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A Regularity Theory for a General Class of Quasilinear Elliptic Partial Differential Equations and Obstacle Problems

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1. Introduction

We consider the following variational inequality involving p-Laplacian functions,

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot (\nabla v - \nabla u)\, dx \geqq \int_\Omega b(x, u, \nabla u)(v - u)\, dx + \int_\Omega f(x) \cdot \nabla(v - u)\, dx \tag{1}$$

for all $v \in \mathscr{C} = \{v \in W_0^{1,p}(\Omega) + u_0$ and $v(x) \geqq \psi(x)$ a.e. in $\Omega\}$. Here Ω is a bounded domain in $\mathbb{R}^n$ and $u_0 \in W^{1,p}(\Omega)$ with $u_0(x) \geqq \psi(x)$ a.e. in Ω. Naturally $1 < p < \infty$ and $u \in W^{1,p}(\Omega)$.

When $f = 0$ and b satisfies a growth condition of Serrin type [12], namely

$$|b(x, u, h)| \leqq c_1 |h|^{p-1} + c_2 |u|^{p-1} + c_3,$$

where c_1, c_2 and c_3 are positive constants, MICHAEL & ZIEMER [10] have proved that u is Hölder continuous when ψ is Hölder continuous.

Recently FUCHS [3], LINDQUIST [7], and NORANDO [11] proved the $C^{1,\alpha}$-regularity of u under various restrictions, LINDQUIST assuming that $n = 2$, $p \geqq 2$, and FUCHS and NORANDO assuming that $p \geqq 2$, $b \equiv 0$ and $\psi \in W^{2,\infty}$. CHOE & LEWIS [1] moreover have obtained $C^{1,\alpha}$-regularity for bounded solutions u when $\psi \in W^{2,n+\varepsilon}(\Omega)$ and b satisfies the natural growth condition $b(x, u, h) \leqq c(g(x) + |h|^p)$ where $g \in L^{n+\varepsilon}$, $\varepsilon > 0$.

Here we shall prove that solutions of (1) have $C^{0,\alpha}_{\text{loc}}$- or $C^{1,\alpha}_{\text{loc}}$-regularity under various conditions on b, f, ψ in the spirit of [12]. Roughly speaking, we follow the principle that solutions of the obstacle problem (1) should be as regular as solutions of the equation

$$\operatorname{div}(|\nabla u|^{p-2} \nabla u) = \operatorname{div}(|\nabla \psi|^{p-2} \nabla \psi).$$

Since we shall use well-known integral inequalities (e.g., the Morrey growth condition and the Campanato growth condition) for solutions of elliptic equations, and a comparison principle, our proof thus lies closer to the standard context of elliptic partial differential equation theory than the demonstrations given in [3], [7] and [11]. Moreover our $C^{1,\alpha}_{\text{loc}}$-regularity result under the assumption $\psi \in C^{1,\beta}$, $\beta > 0$, is new. We show in Section 3 that the condition $\psi \in C^{1,\beta}$ is necessary for $C^{1,\alpha}_{\text{loc}}$-regularity for u, while the condition $\psi \in W^{2,n+\varepsilon}$, $\varepsilon > 0$, implies that $\psi \in C^{1,\beta}$ for some $\beta > 0$, by the Sobolev imbedding theorem. Finally we shall examine in detail how the regularity of u depends on the regularity assumptions for b, f and ψ.

We define $\|f\|_S$ as the norm of f in the space S and let $B(r) = B(x_0, r) = \{x \in \mathbb{R};\ |x - x_0| < r\}$ with a typical point $x_0 \in \Omega$. Also we define $(u)_r = \frac{1}{|B(r)|} \int_{B(r)} u\, dx$. Throughout the paper c denotes a given constant depending on n, p and various exterior data.

2. $C^{0,\alpha}$-regularity

In this section we shall prove the $C^{0,\alpha}_{\text{loc}}$-regularity of solutions of the variational inequality (1). We can assume that $1 < p \leqq n$, since $C^{0,\alpha}$-regularity is immediate and trivial when $p > n$ in view of the Sobolev Imbedding Theorem. First we prove a Morrey-type growth condition for solutions $w \in W^{1,p}(\Omega)$ of the differential relations

$$\int_\Omega |\nabla w|^{p-2} \nabla w \cdot \nabla \phi\, dx = \int_\Omega b(x, w, \nabla w)\, \phi\, dx + \int_\Omega f \cdot \nabla \phi\, dx \tag{2}$$

for all $\phi \in C_0^\infty(\Omega)$, where b and f satisfy the controllable growth condition

$$|b(x, w, h)| \leqq c_1(a(x) + |w|^{p-1} + |h|^{p-1}), \tag{3}$$

$$f(x) \in L^s(\Omega), \quad s > \frac{n}{p-1}; \quad a(x) \in L^t(\Omega), \quad t > \frac{n}{p}$$

for some $c_1 \geqq 0$. As a preliminary to this result, we note that $w \in L^\infty_{\text{loc}}(\Omega)$ by Moser iteration (see Theorem 1 in [12]).

