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A Well-Posed Problem for the Exterior Stokes Equations in Two and Three Dimensions

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Abstract

This paper treats the Stokes problem in exterior Lipschitz-continuous domains of $\mathbb{R}^2$ and $\mathbb{R}^3$. Using the weighted Sobolev spaces of HANOUZET (in $\mathbb{R}^3$) and GIROIRE (in $\mathbb{R}^2$), we establish the inf-sup condition between the velocity and pressure spaces. This fundamental result shows that the variational Stokes problem is well-posed in those spaces. In the last paragraph, we obtain additional regularity of the solution when the data are smoother.

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^n$ ($n = 2$ or 3) with a Lipschitz-continuous boundary Γ, and let Ω' denote the complement of $\bar{\Omega}$. This paper treats the steady-state nonhomogeneous Stokes flow in Ω', which is governed by

$$\text{(S)} \qquad \begin{cases} -\nu\,\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f}, \quad \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega', \\ \boldsymbol{u}|_\Gamma = \boldsymbol{g}, \end{cases}$$

with a condition on $\boldsymbol{u}$ at infinity expressed by

$$\int_{\Omega'} \|\nabla \boldsymbol{u}\|^2 \, dx < +\infty, \qquad \int_{\Omega'} \frac{1}{\omega^2} \|\boldsymbol{u}\|^2 \, dx < +\infty$$

for an appropriate weight function ω that depends upon the dimension. This weight stems very naturally from Hardy's inequalities [Hr], under the implicit assumption that $\boldsymbol{u}$ is the limit of a sequence of smooth functions that vanish at infinity. The force $\boldsymbol{f}$ is given in the dual of the velocity space, and the boundary value $\boldsymbol{g}$ is given in $(H^{\frac{1}{2}}(\Gamma))^n$. As usual, the viscosity ν is a positive constant.

We shall put problem (S) in an equivalent variational form, show that it is well-posed, *i.e.*, that it has a unique solution $(\boldsymbol{u}, p)$ that depends continuously upon the data f and g, and derive additional regularity results when the boundary Γ and the data are smoother. In this case, the weak solution, *i.e.*, the solution of the variational problem, coincides with the strong solution.

The exterior Stokes (and Navier-Stokes) problem is very challenging; many authors, using different approaches such as semi-groups, potential theory, weighted spaces, and weighted equations, have contributed to it. Without being exhaustive, let us cite the work of BABENKO [Ba], FINN $[\mathrm{Fi}]_1$, $[\mathrm{Fi}]_2$, FUJITA [Fu], GILBARG & WEINBERGER [Gl], HEYWOOD $[\mathrm{He}]_1$, $[\mathrm{He}]_2$, LADYZHENSKAYA & SOLONNIKOV [La], LERAY $[\mathrm{Le}]_1$, $[\mathrm{Le}]_2$, MA [Ma], MASUDA [Ms], SMITH [Sm]; and more recently, the work of AMICK [Am], SEQUEIRA $[\mathrm{Se}]_1$, $[\mathrm{Se}]_2$, $[\mathrm{Se}]_3$, GUIRGUIS [Gu], SPECOVIUS-NEUGEBAUER [Sp], SOHR & VARNHORN [So], and BORCHERS & SOHR [Bo]. Our present work follows the approach of $[\mathrm{Se}]_1$, $[\mathrm{Se}]_2$ and completes it. As in the latter references, we seek the solution in the weighted spaces studied by HANOUZET [Ha] in three dimensions and by GIROIRE [Gr] in two dimensions, so that the same analysis applies to two and three dimensions. Moreover, we extend the results of SEQUEIRA to the case of a Lipschitz-continuous boundary and we eliminate the restriction imposed by this reference (and others) on the boundary data, namely that $\int_\Gamma \boldsymbol{g} \cdot \boldsymbol{n} \, ds = 0$. Finally, we derive further regularity results when the data and boundary are more regular. Our proofs are simple and concise because we make constant use of sharp isomorphisms established by [Gr], as well as general results concerning saddle-point problems.

This variational formulation has the advantage of being well adapted to numerical solution by finite elements coupled with boundary integrals. The numerical implementation can be found in $[\mathrm{Se}]_3$ for the Stokes problem, following a technique introduced by JOHNSON & NEDELEC [Jo] for the Laplace equation.

2. Notations and Preliminary Results

As mentioned above, to define a suitable functional setting for the variational solution of the exterior Stokes problem we need to use weighted Sobolev spaces. In this section we give some notation and a brief survey of the most important results we shall need in the sequel.

From now on, let Ω be a bounded open set in $\mathbb{R}^n$ ($n = 2$ or 3) with a Lipschitz-continuous boundary Γ, and let Ω' be the complement of its closure in $\mathbb{R}^n$. We denote by $\boldsymbol{n}$ the unit normal to Γ, pointing outside Ω, which exists almost everywhere on Γ. Furthermore, let $x = (x_i)$, $i = 1, \ldots,$ let n be a typical point in $\mathbb{R}_n$, and let $r = r(x)$ be its distance to the origin. We use the customary multi-index notation

$$|\lambda| = \sum_{i=1}^{n} \lambda_i, \qquad D^\lambda = \frac{\partial^{|\lambda|}}{\partial x_1^{\lambda_1} \ldots \partial x_n^{\lambda_n}}$$

for any nonnegative integers λ_i. Let $\varrho(r) = (1 + r^2)^{\frac{1}{2}}$ and $\lg r = \ln(2 + r^2)$. For any nonnegative integer m and for any $\alpha \in \mathbb{R}$ we define the weighted Sobolev space $W^m_\alpha(\Omega')$ by

$$W^m_\alpha(\Omega') = \{u \in \mathscr{D}'(\Omega') : \varrho(r)^{\alpha-m+|\lambda|} (\lg r)^{-1} D^\lambda u \in L^2(\Omega'), \forall\, \lambda, 0 \leqq |\lambda| \leqq k;$$
$$\varrho(r)^{\alpha-m+|\lambda|} D^\lambda u \in L^2(\Omega'), \forall\, \lambda, k+1 \leqq |\lambda| \leqq m\},$$

where k is such that

$$k = \begin{cases} m - \left(\frac{n}{2} + \alpha\right) & \text{if } \frac{n}{2} + \alpha = 1, 2, \ldots, m; \\ -1 & \text{otherwise.} \end{cases}$$

These spaces have been introduced in [Ha]. We briefly mention some basic properties we shall need. The details can be found in [Ha], [Gr].

1. $W^m_\alpha(\Omega')$ is a Hilbert space, provided with its natural norm

$$\|u\|_{m,\alpha,\Omega'} = \left[\sum_{0 \leqq |\lambda| \leqq k} \|\varrho(r)^{\alpha-m+|\lambda|} (\lg r)^{-1} D^\lambda u\|^2_{L^2(\Omega')} + \sum_{k+1 \leqq |\lambda| \leqq m} \|\varrho(r)^{\alpha-m+|\lambda|} D^\lambda u\|^2_{L^2(\Omega')}\right]^{\frac{1}{2}}$$

and associated seminorm

$$|u|_{m,\alpha,\Omega'} = \left[\sum_{|\lambda|=m} \|\varrho(r)^\alpha D^\lambda u\|^2_{L^2(\Omega')}\right]^{\frac{1}{2}}.$$

2. The following imbeddings are continuous

$$W^m_\alpha(\Omega') \hookrightarrow W^{m-1}_{\alpha-1}(\Omega') \hookrightarrow \ldots \hookrightarrow W^0_{\alpha-m}(\Omega').$$

3. The space $\mathscr{D}(\bar{\Omega}')$ of indefinitely differentiable functions with compact support in $\bar{\Omega}'$ is dense in $W^m_\alpha(\Omega')$.
4. Multiplication by a function of $\mathscr{D}(\bar{\Omega}')$ is a linear continuous mapping from $W^m_\alpha(\Omega')$ into $H^m(\Omega')$. This fact allows us to use the standard properties of the usual Sobolev spaces, especially the trace theorems.
5. The completion of the space $\mathscr{D}(\Omega')$ in $W^m_\alpha(\Omega')$ for the appropriate norm is

$$\mathring{W}^m_\alpha(\Omega') = \left\{u \in W^m_\alpha(\Omega'); \left.\frac{\partial^j u}{\partial n^j}\right|_\Gamma = 0,\ j = 0, \ldots, m-1\right\},$$

where $\frac{\partial}{\partial n}$ denotes the normal derivative. Its dual space is $W^{-m}_{-\alpha}(\Omega')$ with the norm

$$\|u\|_{-m,-\alpha,\Omega'} = \sup_{v \in \mathring{W}^m_\alpha(\Omega')} \frac{\langle u, v\rangle}{\|v\|_{m,\alpha,\Omega'}}.$$

6. For any nonnegative integer m and for any real number β, multiplication by $\varrho(r)^\beta$ is an isomorphism from $W^m_\alpha(\Omega')$ onto $W^m_{\alpha-\beta}(\Omega')$ and from $\mathring{W}^m_\alpha(\Omega')$ onto

$\overset{\circ}{W}{}^{m}_{\alpha-\beta}(\Omega')$, provided that neither $n/2+\alpha$ nor $n/2+\alpha-\beta$ belongs to $\{1, 2, \ldots, m\}$.

