A Stochastic System of Particles Modelling the Euler Equations

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Abstract

We consider a system of N spheres interacting through elastic collisions at a stochastic distance. In the limit $N \to \infty$, for a suitable rescaling of the interaction parameters, we prove that the one-particle distribution function converges to a local Maxwellian, whose gross density, velocity, and temperature satisfy the Euler equation.

1. Introduction

In this paper we prove that solutions of the Euler equation for compressible fluids can be approximated by solutions of an equation describing the dynamics of suitable systems of particles. To be more specific consider density, velocity, and temperature fields, ϱ, u and T, which constitute a smooth solution of the Euler equations (up to some time t_0, before the appearance of the first singularity) and construct a local Maxwellian M whose mean density, velocity, and temperature are given by ϱ, u and T, respectively.

Consider also a system of particles interacting through elastic collisions with stochastic distance of interaction. We prove that M is well approximated by the time dependent one-particle distribution function of our system, provided that the number of particles is sufficiently large and that the initial distribution and the parameters of collision are suitably chosen.

For two different reasons we do not claim that this result is a "derivation" of the Euler equation. The first is that our starting point is a stochastic system, so that we are not really able to construct the hydrodynamical picture from Newton's laws of motion. The second is that we are not following the correct physical procedure in deriving the Euler equations. This derivation is believed to hold (starting from realistic physical systems) under a hyperbolic space-time scaling. A fluid is in fact a continuum of points, each of which is, microscopically, a system consisting of a large number of particles in (local) thermal equilibrium. The parameters of local equilibrium, ϱ, u, T, then evolve according to the Euler equation.

Thus to describe a fluid in terms of the molecules one has to scale the space (to localize the microscopic structure) and the time (to reach the local equilibrium).

We direct the reader to [4] for a survey of the hydrodynamical behavior of many-particle systems.

An intermediate regime is that described by the Boltzmann equation in which, with the space-time scaling, the interaction is made weaker and weaker. This corresponds, from a physical point of view, to a rarefaction hypothesis and is equivalent to the well known Boltzmann-Grad limit.

The regime of fluid dynamics can also be recovered from the Boltzmann equation in the limit $\varepsilon \to 0$, where ε is the mean free path.

Rigorous results in this last direction are known (see [2], [8]), while there has been no significant progress in deriving the Euler equations on the basis of Hamiltonian dynamics.

Very little is known even for the problem (which is in principle easier) of deriving the Boltzmann equation [5], [6], [10], [13].

Our result is obtained through a sequence of steps which will be illustrated in Section 3, where also we establish the main result.

The model and related equations will be introduced in Section 2. Section 4 and 5 are devoted to the proofs.

2. The Model

We consider a system of N particles located at the points $x_1, \ldots, x_N$ on a rectangular domain $\Omega \subset \mathbb{R}^3$. The dynamics of the system is the following: the particles move freely unless a pair of them undergo an elastic collision, as expressed by the following formula:

$$\begin{aligned} v_i' &= v_i - n_{ij}(n_{ij} \cdot (v_i - v_j)), \\ v_j' &= v_j + n_{ij}(n_{ij} \cdot (v_i - v_j)) \end{aligned} \tag{2.1}$$

where $n_{ij} = \dfrac{x_i - x_j}{|x_i - x_j|}$.

Here v_i' and v_j' denote the outgoing velocities, where the ingoing velocities are given by v_i and v_j provided that $n_{ij} \cdot (v_i - v_j) < 0$.

Each binary collision takes place according to a stochastic law: the collision times for each pair i and j of particles are independent Poisson processes with intensity given by the function $\varphi(x_i, x_j; v_i, v_j)\, |n_{ij} \cdot (v_i - v_j)|$ and φ is given by

$$\varphi(x_i, x_j; v_i, v_j) = \omega\chi(|x_i - x_j| \in I)\, \chi(|v_i - v_j| \leqq \theta) \tag{2.2}$$

where $I \subset \mathbb{R}^1$ is an interval, ω and $\theta > 0$ are parameters and χ (something) is the characteristic function of the set of variables in which "something" happens.

