PAVEL TICHÝ

Indiscernibility of Identicals

Abstract. It is well known that the manner in which a definitely descriptive term contributes to the meaning of a sentence depends on the place the term occupies in the sentence. A distinction is accordingly drawn between ordinary contexts and contexts variously termed 'non-referential', 'intensional', 'oblique', or 'opaque'. The aim of the present article is to offer a general account of the phenomenon, based on transparent intensional logic. It turns out that on this approach there is no need to say (as Frege does) that descriptive terms are referentially ambiguous or to deny (as Russell does) that descriptive terms represent self-contained units of meaning. There is also no need to tolerate (as Montague does) exceptions to the Principle of Functionality. The notion of an ordinary (i.e., 'non-intensional') context is explicated exclusively in terms of logical structure and it is argued that two aspects of ordinariness (termed 'hospitality' and 'exposure') must be distinguished.

0. Introduction

A great deal of recent logical research has revolved around the disconcerting fact that the plausibly looking argument form

(*) The φ = the ψ; A(the φ/z) $\therefore$ A(the ψ/z)

[where A(Z/z) is the result of substituting Z for free occurrences of z in the sentence matrix A] is invalid. For let φ be 'man who lives next door' and ψ 'man who runs the city'. Then although (*) is valid if A is 'z is sick', it is invalid if A is 'Muhammad Ali believes that z is sick'.

The fact is puzzling because it seems undeniable that whatever A itself says about the unspecified individual z, A(the φ/z) says about the referent of 'the φ' and A(the ψ/z) about the referent of 'the ψ'. Thus when those referents are one and the same, the latter two sentences are bound to say one and the same thing about one and the same object. How then can they differ in truth-value?

The conventional answer to the problem is to deny that an expression like 'the φ' or 'the ψ' is what it seems, namely a univocally referring term. Some semanticists maintain that such a term never refers to anything, that it does not represent a self-contained unit of meaning. Others grant it the status of a referring term but insist that it refers ambiguously: it designates one object in some syntactic contexts and a different object — or nothing at all — in other contexts.

According to Russell, the originator of the first of the two theories, the subject matter of a sentence containing 'the ψ' is not the unique individual (if any) instantiating φ, but the property φ itself. From a logical point of view, the sentences 'The φ is sick' and 'The ψ is sick' are not of the form 'a is sick' and 'b is sick'; their logical (as distinct from syntactical) form is represented by the formulas

(1a) $(\exists z)\big((\forall y)(\varphi y \equiv .y = z) \,\&\, z \text{ is sick}\big)$
(1b) $(\exists z)\big((\forall y)(\psi y \equiv .y = z) \,\&\, z \text{ is sick}\big)$,

no part of which names any individual. Similarly, the logical form of 'Ali believes that the φ is sick' and 'Ali believes that the ψ is sick' is (on at least one reading)

(2a) Ali believes that $(\exists z)\big((\forall y)(\varphi y \equiv .y = z) \,\&\, z \text{ is sick}\big)$
(2b) Ali believes that $(\exists z)\big((\forall y)(\psi y \equiv .y = z) \,\&\, z \text{ is sick}\big)$,

where the only name of an individual is 'Ali'. The logical form of 'The φ = the ψ', according to Russell, is not '$a = b$' but

(3) $(\exists z)\big((\forall y)(\varphi y \equiv .y = z) \,\&\, (\forall y)(\psi y \equiv .y = z)\big)$.

It is easily seen that while (3) and (1a) imply (1b), (3) and (2a) do not imply (2b).

Frege, the originator of the second of the two theories, did not question the status of 'the φ' and 'the ψ' as names, but maintained that they are ambiguous. The reference of 'the φ' in A(the φ/z) depends, according to Frege, on the nature of the context A: if A is 'z is sick' or 'z = x', for example, then 'the φ' refers to the only individual (if any) instantiating φ; if A is 'Ali believes that z is sick', the same term refers rather to an abstract entity which Frege calls the (ordinary) *sense* of 'the φ'. On Frege's view, (*) is a perfectly valid inference schema. But, as with any such schema, one must beware of equivocation. If A is 'Ali believes that z is sick', the term 'the ψ' means one thing in the first premise and a completely different thing in the conclusion. Thus the fact that in this case the conclusion does not follow from the premises is nothing against the soundness of the schema.

Neither of the two theories is satisfactory. Russell's contention that 'the φ' refers to nothing and is meaningless in isolation is difficult to square with linguistic evidence, such as the facts that the term is invariably pronominalizable and that it constitutes, by itself, a complete answer to a question (e.g., 'Who is sick?'). Besides, Russell's procedure for eliminating such terms fails when applied to sentences expressing notional attitudes like 'Ali contemplates the φ', 'Ali worships the φ' and the like, where the descriptive term carries no existential import and yet does not occur in a subordinate clause.