7. The seminorm $|\cdot|_{m,\alpha,\Omega'}$ is a norm on $\overset{\circ}{W}{}^{m}_{\alpha}(\Omega')$ equivalent to $\|\cdot\|_{m,\alpha,\Omega'}$. (This result follows from Hardy's inequality; the proof can be found in [Gr] or [Ha].) In particular,

$$\|u\|_{1,0,\Omega'} \leqq C\,|u|_{1,0,\Omega'}, \quad \forall\, u \in \overset{\circ}{W}{}^{1}_{0}(\Omega'). \tag{2.1}$$

8. Let $m - \frac{n}{2} - \alpha \geqq 0$. Then

$$W^m_\alpha(\Omega') \supset \mathbb{P}_{q'},$$

where $\mathbb{P}_{q'}$ is the space of polynomials of degree less than or equal to q', with

$$q' = \begin{cases} m - \dfrac{n}{2} - \alpha & \text{if } k \neq -1, \\ \text{largest integer strictly less than } m - \dfrac{n}{2} - \alpha & \text{if } k = -1. \end{cases}$$

Thus setting $q = \min(q', m-1)$ we have in particular

$$W^m_\alpha(\Omega') \supset \mathbb{P}_q.$$

9. The seminorm $|\cdot|_{m,\alpha,\Omega'}$ is a norm on $W^m_\alpha(\Omega')/\mathbb{P}_q$ equivalent to the quotient norm.

Remark 2.1. $W^0_0(\Omega') = L^2(\Omega')$. In the sequel we shall most frequently use the following particular spaces:

For $n = 2$,

$$W^1_0(\Omega') = \{u \in \mathscr{D}'(\Omega') : \varrho(r)^{-1}(\lg r)^{-1}\, u \in L^2(\Omega'),\ \nabla u \in L^2(\Omega')\},$$

$$W^2_0(\Omega') = \{u \in \mathscr{D}'(\Omega') : \varrho(r)^{-2}(\lg r)^{-1}\, u \in L^2(\Omega'),\ \varrho(r)^{-1}(\lg r)^{-1}\,\nabla u \in L^2(\Omega'),\ D^2 u \in L^2(\Omega')\};$$

For $n = 3$,

$$W^1_0(\Omega') = \{u \in \mathscr{D}'(\Omega') : \varrho(r)^{-1}\, u \in L^2(\Omega'),\ \nabla u \in L^2(\Omega')\},$$

$$W^2_0(\Omega') = \{u \in \mathscr{D}'(\Omega') : \varrho(r)^{-2}\, u \in L^2(\Omega'),\ \varrho(r)^{-1}\,\nabla u \in L^2(\Omega'),\ D^2 u \in L^2(\Omega')\};$$

For $n = 2$ or 3,

$$W^1_1(\Omega') = \{u \in \mathscr{D}'(\Omega') : u \in L^2(\Omega'),\ \varrho(r)\,\nabla u \in L^2(\Omega')\},$$

$$W^1_{-1}(\Omega') = \{u \in \mathscr{D}'(\Omega') : \varrho(r)^{-2}\, u \in L^2(\Omega'),\ \varrho(r)^{-1}\,\nabla u \in L^2(\Omega')\}.$$

All these spaces are equipped with their natural norms and seminorms.

Remark 2.2. Property 8 implies that $W^1_1(\Omega')$ (for $n = 2$) and $W^1_1(\Omega')$ and $W^1_0(\Omega')$ (for $n = 3$) contain no polynomials, that $W^1_0(\Omega')$ and $W^1_{-1}(\Omega')$ (for $n = 2$) and $W^1_{-1}(\Omega')$ and $W^2_0(\Omega')$ (for $n = 3$) contain $\mathbb{P}_0$, and that $W^2_0(\Omega')$ (for $n = 2$) contains $\mathbb{P}_1$.

3. Variational Formulation of the Exterior Stokes Problem in the Primitive Variables

From now on we shall often deal with vector-valued functions and extend naturally all the previous norms to vectors as follows: If $\boldsymbol{v} = (v_1, \ldots, v_n)$, then

$$\|\boldsymbol{v}\|_{m,\alpha,\Omega'} = \left(\sum_{i=1}^{n} \|v_i\|^2_{m,\alpha,\Omega'}\right)^{\frac{1}{2}}.$$

For such vectors we recall that the divergence operator is defined by

$$\operatorname{div} \boldsymbol{v} = \sum_{i=1}^{n} \frac{\partial v_i}{\partial x_i}$$

and we note the identity

$$\operatorname{div}(\nabla v) = \Delta v.$$

Let us introduce the Hilbert spaces

$$(3.1) \qquad \begin{aligned} M &= L^2(\Omega') && \text{normed by } \|\cdot\|_M = \|\cdot\|_{0,\Omega'}, \\ X &= (\mathring{W}^1_0(\Omega'))^n && \text{normed by } \|\cdot\|_X = \|\cdot\|_{1,0\Omega'}, \end{aligned}$$

and let $M' = L^2(\Omega')$ and $X' = (W_0^{-1}(\Omega'))^n$ be their corresponding dual spaces with norms $\|\cdot\|_{M'} = \|\cdot\|_{0,\Omega'}$ and $\|\cdot\|_{X'} = \|\cdot\|_{-1,0,\Omega'}$, respectively. As usual, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between the spaces M and M' or X and X'.

Furthermore we require the following Hilbert spaces:

$$(3.2) \qquad \begin{aligned} V &= \{\boldsymbol{v} \in (\mathring{W}^1_0(\Omega'))^n : \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega'\}, \\ V^\perp &= \{\boldsymbol{v} \in (\mathring{W}^1_0(\Omega'))^n : (\nabla \boldsymbol{v}, \nabla \boldsymbol{w}) = 0, \forall\, \boldsymbol{w} \in V\}, \\ V^0 &= \{\boldsymbol{f} \in (W_0^{-1}(\Omega'))^n : \langle \boldsymbol{f}, \boldsymbol{w}\rangle = 0, \forall\, \boldsymbol{w} \in V\}. \end{aligned}$$

In order to study the nonhomogeneous Dirichlet problem (S) in a velocity-pressure formulation, we begin by lifting boundary values with divergence-free functions in Ω'.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^n$ ($n = 2$ or 3) have a Lipschitz-continuous boundary Γ that is not necessarily connected, but has no interior connected component (i.e., Ω has no "holes"). Then, for each $\boldsymbol{g} \in (H^{\frac{1}{2}}(\Gamma))^n$ such that*

$$(3.3) \qquad \int_\Gamma \boldsymbol{g} \cdot \boldsymbol{n}\, ds = 0,$$

there exists a function $\boldsymbol{u}_0 \in (H^1(\Omega'))^n$ with compact support satisfying

$$(3.4) \qquad \operatorname{div} \boldsymbol{u}_0 = 0 \quad \text{in } \Omega', \qquad \boldsymbol{u}_{0|_\Gamma} = \boldsymbol{g},$$

$$(3.5) \qquad \|\boldsymbol{u}_0\|_{1,\Omega'} \leqq C\,\|\boldsymbol{g}\|_{\frac{1}{2},\Gamma},$$

where the constant $C > 0$ is independent of $\boldsymbol{u}_0$ and $\boldsymbol{g}$. Moreover, the mapping $\boldsymbol{g} \to \boldsymbol{u}_0$ is linear.