Define $\delta_1 = n + \frac{p}{p-1} - \frac{np}{t(p-1)}$ and $\delta_2 = n - \frac{np}{s(p-1)}$.

Lemma 1. *Suppose that $B(2r) \subset \Omega$ and $0 < \varepsilon < \min(\delta_1 - n + p, \delta_2 - n + p)$. Then ∇w satisfies the following inequality*

$$\int_{B(\varrho)} |\nabla w|^p\, dx \leqq c_2 \left[r^{\varepsilon(p-1)} + \left(\frac{\varrho}{r}\right)^n \right] \int_{B(r)} |\nabla w|^p\, dx + c_3 r^{\delta_1 - \varepsilon} + c_4 r^{\delta_2 - \varepsilon} \tag{4}$$

for all $\varrho < r/2$, where c_2 depends only on n, p and where c_3, c_4 depend on n, p, $\|w\|_{L^\infty(B(r))}$, $\|f\|_{L^s}$, $\|a\|_{L^t}$ and c_1.

Proof. Let $\overline{w} \in W^{1,p}(B(r))$ be the solution of

$$\int_{B(r)} |\nabla \overline{w}|^{p-2} \nabla \overline{w} \cdot \nabla \phi \, dx = 0 \tag{5}$$

satisfying $\overline{w} = w$ on $\partial B(r)$. By the weak Harnack inequality for $\nabla \overline{w}$, we have

$$\int_{B(\varrho)} |\nabla \overline{w}|^p \, dx \leqq c \left(\frac{\varrho}{r}\right)^n \int_{B(r)} |\nabla \overline{w}|^p \, dx \leqq c \left(\frac{\varrho}{r}\right)^n \int_{B(r)} |\nabla w|^p \, dx \tag{6}$$

for all $\varrho < r/2$, where c depends only on n, p. In turn,

$$\begin{aligned} \int_{B(\varrho)} |\nabla w|^p \, dx &\leqq 2^p \int_{B(\varrho)} |\nabla \overline{w}|^p \, dx + 2^p \int_{B(\varrho)} |\nabla w - \nabla \overline{w}|^p \, dx \\ &\leqq c \left(\frac{\varrho}{r}\right)^n \int_{B(\varrho)} |\nabla w|^p \, dx + 2^p \int_{B(\varrho)} |\nabla w - \nabla \overline{w}|^p \, dx. \end{aligned} \tag{7}$$

If we assume that $2 \leqq p < \infty$, then the last term in (7) can be estimated as follows:

$$\begin{aligned} \int_{B(\varrho)} |\nabla w - \nabla \overline{w}|^p \, dx &\leqq \int_{B(r)} |\nabla w - \nabla \overline{w}|^p \, dx \\ &\leqq c \int_{B(r)} [|\nabla w|^{p-2} \nabla w - |\nabla \overline{w}|^{p-2} \nabla \overline{w}] \cdot [\nabla w - \nabla \overline{w}] \, dx \\ &= c \int_{B(r)} b(x, w, \nabla w) \cdot (w - \overline{w}) + [f - (f)_r] \cdot (\nabla w - \nabla \overline{w}) \, dx \\ &\leqq c \int_{B(r)} (a(x) + 1 + |\nabla w|^{p-1}) \, |w - \overline{w}| \, dx \\ &\quad + c \int_{B(r)} |f - (f)_r| \, |\nabla w - \nabla \overline{w}| \, dx \\ &= \mathrm{I} + \mathrm{II}. \end{aligned} \tag{8}$$

Now assume that $2 \leqq p < n$. By Hölder's inequality,

$$\begin{aligned} \mathrm{I} \leqq c &\left[\int_{B(r)} (a(x) + 1)^{\frac{np}{np+p-n}} \, dx\right]^{\frac{np+p-n}{np}} \left[\int_{B(r)} |w - \overline{w}|^{\frac{np}{n-p}} \, dx\right]^{\frac{n-p}{np}} \\ &+ c \left[\int_{B(r)} |w - \overline{w}|^p \, dx\right]^{\frac{1}{p}} \left[\int_{B(r)} |\nabla w|^p \, dx\right]^{\frac{p-1}{p}}. \end{aligned} \tag{9}$$