The basic evolution equation has the following form:

$$\begin{aligned} D_t f^N(t; x_1, v_1, \dots, x_N, v_N) &= \tfrac{1}{2} \sum_{i,j=1}^{N} \varphi(x_i, x_j; v_i, v_j) \\ &\times |n_{ij}(v_i - v_j)| \{H(n_{ij} \cdot (v_i - v_j)) f^N(t; x_1, v_1, \dots, x_i, v_i', \dots, x_j, v_j', \dots, x_N, v_N) \\ &- H(-n_{ij}(v_i - v_j)) f^N(t; x_1, v_1, \dots, x_N, v_N)\} \\ &\equiv (L_N f^N)(t; x_1, v_1, \dots, x_N, v_N) \end{aligned} \tag{2.3}$$

where f^N is the N-particle distribution function describing the probability density for finding the N particles in the points $x_1, \dots, x_N \in \Omega$ with velocities $v_1, \dots, v_N \in \mathbb{R}^3$ at the time $t \geqq 0$. H is the usual step function

$$H(x) = \begin{cases} 1 & \text{if } x \geqq 0, \\ 0 & \text{if } x < 0. \end{cases} \tag{2.4}$$

Finally we put

$$D_t = \partial_t + \sum_i v_i \partial_{x_i}. \tag{2.5}$$

To avoid the boundary conditions we assume that all functions are periodic with respect to $x_1, \dots, x_N$, which means that Ω is a torus. Consequently in the previous formulas $|x - y|$ means the distance on the torus of two points x and y.

Moreover we assume that all functions are symmetric with respect to exchange of the particles.

The s-particle distribution functions are defined as

$$f^{N,s}(t; x_1, v_1, \dots, x_s, v_s) = \int f^N(t; x_1, v_1, \dots, x_N, v_N)\, dx_{s+1}\, dv_{s+1} \dots dx_N\, dv_N. \tag{2.6}$$

By a standard argument we can show that $\{f^{N,s}\}_{s=1}^{N-1}$ satisfies the following finite hierarchy of equations:

$$\begin{aligned} D_t f^{N,s}(t; x_1, v_1, \dots, x_s, v_s) &= (L_s f^{N,s})(t; x_1, v_1, \dots, x_s, v_s) \\ &+ (N - s) \sum_{i=1}^{s} dx_{s+1}\, dv_{s+1}\, \varphi(x_i, x_{s+1}; v_i, v_{s+1}) |n_{i,s+1}(v_i - v_{s+1})| \\ &\times \{H_{i,s+1} f^{N,s+1}(t; x_1, v_1, \dots, x_i, v_i', \dots x_{s+1}, v_{s+1}') \\ &- H_{i,s+1}^- f^{N,s+1}(t; x_1, v_1, \dots, x_{s+1}, v_{s+1})\}. \end{aligned} \tag{2.7}$$

Here L_s denotes the operator defined in (2.3) and we have used the shorthand notation

$$H_{i,j} = H(n_{i,j} \cdot (v_i - v_j)), \qquad H^-(x) = H(-x).$$

The hierarchy (2.7) corresponds to the usual BBKGY hierarchy (*cfr.* [5], [6], [7], [10], [13], [14]) for hard spheres. Actually the hierarchy defined in (2.7) converges, at least formally, to the usual BBKGY hierarchy for hard spheres of diameter d, when $\theta = +\infty$, $I = [d, d + \eta]$, $w = 1/\eta$, in the limit $\eta \to 0$.

Since we are interested in other kinds of asymptotic behavior we shall not discuss this point further.

From now on we shall use the definitions and notations

$$I = [0, \delta], \quad \delta > 0, \quad \omega = \frac{3}{N\delta^3} \cdot \frac{1}{\varepsilon}, \quad \varepsilon > 0. \tag{2.8}$$

From a physical point of view ε is a measure of the mean free path.

Formally making N tend to infinity, we obtain the following hierarchy of equations:

$$\begin{aligned}(D_t f^s)(t; x_1, v_1, \ldots, x_s, v_s) &= \frac{3}{\varepsilon\delta^3} \sum_{i=1}^{s} \int \chi(|x_i - x_{s+1}| \leqq \delta)\, \chi(|v_i - v_{s+1}| \leqq \theta) \\ &\times |n_{ij} \cdot (v_i - v_j)| \{H_{i,s+1} f^{N,s+1}(t; x_1, v_1, \ldots, x_i, v_i', \ldots x_{s+1}, v_{s+1}') \\ &- H_{i,s+1}^{-} f^{N,s+1}(t; x_1, v_1, \ldots, x_{s+1}, v_{s+1})\}\, dx_{s+1}\, dv_{s+1}. \end{aligned} \tag{2.9}$$

