.1), we further find

$$\int_0^\infty V(t,x)\,dx=\int_0^\infty V(0,x)\,dx+\tfrac{1}{2}tE(V(0,x)).$$

The value of $u(t,x)$ is increasing with respect to x, and equal to $\int_0^\infty V(t,x)\,dx$ when x is larger than the characteristic $X(t;0,1)$ starting at $t=0$, $x=1$, which we find to be

$$X(t;0,1)=1+t\int_0^\infty V(0,x)\,dx+\tfrac{1}{4}t^2E(V(0,\cdot)).$$

So the difference between V and U can be split into two parts. The first is in the strip $S_1:\frac{1}{4}t^2E(V(0,\cdot))<x<X(t;0,1)$, and the second is in the region $S_2:0<x<\frac{1}{4}t^2E(V(0,\cdot))$. Inside the first strip S_1, we have $U=0$ and

$$0\leqq V=\frac{2}{t}-\frac{4}{t[2+tV(0,x_0(t,x))]}\leqq\frac{2}{t}.$$

Thus

$$\int_{S_1}|V-U|^2\,dx=O(1/t).$$

Inside the second region S_2, we have

$$V-U=-\frac{4}{t[2+tV(0,x_0(t,x))]}.$$

We see that this difference decays like $-2/t$ at points where $V(0,x_0(t,x))=0$. A simple bound such as $|V-U|\leqq 2/t$ is not enough, because the width of the domain of integration grows like $t^2E(V(0,\cdot))/4$. To overcome this difficulty, we note that intervals $(a_1,a_2)\subset(0,1)$ in which $V(0,x_0(t,x))=0$ at $t=0$ do not expand (or shrink) with time. That is, the two characteristics $X(t;0,a_1)$ and $X(t;0,a_2)$ have a constant distance $X(t;0,a_1)-X(t;0,a_2)=a_1-a_2$ for all time $t>0$ if $V(0,x)=0$ in (a_1,a_2). So the simple bound $|V-U|\leqq 2/t$ suffices in regions where $V(t,x)=0$. We further note that the distance between any two characterstics $X(t;0,a_i)$, $i=1,2$ expands at a rate at most $O(t^2)$; and $|V-U|\leqq c/t^2$ hold in regions where $V(t,x)\geqq c$. From this analysis and a coordinate transform $x \to x_0$, we find that

$$\int_0^{\frac{1}{4}t^2E(V(0,\cdot))} |V-U|^2\,dx \leqq 16 \int_0^{X(t;0,1)} \frac{1}{t^2[2+tV(0,x_0(t,x))]^2}\,dx$$

$$\leqq \int_{S_0} \frac{dx_0}{(2+tV(0,x_0))^2} + O(1/t^2)$$

where $S_0 = \{x_0 \in (0,1) \mid V(0,x_0) \neq 0\}$. Splitting the integral over S_0 into two parts $S_3 = \{x_0 \in S_0 \mid V(0,x_0) < 1/\sqrt{t}\}$ and $S_4 = \{x_0 \in S_0 \mid V(0,x_0) \geqq 1/\sqrt{t}\}$, we find that

$$\int_0^{\frac{1}{4}t^2E(V(0,\cdot))} |V-U|^2\,dx \leqq \frac{c}{4}|S_3 \cap S_0| + \frac{1}{(2+\sqrt{t})^2} + O(1/t^2) = o(1).$$

So we have established (5.2) in the second case $V(0,x) \geqq 0$.

For the general case, we split V into two parts

$$V(t,x) = V^+(t,x) - V^-(t,x)$$

where both V^+ and V^- are nonnegative. As in case (i), we obtain

$$\int_0^\infty |V^-(t,x)|^2\,dx \leqq \frac{4}{t^2}.$$

So we only need to find the asymptotic behavior for V^+. By using V^n and strong L^2 convergence, we find that

$$\int_0^\infty |V^+(t,x)|^2\,dx = \int_0^\infty |V^+(0,x)|^2\,dx \equiv E(V^+(0,x)).$$