By Sobolev's inequality and Hölder's inequality applied to the first term of the right-hand side of (9) and by Poincare's inequality applied to the second term, we have

$$\begin{aligned} \mathrm{I} \leqq c r^{\delta_1\left(\frac{p-1}{p}\right)} &\|a + 1\|_{L^t} \left[\int_{B(r)} |\nabla w - \nabla \overline{w}|^p \, dx\right]^{\frac{1}{p}} \\ &+ c r \left[\int_{B(r)} |\nabla w|^p \, dx\right]^{\frac{p-1}{p}} \left[\int_{B(r)} |\nabla w - \nabla \overline{w}|^p \, dx\right]^{\frac{1}{p}}. \end{aligned} \tag{10}$$

When $p = n$, we see that

$$\int_{B(r)} (a(x)+1)\,|w-\overline{w}|\,dx \leqq \|a+1\|_{L^t}\,\|w-\overline{w}\|_{L^{\frac{t}{t-1}}(B(r))}$$
$$\leqq r^{n\left(\frac{t-1}{t}-\frac{1}{\mu}\right)} \|a+1\|_{L^t} \left[\int_{B(r)} |w-\overline{w}|^\mu\,dx\right]^{\frac{1}{\mu}}$$

for all $n < \mu < \infty$ by Sobolev's inequality and Hölder's inequality. Thus when $p = n$, we obtain (10) with δ_1 replaced by $\delta_1 - \varepsilon'$, for some small $\varepsilon' > 0$.

By Hölder's inequality,

$$\mathrm{II} \leqq cr^{\frac{n(p-1)}{p}-\frac{n}{s}} \|f-(f)_r\|_{L^s} \left[\int_{B(r)} |\nabla w - \nabla\overline{w}|^p\,dx\right]^{\frac{1}{p}}. \tag{11}$$

Combining (8), (10) and (11) and using Young's inequality, we obtain

$$\int_{B(r)} |\nabla w-\nabla\overline{w}|^p\,dx \leqq cr^{\delta_1-\varepsilon'}\|a+1\|_{L^t}^{\frac{p}{p-1}} + cr^{\delta_2}\|f-(f)\|_{L^s}^{\frac{p}{p-1}} + cr^{\frac{p}{p-1}} \int_{B(r)} |\nabla w|^p\,dx, \tag{12}$$

where $\varepsilon' = 0$ when $1 < p < n$ and $\varepsilon' > 0$ when $p = n$. Combining (7) and (12) gives Lemma 1 when $p \geqq 2$.

Now assume that $1 < p < 2$. Then by Hölder's inequality we have

$$\int_{B(r)} |\nabla w - \nabla\overline{w}|^p\,dx \tag{13}$$
$$\leqq c\left[\int_{B(r)} (|\nabla w| + |\nabla\overline{w}|)^p\,dx\right]^{\frac{2-p}{p}} \left[\int_{B(r)} (|\nabla w| + |\nabla\overline{w}|)^{p-2}\,|\nabla w-\nabla\overline{w}|^2\,dx\right]^{\frac{p}{2}}$$
$$\leqq c\left[\int_{B(r)} |\nabla w|^p\,dx\right]^{\frac{2-p}{2}} \left[\int_{B(r)} (|\nabla w|^{p-2}\nabla w - |\nabla\overline{w}|^{p-2}\nabla\overline{w})\cdot(\nabla w-\nabla\overline{w})\,dx\right]^{\frac{p}{2}}.$$

As in the case $p \geqq 2$, the last term in (13) can be estimated as follows:

$$\int_{B(r)} (|\nabla w|^{p-2}\nabla w - |\nabla\overline{w}|^{p-2}\nabla\overline{w})\cdot(\nabla w-\nabla\overline{w})\,dx$$
$$\leqq cr^{\frac{p}{p-1}} \int_{B(r)} |\nabla w|^p\,dx \tag{14}$$
$$+ c\left[r^{\frac{np+p-n}{p}-\frac{n}{t}-\varepsilon'}\|a+1\|_{L^t} + r^{\frac{n(p-1)}{p}-\frac{n}{s}}\|f-(f)_r\|_{L^s}\right]\left[\int_{B(r)} |\nabla w|^p\,dx\right]^{\frac{1}{p}}.$$

Then by Young's inequality, we obtain

$$\int_{B(r)} |\nabla w-\nabla\overline{w}|^p\,dx \leqq cr^{\varepsilon(p-1)} \int_{B(r)} |\nabla w|^p\,dx + cr^{\delta_1-\varepsilon}\|a+1\|_{L^t}^{\frac{p}{p-1}} + cr^{\delta_2-\varepsilon}\|f-(f)_r\|_{L^s}^{\frac{p}{p-1}} \tag{15}$$

for each $0 < \varepsilon < \min(\delta_1 - n + p, \delta_2 - n + p)$. Combining (7) and (15) completes the proof. □

Remark 1. If $t > n$ and $f \in C^{0,\alpha}$, $\alpha > 0$, then $\delta_1 > n$ and $\delta_2 > n$.