As we do for the Boltzmann hierarchy (*cfr.* [10], [13], [14]), we expect that the propagation of chaos holds for the above hierarchy. Namely, suppose the particles are initially identically and independently distributed according to a distribution density $F = F(x, v)$; then at later times, they are identically and independently distributed according to a solution of the following Povzner equation (*cfr.* [12]) with a cutoff of the collision kernel:

$$\begin{aligned}(D_t f_{P,\theta})(t; x_1, v_1) &= \frac{3}{\delta^3} \int \chi(|x_1 - x_2| \leqq \delta)\, \chi(|v_1 - v_2| \leqq \theta) \\ &\times |n_{1,2} \cdot (v_1 - v_2)| \{H_{1,2} f_{P,\theta}(t; x_1, v_1') f_{P,\theta}(t; x_2, v_2') \\ &- H_{1,2}^{-} f_{P,\theta}(t; x_1, v_1) f_{P,\theta}(t; x_2, v_2)\}\, dx_2\, dv_2 \end{aligned} \tag{2.10}$$

and with initial datum $f_{P,\theta}(0, x, v) = F(x, v)$.

We also consider the function f_P, a solution of the Povzner equation, obtained by putting $\theta = +\infty$ in (2.10). The Povzner equation is a certain modified Boltzmann equation in which the stochastic nature of the collisions in the underlying model results in a spatial smearing, whereas in the Boltzmann equation the deterministic collision law gives rise to a strictly local interaction. Formally, the Boltzmann equation is obtained from the Povzner equation by making δ tend to zero:

$$\begin{aligned}(D_t f_B)(t; x, v_1) &= \frac{1}{\varepsilon} \int dv_2 \int dn_{1,2} n_{1,2} \cdot (v_1 - v_2)\, H(n_{1,2} \cdot (v_1 - v_2)) \\ &\times \{f_B(t; x, v_1') f_B(t; x, v_2') - f_B(t; x, v_1) f_B(t; x, v_2)\}. \end{aligned} \tag{2.11}$$

The right-hand side of Equation (2.11) vanishes only on the local Maxwellians, *i.e.* on functions of the kind

$$M(t; x, v) = \varrho(t, x)\, (2\pi T(t, x))^{-\frac{3}{2}} \exp\left(-\frac{(v - u(t, x))^2}{2T(t, x)}\right). \tag{2.12}$$

In this paper we wish to compare the time evolution of the one-particle distribution function $f^{N,1}$, obeying the first equation of the hierarchy (2.7), with a local Maxwellian (2.12), where the hydrodynamical parameters ϱ, u, T satisfy the Euler equations. We shall prove that the difference can be made arbitrarily small provided that the parameters N, δ, ε, θ are suitably chosen.

3. Main Result and Outline of the Proof

In this paper the letter c (with or without subscripts) is reserved for positive constants, independent of all the relevant variables and the parameters N, s, θ, δ and ε. When the letter c occurs in a formula, the formula is assumed to be valid for some particular c.

We shall denote by $\|\cdot\|_1$ the L_1 norm with respect to Lebesgue measure.

The starting point of our analysis is the existence of unique solutions in the interval $[0, t_0]$:

$$\varrho, u, T \in C^m([0, t_0]; C^n(\Omega)), \tag{3.1}$$

of the system of Euler equations for compressible fluids such that

$$\inf_{[0,t_0]\times\Omega} \varrho > 0, \inf_{[0,t_0]\times\Omega} T > 0. \tag{3.2}$$

The indices m and n can be arbitrarily large if the initial conditions are sufficiently smooth. (See, for example [11].)

We denote by M the local Maxwellian (2.12) with ϱ, u, T given by such a solution of the Euler equation and by M_+ a global Maxwellian (*i.e.* a Maxwellian with constant parameters) such that

$$\sup_{\substack{t\in[0,t_0]\\ x\in\Omega}} (1+v^2)^{\alpha/2} M(t, x, v) \leqq c_\alpha M_+(v) \tag{3.3}$$

for all $\alpha \in \mathbb{R}^1$.

We are now im position to formulate our main result.