Proof. Let B_1 denote an open ball with boundary Σ, such that $\bar{\Omega} \subset B_1$. We set $\Omega_1 = B_1 \cap \Omega'$ (*cf.* Fig. 1). Since Ω_1 is a bounded open set with boundary $\Gamma \cup \Sigma$, it follows from hypothesis (3.3), that there exists a $\boldsymbol{u}_0 \in (H^1(\Omega_1))^n$ with $\operatorname{div} \boldsymbol{u}_0 = 0$ in Ω_1, $\boldsymbol{u}_{0|\Gamma} = \boldsymbol{g}$, $\boldsymbol{u}_{0|\Sigma} = \boldsymbol{0}$ satisfying $\|\boldsymbol{u}_0\|_{1,\Omega_1} \leqq C \|\boldsymbol{g}\|_{\frac{1}{2},\Gamma}$. Clearly $\boldsymbol{u}_0$ can be constructed so that the mapping $\boldsymbol{g} \mapsto \boldsymbol{u}_0$ is linear.

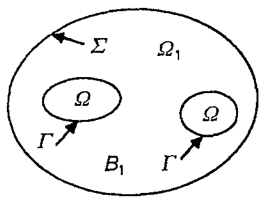

Fig. 1

Now we extend $\boldsymbol{u}_0$ to Ω' so that it remains divergence-free. In fact the extension

$$\tilde{\boldsymbol{u}}_0 = \begin{cases} \boldsymbol{u}_0 & \text{in } \Omega_1, \\ \boldsymbol{0} & \text{in } B_1' \end{cases}$$

belongs to $(H^1(\Omega'))^n$, has compact support, and satisfies

$$\operatorname{div} \tilde{\boldsymbol{u}}_0 = 0 \quad \text{in } \Omega', \qquad \tilde{\boldsymbol{u}}_{0|\Gamma} = \boldsymbol{g},$$

with

$$\|\tilde{\boldsymbol{u}}_0\|_{1,\Omega'} \leqq C \|\boldsymbol{g}\|_{\frac{1}{2},\Gamma}. \quad \square$$

Remark 3.1. Suppose that Ω has "holes": $\Gamma = \bigcup_{i=0}^{p} \Gamma_i$, where Γ_0 denotes the exterior boundary of Ω and Γ_i, $1 \leqq i \leqq p$ are the other "interior" components of Γ. Then the statement of Lemma 3.1 is still valid provided that the boundary value $\boldsymbol{g}$ satisfies the conditions

$$\int_{\Gamma_i} \boldsymbol{g} \cdot \boldsymbol{n} \, ds = 0, \qquad 0 \leqq i \leqq p.$$

The theorem below is a fundamental tool in the theory of exterior Stokes equations; its proof can also be found in [Se]$_1$.

Theorem 3.1. *Let Ω be as in Lemma 3.1 and let the boundary Γ be sufficiently smooth (in C^2 for example). Then for each $p \in L^2(\Omega')$, there exists $\boldsymbol{w} \in (\mathring{W}_0^1(\Omega'))^n$ such that*

$$\begin{aligned} &\operatorname{div} \boldsymbol{w} = p \quad \text{in } \Omega', \\ &\|\boldsymbol{w}\|_{1,0,\Omega'} \leqq K \|p\|_{0,\Omega'}. \end{aligned} \tag{3.6}$$

Proof. For $p \in L^2(\Omega')$, let us solve the exterior Neumann problem

$$\Delta\varphi = p \ \text{ in } \Omega', \quad \left.\frac{\partial\varphi}{\partial n}\right|_{\Gamma} = 0.$$

We set $\boldsymbol{v} = \nabla\varphi$. It is shown ([Gr] Th. 7.13, (for $n = 2$) and Th. 4.12. (for $n = 3$)) that this problem has a unique solution $\varphi \in W_0^2(\Omega')/\mathbb{R}$ and that the mapping $p \mapsto \varphi$ is continuous: In particular $\|\boldsymbol{v}\|_{1,0,\Omega'} \leqq C_1 \|p\|_{0,\Omega'}$. Moreover, $\boldsymbol{v}_{|\Gamma} \in (H^{\frac{1}{2}}(\Gamma))^n$ and obviously satisfies (3.3). Then, according to Lemma 3.1, there exists a $\boldsymbol{v}_0 \in (H^1(\Omega'))^n$ such that

$$\operatorname{div} \boldsymbol{v}_0 = 0 \ \text{ in } \Omega', \quad \boldsymbol{v}_{0|\Gamma} = \boldsymbol{v}_{|\Gamma},$$

$$\|\boldsymbol{v}_0\|_{1,\Omega'} \leqq C_2 \|\boldsymbol{v}\|_{\frac{1}{2},\Gamma} \leqq C_3 \|\boldsymbol{v}\|_{1,0,\Omega'} \leqq C_4 \|p\|_{0,\Omega'}.$$

The theorem follows by choosing $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{v}_0$. □

As an immediate consequence, we derive

Corollary 3.1. *Under the assumptions of Theorem 3.1 the following* inf-sup *condition (also called the Babuška-Brezzi condition,* [Bu], [Br]) *holds:*

$$\inf_{p\in L^2(\Omega')} \sup_{\boldsymbol{w}\in(\mathring{W}_0^1(\Omega'))^n} \frac{\int_{\Omega'} p \operatorname{div} \boldsymbol{w}\, dx}{\|p\|_{0,\Omega'} \|\boldsymbol{w}\|_{1,0,\Omega'}} \geqq \frac{1}{K}, \tag{3.7}$$

where K is the constant in inequality (3.6).

We introduce the continuous bilinear form

$$b(\cdot,\cdot): X \times M \to \mathbb{R}$$

defined by

$$b(\boldsymbol{w}, q) = -\int_{\Omega'} q \operatorname{div} \boldsymbol{w}\, dx = \langle \boldsymbol{w}, \nabla q\rangle, \quad \forall\, \boldsymbol{w} \in (\mathring{W}_0^1(\Omega'))^n, \ \forall\, q \in L^2(\Omega'). \tag{3.8}$$

Let $B \in \mathscr{L}(X, M')$ be the associated linear operator and let $B' \in \mathscr{L}(M, X')$ be the dual operator of B, *i.e.*,

$$b(\boldsymbol{w}, q) = \langle B\boldsymbol{w}, q\rangle = \langle \boldsymbol{w}, B'q\rangle \tag{3.9}$$

with

$$B\boldsymbol{w} = -\operatorname{div} \boldsymbol{w}; \quad B'q = \nabla q, \quad \forall\, \boldsymbol{w} \in X, \forall\, q \in M. \tag{3.10}$$

In abstract terms (*cf.*, § 4 [Gi]), we know that B is an isomorphism from $V^\perp$ onto M' if and only if B' is an isomorphism from M onto V^0; these properties are equivalent to the inf-sup condition for the bilinear form $b(\cdot,\cdot)$. More precisely, we have

Corollary 3.2. *Let the hypotheses of Theorem 3.1 hold. Then the operator* div *is an isomorphism from* $V^\perp$ *onto* M'. *The operator* grad *is an isomorphism from* M *onto* V^0.

It follows from Theorem 3.1 that the constraint (3.3), which leads to a simple proof of Lemma 3.1, can be eliminated. To show this we first suppose that the boundary Γ is more regular.