Remark 2. Suppose $0 < \nu < \min(\delta_1 - \varepsilon, \delta_2 - \varepsilon, n)$. By iteration [4] there exists a constant r_0 depending on c_1, n, p, ε, s, t, $\|u\|_{L^\infty(B(r_0))}$, $\|a\|_{L^t}$, $\|f\|_{L^s}$ and ν such that, for each $0 < r < r_0$,

$$\int_{B(r)} |\nabla w|^p \, dx \leqq c r^\nu$$

where c is a constant independent of r.

Remark 3. By the imbedding theorem of MORREY we have $w \in C^{0,\alpha}_{\text{loc}}$ for some $\alpha > 0$.

We are now in position to consider the obstacle problem. Suppose $\psi \in W^{1,m}(\Omega)$, $m > n$, and let $u \in W^{1,p}(\Omega)$ satisfy the variational inequality

$$\begin{aligned}\int_\Omega |\nabla u|^{p-2} \nabla u \cdot (\nabla v - \nabla u) \, dx \geqq & \int_\Omega b(x, u, \nabla u)(v - u) \, dx \\ & + \int_\Omega f \cdot (\nabla v - \nabla u) \, dx\end{aligned} \tag{16}$$

for all $v \in \mathscr{C}$. We show that u is locally bounded by following a well-known truncation idea going back to DE GIORGI.

Lemma 2. $u \in L^\infty_{\text{loc}}(\Omega)$.

Proof. Let $k \geqq \sup_\Omega \psi(x)$ and $u_k = (u - k)^+$. Then

$$v(x) = u(x) - u_k(x) \, \eta^p(x) \geqq \psi(x)$$

for all $\eta \in C_0^\infty(\Omega)$, $0 \leqq \eta \leqq 1$, and $v \in \mathscr{C}$. Then from (16) we get

$$\begin{aligned}\int_{u \geqq k} |\nabla u|^p \eta^p fx \leqq c & \int_{u \geqq k} |\nabla \eta|^p |u_k|^p \, dx + c \int_{u \geqq k} a^{\frac{p}{p-1}} \, dx \\ & + c \int_{u \geqq k} (|u_k|^p + k^p) \, dx + c \int_{u \geqq k} |f|^{\frac{p}{p-1}} \, dx.\end{aligned}$$

Since $t > \frac{n}{p}$ and $s > \frac{n}{p-1}$, we can use Lemma 5.4 in [8] to show that u is locally bounded. □

Define $\delta_3 = n - \frac{np}{m} > n - p$.

Theorem 1. *Suppose that* $B(2R_0) \subset \Omega$ *and* ε *is such that* $0 < \varepsilon < \min(\delta_1, \delta_2, \delta_3) - n + p$. *Then there exists an* $r_0 \in (0, R_0)$ *such that for each* $r < r_0$,

$$\int_{B(\varrho)} |\nabla u|^p \, dx \leqq c_5 \left(r^{\varepsilon(p-1)} + \left(\frac{\varrho}{r}\right)^n \right) \int_{B(r)} |\nabla u|^p \, dx + c_6 r^{\delta_1 - \varepsilon} + c_7 r^{\delta_2 - \varepsilon} + c_8 r^{\delta_3 - \varepsilon} \tag{17}$$

for all $0 < \varrho < \frac{r}{2}$, *where* c_5 *depends on* p *and* n, c_6 *depends on* $\|u\|_{L^\infty(B(R_0))}$ *for and* $\|a\|_{L^t}$, c_7 *depends on* $\|f\|_{L^s}$, *and* c_8 *depends on* $\|\nabla \psi\|_{L^m}$. *Consequently* $u \in C^{0,\alpha}_{\text{loc}}$ *for some* $\alpha > 0$.

Proof. Let $\bar{u} \in W^{1,p}(B(r))$ satisfy

$$\int_{B(r)} |\nabla \bar{u}|^{p-2} \nabla \bar{u} \cdot \nabla \phi \, dx = \int_{B(r)} |\nabla \psi|^{p-2} \nabla \psi \cdot \nabla \phi \, dx \tag{18}$$

for all $\phi \in C_0^\infty(B(r))$ and $\bar{u} = u$ on $\partial B(r)$. Then by the maximum principle, $\bar{u}(x) \geqq \psi(x)$ in $B(r)$, since $u \geqq \psi$ on $\partial B(r)$. We also have

$$\int_{B(r)} |\nabla u|^{p-2} \nabla u \cdot (\nabla \bar{u} - \nabla u) \, dx \geqq \int_{B(r)} b(x, u, \nabla u)(\bar{u} - u) + f \cdot (\nabla \bar{u} - \nabla u) \, dx,$$

since $u - \bar{u} \in W_0^{1,p}(B(r))$ and $\bar{u} \geqq \psi$ in $B(r)$.