Theorem 1. *Let m and n be sufficiently large. Then for all $\sigma > 0$, there exist $\varepsilon_0(\sigma)$, $\delta_0(\sigma, \varepsilon)$, $\theta_0(\sigma, \varepsilon, \delta)$ and $N_0(\sigma, \varepsilon, \delta, \theta)$ such that if $\varepsilon \leqq \varepsilon_0$, $\delta \leqq \delta_0$, $\theta \geqq \theta_0$, $N \geqq N_0$,* then

$$\sup_{[0,t_0]} \|M - f^{N,1}\|_1 < \sigma, \tag{3.4}$$

where $\{f^{N,s}\}_{s=1}^N$ is the solution of the hierarchy (2.7) *with initial condition*

$$f^{N,s}(0; x_1, v_1, \ldots, x_s, v_s) = \prod_{j=1}^{s} M(0; x_j, v_j).$$

The proof of Theorem 1 is carried out in the following steps.

First we investigate the Boltzmann-Grad limit for our particle system and prove

Theorem 2. *Let* $0 < \delta, \varepsilon, \theta < \infty$, $0 \leqq F \in L_1(\Omega \times R^3)$, $\|F\|_1 = 1$, $F \geqq 0$. *Then there exist unique, nonnegative functions* $f^{N,s}$ *forming the mild solution of* (2.7) *with initial data*

$$f^{N,s}(0; x_1, v_1, \dots x_s, v_s) = \prod_{i=1}^{s} F(x_i, v_i) \tag{3.5}$$

and a unique solution $f_{P,\theta}$ *of the cutoff Povzner equation* (2.10), *such that*

$$\sup_{[0,t_0]} \|f^{N,1} - f_{P,\theta}\|_1 \leqq \exp\{-2^{-2(1+\Gamma t_0)} \ln N + ct_0\Gamma\}, \tag{3.6}$$

where $\Gamma = \dfrac{24\theta}{\varepsilon\delta^3}$.

A similar result has already been proven in [3] in a weak sense and by means of a compactness argument. For our purposes the present form of the result is needed.

We remark that it is only in the proof of Theorem 2 that the stochastic nature of the basic system is used to smooth the interaction of the particles.

The proof of Theorem 2 will be given in Section 4. A second step removes the cutoff θ.

Theorem 3. *Suppose* $0 \leqq F \leqq M^{\frac{1}{2}}$. *Then there exist a unique positive solution* f_P *of the Povzner equation, with initial datum F, and two positive constants* $A(\varepsilon, \delta)$, $B(\varepsilon, \delta)$ *depending only on* ε *and* δ, *such that*

$$\sup_{[0,t_0]} \|f_P - f_{P,\theta}\|_1 \leqq \frac{A(\varepsilon, \delta)}{\theta} \exp(B(\varepsilon, \delta)\, t_0). \tag{3.7}$$

The proof of Theorem 3 follows the ideas of ARKERYD [1] developed in the context of the homogeneous Boltzmann equation. For the proof see Section 5.

As a third step we compare f_P with a suitable solution f_B of the Boltzmann equation exhibiting hydrodynamical behavior. The existence of such a solution has been proved in [2], [8]. To formulate precisely the result we need to introduce the following space. Let

$$\|f\|_{\beta,l} = \sup_v \left((1 + v^2)^{\beta/2} M_+^{-\frac{1}{2}}(v) \|f(\cdot, v)\|_{H_2^l(\Omega)}\right) \tag{3.8}$$

where $H_2^l(\Omega)$ is the usual Sobolev norm. Denote by $X_{\beta,l}$ the Banach space equipped with the norm (3.8). Then

Theorem 4. *There exist* $\varepsilon_0 > 0$ *and a unique, nonnegative, classical solution* f_B *of the Boltzmann equation* (2.11), *for* $0 < \varepsilon \leqq \varepsilon_0$, *with initial datum* $M_0(x, v) = M(0; x, x)$, *of the form*

$$f_B = M + \varepsilon\Phi \tag{3.9}$$

where Φ *is such that*

$$\sup_{[0,t_0]} \|\Phi\|_{\beta,l} \leqq c_\beta \tag{3.10}$$

for all β *and sufficiently large* l, *depending on* m *and* n.

The proof of Theorem 4 follows by the analysis given in [8] where a stronger result, including the initial layer effect, was obtained.

We remark that f_B is nonnegative as follows by general arguments.