So

$$\int_0^\infty V^2\,dx = \int_0^\infty |V^+(0,x)|^2\,dx + O(1/t^2).$$

The characteristic for $V(t,x)$ starting at $t=0$, $x=1$ is

$$X(t;0,1) = 1 + t\int_0^\infty V(0,x)\,dx + \frac{t^2}{4}E(V^+) + O(\log t).$$

The integral $\int_0^\infty V(0,x)\,dx$ can be assumed to be nonnegative without loss of generality, because $\int_0^\infty V^-(t,x)\,dx$ vanishes as $t \to \infty$ and because we can start the problem (2.1) at any later time $t_0 > 0$ without changing the asymptotic behavior. Therefore we can assume that $X(t;0,1) > t^2E(V^+)/4$ for sufficiently large t. So the difference of $V^+ - U$ in the strip $t^2E(V^+)/4 < x < X(t;0,1)$ tends to zero as in case (ii). We note that inside the strip $0 < x < t^2E(V^+)/4$, the portion of $\{x : V^-(t,x) \neq 0\}$ has a length which is decreasing with time; thus it is less than its initial length. Hence the length of $\{x : V^+(t, x) = 0\}$ is at most 1. Thus the argument in case (ii) still applies to give

$$\int_0^{t^2 E(V^+)/4} |V^+ - U|^2 \, dx = o(1),$$

and (5.2) is proved in the general case. The sketch of the proof of Theorem 5.1 is complete.

Appendix A. Proof of Lemma 3.1

Lemma 3.1. *For any* $f(x) \in W([0, 1])$, *there exists a sequence of step functions* $\{f^n(x)\}_{n=1}^{\infty} \subset L^2[0, 1]$ *with the properties*:

(1) $f^n(x) \to f(x)$ *in* $L^2([0, 1])$,

(2) $\|(f^n)_M\|_{BV[0,1]} \leqq \|f_M\|_{BV[0,1]}$ $\forall M = 1, 2, \ldots$ $\forall n = 1, 2, \ldots$.

Proof of Lemma 3.1. Let $f(x) \in W[0, 1]$ be given. Suppose $g(x)$ is a step function on $[0, 1]$ and can be written as

$$g(x) = g_i, \quad x \in (a_{i-1}, a_i), \; i = 1, 2, \ldots, k,$$

where $0 = a_0 < a_1 < \ldots < a_k = 1$ and g_i are constants. If on each interval (a_{i-1}, a_i), $i = 1, 2, \ldots, k$, there exists a point $x_i \in (a_{i-1}, a_i)$ such that

$$f(x_i) = g_i$$

or there are at least two points $y_i, z_i \in (a_{i-1}, a_i)$ such that

$$f(y_i) > g_i > f(z_i),$$

then we say that g is *twisted* with f. It follows that

$$\|g\|_{BV} \leqq \|f\|_{BV}$$

by the definition of BV.

Now for any $M = 1, 2, \ldots, f_M$ is in BV by the definition of $W([0, 1])$ in Section 3.2. There exists a sequence of step functions $\{f_M{}^n(x)\}_{n=1}^{\infty}$, each of which is twisted with f_M, and

$$\|f_M - f_M{}^n\|_{L^2(0,1)} \leqq \frac{1}{2^n},$$

$$\|f_M{}^n\|_{BV} \leqq \|f_M\|.$$

We assert that $f^n \equiv f_n{}^n$ satisfies all the conditions of the lemma. Indeed, f^n is a step function, is bounded from below by $-n$ and is in $L^2(0, 1)$. Furthermore,

$$\|f^n - f\|_{L^2} \leqq \|f_n{}^n - f_n\|_{L^2} + \|f_n - f\|_{L^2}$$

$$\leqq \frac{1}{2^n} + \|f_n - f\|_{L^2}.$$

Therefore (1) is satisfied. To verify (2), we note that

$$(f^n)_M = f_n^n \quad \text{if } M \geqq n,$$

so

$$\|(f^n)_M\|_{BV} \leqq \|f_n^n\|_{BV} \leqq \|f_n\|_{BV} \leqq \|f_M\|_{BV} \quad \text{if } M > \text{n}.$$

For $M < n$, it can be verified that $(f_n^n)_M$ is twisted with f_M, and thus

$$\|(f^n)_M\|_{BV} \leqq \|(f_n^n)_M\|_{BV} \leqq \|f_M\|_{BV}.$$

So (2) is verified. This completes the proof of Lemma 3.1.