Applying Lemma 1 to $\bar{u}$ we have

$$\int_{B(\varrho)} |\nabla \bar{u}|^p \, dx \leqq c \left(r^{\varepsilon(p-1)} + \left(\frac{\varrho}{r}\right)^n \right) \int_{B(r)} |\nabla \bar{u}|^p \, dx + c_8 r^{\delta_3 - \varepsilon}, \tag{19}$$

where c_8 depends on n, p and $\|\nabla \psi\|_{L^m}$. With $\phi = \bar{u} - u$ in (18), an application of Young's inequality and Hölder's inequality yields

$$\begin{aligned} \int_{B(r)} |\nabla \bar{u}|^p \, dx &\leqq c \int_{B(r)} |\nabla u|^p \, dx + c \int_{B(r)} |\nabla \psi|^p \, dx \\ &\leqq c \int_{B(r)} |\nabla u|^p \, dx + c r^{\delta_3} \|\nabla \psi\|_{L^m}^p. \end{aligned} \tag{20}$$

Now from (19) and (20),

$$\begin{aligned} \int_{B(\varrho)} |\nabla u|^p \, dx &\leqq c \int_{B(\varrho)} |\nabla \bar{u}|^p \, dx + c \int_{B(\varrho)} |\nabla u - \nabla \bar{u}|^p \, dx \\ &\leqq c \left(r^{\varepsilon(p-1)} + \left(\frac{\varrho}{r}\right)^n \right) \left[\int_{B(r)} |\nabla u|^p \, dx + r^{\delta_3} \|\nabla \psi\|_m^p \right] \qquad (21) \\ &\quad + c r^{\delta_3 - \varepsilon} + c \int_{B(\varrho)} |\nabla u - \nabla \bar{u}|^p \, dx. \end{aligned}$$

Assume $p \geqq 2$. Then

$$
\begin{aligned}
\int_{B(r)} |\nabla u - \nabla \bar{u}|^p \, dx &\leqq c \int_{B(r)} [|\nabla u|^{p-2} \nabla u - |\nabla \bar{u}|^{p-2} \nabla \bar{u}] \cdot [\nabla u - \nabla \bar{u}] \, dx \\
&\leqq c \int_{B(r)} b(x, u, \nabla u)(u - \bar{u}) + f \cdot (\nabla u - \nabla \bar{u}) \, dx \\
&\quad - \int_{B(r)} |\nabla \psi|^{p-2} \nabla \psi \cdot (\nabla u - \nabla \bar{u}) \, dx \\
&= \mathrm{III} + \mathrm{IV}.
\end{aligned} \tag{22}
$$

As in Lemma 1, we see that

$$
\mathrm{III} \leqq \mu \int_{B(r)} |\nabla u - \nabla \bar{u}|^p \, dx + cr^{\frac{p}{p-1}} \int_{B(r)} |\nabla u|^p \, dx + cr^{\delta_1 - \varepsilon'} + cr^{\delta_2} \tag{23}
$$

and

$$
\mathrm{IV} \leqq \mu \int_{B(r)} |\nabla u - \nabla \bar{u}|^p \, dx + cr^{\delta_3}, \tag{24}
$$

for some small μ depending only on n, p, where $\varepsilon' = 0$ for $p < n$ and $\varepsilon' > 0$ when $p = n$. Combining (21) through (24) we obtain Theorem 1 when $p \geqq 2$.

Now assume that $1 < p < 2$. By Young's inequality,

$$
\begin{aligned}
&\int_{B(\varrho)} |\nabla u - \nabla \bar{u}|^p \, dx \\
&\quad \leqq \int_{B(r)} |\nabla u - \nabla \bar{u}|^p \, dx \qquad\qquad (25) \\
&\quad \leqq c \left[\int_{B(r)} |\nabla u|^p + |\nabla \bar{u}|^p \, dx \right]^{\frac{2-p}{p}} \left[\int_{B(r)} (|\nabla u| + |\nabla \bar{u}|)^{p-2} |\nabla u - \nabla \bar{u}|^2 \, dx \right]^{\frac{p}{2}} \\
&\quad \leqq c \left[\int_{B(r)} |\nabla u|^p \, dx + r^{\delta_3} \right]^{\frac{2-p}{2}} \left[\int_{B(r)} (|\nabla u| + |\nabla \bar{u}|)^{p-2} |\nabla u - \nabla \bar{u}|^2 \, dx \right]^{\frac{p}{2}}.
\end{aligned}
$$

As in the proof of Lemma 1, we also have

$$
\int_{B(r)} |\nabla u - \nabla \bar{u}|^p \, dx \leqq cr^{\varepsilon(p-1)} \int_{B(r)} |\nabla u|^p \, dx + cr^{\delta_1 - \varepsilon} + cr^{\delta_2 - \varepsilon} + cr^{\delta_3 - \varepsilon}. \tag{26}
$$