Now we are in position to formulate the last step; namely to compare f_P with f_B.

Theorem 5. *Let f_B be as in Theorem 4 and f_P be a solution of the Povzner equation with the same initial datum. Then if* $\delta \leqq \delta_0(\varepsilon)$,

$$\sup_{[0,t_0]} \|f_B - f_P\|_1 \leqq c\sqrt{\delta}. \tag{3.11}$$

The proof of Theorem 1 follows easily by collecting all the error terms as given by the estimates in Theorems 2, 3, 4, 5.

4. Proof of Theorem 2

First we consider the basic evolution equation (2.3) in the following integral form:

$$f^N(t) \circ S_N(t) = F^N + \int_0^t (L^N f^N)(t_1) \circ S(t_1)\, dt_1, \tag{4.1}$$

where

$$F^N(x_1, v_1, \ldots, x_N, v_N) = \prod_{i=1}^{N} F(x_i, v_i) \tag{4.2}$$

is the initial datum and

$$\begin{gathered} S_N(t): (\Omega \times \mathbb{R}^3)^N \to (\Omega \times \mathbb{R}^3)^N, \\ S_N(t)(x_1, v_1, \ldots, x_N, v_N) = (x_1 + v_1 t, v_1, \ldots, x_N + v_N t, v_N). \end{gathered} \tag{4.3}$$

Since L^N is a linear, bounded operator in $L_1((\Omega \times R^3)^N)$, existence and uniqueness for the equation (4.1) is a trivial problem.

Consider now the (infinite) hierarchy (2.9) and denote by $\mathfrak{A}^{s+1}$ the operator appearing in the right-hand side, *i.e.*

$$D_t f^s(t) = \mathfrak{A}^{s+1} f^{s+1}(t). \tag{4.4}$$

We have

$$\|\mathfrak{A}^{s+1} f^{s+1}\|_1 \leqq \frac{6\theta s}{\varepsilon\, \delta^3} \|f^{s+1}\|_1. \tag{4.5}$$

The mild form of Equation (4.4) has a solution given by the formal perturbation series

$$f^s(t) = F^s \circ S_s(t) + \sum_{n=1}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n \tag{4.6}$$

$$(\mathfrak{A}^{s+1} \ldots (\mathfrak{A}^{s+n}(F^{s+n} \circ S_{s+n}(t_n)) \circ S_{s+n-1}(t_n - t_{n-1}) \ldots) \circ S_{s+1}(t - t_1)$$

where F^s is the initial condition assumed to be a product of identical densities:

$$F^s(x_{,1}, v_1, \ldots, x_s, v_s) = \prod_{j=1}^{s} F(x_j, v_j). \tag{4.7}$$

The L_1 norm of the n^{th} term of the series (4.6) can be estimated by

$$\left(\frac{6\theta}{\varepsilon\,\delta^3}\right)^n t^n \frac{1}{n!} \frac{(2s+n-1)!}{(s-1)!}. \tag{4.8}$$

Choosing $t_* = \dfrac{\varepsilon\,\delta^3}{24\theta}$, we can see that the series converges uniformly for $t \in [0, t_*]$.

By the same method we can prove that the solution we have constructed for $t \in [0, t^*]$ is unique.

It is easy to realize that the series expansion (4.6) also provides a solution for products of the cutoff Povzner equation (2.10) so that we have

$$f^s(t; x_1, v_1, \ldots, x_s, v_s) = \prod_{j=1}^{s} f_{P,\theta}(t; x_j, v_j), \tag{4.9}$$

The positivity of $f_{P,\theta}$ (and hence of f^s) follows by general arguments (see for example [1]).
Moreover, since

$$\int \mathfrak{A}^s f^s \, dx_1 \, fv_1 \ldots dx_s \, dv_s = 0 \tag{4.10}$$

the L_1 norm of $f^s(t)$ is preserved during the motion and hence, assuming $f^s(t_*)$ as initial datum, we can prove the convergence of the series for $t \in [t_*, 2t_*]$ and so on.