Lemma 3.2. *Let the open set* $\Omega \subset \mathbb{R}^n$ ($n = 2$ or 3) *be as in Lemma 3.1, but let its boundary* Γ *be sufficiently smooth. Then there exists a lifting operator* $R \in \mathscr{L}((H^{\frac{1}{2}}(\Gamma))^n; (W_0^1(\Omega'))^n)$ *such that*

$$Rg_{|\Gamma} = g, \quad \operatorname{div}(Rg) = 0 \quad \text{in } \Omega',$$

i.e., for each $g \in (H^{\frac{1}{2}}(\Gamma))^n$, *there exists a* $\boldsymbol{u}_0 \in (W_0^1(\Omega'))^n$ *satisfying*

$$\boldsymbol{u}_{0|\Gamma} = g, \quad \operatorname{div} \boldsymbol{u}_0 = 0, \quad \|\boldsymbol{u}_0\|_{1,0,\Omega'} \leqq C\,\|g\|_{\frac{1}{2},\Gamma}.$$

Proof. We retain the notations of Lemma 3.1. For $g \in (H^{\frac{1}{2}}(\Gamma))^n$, let $\boldsymbol{w}_0 \in (H^1(\Omega))^n$ be the unique solution of the Dirichlet problem

$$\Delta \boldsymbol{w}_0 = \boldsymbol{0} \quad \text{in } \Omega_1, \quad \boldsymbol{w}_{0|\Sigma} = \boldsymbol{0}, \quad \boldsymbol{w}_{0|\Gamma} = g. \tag{3.11}$$

Its extension by zero outside B_1 (still denoted $\boldsymbol{w}_0$) satisfies

$$\|\boldsymbol{w}_0\|_{1,0,\Omega'} \leqq C_1\,\|g\|_{\frac{1}{2},\Gamma}. \tag{3.12}$$

Moreover, $\operatorname{div} \boldsymbol{w}_0 \in L^2(\Omega')$, and by Corollary 3.2 there exists a unique function $\boldsymbol{w} \in V^\perp$ such that

$$\operatorname{div} \boldsymbol{w} = \operatorname{div} \boldsymbol{w}_0, \quad \|\boldsymbol{w}\|_{1,0,\Omega'} \leqq K\,\|\operatorname{div} \boldsymbol{w}_0\|_{0,\Omega'}.$$

From (3.12) we can deduce that

$$\|\boldsymbol{w}\|_{1,0,\Omega'} \leqq K\sqrt{n}\;|\boldsymbol{w}_0|_{1,0,\Omega'} \leqq KC_1\sqrt{n}\;\|g\|_{\frac{1}{2},\Gamma}. \tag{3.13}$$

Hence

$$\boldsymbol{u}_0 = \boldsymbol{w}_0 - \boldsymbol{w}$$

is the required function, since $\boldsymbol{u}_0 \in (W_0^1(\Omega'))^n$,

$$\operatorname{div} \boldsymbol{u}_0 = 0 \quad \text{in } \Omega', \quad \boldsymbol{u}_{0|\Gamma} = g.$$

Combining the inequalities (3.12) and (3.13) we get

$$\|\boldsymbol{u}_0\|_{1,0,\Omega'} \leqq C_1(1 + K\sqrt{n})\;\|g\|_{\frac{1}{2},\Gamma}.$$

We finally note the linearity of the mappings

$$g \mapsto \boldsymbol{w}_0 \mapsto \boldsymbol{w} \mapsto \boldsymbol{u}_0. \quad \square$$

Remark 3.2. Note that in removing the constraint

$$\int_{\Gamma} g \cdot n \, ds = 0, \tag{3.3}$$

we have automatically lost the compact support of the lift u_0. Obviously (3.3) is necessary to obtain a divergence-free lift with compact support.

Lemma 3.3. *Let the assumptions of Lemma 3.2 hold. For each* $g \in (H^{\frac{1}{2}}(\Gamma))^n$ *and* $h \in L^2(\Omega')$, *there exists a* $v \in (W_0^1(\Omega'))^n$ *such that*

$$\operatorname{div} v = h \quad in \ \Omega', \qquad v_{|\Gamma} = g$$

and such that the mapping $\{g, h\} \to v$ *is continuous:*

$$\|v\|_{1,0,\Omega'} \leqq C[\|h\|_{0,\Omega'} + \|g\|_{\frac{1}{2},\Gamma}].$$

The proof follows immediately from Lemma 3.2 and Theorem 3.1.

Now observe that when the domain is bounded, the Stokes problem is well-posed when the boundary is only Lipschitz-continuous. Intuitively, this property should remain true in exterior domains, since it should not affect the behaviour of the solution at infinity, Indeed, this is the case here. The theorem below, analogous to Theorem 3.1, eliminates the regularity assumption on Γ.

Theorem 3.2. *Let* Ω *be as in Lemma 3.1. Then for each* $p \in L^2(\Omega')$ *there exists* $u \in (\overset{\circ}{W}{}_0^1(\Omega'))^n$ *such that*

$$\begin{aligned} &\operatorname{div} u = p \quad in \ \Omega', \\ &\|u\|_{1,0,\Omega'} \leqq K' \, \|p\|_{0,\Omega'}. \end{aligned} \tag{3.6'}$$

Proof. Again we keep the notations of Lemma 3.1. Let B_2 denote another open ball such that $B_2 \supset \overline{B}_1$ and set $\Omega_2 = B_2 \cap B_1'$, as in Fig. 2. Given $p \in L^2(\Omega')$, let us define a function $\tilde{p}$ in $\Omega_1 \cup \Omega_2$ by

$$\tilde{p} = \begin{cases} p & \text{in } \Omega_1, \\ C & \text{in } \Omega_2, \end{cases}$$

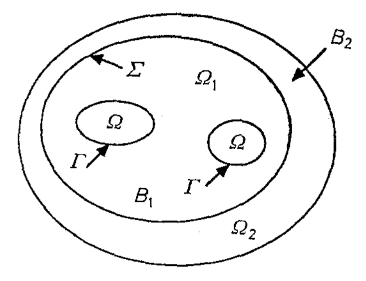

Fig. 2

where the constant C is chosen so that

$$\int_{\Omega_1 \cup \Omega_2} \tilde{p}\, dx = 0,$$

i.e.,

$$C = -\frac{1}{\text{mes}\,(\Omega_2)} \int_{\Omega_1} p\, dx.$$

A simple calculation gives

$$\|\tilde{p}\|^2_{0,\Omega_1 \cup \Omega_2} \leqq \|p\|^2_{0,\Omega_1} + \frac{\text{mes}\,(\Omega_1)}{\text{mes}\,(\Omega_2)} \|p\|^2_{0,\Omega_1} \leqq C_1^2 \|p\|^2_{0,\Omega_1}. \tag{3.14}$$

Since $\tilde{p}$ has zero mean value in the bounded domain $\Omega_1 \cup \Omega_2$, we know there exists $\boldsymbol{w} \in (H_0^1(\Omega_1 \cup \Omega_2))^n$ such that

$$\text{div}\, \boldsymbol{w} = \tilde{p} \quad \text{in } \Omega_1 \cup \Omega_2$$

with

$$\|\boldsymbol{w}\|_{1,\Omega_1 \cup \Omega_2} \leqq C_2 \|\tilde{p}\|_{0,\Omega_1 \cup \Omega_2}. \tag{3.15}$$

By combining inequalities (3.14) and (3.15) we immediately obtain

$$\boldsymbol{w}_{|\Gamma} = \boldsymbol{0}, \quad \|\boldsymbol{w}\|_{1,\Omega_1 \cup \Omega_2} \leqq C_1 C_2 \|p\|_{0,\Omega_1}. \tag{3.16}$$

On the other hand, since the boundary Σ is smooth, it follows from Lemma 3.3 that there exists $\boldsymbol{v} \in (W_0^1(B_1'))^n$ such that

$$\text{div}\, \boldsymbol{v} = p \quad \text{in } B_1', \quad \boldsymbol{v}_{|\Sigma} = \boldsymbol{w}_{|\Sigma},$$

$$\|\boldsymbol{v}\|_{1,0,B_1'} \leqq C_3[\|p\|_{0,B_1'} + \|\boldsymbol{w}\|_{\frac{1}{2},\Sigma}].$$

Using the trace theorem and inequality (3.16), we have

$$\begin{aligned} \|\boldsymbol{v}\|_{1,0,B_1'} &\leqq C_3[\|p\|_{0,B_1'} + C_4 \|\boldsymbol{w}\|_{1,\Omega_1}] \\ &\leq C_3[\|p\|_{0,B_1'} + C_1 C_2 C_4 \|p\|_{0,\Omega_1}]. \end{aligned}$$

Finally, the function $\boldsymbol{u}$ defined by

$$\boldsymbol{u} = \begin{cases} \boldsymbol{w} & \text{on } \Omega_1, \\ \boldsymbol{v} & \text{on } B_1' \end{cases}$$

belongs to $(\mathring{W}_0^1(\Omega'))^n$ and satisfies

$$\text{div}\, \boldsymbol{u} = p \quad \text{in } \Omega',$$

$$\|\boldsymbol{u}\|_{1,0,\Omega'} \leqq C_5 \|p\|_{0,\Omega'}. \quad \square$$

Remark 3.3. Now it is a simple matter to check that the lifting operators of Lemmas 3.2 and 3.3 still exist in the case of a Lipschitz-continuous boundary Γ.