Combining (21), (25) and (26) completes the proof of the theorem. □

3. $C^{1,\alpha}_{\mathrm{loc}}$-regularity

First we prove a Campanato-type growth condition for solutions $w \in W^{1,p}(\Omega)$ of

$$
\int_{\Omega} |\nabla w|^{p-2} \nabla w \cdot \nabla \phi \, dx = \int_{\Omega} b(x, w, \nabla w) \phi \, dx + \int_{\Omega} f \cdot \nabla \phi \, dx, \tag{27}
$$

where b and f satisfy a controllable growth condition

$$
\begin{gathered}
|b(x, w, h)| \leqq c_1(a(x) + |w|^{p-1} + |h|^{p-1}), \qquad (28) \\
f(x) \in C^\beta(\Omega), \ \beta > 0; \quad a(x) \in L^t(\Omega), \ t > n,
\end{gathered}
$$

for some $c_1 \geqq 0$. Suppose $\nu \in (0, n)$ is fixed number and define

$$\delta_4 = n + \frac{p}{p-1}\left(1 - \frac{n}{t}\right), \quad \delta_5 = n + \frac{\beta p}{p-1}, \quad \delta_6 = n + \frac{p}{p-1} + (\nu - n)$$

when $p \geqq 2$, and

$$\delta_4 = n + p\left(1 - \frac{n}{t}\right) + (\nu - n)(2 - p), \quad \delta_5 = n + \beta p + (\nu - n)(2 - p),$$

$$\delta_6 = n + p + (\nu - n)$$

when $1 < p < 2$.

Lemma 3. *Suppose $B(2R_0) \subset \Omega$. Then there exists an $r_0 \in (0, R_0)$, depending on ν, such that for each $r < r_0$ and $\varrho < \frac{r}{2}$,*

$$\int_{B(\varrho)} |\nabla w - (\nabla w)_\varrho|^p \, dx \leqq c_2 \left(\frac{\varrho}{r}\right)^{n+\delta} \int_{B(r)} |\nabla w - (\nabla w)_r|^p \, dx + c_3 r^{\delta_4} + c_4 r^{\delta_5} + c_5 r^{\delta_6} \tag{29}$$

for some $\delta > 0$, where c_2 depends only on n and p, and c_3, c_4, c_5 depend on n, p, ε, $\|f\|_{C^\beta}$, and c_1.

Proof. As in Lemma 1 we let $\bar{w} \in W^{1,p}(B(r))$ satisfy

$$\int_{B(r)} |\nabla \bar{w}|^p \nabla \bar{w} \cdot \nabla \phi \, dx = 0, \qquad \bar{w} = w \text{ on } \partial B(r)$$

for all $\phi \in C_0^\infty(B(r))$. By following a method due to G. LIEBERMAN (see the proof of Lemma 5.1 in [6]), we obtain

$$\int_{B(\varrho)} |\nabla \bar{w} - (\nabla \bar{w})_\varrho|^p \, dx \leqq c \left(\frac{\varrho}{r}\right)^{n+\delta} \int_{B(r)} |\nabla \bar{w} - (\nabla \bar{w})_r|^p \, dx \tag{30}$$

for some $\delta > 0$ and

$$\int_{B(r)} |\nabla \bar{w} - (\nabla \bar{w})_r|^p \, dx \leqq c \int_{B(r)} |\nabla w - (\nabla w)_r|^p \, dx + c \int_{B(r)} |\nabla w - \nabla \bar{w}|^p \, dx. \tag{31}$$

Hence,

$$\begin{aligned} \int_{B(\varrho)} |\nabla w - (\nabla w)_\varrho|^p \, dx & \\ \leqq c \int_{B(\varrho)} |\nabla \bar{w} - (\nabla \bar{w})_\varrho|^p \, dx + c \int_{B(r)} |\nabla w - \nabla \bar{w}|^p \, dx & \quad (32) \\ \leqq c \left(\frac{\varrho}{r}\right)^{n+\delta} \int_{B(r)} |\nabla w - (\nabla w)_r|^p \, dx + c \int_{B(r)} |\nabla w - \nabla \bar{w}|^p \, dx. & \end{aligned}$$

When $p \geqq 2$, we can show as in the proof of Lemma 1 that

$$\int_{B(r)} |\nabla w - \nabla \overline{w}|^p \, dx \leqq cr^{\delta_4} + cr^{\delta_5} + cr^{\frac{p}{p-1}} \int_{B(r)} |\nabla w|^p \, dx. \tag{33}$$

By Remark 2 we see that for each $0 < \nu < n$ there exists an r_0 such that

$$\int_{B(r)} |\nabla w|^p \, dx \leqq cr^{\nu} \tag{34}$$

for all $r < r_0$. Lemma 3 for $p \geqq 2$ now follows by combining (32), (33) and (34).