Appendix B. Verification of Condition (c)

From equation (e) of Definition 2.1, we have

$$\text{(B.1)} \qquad \iint \psi'(t)\varphi(x)V\,dx\,dt = -\iint \psi\varphi' uV\,dx\,dt - \tfrac{1}{2}\iint \psi\varphi V^2\,dx\,dt$$

for all test functions $\psi \in C_0^1(\mathbb{R}^+)$ and $\varphi \in C_0^1(\mathbb{R}^+)$. For any $t_1, t_2 \in (0, T)$, we can choose a sequence $\{\psi_j\}_{j=1}^{\infty} \subset C_0^1(\mathbb{R}^+)$ such that

$$\psi_j(t) \to 1_{[t_1,t_2]} \text{ a.e.}$$

where $1_{[t_1,t_2]}$ is the characteristic function of the set $[t_1, t_2]$. Using ψ_j in (B.1) and taking the limit $j \to \infty$, we find that

$$\text{(B.2)} \qquad \int (\varphi(x)V(t_2,x) - \varphi(x)V(t_1,x))\,dx = -\int_{t_1}^{t_2}\int_0^{\infty} (\varphi' uV + \tfrac{1}{2}\varphi V^2)\,dx\,dt$$

for almost all $t_1, t_2 \in (0, T)$ by using the Lebesgue point and dominated convergence theorems. We can estimate the right-hand side of (B.2) by

$$\begin{aligned}
&\left|\int_{t_1}^{t_2}\int_0^{\infty} (\varphi' uV + \tfrac{1}{2}\varphi V^2)\,dx\,dt\right| \\
&\qquad \leqq |t_2 - t_1|(\|\varphi'\|_{L^2}\|u\|_{L^\infty((0,T)\times\mathbb{R}^+)}\|V\|_{L^2(\mathbb{R}^+)} + \tfrac{1}{2}\|\varphi\|_{L^\infty}\|V\|_{L^2(\mathbb{R}^+)}^2) \\
&\qquad \leqq |t_2 - t_1|C_T\|\varphi\|_{H^1(\mathbb{R}^+)}.
\end{aligned}$$

Therefore

$$\left|\int (\varphi(x)V(t_2,x) - \varphi(x)V(t_1,x))\,dx\right| \leqq |t_2 - t_1|C_T\|\varphi\|_{H^1(\mathbb{R}^+)};$$

that is,

$$\|V(t_2,\cdot) - V(t_1,\cdot)\|_{H^{-1}} \leqq |t_2 - t_1|C_T$$

for almost all $t_1, t_2 \in (0, T)$. This becomes the Lipschitz continuity condition in (c) when we redefine V on a null set of t. To prove the initial condition in (c), we start with the equation

$$V_t^n = -(u^n V^n)_x + \tfrac{1}{2}(V^n)^2, \quad V^n(0,x) = V_0^n(x)$$

in the distributional sense. We find similarly that

$$\left|\int_0^{\infty} \varphi(x)(V^n(t_2,x) - V_0^n(x))\,dx\right| \leqq t_2 C_T\|\varphi\|_{H^1(\mathbb{R}^+)}.$$

Letting $n \to \infty$, we obtain

$$\left| \int_0^\infty \varphi(x)(V(t_2, x) - V_0(x))\, dx \right| \leq t_2 C_T \| \varphi \|_{H^1(\mathbb{R}^+)}$$

for almost all $t_2 \in (0, T)$. Therefore

$$V(t, x) \to V_0(x) \quad \text{in } H^{-1}(\mathbb{R}^+)$$

as $t \to 0+$. Since $V(t, x)$ is Lipschitz continuous on $[0, T]$, $V(0, x)$ is defined as the limit $\lim_{t \to 0} V(t, x)$ in $H^{-1}(\mathbb{R}^+)$. Therefore $V(0, x) = V_0(x)$ in $H^{-1}(\mathbb{R}^+)$. The verification of condition (c) is complete.

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