Frege's theory is also highly implausible. The term 'the man who lives next door' does not seem ambiguous and does not seem to undergo a change in meaning when transplanted from 'The man who lives next door is sick' to 'Ali believes that the man who lives next door is sick'. Indeed the two sentences can be conjoined and the second occurrence of the term pronominalized to avoid the anaphora: 'The man who lives next door is sick and Ali believes that *he* is'. But if Frege is right, the 'he' has no semantic antecedent to refer back to: what it refers to (namely a sense) does not receive reference in the foregoing context.

There is a simple way of avoiding the shortcomings of the two theories while retaining all their virtues. We have seen that Frege construes 'the φ' as a name of a sense when it appears in special contexts like 'Ali believes that...'. It is widely agreed that Frege's *Sinn* is best explicated as what in possible-world semantics goes by the name of *intension*: a function taking world-times to objects of some type. In particular, what Frege calls the sense of 'the φ' is plausibly seen as a function taking each world-time to the unique individual (if any) which instantiates φ in that world at that time. The function is fittingly spoken of as the *office* of the φ, and its value at a world-time as the *holder* or *occupant* of the office in that world at that time. It is thus this function or office that the term 'the φ' refers to in epistemic and other 'oblique' contexts according to this interpretation of Frege's theory. Now all we need to do to remedy the shortcomings of the two semantic theories under consideration is to transfer this brilliant insight of Frege's to all other contexts. Why not say that in the sentence 'The man next door is sick' the descriptive term also refers to the office of the man next door? Note that this move immediately resolves the anaphora problem mentioned above. But it forces itself upon us on independent and much more general grounds.

It is hard to deny that the sentence tells us something about the office of the man who lives next door: it tells us that the office has a sick holder. The sentence cannot possibly be true without the office satisfying this condition. It tells us, on the other hand, nothing at all about Ali, who, we may suppose, is the only man who lives next door. In particular, the sentence does not tell us that Ali is sick, for it is perfectly possible for it to be true without Ali being sick. It does not tell us that Ali lives next door either, for it is perfectly possible for the sentence to be true without Ali living next door. And it is hard to see what other information about Ali the sentence might possibly have to offer. Russell was thus right in maintaining that the term 'the man who lives next door' is not a name of Ali even if it is Ali who lives next door. A name of an object in a sentence which has nothing whatsoever to say about that object would clearly be out of place.

Similar considerations apply to identity sentences like 'The man who lives next door is (identical with) the man who runs the city'. Frege was puzzled by the fact that such sentences are factually informative and to resolve the puzzle he proposed his famous theory of *Sinn* and *Bedeutung*.

It is quite remarkable how long it has gone unnoticed that Frege's theory does not solve the problem at all, for it leaves us with no coherent account of the assertive content of such sentences. To see this, suppose that Ali is unique not only in living next door but also in running the city. Clearly any semantic account of the sentence 'The man who lives next door is the man who runs the city' must yield an answer to the question, Does the sentence convey any information about Ali? If Frege's answer is in the negative, has owes us an explanation of his insistence that the sentence contains two names of that man. If, on the other hand, the answer is in the affirmative, we are surely entitled to ask exactly what it is that the sentence tells us about Ali. Should it be replied that the information is to the effect that Ali is identical with himself, we would be back at square one with puzzle still unexplained. And should i tbe replied that what the sentence tells us about Ali is [perhaps *inter alia*] that he runs the city, the question arises as to how it is that no amount of deductive ingenuity will coax this information from the sentecnce.

But a perfectly natural account of the factual content of the sentence suggests itself. The sentence conveys information about two offices, that of the man who lives next door and that of the man who runs the city. It gives us no clue as to who occupies those offices. But it tells us nevertheless something about them that might not have been the case: namely that they are co-occupied, that some individual or other holds them both. We have seen that an office is a function whose value at a world-time is the occupant (if any) of the office in that world at that time. The assertive content of an identity sentence like the one just considered is simply to the effect that two such functions happen to take the same value in the actual world at the present time. Note that, quite generally, when two functions are said to take the same value at a certain argument, it is the functions themselves that are being sIpoken of, not the common value of those functions at that argument. When we are told that $\tan 45° = \cot 45°$, for example, we learn something about the tangent and cotangent functions, not about the number one, which *is* the common value of those functions at 45°.

Those followers of Frege who might agree with our analysis of the assertive content of the sentence 'The man who lives next door is the man who runs the city' are saddled with the problem of squaring the analysis with their view that the terms flanking the identity copula 'is' are names of Ali. Why is it that a sentence whose sole purpose is to tell us something about two objects of a certain kind contains no expressions referring to those objects but contains instead not one but *two* names of an object of a completely different kind, one which it has nothing to tell us about at all? In other words, adoption of the analysis renders it quite untenable to resist the obvious move of construing the terms 'the man who lives next door' and 'the man who runs the city' as names of the respective offices[1].