When $1 < p < 2$, we see that

$$\int_{B(r)} |\nabla w - \nabla \overline{w}|^p \, dx \tag{35}$$
$$\leqq \left(\int_{B(r)} (|\nabla w| + |\nabla \overline{w}|)^p \, dx \right)^{\frac{2-p}{2}} \left(\int_{B(r)} (|\nabla w| + |\nabla \overline{w}|)^{p-2} |\nabla w - \nabla \overline{w}|^2 \, dx \right)^{\frac{p}{2}}.$$

Also, as in Lemma 1,

$$\int_{B(r)} (|\nabla w| + |\nabla \overline{w}|)^p \, dx \leqq c \int_{B(r)} |\nabla w|^p \, dx \leqq cr^{\nu}, \tag{36}$$

$$\int_{B(r)} (|\nabla w| + |\nabla \overline{w}|)^{p-2} |\nabla w - \nabla \overline{w}|^2 \, dx \tag{37}$$
$$\leqq c \int_{B(r)} (a(x) + 1 + |\nabla w|^{p-1}) \, |w - \overline{w}| \, dx + \int_{B(r)} |f - (f)_r| \, |\nabla w - \nabla \overline{w}| \, dx$$
$$= \text{V} + \text{VI}.$$

Clearly, as before (see Lemma 1),

$$\begin{aligned} \text{V} \leqq c \, \|a + 1\|_{L^t} \, r^{n+1-\frac{n}{p}-\frac{n}{t}} \left[\int_{B(r)} |\nabla w - \nabla \overline{w}|^p \, dx \right]^{1/p} \\ + c \cdot r^{\nu\frac{p-1}{p}} \cdot r \left(\int_{B(r)} |\nabla w - \nabla \overline{w}|^p \, dx \right)^{1/p}, \end{aligned} \tag{38}$$

where we used Poincaré's inequality for the last term.

Similarly VI can be estimated by

$$\begin{aligned} \text{VI} &\leqq \left[\int_{B(r)} |f - (f)_r|^{\frac{p}{p-1}} \right]^{\frac{p-1}{p}} \left[\int_{B(r)} |\nabla w - \nabla \overline{w}|^p \, dx \right]^{1/p} \\ &\leqq cr^{n\frac{p-1}{p}+\beta} \left[\int_{B(r)} |\nabla w - \nabla \overline{w}|^p \, dx \right]^{1/p}. \end{aligned} \tag{39}$$

Combining (35)–(39), we have

$$\begin{aligned} \int_{B(r)} |\nabla w - \nabla \overline{w}|^p \, dx \leqq cr^{\nu\left(\frac{2-p}{2}\right)} \Big(r^{n+1-\frac{n}{p}-\frac{n}{t}} + r^{\nu\frac{p-1}{p}+1} \\ + r^{n\frac{p-1}{p}+\beta} \Big)^{\frac{p}{2}} \left(\int_{B(r)} |\nabla w - \nabla \overline{w}|^p \, dx \right)^{1/2}, \end{aligned} \tag{40}$$

whence by Young's inequality

$$\int_{B(r)} |\nabla w - \nabla \bar{w}|^p \, dx \leqq cr^{n+p-\frac{n}{t}p+(\nu-n)(2-p)} + cr^{n+p+(\nu-n)} + cr^{n+\beta p+(\nu-n)(2-p)}. \tag{41}$$

The required estimate now follows from (32) and (41). □

By using Lemma 3 we can show $C^{1,\alpha}_{\text{loc}}$-regularity for obstacle problems. Suppose $\psi \in C^{1,\gamma}(\Omega)$, $\gamma > 0$, and let $u \in W^{1,p}(\Omega)$ satisfy the variational inequality

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot (\nabla v - \nabla u) \, dx \geqq \int_\Omega b(x, u, \nabla u)(v - u) \, dx + f \cdot (\nabla v - \nabla u) \, dx \tag{42}$$

for all $v \in \mathscr{C}$. Again we assume that $\nu \in (0, n)$. Define $\delta_7 = n + \gamma \dfrac{p}{p-1}$ when $2 \leqq p < \infty$, and $\delta_7 = n + \gamma p(p-1) + (\nu - n)(2-p)$ when $1 < p < 2$.