This establishes the inf-sup condition for the exterior Stokes problem.

Theorem 3.3. *Let Ω be as in Lemma 3.1. Then there exists a constant $K' > 0$ (the constant of inequality* (3.6)′) *such that*

$$\inf_{p \in L^2(\Omega')} \sup_{w \in (\mathring{W}_0^1(\Omega'))^n} \frac{\int_{\Omega'} p \operatorname{div} w \, dx}{\|p\|_{0,\Omega'} \|w\|_{1,0,\Omega'}} \geqq \frac{1}{K'}. \tag{3.7′}$$

Since the assertions of Corollary 3.2 are still vallid, we readily prove the following result.

Corollary 3.3. *Under the assumptions of Lemma 3.1, the mapping*

$$p \mapsto \|\nabla p\|_{-1,0,\Omega'}$$

is a norm on $L^2(\Omega')$ equivalent to the usual norm $\|\cdot\|_{0,\Omega'}$.

Proof. Let $p \in L^2(\Omega')$. We know that $\nabla p \in (W_0^{-1}(\Omega'))^n$ with

$$\|\nabla p\|_{-1,0,\Omega'} = \sup_{w \in (\mathring{W}_0^1(\Omega'))^n} \frac{\langle \nabla p \;\; w \rangle}{\|w\|_{1,0,\Omega'}}. \tag{3.17}$$

Moreover,

$$\langle \nabla p, w \rangle = - \int_{\Omega'} p \operatorname{div} w \, dx \quad \forall\, w \in (\mathring{W}_0^1(\Omega'))^n. \tag{3.18}$$

Now applying the Cauchy-Schwarz inequality we get

$$\left| \int_{\Omega'} p \operatorname{div} w \, dx \right| \leqq \|p\|_{0,\Omega'} \sqrt{n}\, |w|_{1,0,\Omega'}; \tag{3.19}$$

and by the inf-sup condition (3.7′) we obtain

$$\sup_{w \in (\mathring{W}_0^1(\Omega'))^n} \frac{1}{\|w\|_{1,0,\Omega'}} \int_{\Omega'} p \operatorname{div} w \, dx \geqq \frac{\|p\|_{0,\Omega'}}{K'}. \tag{3.20}$$

Hence when we combine (3.17)–(3.20), the conclusion follows at once from the bounds

$$\frac{1}{K'} \|p\|_{0,\Omega'} \leqq \|\nabla p\|_{-1,0,\Omega'} \leqq \sqrt{n}\, \|p\|_{0,\Omega'}. \qquad \square$$

Now we turn to the nonhomogeneous exterior Stokes problem (S). More precisely, given $f \in (W_0^{-1}(\Omega'))^n$, $g \in (H^{\frac{1}{2}}(\Gamma))^n$ and $\nu > 0$, we want to find $u \in (W_0^1(\Omega'))^n$ and $p \in L^2(\Omega')$ such that

$$\begin{aligned} -\nu \Delta u + \nabla p &= f \quad \text{in } \Omega', \\ \operatorname{div} u &= 0 \quad \text{in } \Omega', \\ u_{|\Gamma} &= g. \end{aligned} \tag{S}$$

As in the bounded case, it is easy to see that this problem has the equivalent variational formulation: Find $u \in (W_0^1(\Omega'))^n$ and $p \in L^2(\Omega')$ such that

$$\text{(Q)} \qquad \begin{aligned} \nu(\nabla u, \nabla v) - (p, \operatorname{div} v) &= \langle f, v \rangle \quad \forall v \in (\overset{\circ}{W}{}_0^1(\Omega'))^n, \\ \operatorname{div} u &= 0 \quad \text{in } \Omega', \\ u_{|\Gamma} &= g. \end{aligned}$$

In view of the inf-sup condition (3.7'), it follows from Corollary 3.2 that this problem is also equivalent to the problem:

Find $u \in (W_0^1(\Omega'))^n$ such that

$$\text{(P)} \qquad \begin{aligned} \nu(\nabla u, \nabla v) &= \langle f, v \rangle \quad \forall v \in V, \\ \operatorname{div} u &= 0 \quad \text{in } \Omega', \\ u_{|\Gamma} &= g. \end{aligned}$$

Clearly problem (P) has a unique solution. Indeed let u_0 be the divergence-free lift of g:

$$u_{0|\Gamma} = g, \quad \operatorname{div} u_0 = 0, \quad u_0 \in (W_0^1(\Omega'))^n.$$

Then $w = u - u_0$ satisfies

$$\begin{aligned} \nu(\nabla w, \nabla v) &= \langle f, v \rangle - \nu(\nabla u_0, \nabla v) \quad \forall v \in V, \\ \operatorname{div} w &= 0, \\ w_{|\Gamma} &= \mathbf{0}, \end{aligned}$$

that is, w belongs to V.

By the Lax-Milgram Theorem, this problem is well-posed and we have

$$\|w\|_{1,0,\Omega'} \leqq \frac{C^2}{\nu} \|f\|_{-1,0,\Omega'} + C\,|u_0|_{1,0,\Omega'},$$

where C is the constant of (2.1). Therefore

$$\begin{aligned} \|u\|_{1,0,\Omega'} &\leqq C_1 \left[\|u_0\|_{1,0,\Omega'} + \frac{1}{\nu} \|f\|_{-1,0,\Omega'} \right] \\ &\leqq C_2 \left[\|g\|_{\frac{1}{2},\Gamma} + \frac{1}{\nu} \|f\|_{-1,0,\Omega'} \right], \end{aligned}$$

with a similar estimate for p, thanks to the inf-sup condition. Thus we have proved our main result:

Theorem 3.4. *Suppose that $\Omega \subset \mathbb{R}^n$ ($n = 2$ or 3) has a Lipschitz-continuous boundary Γ that is not necessarily connected, but has no interior connected component. Then for f given in $(W_0^{-1}(\Omega'))^n$ and g given in $(H^{\frac{1}{2}}(\Gamma))^n$, the Stokes problem* (S) *has a unique solution $(u, p) \in (W_0^1(\Omega'))^n \times L^2(\Omega')$ which depends continuously on the data, i.e.,*

$$\|u\|_{1,0,\Omega'} + \|p\|_{0,\Omega'} \leqq C[\|f\|_{-1,0,\Omega'} + \|g\|_{\frac{1}{2},\Gamma}].$$

Remark 3.4. It clearly follows from Lemma 3.3 and Remark 3.3 that the following nonhomogeneous problem is also well-posed: Given $f \in (W_0^{-1}(\Omega'))^n$, $h \in L^2(\Omega')$, $g \in (H^{\frac{1}{2}}(\Gamma))^n$, find $u \in (W_0^1(\Omega'))^n$ and $p \in L^2(\Omega')$ such that

$$-\nu\,\Delta u + \nabla p = f \quad \text{in } \Omega',$$
$$\operatorname{div} u = h \quad \text{in } \Omega',$$
$$u_{|\Gamma} = g.$$

4. Regularity of the Solution

In this section we show that if the boundary and data are smoother, then so is the solution $\{u, p\}$ of the Stokes problem. To prove this, we shall apply a technique used by Hanouzet and Giroire: First derive the desired regularity result in the whole space; then the same problem in the exterior domain Ω' can be reduced, by appropriate cut-off functions, to a problem in a bounded domain and a problem in the whole space. We shall treat the cases of $\mathbb{R}^2$ and $\mathbb{R}^3$ separately because the spaces involved are slightly different, the case of $\mathbb{R}^2$ being more technical. But once the problem is solved in the whole space, the reduction technique to an exterior domain is exactly the same, whatever the dimension. Therefore, to simplify the discussion, we shall describe the reduction technique only in the three-dimensional case.