Once made, the move results in a radical simplification of semantic theory. The fallacious dogma that a descriptive term normally serves to refer to the object (if any) which answers to the description is so entrenched that most semanticists no longer take it for a piece of theory that it is, but for a brute fact that any theory must accommodate on pain of inadequacy. Attempts to square the dogma with hard linguistic data have lead to a cluster of spurious, *ad hoc* distinctions resulting in a chronic obfuscation of semantic theory.

One case in point is Frege's distinction between *Sinn* and *Bedeutung*, which has itself gained the status of an unquestioned presupposition of current semantic debate. Frege postulated the distinction in an attempt to reconcile the assumption that a descriptive term is a name of the object described, with the undeniable fact that to understand the term it is not enough to know which object it is. To solve the problem, Frege ascribed the term two semantic functions: to *refer* to the object and to *express* a *Sinn*. Once it is realized, however, that the term *names* what Frege called *Sinn*, or something very much like it, the need to postulate a second dimension of meaning disappears. We can say that the sole function of any meaningful expression is to name a definite object and that to understand the expression consists simply in knowing which object it is.

The recently popular distinction, due to Kripke, between rigid and non-rigid designators can also be set aside. If the term 'the φ' does not stand for the only φ-er but rather for the *office* of the only φ-er, its designatum is independent of world and time. We can say that *all* designators, without exception, are rigid. A designator which stands for an object in one world at one time, stands for it in all worlds at all times; or better still, designation, i.e. the correspondence between names and what they stand for in a given language, is absolute, not relative to world and time. What a sentence of a given language says and what it says it about is invariably an *a priori* matter; extralinguistic facts only come in when it is to be decided whether what the sentence says is the case.

Finally, the widely accepted distinction between so-called 'referential' and 'non-referential' contexts also turns out to be uncalled for. Authors invoke this distinction in order to deal with the problem raised by the schema (∗) and also with a related problem occasioned by the schema

(∗∗) $\quad$ A(the φ/z) ∴ There is a z such that A.

---

[1] Further arguments for this thesis can be found in [1]–[5]. For a detailed philosophical defence, see [7].

Having conjectured that a term like 'the φ' refers in some contexts and not in others, the authors then assert that (*) and (**) are valid if A is a context of the former kind (with respect to z) and otherwise invalid.

Two flaws make this theory less than satisfactory. One is that it leaves us in the dark as to what the semantic role of a descriptive term is in contexts where it is said not to refer. The other, more serious, flaw is that the theory is viciously circular. It invokes the notion of a referential context to circumscribe the class of contexts A for which (*) and (**) are valid. Yet the only explanation of the notion of referentiality it has to offer is *via* (*) and (**) themselves: a context is referential, we are told, if the conclusions of (*) and (**) can be validly drawn from the premises. This is hardly more enlightening than the theory that a substance will put one to sleep just in case it is possessed of *virtus dormitiva*.

If the theory suggested above is correct, there are no contexts (other than quotational ones) in which 'the φ' fails to refer: the term invariably refers to the office of the (only) φ-er. There are no exceptions to the Functionality Principle: what a compound expression refers to depends on only one feature of its constituent terms, namely on what *they* refer to. A distinction is to be drawn, of course, between contexts which generate valid inferences of the form (*) and (**) and those which do not. But it cannot be drawn in terms of the alleged effect such contexts have on the semantic behaviour of expressions like 'the φ'. Rather, it must be drawn in a Russellian spirit, by reference to logical structure, i.e., to the mode in which the fixed referent of the descriptive term is embedded in the logical construction expressed by the whole context. The aim of the present article is to develop a logical theory in which the distinction can be drawn rigorously.

1. Frames

Logical problems can be given rigorous, mathematical treatment only if they arise within a conceptual scheme which is itself mathematically rigorous. In order to apply exact methods, therefore, we have to assume that our conceptual scheme has been *explicated*, i.e., that it has been specified exactly which objects belong to our universe of discourse, exactly which systems of extensions for our concepts are to count as logically possible, etc.

The conceptual system in which we actually think has not, needless to say, been explicated in this way. Hence to make the above assumption is to idealize the epistemic situation we find ourselves in. But idealization has proven a fruitful method in natural as well as social sciences and there is little reason to think that logic is an exception.

The notion of 'frame' introduced in this section is a mathematically rigorous tool for explicating intuitive conceptual systems.

A *frame* is a family of four non-empty collections, o, τ, ι, and ω. Collection o has two elements, T (truth) and F (falsehood), called *truth-values*. τ is the linearly ordered collection of *moments of time*; as moments of time are uniquely representable by real numbers, τ can be looked upon simply as the set of reals. ι and ω differ from frame to frame; the elements of ι are called *individuals* and the elements of ω *possible worlds*.

Let B be the frame $\{o, \tau, \iota, \omega\}$. Each member of B is called a *type over* B. Moreover, if $\xi^1, \ldots, \xi^m$ and η are *types* over B then the collection $(\eta\xi^1 \ldots \xi^m)$ of all m-argument (total and partial) functions from $\xi^1, \ldots, \xi^m$ into η is also a *type over* B. (Nothing is a *type over* B unless it so follows from the above[2].) Elements of a type ξ over B will be called *ξ-objects over* B.