The difference

$$\Delta^{N,s}(t) = f^{N,s}(t) - f^s(t) \tag{4.11}$$

satisfies the following equation (for $t_1 \leqq t$ and $s \leqq N$):

$$\Delta^{N,s}(t) = \mathfrak{M}^{N,s}(t, t_1) + \int_{t_1}^{t} dt_2 (\mathfrak{A}^{s+1} \Delta^{N,s+1}(t_2)) \circ S_s(t_2 - t) \, dt_2 \tag{4.12}$$

where

$$\mathfrak{M}^{N,s}(t, t_1) = \Delta^{N,s}(t_1) \circ S_s(t - t_1)$$
$$+ \int_{t_1}^{t} dt_2 \left\{ L^s f^{N,s} - \frac{s}{N} \mathfrak{A}^{s+1} f^{N,s+1} \right\} (t_2) \circ S_s(t_2 - t_1). \tag{4.13}$$

Letting $t_k = t_* k$ and

$$a_k^s = \sup_{t \in [t_{k-1}, t_k]} \| \Delta^{N,s}(t) \|_1 \tag{4.14}$$

we shall prove the bound

$$a_k^s \leqq \exp\{-\varphi_k \ln N + Z_k\} \quad \text{for } s \leqq \beta_k \ln N \tag{4.15}$$

where

$$\varphi_k = 2^{-2(k+1)}, \quad \beta_k = \frac{\varphi_k}{2 \ln 2}, \quad Z_k = (5 \ln 2)\, k. \tag{4.16}$$

The bound (4.15) will proven by iteration on k. It will be assumed for $k-1$ if $k>1$, and otherwise we shall use $\Delta^{N,s}(t_0)=0$. By (4.13) for $t\in[t_{k-1}, t_k]$

$$\|\mathfrak{M}^{N,s}(t, t_{k-1})\|_1 \leqq \|\Delta^{N,s}(t_{k-1})\|_1 + \frac{s^2}{N} t_* \left(\frac{12\theta}{\varepsilon\,\delta^3}\right). \tag{4.17}$$

By the hypothesis of induction, since for $s \leqq \beta_k \ln N$, $k \leqq \left[\frac{t_0}{t_*}\right]+1$ and N sufficiently large

$$\frac{s^2}{N} \leqq \frac{1}{N} 2^{-4(k+1)} \left(\frac{\ln N}{2\ln 2}\right)^2 \leqq c^{-\varphi_{k-1}\ln N + Z_{k-1}} \equiv \alpha_{k-1}, \tag{4.18}$$

we have

$$\|\mathfrak{M}^{N,s}(t, t_{k-1})\| \leqq 2\alpha_{k-1}. \tag{4.19}$$

Therefore, by (4.12),

$$\|\Delta^{N,s}(t)\|_1 \leqq 2\alpha_{k-1} + \int_{t_{k-1}}^{t} dt_1 s \left(\frac{6\theta}{\varepsilon\,\delta^3}\right) \|\Delta^{N,s+1}(t_1)\|_1. \tag{4.20}$$

We iterate the above inequality up to the largest n for which $s+n \leqq \beta_{k-1}\ln N$ (in order to satiafy the hypothesis of induction). Thus we obtain

$$\|\Delta^{N,s}(t)\|_1 \leqq 2\alpha_{k-1} \sum_{l=0}^{n} \frac{(\Gamma t_*/4)^l}{l!} s(s+1)\ldots(s+1) + \frac{2s(s+1)\ldots(s+n)}{(n+1)!}\left(\frac{\Gamma t_*}{4}\right)^n \tag{4.21}$$

where the last term on the right-hand side of (4.21) is obtained by using the obvious bound

$$\|\Delta^{N,s}(t)\|_1 \leqq 2.$$

Therefore

$$a_k^s \leqq 4\alpha_{k-1}2^s + 2(\tfrac{1}{2})^n\, 2^s. \tag{4.22}$$

For $s \leqq \beta_k \ln N$ we have

$$\begin{aligned} a_k^s \leqq{} & \exp\{-\varphi_{k-1}\ln N + Z_{k-1} + 2\ln 2 + \ln 2\,\beta_k \ln N\} \\ & + \exp\{2\ln 2 + 2\ln 2\beta_k \ln N - \ln 2\beta_{k-1}\ln N\} \end{aligned} \tag{4.23}$$

By our choice of φ_k, β_k and Z_k we easily obtain the estimate (4.15) and from this, we complete the proof of Theorem 2.