4.1. The Stokes Problem in $\mathbb{R}^3$

Let us first consider the homogeneous Stokes problem:

Theorem 4.1. *Let f be given in $(W_1^0(\mathbb{R}^3))^3$. Then the homogeneous Stokes problem*

$$(4.1)\qquad \begin{aligned} -\nu\,\Delta u + \nabla p &= f \quad \text{in } \mathbb{R}^3,\\ \operatorname{div} u &= 0 \quad \text{in } \mathbb{R}^3 \end{aligned}$$

has a unique solution, $u \in (W_1^2(\mathbb{R}^3))^3$ and $p \in W_1^1(\mathbb{R}^3)$, with

$$(4.2)\qquad \|u\|_{2,1,\mathbb{R}^3} + \|p\|_{1,1,\mathbb{R}^3} \leqq C\,\|f\|_{0,1,\mathbb{R}^3}.$$

Proof. First note that since $f \in (W_1^0(\mathbb{R}^3))^3 \hookrightarrow (W_0^{-1}(\mathbb{R}^3))^3$, the problem (4.1) has a unique solution $u \in (W_0^1(\mathbb{R}^3))^3$ and $p \in L^2(\mathbb{R}^3)$.

Now, by taking the divergence of (4.1), we reduce the Stokes problem to a Poisson equation for the pressure

$$(4.3)\qquad \Delta p = \operatorname{div} f \quad \text{in } \mathbb{R}^3.$$

Since $f \in (W_1^0(\mathbb{R}^3))^3$, it follows that $\operatorname{div} f \in W_1^{-1}(\mathbb{R}^3)$ and clearly, $\operatorname{div} f$ is orthogonal to constants:

$$\langle \operatorname{div} f, c\rangle = 0 \quad \forall\, c \in \mathbb{R}.$$

Therefore, $\operatorname{div} f \in (W^1_{-1}(\mathbb{R}^3)/\mathbb{R})'$. Hence, as the operator Δ is self-adjoint, we can use the following result, which is the dual of a proposition established by GIROIRE [Gr, Prop. 2.11]: *The operator* Δ *is an isomorphism from*

$$(W^{-1}_{-1}(\mathbb{R}^3))' \quad \text{onto} \quad (W^1_{-1}(\mathbb{R}^3)/\mathbb{R})'. \tag{4.4}$$

Since $(W^{-1}_{-1}(\mathbb{R}^3))' = W^1_1(\mathbb{R}^3)$, (4.4) implies that $p \in W^1_1(\mathbb{R}^3)$ and

$$\|p\|_{1,1,\mathbb{R}^3} \leqq C_1 \|\operatorname{div} f\|_{-1,1,\mathbb{R}^3} \leqq C_2 \|f\|_{0,1,\mathbb{R}^3}.$$

Hence equation (4.1) implies that each component of u is the solution of the Laplace equation in $\mathbb{R}^3$ with right-hand side in $W^0_1(\mathbb{R}^3)$. Then applying Proposition 2.2 of [Gr], which states that *the operator* Δ *is an isomorphism* from

$$W^2_1(\mathbb{R}^3) \quad \text{onto} \quad W^0_1(\mathbb{R}^3), \tag{4.5}$$

we find immediately that $u \in (W^2_1(\mathbb{R}^3))^3$ and

$$\|u\|_{2,1,\mathbb{R}^3} \leqq C_3 \frac{1}{\nu} [\|f\|_{0,1,\mathbb{R}^3} + \|\nabla p\|_{0,1,\mathbb{R}^3}] \leqq \frac{C_4}{\nu} \|f\|_{0,1,\mathbb{R}^3}. \quad \square$$

Now let us solve the nonhomogeneous Stokes problem.

Theorem 4.2. *Let* $f \in (W^0_1(\mathbb{R}^3))^3$ *and* $h \in W^1_1(\mathbb{R}^3)$. *Then the nonhomogeneous Stokes problem*

$$\begin{aligned} -\nu \Delta u + \nabla p &= f \quad \text{in } \mathbb{R}^3, \\ \operatorname{div} u &= h \quad \text{in } \mathbb{R}^3 \end{aligned} \tag{4.6}$$

has a unique solution $u \in (W^2_1(\mathbb{R}^3))^3$ *and* $p \in W^1_1(\mathbb{R}^3)$, *with*

$$\|u\|_{2,1,\mathbb{R}^3} + \|p\|_{1,1,\mathbb{R}^3} \leqq C[\|f\|_{0,1,\mathbb{R}^3} + \|h\|_{1,1,\mathbb{R}^3}]. \tag{4.7}$$

Proof. Let us reduce problem (4.6) to a homogeneous problem. For this, we solve the Neumann problem

$$\Delta\varphi = h \quad \text{in } \mathbb{R}^3. \tag{4.8}$$

Since $h \in W^1_1(\mathbb{R}^3)$, Proposition 2.13 of [Gr], which states that *the operator* Δ *is an isomorphism from*

$$W^3_1(\mathbb{R}^3)/\mathbb{R} \quad \text{onto} \quad W^1_1(\mathbb{R}^3), \tag{4.9}$$

implies that this problem has exactly one solution $\varphi \in W^3_1(\mathbb{R}^3)/\mathbb{R}$ and

$$\|\varphi\|_{W^3_1(\mathbb{R}^3)/\mathbb{R}} \leqq C_1 \|h\|_{1,1,\mathbb{R}^3}.$$

Let us take $u_0 = \nabla\varphi$. Then $u_0 \in (W^2_1(\mathbb{R}^3))^3$, $\operatorname{div} u_0 = h$, and

$$\|u_0\|_{2,1,\mathbb{R}^3} \leqq C_1 \|h\|_{1,1,\mathbb{R}^3}.$$

Now, the function $w = u - u_0$ is a solution of the homogeneous Stokes problem

$$(4.10)\qquad \begin{aligned} -\nu \Delta w + \nabla p = f + \nu \Delta u_0 \quad \text{in } \mathbb{R}^3, \\ \operatorname{div} w = 0 \quad \text{in } \mathbb{R}^3, \end{aligned}$$

with right-hand side in $(W_1^0(\mathbb{R}^3))^3$. Then the desired result follows from Theorem 4.1. □

4.2. The Case of an Exterior Domain

Let us assume that Ω is as in Lemma 3.1, except that its boundary Γ is smooth (of class C^2) and, as usual, let Ω' denote the complement of $\bar{\Omega}$. Given f in $(W_1^0(\Omega'))^3$ and g in $(H^{3/2}(\Gamma))^{3/2}$, we want to show that the solution $u \in (W_0^1(\Omega'))^3$, $p \in L^2(\Omega')$ of

$$(4.11)\qquad \begin{aligned} -\nu \Delta u + \nabla p = f, \quad \operatorname{div} u = 0 \quad \text{in } \Omega', \\ u_{|\Gamma} = g \end{aligned}$$

is such that $u \in (W_1^2(\Omega'))^3$ and $p \in W_1^1(\Omega')$.

Let $R \in \mathbb{N}$ be so large that Ω is contained in $\bar{B}_R$. (B_R is the ball with center 0 and radius R.) Consider the following partition of unity:

$$(4.12)\qquad \begin{aligned} \varphi, \psi \in C^\infty(\mathbb{R}^3), \quad \varphi + \psi \equiv 1 \quad \text{in } \mathbb{R}^3, 0 \leqq \varphi, \psi \leqq 1 \text{ in } \mathbb{R}^3, \\ \varphi_{|B_R} = 1, \quad \varphi_{|B'_{R+1}} = 0, \\ \psi_{|B_R} = 0, \quad \psi_{|B'_{R+1}} = 1. \end{aligned}$$

Then $u = u\varphi + u\psi$ and $p = p\varphi + p\psi$, where $u\varphi$ and $u\psi$ have the same regularity as u, and similarly, $p\varphi$ and $p\psi$ have the same regularity as p.