Theorem 2. *Suppose $B(2R_0) \subset \Omega$. Then there exists an $r_0 \in (0, R_0)$ depending on ν, such that for each $r < r_0$*

$$\int_{B(\varrho)} |\nabla u - (\nabla u)_\varrho|^p \, dx \tag{43}$$
$$\leqq c \left(\frac{\varrho}{r}\right)^{n+\delta} \int_{B(r)} |\nabla u - (\nabla u)_r|^p \, dx + cr^{\delta_4} + cr^{\delta_5} + cr^{\delta_6} + cr^{\delta_7}$$

for all $0 < \varrho < r/2$. Consequently $u \in C^{1,\alpha}_{\text{loc}}(\Omega)$ for some $\alpha > 0$.

Proof. As in the proof of Theorem 1, we define $\bar{u} \in W^{1,p}(B(r))$ to be the solution to

$$\int_{B(r)} |\nabla \bar{u}|^{p-2} \nabla \bar{u} \cdot \nabla \phi \, dx = \int |\nabla \psi|^{p-2} \nabla \psi \cdot \nabla \phi \, dx$$

for all $\phi \in C_0^\infty(B(r))$, with $\bar{u} = u$ on $\partial B(r)$. Then by Lemma 3 we have

$$\int_{B(\varrho)} |\nabla \bar{u} - (\nabla \bar{u})_\varrho|^p \, dx \leqq c \left(\frac{\varrho}{r}\right)^{n+\delta} \int_{B(r)} |\nabla \bar{u} - (\nabla \bar{u})_r|^p \, dx + cr^{\delta_7}. \tag{44}$$

As before,

$$\int_{B(\varrho)} |\nabla u - \nabla \bar{u}|^p \, dx \leqq c \left(\frac{\varrho}{r}\right)^{n+\delta} \int_{B(r)} |\nabla u - (\nabla u)_r|^p dx + c \int_{B(r)} |\nabla u - \nabla \bar{u}|^p \, dx. \tag{45}$$

As in the proof of Theorem 1, it is evident that $\bar{u}$ is an admissible competing function in the class $\mathscr{C}$ for the domain $B(r)$. Assume that $2 \leqq p < \infty$. In this case $|\nabla\psi|^{p-2}\nabla\psi \in C^{0,\gamma}$. Hence

$$\begin{aligned}\int_{B(r)} |\nabla u - \nabla\bar{u}|^p\,dx &\leqq c \int_{B(r)} (|\nabla u|^{p-2}\nabla u - |\nabla\bar{u}|^{p-2}\nabla\bar{u})\cdot(\nabla u - \nabla\bar{u})\,dx \qquad (46)\\ &\leqq c \int_{B(r)} b(x,u,\nabla u)\,(u-\bar{u})\,dx + c\int_{B(r)} f\cdot(\nabla u - \nabla\bar{u})\,dx\\ &\quad - \int_{B(r)} [|\nabla\psi|^{p-2}\nabla\psi - (|\nabla\psi|^{p-2}\nabla\psi)_r]\cdot(\nabla u - \nabla\bar{u})\,dx.\end{aligned}$$

An estimate of the right-hand side follows exactly as in the proof of Lemma 3, it yields the proof of Theorem 2.

Now assume $1 < p < 2$. It is easy to see that $|\nabla\psi|^{p-2}\nabla\psi \in C^{0,\gamma(p-1)}$. Again using Hölder's inequality, we have

$$\begin{aligned}\int_{B(r)} |\nabla u - \nabla\bar{u}|^p\,dx &\leqq \left[\int_{B(r)} (|\nabla u| + |\nabla\bar{u}|)^p\,dx\right]^{\frac{2-p}{2}} \qquad (47)\\ &\quad\times\left[\int_{B(r)} (|\nabla u|^{p-2}\nabla u - |\nabla\bar{u}|^{p-2}\nabla\bar{u})\cdot(\nabla u - \nabla\bar{u})\right]^{p/2}.\end{aligned}$$

We already know that

$$\int_{B(r)} (|\nabla u| + |\nabla\bar{u}|)^p\,dx \leqq c\int_{B(r)} |\nabla u|^p\,dx + c\int_{B(r)} |\nabla\psi|^p\,dx \leqq c\cdot r^\nu \qquad (48)$$

for some $R_0 > 0$ and for all $0 < r < R_0$. Also

$$\begin{aligned}\int_{B(r)} (|\nabla u|^{p-2}\nabla u - |\nabla\bar{u}|^{p-2}\nabla\bar{u})\cdot(\nabla u - \nabla\bar{u})\,dx &\leqq \int b(x,u,\nabla u)\cdot(u-\bar{u})\,dx + \int f\cdot(\nabla u - \nabla\bar{u})\,dx\\ &\quad - \int [|\nabla\psi|^{p-2}\nabla\psi - (|\nabla\psi|^{p-2}\nabla\psi)_r]\cdot(\nabla u - \nabla\bar{u})\,dx.\end{aligned}$$

The right-hand side of (49) can be estimated as in the proof of Lemma 3; this estimate yields the proof of Theorem 2. □

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