For any type ξ, $(o\xi)$-objects will be also called *ξ-classes*, $(\xi\tau)$-objects *ξ-chronologies*, and $((\xi\tau)\omega)$-objects *ξ-intensions*. o-intensions are also known as *propositions*, and their type $((o\tau)\omega)$ will be written, alternatively, as π; $(o\xi)$-intensions are also known as *properties* of ξ-objects, and $(o\xi^1 \ldots \xi^m)$-intensions as m-ary *relations* of objects of the types $\xi^1, \ldots, \xi^m$.

For the sake of a simple example, consider the miniature frame B_0 consisting of o, τ, ι, and ω, where ι has two members, X^1 and X^2, and ω has also two members, W^1 and W^2. The function C^1 which takes X^1 to T and X^2 to F, and the function C^2 which takes both X^1 and X^2 to F, are two of altogether 9 ι-classes over B_0. The function D^1 which takes every moment of time to T, and the function D^2 which takes all moments prior to 1/1/1985 to T and all other moments to F are two of continuum many o-chronologies over B_0. The function P^1 which takes both W^1 and W^2 to D^1, and the function which takes W^1 to D^1 and W^2 to D^2, are two of continuum many propositions over B_0. The function E^1 which takes all moments to C^1, and the function E^2 which takes all moments prior to 1/1/85 to C^1 and all other moments to C^2, are two of continuum many $(o\iota)$-chronologies over B_0. The function G^1 which takes W^1 to E^1 and W^2 to E^2, is one of continuum many ι-properties over B^1.

A ξ-intension I is said to be *embodied* by ξ-object X at time T in world W iff I takes W to a ξ-chronology which in turn takes T to X. X is said to *instantiate* property S in W at T iff S is embodied, in W at T, by a class which takes X to T. X is said to *bear* relation R *to* Y in W at T if R is embodied, in W at T, by a function which takes X and Y to T. Finally, a proposition P is said to be *true* (*or false*) in W at T if it is embodied, in W at T, by T (or F). A class of propositions is said to *imply* proposition P iff P is true in every world at every time at which every member of the class is true.

2. Explication

A pretheoretical conceptual scheme consists of intuitive concepts which can be looked upon as various features or traits which objects of various types can have or lack (either absolutely or relative to other

[2] Indirect clauses like the present one, which should, strictly speaking, be part of every inductive definition, will be omitted in what follows.

objects). A collection of such features will be called *intensional base.* To explicate an intensional base in a frame B = $\{o, \tau, \iota, \omega\}$ is to assign to each world-time couple (i.e., to each couple consisting of a member of ω and a member of τ) a unique, fully specific distribution of those features through the objects of appropriate type over B. Such an assignment is called an *explication* of the intensional base *in* B. An intensional base together with its explication in a frame form an *epistemic framework.*

For the sake of a simple example let us consider the intensional base whose only member is the notion of belief, the intuitive relation which an individual bears to the propositions he accepts as true. Let us explicate this base in the miniature frame B_0. All we have to do is to stipulate which individual believes which proposition in which world (over B_0). Let us stipulate as follows: in W^1, X^1 always believes P^1 and X^2 always believes P^2, and in W^2, X^1 believes P^1 until 1/1/85, then changes his mind and believes P^2 instead, while X^2 always believes P^2. Call the epistemic framework thus defined F_0.

By specifying which particular, intuitively understood, course of events is represented by each member of ω, an explication gives intuitive content not only to the possible worlds but also to every intension over the frame. In particular, it specifies exactly what condition an object must satisfy to instantiate a given property, or to bear a given relation to another object. Most importantly, it specifies for each proposition over the frame under what circumstances it comes out true; in other words, it specifies exactly what those propositions *say.*

In the framework F_0 an individual instantiates G^1 just in case he believes proposition P^1; P^1 is true just in case (and hence says that) X^2 believes P^2; and P^2 is true just in case (and hence says that) X^1 believes P^1. Observe that a unique $(o\iota\pi)_{\tau\omega}$-object is a rigorous counterpart (the *explicans* in F_0 of) the intuitive concept of belief (the *explicandum*): it is the (unique) relation which in W^1 is borne at all times by X^1 to P^1 and by X^2 to P^2, and which in W^2 is borne by X^1 to P^1 until 1/1/85 and then to P^2, and by X^2 at all times to P^2.

Let us assume that an epistemic framework F has been fixed; we can then speak simply of types, objects, etc., meaning types, objects, etc. over the frame F.

3. Constructions

Objects can be arrived at, singled out, or, as we shall say, *constructed* by means of other objects. For example, if F is a function which is defined at X, then a definite object, namely the value of F at X, can be arrived at or singled out by applying F to X; we shall speak of the application of F to X as a specific construction of that object.