Let us first study $u\varphi$:

$$u\varphi \in (W_0^1(\Omega'))^3, \quad u\varphi_{|B'_{R+1}} = \mathbf{0}.$$

Hence $u\varphi \in (H^1(\Omega' \cap B_{R+1}))^3$; likewise, $p\varphi \in L^2(\Omega' \cap B_{R+1})$ and the pair $\{u\varphi, p\varphi\}$ satisfies the equations

$$(4.13)\qquad \begin{aligned} -\nu \Delta(u\varphi) + \nabla(p\varphi) = f + \nu \Delta(u\psi) - \nabla(p\psi) \quad \text{in } \Omega' \cap B_{R+1}, \\ \operatorname{div}(u\varphi) = -\operatorname{div}(u\psi) \quad \text{in } \Omega' \cap B_{R+1}, \\ u\varphi_{|\Gamma} = g, \quad u\varphi_{|\partial B_{R+1}} = \mathbf{0}. \end{aligned}$$

This is a nonhomogeneous Stokes problem on a bounded domain with a smooth boundary. After some rearrangement, the right-hand side of the state equation has the expression

$$F = \varphi f + 2\nu\, \partial_i u\, \partial_i \psi + \nu u \Delta\psi - p \nabla\psi.$$

Since φ and ψ are smooth, it is easily checked that

$$F \in (L^2(\Omega' \cap B_{R+1}))^3.$$

Likewise,

$$\operatorname{div}(\boldsymbol{u}\varphi) = -\boldsymbol{u} \cdot \nabla\varphi \in H^1(\Omega' \cap B_{R+1}).$$

Therefore, it follows from the regularity results of CATTABRIGA [Ca] that the solution pair $\{\boldsymbol{u}\varphi, p\varphi\}$ belongs to $(H^2(\Omega' \cap B_{R+1}))^3 \times H^1(\Omega' \cap B_{R+1})$ and satisfies the bound

$$\begin{aligned}\|\boldsymbol{u}\varphi\|_{2,\Omega'\cap B_{R+1}} + \|p\varphi\|_{1,\Omega'\cap B_{R+1}} &\\ \leqq C_1\{\|\boldsymbol{f}\|_{0,\Omega'\cap B_{R+1}} &+ \|\boldsymbol{u}\|_{1,\Omega'\cap B_{R+1}} \\ &+ \|p\|_{0,\Omega'\cap B_{R+1}} + \|\boldsymbol{g}\|_{3/2,\Gamma}\}.\end{aligned}$$

As φ is very smooth and has its support in B_{R+1}, we can extend $\boldsymbol{u}\varphi$ and $p\varphi$ by zero in B'_{R+1}; the extended functions belong respectively to $(W_1^2(\Omega'))^3$ and $W_1^1(\Omega')$ and satisfy the above bound in Ω'.

Now, we consider $\boldsymbol{u}\psi$. Since ψ vanishes in B_R, we can extend $\boldsymbol{u}\psi$ and $p\psi$ by zero in B_R, and the extended functions belong respectively to $(\mathring{W}_0^1(\mathbb{R}^3))^3$ and $L^2(\mathbb{R}^3)$. Then as in the preceding case, we can easily check that

$$-\nu\,\Delta(\boldsymbol{u}\psi) + \nabla(p\psi) = \boldsymbol{F} = \boldsymbol{f}\psi - \nu(2\partial_j\psi\,\partial_j\boldsymbol{u} + \Delta\psi\boldsymbol{u}) + p\,\nabla\psi,$$

$$\operatorname{div}(\boldsymbol{u}\psi) = H = \boldsymbol{u}\cdot\nabla\psi.$$

Therefore the pair $\{\boldsymbol{u}\psi, p\psi\} \in (\mathring{W}_0^1(\mathbb{R}^3))^3 \times L^2(\mathbb{R}^3)$ is the solution of the Stokes problem

$$-\nu\,\Delta(\boldsymbol{u}\psi) + \nabla(p\psi) = \boldsymbol{F}, \quad \operatorname{div}(\boldsymbol{u}\psi) = H \quad \text{in } \mathbb{R}^3.$$

Since $\boldsymbol{F}$ belongs to $(W_1^0(\mathbb{R}^3))^3$ and H belongs to $H^1(\mathbb{R}^3)$ with compact support, so that H is in $W_1^1(\mathbb{R}^3)$, we can apply Theorem 4.2, which yields

$$\boldsymbol{u}\psi \in (W_1^2(\mathbb{R}^3))^3, \quad p\psi \in W_1^1(\mathbb{R}^3),$$

$$\|\boldsymbol{u}\psi\|_{2,1,\mathbb{R}^3} + \|p\psi\|_{1,1,\mathbb{R}^3} \leqq C_2\{\|\boldsymbol{f}\|_{0,1,B'_R} + \|\boldsymbol{u}\|_{1,\Omega'\cap B_{R+1}} + \|p\|_{0,\Omega'\cap B_{R+1}}\}.$$

Finally, combining the last two inequalities and Theorem 3.4, we obtain the regularity announced for the solution $(\boldsymbol{u}, p)$ of Problem (4.11):

$$\boldsymbol{u} \in (W_1^2(\Omega'))^3, \quad p \in W_1^1(\Omega'),$$

$$\|\boldsymbol{u}\|_{2,1,\Omega'} + \|p\|_{1,1,\Omega'} \leqq C_3\{\|\boldsymbol{f}\|_{0,1,\Omega'} + \|\boldsymbol{g}\|_{3/2\Gamma}\}.$$

4.3. The Stokes problem in $\mathbb{R}^2$

The situation is more delicate in $\mathbb{R}^2$ because some of the regularity results that we need do not hold in the W_p^m spaces. Instead, we shall work with a slightly different family of spaces, the X_p^{m+p} spaces (*cf.* [Gr]). For the sake of simplicity, we do not define the most general X_p^{m+p} space, we only introduce the specific spaces that we require, namely

$$X_1^0(\mathbb{R}^2) = \{f \in W_0^{-1}(\mathbb{R}^2);\, x_i f \in L^2(\mathbb{R}^2),\, i = 1, 2;\, f \in L^2_{\mathrm{loc}}(\mathbb{R}^2)\},$$

$$X_1^{-1}(\mathbb{R}^2) = \{f \in W_0^{-2}(\mathbb{R}^2);\, x_i f \in W_0^{-1}(\mathbb{R}^2),\, i = 1, 2;\, f \in H^{-1}_{\mathrm{loc}}(\mathbb{R}^2)\},$$

and their dual spaces, $X^0_{-1}(\mathbb{R}^2)$ and $X^1_{-1}(\mathbb{R}^2)$, respectively.

The properties of these spaces are established by [Gr], who proves (in Proposition 10.1) that

$$X_1^0(\mathbb{R}^2) = \{f \in W_0^{-1}(\mathbb{R}^2);\, f \in W_1^0(\mathbb{R}^2)\},$$

and therefore $X_1^0(\mathbb{R}^2)$ is a proper subspace of $W_1^0(\mathbb{R}^2)$ because $W_1^0(\mathbb{R}^2) \not\subset W_0^{-1}(\mathbb{R}^2)$. Moreover, it is established that the polynomial space $\mathbb{P}_1$ is contained in $X^1_{-1}(\mathbb{R}^2)$ (*cf.* Lemma 5.12 of [Gr]). Likewise, a straightforward argument shows that the constants are contained in $X^0_{-1}(\mathbb{R}^2)$.

We shall prove the following result:

$$\text{If } f \in (X_1^0(\mathbb{R}^2))^2, \text{ then } \boldsymbol{u} \in (W_1^2(\mathbb{R}^2))^2 \text{ and } p \in W_1^1(\mathbb{R}^2).$$

In other words, we shall deduce the *same regularity as in* $\mathbb{R}^3$, provided we start with a slightly more regular right-hand side.

