The Value of the Critical Exponent for Reaction-Diffusion Equations in Cones

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Abstract

Let $D \subset R^N$ be a cone with vertex at the origin *i.e.*, $D = (0, \infty) \times \Omega$ where $\Omega \subset S^{N-1}$ and $x \in D$ if and only if $x = (r, \theta)$ with $r = |x|$, $\theta \in \Omega$. We consider the initial boundary value problem: $u_t = \Delta u + u^p$ in $D \times (0, T)$, $u = 0$ on $\partial D \times (0, T)$ with $u(x, 0) = u_0(x) \geqq 0$. Let ω_1 denote the smallest Dirichlet eigenvalue for the Laplace-Beltrami operator on Ω and let γ_+ denote the positive root of $\gamma(\gamma + N - 2) = \omega_1$. Let $p^* = 1 + 2/(N + \gamma_+)$. If $1 < p < p^*$, no positive global solution exists. If $p > p^*$, positive global solutions do exist. Extensions are given to the same problem for $u_t = \Delta u + |x|^\sigma u^p$.

I. Introduction

Let $D \subset R^N$ be a domain with a piecewise smooth boundary or all of R^N. We consider nonnegative classical solutions of

$$\begin{aligned} &\frac{\partial u}{\partial t} = \Delta u + u^p \qquad (x, t) \in D \times (0, T), \qquad\qquad (P)\\ &u(x, t) = 0 \qquad (x, t) \in \partial D \times (0, T),\\ &u(x, 0) = u_0(x) \qquad x \in D \end{aligned}$$

where $u_0 \geqq 0$ and $p > 1$. It is well known that not all solutions of (P) are global. This follows from several sources. See [5, 6, 9, 12] for example.

When $D = R^N$, FUJITA proved that if $1 < p < 1 + 2/N \equiv p^*$, no positive global solutions exist. He also showed that if $p > p^*$, positive global solutions do exist. Later several authors proved that p^* belongs to the former case [1, 8, 10, 11, 20].

In [15], MEIER proved that if $k \in \{1, \ldots, N\}$ is fixed and

$$D = D_k = \{(x_1, \ldots, x_N) \mid x_1 > 0, \ldots, x_k > 0\}$$

and if $p^* \equiv 1 + 2/(N + k)$, then both statements of FUJITA hold in this case with this value of p^*.

More recently BANDLE & LEVINE [2] undertook the study of (P) when D is a cone with vertex at the origin. That is, $x \in D$ if and only if $x = (r, \theta)$, where $r = |x|$ and $\theta \in \Omega$ where $\Omega \subset S^{N-1}$ is a region with boundary, $\partial\Omega$. We assume $\partial\Omega$ is smooth enough to permit integration by parts. Let ω_1 denote the smallest Dirichlet eigenvalue for the Laplace-Beltrami operator on Ω and γ_+ denote the positive root of $\gamma(\gamma + N - 2) = \omega_1$. Let

$$\underline{p} = 1 + 2/(N + \gamma_+),$$
$$\bar{p} = \min\,[1 + 2/(N - 2 + \gamma_+), 1 + 2/N].$$

They showed that if $1 < p < \underline{p}$, no positive global solutions were possible. They also showed that if

$$\bar{p} < p < \begin{cases} (N+1)/(N-3) & \text{if } N > 3 \text{ and } \bar{p} < 1 + 2/N, \\ \infty & \text{if } N = 2, 3 \text{ or } \bar{p} = 1 + 2/N \end{cases}$$

then positive global solutions do exist. When $N = 2, 3$ or $\bar{p} < 1 + 2/N$, they showed more but those results need not concern us here.

In view of MEIER's result, he conjectured that in the case of a general cone, $\underline{p} = p^*$, i.e., $\underline{p}$ is the cutoff between the blow up case and the global existence case. He had shown this in some special cases in [14, 15, 16].

It is the purpose of this note to establish MEIER's conjecture. In view of the fact that the works of FUJITA and of MEIER [5, 14, 15, 16], depend heavily on the availability of explicit formulas for the Green's function for the heat equation in D, it is desirable to have proofs that avoid the Green's function when dealing with general regions.

The plan of the paper is as follows. In § II we define precisely what we mean by a solution. In § III we prove our principle result. Finally we indicate some simple extensions of the result to other problems.