For each type ξ there is an infinite supply of variables ranging over ξ, called *ξ-variables.* An assignment of an object of appropriate type to each variable is called a *valuation.* If v is a valuation and $X^1, \ldots, X^m$ objects of the same respective types as the distinct variables $x^1, \ldots, x^m$ then $v(X^1, \ldots, X^m/x^1, \ldots, x^m)$ is just like v except for assigning $X^1, \ldots, X^m$ respectively to $x^1, \ldots, x^m$.

Objects and variables will be called, collectively, *atoms*. Atoms can be combined into constructions. Constructions, their types, ranks, and what they construct are defined as follows. Any atom of type ξ is a *ξ-construction* of *rank* 0; a ξ-object *v-constructs* itself and a ξ-variable v-constructs the ξ-object assigned to it by the valuation v. Let $\mathrm{X}^0, \mathrm{X}^1, \ldots, \mathrm{X}^m$ $(0 < m)$ be constructions of the respective types $(\eta\xi^1 \ldots \xi^m)$, $\xi^1, \ldots, \xi^m$. Then the application $[\mathrm{X}^0\mathrm{X}^1 \ldots \mathrm{X}^m]$ of X^0 to $\mathrm{X}^1, \ldots, \mathrm{X}^m$ is an *η-construction* of *rank* $r+1$, where r is the largest among the ranks of $\mathrm{X}^0, \mathrm{X}^1, \ldots, \mathrm{X}^m$. $[\mathrm{X}^0\mathrm{X}^1 \ldots \mathrm{X}^m]$ is v-improper (i.e., constructs nothing on v) if one of $\mathrm{X}^0, \mathrm{X}^1, \ldots, \mathrm{X}^m$ is v-improper; otherwise let $X^0, X^1, \ldots, X^m$ be the objects *v-constructed* by $\mathrm{X}^0, \mathrm{X}^1, \ldots, \mathrm{X}^m$ respectively. If X^0 is not defined at $X^0, \ldots, X^m$ then $[\mathrm{X}^0\mathrm{X}^1 \ldots \mathrm{X}^m]$ is v-improper; otherwise it *v-constructs* the value of X^0 at $X^1, \ldots, X^m$. Let Y be an *η-construction* of *rank* r and $x^1, \ldots, x^m$ $(0 < m)$ distinct variables of the respective types $\xi^1, \ldots, \xi^m$. Then the closure $[\lambda x^1 \ldots x^m \mathrm{Y}]$ of Y on $x^1, \ldots, x^m$ is an *$(\eta\xi^1 \ldots \xi^m)$-construction* of *rank* $r+1$ and *v-constructs* the following function F: for any objects $X^1, \ldots, X^m$ of the respective types $\xi^1, \ldots, \xi^m$, if Y is $v(X^1, \ldots, X^m/x^1, \ldots, x^m)$-improper then F is not defined at $X^1, \ldots, X^m$; otherwise the value of F at $X^1, \ldots, X^m$ is the object $v(X^1, \ldots, X^m/x^1, \ldots, x^m)$*-constructed* by Y.

The use of letter types exemplified by the foregoing paragraph will be adhered to in the rest of the paper. In particular, capital Romans will be used to refer to unspecified constructions, small Romans to unspecified atoms, capital italics to unspecified objects, and small italics to unspecified variables. (Small italics will be also used as numerical variables, but no confusion will ensue.) Small Greek letters other than 'o', 'τ', 'ι', 'π', and 'ω' will be used for reference to unspecified types. Brackets will be omitted wherever no misunderstanding can arise. A dot will represent a left-hand bracket whose right-hand mate is to be imagined as far to the right as is compatible with other pairs of brackets.

Let $z^1, \ldots, z^m$ be distinct and let $\mathrm{Z}^1, \ldots, \mathrm{Z}^m$ be of the same respective types. By $\mathrm{A}(\mathrm{Z}^1, \ldots, \mathrm{Z}^m/z^1, \ldots, z^m)$ we shall understand the result of supplanting the free occurrences of z^s in A by Z^s, for each $1 \leqslant s \leqslant m$. Z is said to be *free for* z in A if no free occurrence of z in A is part of an occurrence of $\lambda x^1 \ldots x^m \mathrm{Y}$, where one of $x^1, \ldots, x^m$ is free in Z. It is not difficult to prove[3] the following

3.1. THEOREM. $(\mathrm{Z}^1, Z^1, z^1/\zeta^1; \ldots; \mathrm{Z}^m, Z^m, z^m/\zeta^m)$[4]. *If for* $1 \leqslant s \leqslant m$, Z^s *is free for* z^s *in* A *and* Z^s *v-constructs* Z^s, *then* $\mathrm{A}(\mathrm{Z}^1, \ldots, \mathrm{Z}^m/z^1, \ldots, z^m)$ *v-constructs* A *if and only if* A $v(Z^1, \ldots, Z^m/z^1, \ldots, z^m)$*-constructs* A.

In view of 3.1, constructions conform to the following Functionality

[3] For a proof, see [6].