But first of all, let us observe that, in contrast to what happens for $\mathbb{R}^3$ or for exterior domains of $\mathbb{R}^2$, a *necessary condition* for the Stokes problem to have a solution in $\mathbb{R}^2$ is that

$$f \perp \mathbb{R}^2. \tag{4.14}$$

Indeed, let $f \in (W_0^{-1}(\mathbb{R}^2))^2$ be such that

$$f = -\nu\,\Delta \boldsymbol{u} + \nabla p \quad \text{in } \mathbb{R}^2$$

with $\boldsymbol{u} \in (W_0^1(\mathbb{R}^2))^2$ and $p \in L^2(\mathbb{R}^2)$. Then

$$\langle f, \boldsymbol{v}\rangle = \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) - (p, \operatorname{div} \boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in (W_0^1(\mathbb{R}^2))^2,$$

and since $\mathbb{R} \subset W_0^1(\mathbb{R}^2)$, this equation implies that

$$\langle f, \boldsymbol{c}\rangle = 0, \quad \forall\, \boldsymbol{c} \in \mathbb{R}^2. \tag{4.15}$$

Furthermore, the velocity $\boldsymbol{u}$ can only be determined up to an additive constant since the mapping $v \mapsto \|\nabla v\|_{0,\mathbb{R}^2}$ is a norm on the quotient space $W_0^1(\mathbb{R}^2)/\mathbb{R}$, equivalent to the quotient norm. Thus, taking into account these two remarks, we can apply the argument of Section 3 to prove the following lemma.

Lemma 4.1. *Each* $f \in (W_0^{-1}(\mathbb{R}^2))^2$ *satisfying*

$$\langle f, \boldsymbol{c}\rangle = 0, \quad \forall\, \boldsymbol{c} \in \mathbb{R}^2$$

determines a unique $\boldsymbol{u} \in (W_0^1(\mathbb{R}^2)/\mathbb{R})^2$ *and* $p \in L^2(\mathbb{R}^2)$ *such that*

$$\begin{aligned} -\nu\,\Delta \boldsymbol{u} + \nabla p &= f \quad \textit{in } \mathbb{R}^2,\\ \operatorname{div} \boldsymbol{u} &= 0 \quad \textit{in } \mathbb{R}^2, \end{aligned} \tag{4.16}$$

and there exists a constant C_1 such that

$$\|u\|_{(W_0^1(\mathbb{R}^2)/\mathbb{R})^2} + \|p\|_{0,\mathbb{R}^2} \leqq C_1 \|f\|_{-1,0,\mathbb{R}^2}.$$

Now, let us take $f \in (X_1^0(\mathbb{R}^2))^2$ with (4.15). Then f also belongs to $(W_0^{-1}(\mathbb{R}^2))^2$, and therefore the Stokes problem (4.16) has a unique solution $u \in (W_1^0(\mathbb{R}^2)/\mathbb{R})^2$ and $p \in L^2(\mathbb{R}^2)$. As in $\mathbb{R}^3$, we know that p is a solution of Laplace's equation

$$\Delta p = \operatorname{div} f \quad \text{in } \mathbb{R}^2. \tag{4.17}$$

But Proposition 11.1 of [Gr] implies that

$$f \in (X_1^0(\mathbb{R}^2))^2 \Rightarrow \operatorname{div} f \in X_1^{-1}(\mathbb{R}^2).$$

Furthermore, using (4.15), we easily see that

$$\operatorname{div} f \perp \mathbb{P}_1.$$

Therefore, we can apply to $\operatorname{div} f$ the dual proposition of the following isomorphism result established by [Gr, Th. 5.11]: *The Laplace operator Δ is an isomorphism from $X_{-1}^1(\mathbb{R}^2)/\mathbb{P}_1$ onto $X_{-1}^{-1}(\mathbb{R}^2)$.* The dual proposition reads: *Δ is an isomorphism from $X_1^1(\mathbb{R}^2)$ onto the subspace of $X_1^{-1}(\mathbb{R}^2)$ that is orthogonal to $\mathbb{P}_1$.* Since $X_1^1(\mathbb{R}^2) = W_1^1(\mathbb{R}^2)$ (*cf.* [Gr, Prop. 9.1]), this means that Problem (4.17) has a unique solution $p \in W_1^1(\mathbb{R}^2)$ which coincides with the pressure of the Stokes problem) and

$$\|p\|_{1,1,\mathbb{R}^2} \leqq C_2 \|\operatorname{div} f\|_{X_1^{-1}(\mathbb{R}^2)} \leqq C_3 \|f\|_{(X_1^0(\mathbb{R}^2))^2}. \tag{4.18}$$

Thus, the velocity u satisfies Laplace's equation

$$-\nu \Delta u = f - \nabla p \quad \text{in } \mathbb{R}^2, \tag{4.19}$$

with right-hand side $f - \nabla p \in (X_1^0(\mathbb{R}^2))^2$ and

$$\langle f, c\rangle = 0, \quad \langle \nabla p, c\rangle = 0, \quad \forall\, c \in \mathbb{R}^2.$$

Then $f - \nabla p \in ((X_{-1}^0(\mathbb{R}^2)/\mathbb{R})^2)'$, and it follows from [Gr, Prop. 5.2] that (4.19) has a unique solution $u \in (W_1^2(\mathbb{R}^2)/\mathbb{R})^2$ (which coincides with the velocity of the Stokes problem) and

$$\|u\|_{(W_1^2(\mathbb{R}^2)/\mathbb{R})^2} \leqq C_4 \|f - \nabla p\|_{(X_1^0(\mathbb{R}^2))^2} \leqq C_5 \|f\|_{(X_1^0(\mathbb{R}^2))^2}. \tag{4.20}$$

These results furnish the proof of

Theorem 4.3. *Let $f \in (X_1^0(\mathbb{R}^2))^2$ with*

$$\langle f, c\rangle = 0, \quad \forall\, c \in \mathbb{R}^2.$$

Then the solution $\{u, p\}$ of the Stokes problem (4.16) *has the regularity $u \in (W_1^2(\mathbb{R}^2)/\mathbb{R})^2$, $p \in W_1^1(\mathbb{R}^2)$ and*

$$\|u\|_{(W_1^2(\mathbb{R}^2)/\mathbb{R})^2} + \|p\|_{1,1,\mathbb{R}^2} \leqq C_6 \|f\|_{(X_1^0(\mathbb{R}^2))^2}. \tag{4.21}$$

Finally, we can easily derive the same regularity for the nonhomogeneous Stokes problem

$$(4.22) \qquad -\nu\,\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f}, \quad \operatorname{div} \boldsymbol{u} = h \quad \text{in } \mathbb{R}^2,$$

with $\boldsymbol{f}$ as above and h given in $W_1^1(\mathbb{R}^2)$. For this, it suffices to solve Laplace's equation:

$$(4.23) \qquad \Delta\varphi = h \quad \text{in } \mathbb{R}^2.$$

According to [Gr, Lemma 5.14], this problem has a unique solution $\varphi \in W_1^3(\mathbb{R}^2)/\mathbb{P}_1$ such that

$$(4.24) \qquad \|\varphi\|_{W_1^3(\mathbb{R}^2)/\mathbb{P}_1} \leqq C_7\, \|h\|_{1,1,\mathbb{R}^2}.$$

Then $\boldsymbol{u}_0 = \nabla\varphi$ belongs to $(W_1^2(\mathbb{R}^2)/\mathbb{R})^2$ and

$$\operatorname{div} \boldsymbol{u}_0 = h.$$

Therefore $\boldsymbol{u} - \boldsymbol{u}_0$ is the solution of the homogeneous Stokes problem

$$-\nu\,\Delta(\boldsymbol{u} - \boldsymbol{u}_0) + \nabla p = \boldsymbol{f} + \nu\,\Delta\boldsymbol{u}_0, \quad \operatorname{div}(\boldsymbol{u} - \boldsymbol{u}_0) = 0,$$

with right-hand side $\boldsymbol{f} + \nu\,\Delta\boldsymbol{u}_0 \in (X_1^0(\mathbb{R}^2))^2$. Theorem 4.3 and (4.24) then imply

$$\boldsymbol{u} \in (W_1^2(\mathbb{R}^2)/\mathbb{R})^2, \quad p \in W_1^1(\mathbb{R}^2)$$

with the following analogue of (4.21):

$$(4.25) \qquad \|\boldsymbol{u}\|_{(W_1^2(\mathbb{R}^2)/\mathbb{R})^2} + \|p\|_{1,1,\mathbb{R}^2} \leqq C_8[\|\boldsymbol{f}\|_{(X_1^0(\mathbb{R}^2))^2} + \|h\|_{1,1,\mathbb{R}^2}].$$

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