II. Definitions

The earlier terminology being in force here, for each $T > 0$ we let

$$Q_T := D \times (0, T).$$

A (nonnegative) solution of (P) is called quasiregular if

(i) $u \in C^2(Q_T) \cap C^0(\bar{Q}_T - D \times \{T\})$,
(ii) for all $k > 0$, $t \in [0, T)$

$$\lim_{r \to \infty} e^{-kr} \int_\Omega u(r, \theta, t)\, dS_\theta = 0$$

and

$$\lim_{r\to\infty} e^{-kr} \int_\Omega |u_r(r, \theta, t)|\, dS_\theta = 0.$$

A quasiregular solution is called almost regular if for all $t \in [0, T)$ there is a sequence $\{r_n\}_{n=1}^\infty, r_n \to 0$ such that

$$\lim_{n\to\infty} \int_\Omega r_n^{N-1} [u(r_n, \theta, t)/r_n + |u_r(r_n, \theta, t)|]\, dS_\theta = 0.$$

Throughout the remainder of the paper we consider only almost regular solutions of (P).

III. The Global Existence—Global Nonexistence Results

We prove the following theorem.

Theorem 3.1. *Let* $p^* = 1 + 2/(N + \gamma_+)$. *If* $1 < p < p^*$, (P) *has no nontrivial global solution. If* $p > p^*$, *nontrivial global solutions of* (P) *exist.*

We recently learned of some related results of KAVIAN and his co-authors [3, 4, 10] which, taken with our results, show that for (P), p^* belongs to the blow up case if the cone is *convex*. After submission of this article, we found a proof of this for *arbitrary* cones. The proof is based on a modification of WEISSLER'S [20] method and will be published in a separate paper [13].

The first statement of the theorem has already been proved in [2] if $p < p^*$. We include a sketch of that proof for the convenience of the reader.

To prove the first statement, we set

$$\varphi(r, \theta) := C^{-1} r^m e^{-kr} \psi(\theta) \tag{3.1}$$

where $m, k > 0$, ψ is the (positive) eigenfunction of Δ_θ corresponding to ω_1 with

$$\int_\Omega \psi(\theta)\, dS_\theta = 1 \tag{3.2}$$

and where

$$C = k^{-(m+N)} \Gamma(m + N). \tag{3.3}$$

Therefore

$$\int_D \varphi\, dx = 1.$$

It follows that in D

$$\Delta\varphi + \lambda\varphi \geqq 0 \tag{3.4}$$

provided

$$(k^2 + \lambda)(m^2 + (N - 2) m - \omega_1) \geqq k^2(m + \tfrac{1}{2}(N - 1))^2. \tag{3.5}$$

We then define

$$F(t) = \int_D u\varphi\, dx. \tag{3.6}$$

In view of our definition of a solution, we have

$$F'(t) \geqq -\lambda F(t) + (F(t))^p. \tag{3.7}$$

Consequently u will not be global in time if

$$F(0) > \lambda^{1/(p-1)}. \tag{3.8}$$

Now (3.5), (3.7) and (3.8) will hold provided

$$m^2 + (N-2)\,m - \omega_1 > 0, \tag{3.9}$$

$$\frac{\lambda}{k^2} = \beta := \frac{m + \omega_1 + \frac{1}{4}(N-1)^2}{m^2 + (N-2)\,m - \omega_1} \tag{3.10}$$

and

$$k^{-[2/(p-1)-(m+N)]} \int_D \varphi(x)\, u_0(x)\, dx > \Gamma(m+N)\,\beta^{1/(p-1)}. \tag{3.11}$$

Thus, if

$$2 - N - \gamma_- = \gamma_+ < m < 2/(p-1) - N \tag{3.12}$$

we can choose k (and hence λ) so small that (3.9), (3.10) and (3.11) hold so that in turn (3.4), (3.7) and (3.8) also hold. There is m which satisfies (3.12) provided $1 < p < p^*$.