5. Proof of Theorems 3 and 5

Proof of Theorem 3. Following [1] we can construct unique positive solutions $f_{P,\theta}$ of (2.10) and also f_P, solution of the Povzner equation without cutoff θ satisfying the conservation of energy and mass and the following bound:

$$\sup_\theta \sup_t |(f(t)|_4 \leqq C_F(\varepsilon, \delta) \tag{5.1}$$

where $C_F(\varepsilon, \delta)$ is some positive constant depending only on ε, δ and on the initial datum F, and

$$|f|_\alpha = \int dx\, dv (1 + v^2)^{\alpha/2} |f(x, v)| \tag{5.2}$$

To estimate the difference $\Delta_\theta = f_{P,\theta} - f_P$, we introduce the following equation *cf.* [1] Part II, Theorem 1.1):

$$D_t \tilde{f}_{P,\theta} + P\tilde{f}_{P,\theta} = Q_\theta^+(\tilde{f}_{P,\theta}, \tilde{f}_{P,\theta}) - Q^-(\tilde{f}_{P,\theta}, \tilde{f}_{P,\theta}) + R(\tilde{f}_{P,\theta}, \tilde{f}_{P,\theta}), \tag{5.3}$$

where Q_θ^+ and Q^- denote the gain term in the cutoff symmetrized collision operator of the Povzner equation and the loss term in the symmetrized collision operator when there is no cutoff,

$$P(v_1) = \frac{c_+}{\varepsilon\, \delta^3} (1 + v_1^2)\, |F|_2, \quad c_+ \text{ a large constant,} \tag{5.4}$$

and

$$R(f_1, f_2)\,(x_1, v_1) = \frac{c_+}{\varepsilon\, \delta^3} (1 + v_1^2) f_1(x_1, v_1)\, |f_2|_2 . \tag{5.5}$$

The unique mild solution of (5.3) with F as initial datum is such that

$$0 \leqq \tilde{f}_{P,\theta} \leqq \tilde{f}_{P,\theta_1} \leqq \tilde{f}_P \quad \text{for } \theta \leqq \theta_1, \tag{5.6}$$

and

$$0 \leqq \tilde{f}_{P,\theta} \leqq f_{P,\theta} .$$

Therefore

$$\Delta_\theta' = f_{P,\theta} - \tilde{f}_{P,\theta} \quad \text{and} \quad \Delta_\theta'' = f_P - \tilde{f}_{P,\theta} \tag{5.7}$$

are nonnegative functions and Δ_θ can easily be estimated in terms of Δ_θ' and Δ_θ''. We have

$$\begin{aligned} D_t \Delta_\theta' = {} & Q_\theta^+(\Delta_\theta', f_{P,\theta} + \tilde{f}_{P,\theta}) - Q_\theta^-(\Delta_\theta', f_{P,\theta} + \tilde{f}_{P,\theta}) \\ & + Q^-(f_{P,\theta}, f_{P,\theta}) - Q_\theta^-(f_{P,\theta}, f_{P,\theta}) + Pf_{P,\theta} - R(f_{P,\theta}, \tilde{f}_{P,\theta}) \end{aligned} \tag{5.8}$$

with initial datum $\Delta_\theta'(0) = 0$.

Integrating Equation (5.8) with respect to $(1 + v^2)\, dv\, dx$, we obtain

$$\frac{d}{dt} |\Delta_\theta'|_2 \leqq |(Q^- - Q_\theta^-)(\tilde{f}_{P,\theta}, \tilde{f}_{P,\theta})|_2 + \frac{c}{\varepsilon\, \delta^3} |\tilde{f}_{P,\theta}|_4\, |\Delta_\theta'|_2 . \tag{5.9}$$

Here the estimate $\big||F|_2 - |f_{P,\theta}|_2\big| \leqq |\Delta_\theta'|_2$ has been used.

Finally we bound the first term on the right-hand side of (5.9) by

$$\frac{1}{\theta} \frac{c}{\varepsilon\, \delta^3} |f_{P,\theta}(t)|_4^2 . \tag{5.10}$$

An analogous estimate can be obtained for $|\Delta_\theta''|_2$, so that the uniform estimate (in θ) on $|f_{P,\theta}(t)|_4$ and $|f_P(t)|_4$ (see (5.1)) allow us to conclude the proof of Theorem 3.