[4] The notation '$\mathrm{Z}^1, Z^1, z^1/\zeta^1; \ldots; \mathrm{Z}^m, Z^m, z^m/\zeta^m$, is short for: '$\mathrm{Z}^1$, Z^1, and z^1 are ζ^1-constructions; …; Z^m, Z^m, and z^m are ζ^m-constructions'.

Principle: a construction which contains some subconstructions depends, for what it constructs, on a single feature of those subconstructions, namely on what they construct. Briefly, what a construction constructs is a function of what its subconstructions construct.

4. Linguistic constructions

We shall now define a certain class of constructions over the framework F. Constructions belonging to the class will be called 'linguistic', because the expressions of a typical language serve to express constructions of this sort. In this section, however, we do not consider any particular language; the Section is, of course, written in a definite language (English) but what it deals with are constructions themselves, in abstraction from the way they may be expressed in any particular system of communication.

Let w be a fixed ω-variable and t a fixed τ-variable. w and t will be set aside to play a special role in combining linguistic constructions into compound ones. It will be convenient to adopt the following notational conventions: for any construction X of appropriate type, X_{wt}, X_w, and X_t are $[\mathrm{X}w]t$, $\mathrm{X}w$ and $\mathrm{X}t$ respectively; $\mathrm{X}_{(1,1)}$, $\mathrm{X}_{(1,0)}$, $\mathrm{X}_{(0,1)}$ and $\mathrm{X}_{(0,0)}$ are X_{wt}, X_w, X_t, and X respectively; similarly for types: $\xi_{\tau\omega}$, ξ_τ, and ξ_ω are $(\xi\tau)\omega$, $(\xi\tau)$ and $(\xi\omega)$ respectively; $\xi_{(1,1)}$, $\xi_{(1,0)}$, $\xi_{(0,1)}$, and $\xi_{(0,0)}$ are $\xi_{\tau\omega}$, ξ_τ, ξ_ω, and ξ respectively. The class of linguistic constructions is defined inductively as follows.

4.1. Definition. Every atom other than w or t is a *linguistic construction*. Let each of i, j, i^s, j^s $(1 \leqslant s \leqslant m)$ be either 0 or 1. If $\mathscr{X}^0, \mathscr{X}^1, \ldots$ $\ldots, \mathscr{X}^m$ are *linguistic constructions* of the respective types $(\eta\xi^1 \ldots, \xi^m)_{(j^0,i^0)}$, $\xi^1_{(j^1,i^1)}, \ldots, \xi^m_{(j^m,i^m)}$, then

$$\text{(I)} \qquad \lambda w\lambda t.\, \mathscr{X}^0_{(i^0,j^0)}\mathscr{X}^1_{(i^1,j^1)} \ldots \mathscr{X}^m_{(i^m,j^m)}$$

is also a *linguistic construction*. Moreover, if $\mathscr{Y}$ is a *linguistic construction* of type $\theta_{(j,i)}$ and $x^1, \ldots, x^m$ distinct variables other than w or t then

$$\text{(II)} \qquad \lambda w\lambda t\lambda x^1 \ldots x^m \mathscr{Y}_{(i,j)}$$

is also a *linguistic construction*.

Capital script letters $(\mathscr{A}, \mathscr{B}, \ldots)$ will be used to refer to unspecified linguistic constructions. It is readily seen that any linguistic construction is v-proper, and that w and t are not free in it. For each $0 \leqslant s \leqslant m$, the displayed occurrence of $\mathscr{X}^s$ in (I) is a *main constituent* of (I), and (i^s, j^s) is the *supposition* of that main constituent. Similarly, the occurrence of $\mathscr{Y}$ in (II) is the *main constituent* of (II) and its *supposition* is (i, j).

For the sake of illustration we shall consider an epistemic framework $\{o, \iota, \tau, \omega\}$, rich enough to contain objects A/ι, $S/(o\iota)_{\tau\omega}$, $B/(o\iota\pi)_{\tau\omega}$, $E/(o\iota\iota)_{\tau\omega}$ such that A is Muhammad Ali, S is the property (i.e., the explication in the framework of the intuitive notion) of being sick, B is the relation (i.e., the explication in the framework of the intuitive relation) which obtains between believers and the propositions they believe, and E is the relation (i.e., the explication of the intuitive notion) of weight equality. Moreover, let $V/o(o\iota)$ be the class of void ι-classes (i.e., of those classes over the framework which have no members), and let F be the the function which takes every time T to the class of o-chronologies which are frequent at T. For definiteness, let us call a o-chronology (a class of times) *frequent* at time T if for any time T' within six months of T, at least one time within a week of T' belongs to the o-chronology.