The proof of the second statement proceeds by the method of supersolutions [17, 18]. We use an argument similar to that used in [14, 15, 16]. If $w(x, t)$ is a positive solution of $u_t = \Delta u$ in $D \times [0, \infty)$, vanishing on ∂D we let

$$\bar{u}(x, t) = \beta(t)\, w(x, t).$$

Then $\bar{u}$ will be a supersolution of (P) provided

$$\beta'(t) = [\beta(t)]^p \left[\sup_{x \in D} w(x, t)\right]^{p-1}, \quad 0 < t < T. \tag{3.13}$$

The solution of (3.13) with $\beta(0) = \beta_0 > 0$ will be global in t if

$$W_\infty := \int_0^\infty \|w(\cdot, t)\|_\infty^{p-1}\, dt < \infty \tag{3.14}$$

and if

$$0 < \beta_0 < ((p-1)\, W_\infty)^{-1/(p-1)}. \tag{3.15}$$

Thus, it remains to construct $w(x, t)$ such that $W_\infty < \infty$. To do this, we let $r = |x|$ and let $t_0 > 0$ be fixed. We define

$$\nu := \gamma_+ + \tfrac{1}{2}(N-2) = [\omega_1 + (\tfrac{1}{2}(N-2))^2]^{1/2}. \tag{3.16}$$

We let

$$w(r, \theta, t) = (t + t_0)^{-1}\, r^{-\frac{1}{2}(N-2)}\, I_\nu(r/2(t+t_0))\, e^{-(r^2+1)/4(t+t_0)}\, \psi(\theta) \tag{3.17.1}$$

$$= r^{-\frac{1}{2}(N-2)} \int_0^\infty e^{-\lambda(t+t_0)}\, J_\nu(r\sqrt{\lambda})\, J_\nu(\sqrt{\lambda})\, d\lambda \cdot \psi(\theta), \tag{3.17.2}$$

where J_ν, I_ν denote the Bessel function and modified Bessel function of order ν respectively. (See WATSON [18], p. 395.) From the first of these, w is clearly positive and vanishes on ∂D, while from the second, w is seen to be a solution of the heat equation in D. (We use the fact that

$$I_\nu(z) \approx \begin{cases} 2^{-\nu}/\Gamma(\nu+1)\cdot z^\nu & z \to 0^+ \\ (2\pi z)^{-1/2}\, e^z & z \to +\infty \end{cases} \tag{3.18}$$

to see that the boundary condition is satisfied at $r = 0$ and that w vanishes at $r = \infty$.)

Since $\psi(\theta)$ is bounded on $\bar{\Omega}$, in order to show that (3.14) holds for $p > p^*$, it suffices to show that

$$\limsup_{t\to\infty} (t + t_0)^{\frac{1}{2}(N+\gamma_+)} \left[\sup_{r>0} W(r, t)\right] < \infty \tag{3.19}$$

where

$$W(r, t) := (t + t_0)^{-1}\, r^{-\frac{1}{2}(N-2)}\, I_\nu\left(\frac{r}{2(t + t_0)}\right) e^{-(r^2+1)/4(t+t_0)}. \tag{3.20}$$

Now $W(r, t)$ vanishes at $r = 0, \infty$ for each t. Thus a value $r_m(t)$ of r may be found such that $0 < r_m(t) < \infty$ and

$$W(r_m(t), t) = \sup_{r>0} W(r, t).$$

Let

$$\begin{aligned} \mathscr{W}(t) &= (t + t_0)^{\frac{1}{2}(N+\gamma_+)}\, W(r_m(t), t) \\ &= (t + t_0)^{\frac{1}{2}\gamma_+} \left(\frac{r_m(t)}{t + t_0}\right)^{-\frac{1}{2}(N-2)} I_\nu\left(\frac{r_m(t)}{2(t + t_0)}\right) e^{-\left(\frac{r_m^2(t)+1}{4(t+t_0)}\right)}. \end{aligned}$$

Let

$$y_m(t) = \tfrac{1}{2}\, r_m(t)/(t + t_0).$$

Suppose that, on some sequence $\{t_k\}_{k=1}^\infty$, $\mathscr{W}(t_k) \to \infty$ as $t_k \to +\infty$. If, on some (sub) sequence, $y_m(t_k) \to 0$, then

$$\mathscr{W}(t_k) \approx \text{const} \cdot (t_k + t_0)^{\frac{1}{2}\gamma_+}\, y_m^{\gamma_+}\, e^{-y_m^2(t_k+t_0)}.$$