Proof of Theorem 5. Putting

$$f_P = f_B + \sqrt{\delta}\, h, \tag{5.11}$$

we have for h the following weakly nonlinear equation

$$D_t h = \frac{2}{\varepsilon} Q_\delta(f_B, h) + \frac{\sqrt{\delta}}{\varepsilon} Q_\delta(h, h) + \frac{\sqrt{\delta}}{\varepsilon} \hat{Q}_\delta(f_B, f_B), \quad h(0) = 0, \tag{5.12}$$

in which $\frac{1}{\varepsilon} Q_\delta$ denotes the symmetrized Povzner collision operator in which the dependence on δ is explicitly taken into account and

$$\hat{Q}_\delta = \frac{1}{\delta}(Q_\delta - Q), \tag{5.13}$$

where Q is the symmetrized Boltzmann collision operator.

Let us first consider the linear problem associated to (5.12):

$$D_t h + \frac{1}{\varepsilon} \nu_\delta \cdot h = \frac{1}{\varepsilon} K_\delta h + Q_\delta(\Phi, h) + G \tag{5.14}$$

with initial data

$$h\big|_{t=0} = 0, \tag{5.15}$$

where

$$\nu_\delta(x, v) = \frac{1}{\varepsilon\, \delta^3} \int_0^\delta dr\, r^2 \int dv_2 \int dn_{12} H^-(n_{12} \cdot (v_1 - v_2))$$

$$\times M(x_1 + r n_{12}, v_2)\, |n_{12} \cdot (v_1 - v_2)|, \tag{5.16}$$

$$K_\delta f = Q_\delta(M, f) - \nu_\delta \cdot f \tag{5.17}$$

Φ is given in (3.9) and G is considered to be a known function.

It is well known that a solution of the problem

$$D_t h_1 + \frac{1}{\varepsilon} \nu_\delta \cdot h_1 = 0, \quad h_1\big|_{t=0} = h_0 \tag{5.18}$$

satisfies the following estimate:

$$\|h_1(t, \cdot, v)\|_{H_2^l(\Omega)} \leqq c \exp\left(-\frac{ct}{\varepsilon}(1 + |v|)\right) \|h_0(\cdot, v)\|_{H_2^l(\Omega)}. \tag{5.19}$$

Thus by (3.10) a solution h of (5.14) with initial data (5.15) satisfies

$$\|h(t)\|_{\beta,l} \leqq \frac{c}{\varepsilon} \int_0^t \|h(t_1)\|_{\beta,l}\, dt_1 + c \cdot \varepsilon \sup_{[0,t]} \|h\|_{\beta,l} + c \cdot \varepsilon \sup_{[0,t]} \|G\|_{\beta-1,l} \tag{5.20}$$

for all $t \in [0, t_0]$. Choosing ε sufficiently small, we obtain

$$\sup_{[0,t]} \|h\|_{\beta,l} \leqq \frac{c}{\varepsilon} \int_0^t \|h(t_1)\|_{\beta,l}\, dt_1 + c \sup_{[0,t]} \|G\|_{\beta-1}, \tag{5.21}$$

for all $t \in [0, t_0]$. Applying the Gronwall lemma, we obtain

$$\sup_{[0,t_0]} \|h\|_{\beta,l} \leqq c \exp\left(\frac{ct_0}{\varepsilon}\right) \sup_{[0,t_0]} \|G\|_{\beta-1,l}. \tag{5.22}$$

Now assuming that

$$0 < \delta \leqq \delta_1 \exp\left(-\frac{ct_0}{\varepsilon}\right) \tag{5.23}$$

for some properly chosen constants $\delta_1, \delta_2 > 0$, we can prove the existence of a solution of the nonlinear problem (5.12) by the usual successive approximation method as in [8] and [9]. In fact by Grad-type estimation (see [8]) we have

$$\|Q_\delta(h, h)\|_{\beta-1,l} \leqq c \|h\|_{\beta,l} \tag{5.24}$$

provided that $l \geqq 2$. Moreover by Theorem 4 we can consider f_B to be given and sufficiently smooth with respect to the x-variable, so that

$$\|\hat{Q}_\delta(f_B, f_B)\|_{\beta-1,l} \leqq c. \tag{5.25}$$

Thus the nonlinear problem (5.12) has a unique solution h in the space $X_{\beta,l}$ and

$$\|h\|_{\beta,l} \leqq c. \tag{5.26}$$

Now (3.11) immediately follows.

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