Let x and z be distinct variables of type ι. According to the first clause of 4.1, each of x, z, A, S, B, E, V, F is a linguistic construction. By one or more applications of the inductive clauses, the following constructions $\mathscr{L}^1$-$\mathscr{L}^9$ are also linguistic:

($\mathscr{L}^1$) $\lambda w\lambda t . S_{wt}z$
($\mathscr{L}^2$) $\lambda w\lambda t . F_t[\lambda w\lambda t . S_{wt}z]_w$
($\mathscr{L}^3$) $\lambda w\lambda t\lambda x . [\lambda w\lambda t . F_t[\lambda w\lambda t . S_{wt}x]_w]_{wt}$
($\mathscr{L}^4$) $\lambda w\lambda t . [\lambda w\lambda t\lambda x . F_t[\lambda w\lambda t . S_{wt}x]_w]_{wt}z$
($\mathscr{L}^5$) $\lambda w\lambda t . B_{wt}A\lambda w\lambda t . S_{wt}x$
($\mathscr{L}^6$) $\lambda w\lambda t\lambda x . B_{wt}A\lambda w\lambda t . S_{wt}x$
($\mathscr{L}^7$) $\lambda w\lambda t . [\lambda w\lambda t\lambda x . B_{wt}A\lambda w\lambda t . S_{wt}x]_{wt}z$
($\mathscr{L}^8$) $\lambda w\lambda t\lambda x [\lambda w\lambda t . E_{wt}xz]_{wt}$
($\mathscr{L}^9$) $\lambda w\lambda t . V[\lambda w\lambda t\lambda x[\lambda w\lambda t . E_{wt}xz]_{wt}]_{wt}$

$\mathscr{L}^1, \mathscr{L}^2, \mathscr{L}^4, \mathscr{L}^5, \mathscr{L}^7$, and $\mathscr{L}^9$ are of the form (I); in $\mathscr{L}^5$ m is 2 and in the others 1. $\mathscr{L}^3$ and $\mathscr{L}^6$ are of the form (II) with $m = 1$. The main constituents S and z of $\mathscr{L}^1$ have the respective suppositions (1,1) and (0,0). The main constituents F and $\mathscr{L}^1$ of $\mathscr{L}^2$ have the respective suppositions (0,1) and (1,0), the main constituent $\mathscr{L}^2$ of $\mathscr{L}^3$ has the supposition (1,1), etc.

Let v be an arbitrary valuation which assigns individual Z to z. Then the above linguistic constructions v-construct the following objects: $\mathscr{L}^1$ – the proposition that Z is sick, $\mathscr{L}^2$ – the proposition that it is frequently the case that Z is sick, $\mathscr{L}^3$ – the property of being frequently sick, $\mathscr{L}^4$ – the proposition that Z has that property, $\mathscr{L}^5$ – the proposition that Ali believes that Z is sick, $\mathscr{L}^6$ – the property of being believed by Ali to be sick, $\mathscr{L}^7$ – the proposition that Z has that property, $\mathscr{L}^8$ – the property of being equiponderous with Z, and $\mathscr{L}^9$ – the proposition that the property has a void extension.

5. Hospitality

We shall say that supposition (i, j) is (*weakly*) *deeper* than supposition (k, l), symbolically $(k, l) \leqslant (i, j)$, if $k \leqslant i$ and $l \leqslant j$.

5.1. Definition. Let z be a variable other than w or t. If $\mathscr{A}$ is of rank 0, then z is *(k, l)-hospitable in* $\mathscr{A}$ if z is not $\mathscr{A}$. Now let $\mathscr{A}$ be of rank greater than 0. Then z is *(k, l)-hospitable in* $\mathscr{A}$ if each main constituent of $\mathscr{A}$ in which z is free either is z itself and has supposition $(0, 0)$ or has supposition (weakly) deeper than (k, l) and z is (k, l)-hospitable in it.

For example, in $\mathscr{L}^1, \mathscr{L}^3, \mathscr{L}^4$, and $\mathscr{L}^6$–$\mathscr{L}^9$, z is (k, l)-hospitable for any k and l ($\leqslant 1$). In $\mathscr{L}^2$ z is (1,0)- and (0,0)-hospitable only and in $\mathscr{L}^5$ it is (0,0)-hospitable only.

It is easily checked that if z is (k, l)-hospitable in $\mathscr{A}$ nad $\mathscr{X}_{(k,l)}$ is of the same type as z, then $\mathscr{A}(\mathscr{X}_{(k,l)}/z)$ is also a linguistic construction.

For the sake of illustration, let $M/\iota_{\tau\omega}$ be the office of the mayor. Then

$\mathscr{L}^1(M_{wt}/z)$ is $\lambda w\lambda t.S_{wt}M_{wt}$ and constructs the proposition that the mayor is sick,

$\mathscr{L}^2(M_{wt}/z)$ is $\lambda w\lambda t.F_t[\lambda w\lambda t.S_{wt}M_{wt}]_w$ and constructs the proposition that it is frequently the case that the mayor is sick

$\mathscr{L}^4(M_{wt}/z)$ is $\lambda w\lambda t.[\lambda w\lambda t\lambda x.F_t[\lambda w\lambda t.S_{wt}x]_w]_{wt}M_{wt}$ and constructs the proposition that the mayor is such that it is frequently the case that *he* is sick (the *de re* counterpart of the foregoing proposition)

$\mathscr{L}^5(M_{wt}/z)$ is $\lambda w\lambda t.B_{wt}A\lambda w\lambda t.S_{wt}M_{wt}$ and constructs the proposition that Ali believes that the mayor is sick, and

$\mathscr{L}^7(M_{wt}/z)$ is $\lambda w\lambda t.[\lambda w\lambda t\lambda x.B_{wt}A\lambda w\lambda t.S_{wt}x]_{wt}M_{wt}$ and constructs the proposition that the mayor is such that Ali believes that *he* is sick (a *de re* counterpart, with respect to M, of the foregoing proposition).