However, $z^{\gamma_+} e^{-z^2}$ is bounded on $[0, \infty)$ so $\mathscr{W}(t_k) \not\to \infty$ on such a subsequence. If, on the other hand, $y_m(t_k) \to +\infty$ on some (sub) sequence,

$$\mathscr{W}(t_k) \approx \text{const} \cdot y_m^{-\frac{1}{2}(N+2\gamma_+)}\, (y_m^2 \cdot (t_k + t_0))^{\gamma_+/2}\, e^{-y_m^2(t_k+t_0)} \cdot e^{y_m}$$

from which we again conclude that $\mathscr{W}(t_k) \not\to \infty$ on such a sequence. Therefore, if $\mathscr{W}(t_k) \to +\infty$, we must have constants A and B such that

$$0 < A \leqq y_m(t_k) \leqq B < \infty.$$

However, in this case we have

$$0 \leqq \mathscr{W}(t_k) \leqq \text{const} \cdot (t_k + t_0)^{\frac{1}{2}\gamma_+}\, e^{-A^2(t_k+t_0)}$$

so that $\mathscr{W}(t_k) \to +\infty$ is impossible here also. This establishes the theorem. □

Corollary 3.2. *If* $p > p^*$ *and*

$$0 \leqq u(r, \theta, t) \leqq \beta(t)\, w(r, \theta, t) \qquad (r, \theta) \in D, \qquad t > 0$$

for some $t_0 > 0$, *then the corresponding solution of* (P) *satisfies*

$$\limsup_{t \to +\infty} (t + t_0)^{\frac{1}{2}(N+\gamma_+)} \| u(t) \|_{L_\infty} < \infty .$$

Remark 3.2. For any $p > 1$, it is not hard to prove that the blowup of $F(t)$ implies the pointwise unboundedness of the solution in finite time [2].

Remark 3.3. In the case that

$$1 + \frac{2}{(N - 2 + \gamma_+)} < p < \begin{cases} \dfrac{N+1}{N-3}, & N \geqq 4, \\ \infty, & N = 2, 3 \end{cases} \tag{3.21}$$

BANDLE & LEVINE [2] have shown that the stationary problem for (P) possesses singular solutions of the form

$$u_s(r, \theta) = r^{-2/(p-1)}\, \alpha(\theta)$$

and that whenever

$$u(r, \theta, 0) \leqq \min\,(r^\varepsilon, u_s(r, \theta))$$

for some $\varepsilon > 0$, the corresponding solution will be global.

They also showed that if

$$1 < p < 2/(N - 2 + \gamma_+)$$

no stationary solution (or even singular stationary solutions of the above form) exists.

Remark 3.4. A similar result can be obtained for

$$\begin{aligned} u_t &= \Delta u + |x|^\sigma u^p \quad && \text{in } D \times (0, T), \\ u &= 0 && \text{on } \partial D \times (0, T), \\ u(x, 0) &= u_0(x) && \text{on } D. \end{aligned} \tag{P$_\sigma$}$$

(The stationary equation was studied in [7] when $D = R^N$.) Let

$$p^*(\sigma) = 1 + (2 + \sigma)/(N + \gamma_+). \tag{3.22}$$

We have

Theorem 3.3. *If* $1 < p < p^*(\sigma)$ *and* $\sigma \geqq 0$, *no nontrivial global regular solution of* (P_σ) *exists. If*

$$p > p^*(\sigma) \tag{3.23}$$

then there are nontrivial global solutions of (P_σ).

The first statement was proved in [2] for $\sigma > -2$ along with a weaker version of the second statement.

The proof of the second statement proceeds exactly as before. In place of (3.19) we must require

$$\limsup_{t \to +\infty} (t + t_0)^{(N+\gamma_+)/(2+\sigma)} \left[\sup_{r>0} r^{\sigma/(p-1)} W(r, t)\right] < \infty$$

where W is given in (3.20).

The following is an extension of Corollary 3.2.

Corollary 3.4. *If* (3.23) *holds and u solves* (P_σ) *with* $\sigma \geqq 0$ *and*

$$0 \leqq u(r, \theta, t) \leqq \beta(t)\, w(r, \theta, t),$$

then

$$\limsup t^{(N+\gamma_+)/(2+\sigma)} \sup_{\Omega} u(r, \theta, t) \leqq C r^{-\sigma/(p-1)}$$

where C depends only upon $\beta(0)$, t_0 *and geometry.*

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