For any type ξ, let $=^\xi$ be identity between ξ-objects, i.e., the total function of type $o\xi\xi$ which takes T at all and only at couples of the form X, X (for some ξ-object X). We shall conform to the universal practice of writing $X = Y$ for $=^\xi XY$. Linguistic constructions of the form $\lambda w\lambda t.\mathscr{X}_{(k,l)} = \mathscr{X}'_{(k',l')}$ will be spoken of as *identity constructions*.

For example, $\lambda w\lambda t.A = A$, $\lambda w\lambda t.A = M_{wt}$, and $\lambda w\lambda t.N_{wt} = M_{wt}$, where N is the office of my next-door neighbour, are identity constructions. They construct the respective propositions that Ali is Ali (or, equivalently, that Ali is Cassius Clay[5]), that Ali is the mayor, and that my next-door neighbour is the mayor.

The symbol '$\therefore$' will be used to indicate that for any valuation v, the propositions v-constructed by the constructions named to the left of it imply the proposition v-constructed by the construction named on the right.

The following is an easy consequence of Lemma 10.6 (proven in Section 10).

5.2. THEOREM ($\mathscr{A}/\pi$; $\mathscr{X}/\zeta_{(l,k)}$; $\mathscr{X}'/\zeta_{(l',k')}$; z/ζ; $0 \leqslant k, l, k', l' \leqslant 1$). *Let $\mathscr{X}$ and $\mathscr{X}'$ be free for z in $\mathscr{A}$ and let z be both (k, l)- and (k', l')-hospitable in $\mathscr{A}$. Then*

$$\lambda w\lambda t.\mathscr{X}_{(k,l)} = \mathscr{X}'_{(k',l')}, \mathscr{A}(\mathscr{X}_{(k,l)}/z) \therefore \mathscr{A}(\mathscr{X}'_{(k',l')}/z).$$

To illustrate, let us recall that z is (0, 0)-hospitable in $\mathscr{L}^1$. Hence by 5.2, $\lambda w\lambda t.\ A = A$, $\mathscr{L}^1(A/z \therefore \mathscr{L}^1(A/z)$; and indeed, from the premises that Ali is Cassius Clay and that Ali is sick one can validly conclude that Clay is sick. As z is (1,1)-hospitable in $\mathscr{L}^1$ we have $\lambda w\lambda t.\ A = M_{wt}$, $\mathscr{L}^1(A/z) \therefore \mathscr{L}^1(M_{wt}/z)$ and $\lambda w\lambda t.\ N_{wt} = M_{wt}$, $\mathscr{L}^1(N_{wt}/z) \therefore \mathscr{L}^1(M_{wt}/z)$; and indeed, from the premises that Ali is the mayor and that Ali is sick, as well as from the premises that my

[5] The propositions that Ali is Cassius Clay, that Cassius Clay is Cassius Clay, and that Ali is Ali, are, I take it, one and the same, speaking, as they do, of one and the same person (Ali) and saying the very same about him (that he is identical to himself). But see Section 8.

neighbour is the mayor and that my neighbour is sick one can safely conclude that the mayor is sick.

To see the importance of the theorem's hypothesis, recall that z is not (1,1)-hospitable in $\mathscr{L}^2$; thus neither $\lambda w\lambda t.\ A = M_{wt},\ \mathscr{L}^2(A/z \therefore \mathscr{L}^2(M_{wt}/z)$ nor $\lambda w\lambda t.\ N_{wt} = M_{wt},\ \mathscr{L}^2(N_{wt}/z) \therefore \mathscr{L}^2(M_{wt}/z)$ is an instance of 5.2. It is readily seen that the corresponding inferences are indeed fallacious: from the premise that Ali (or my next-door neighbour) is the mayor and that it is frequently the case that Ali (or my next- door neighbour) is sick it does not follow that it is frequently the case that the mayor is sick; if the period of Ali's (or my neighbour's) mayorship is sufficiently brief, the premisses may well be true and the conclusion false.

In the *de re* construction $\mathscr{L}^4$, on the other hand, z is (1,1)-hospitable, hence both $\lambda w\lambda t.A = M_{wt},\ \mathscr{L}^4(A/z) \therefore \mathscr{L}^4(M_{wt}/z)$ and $\lambda w\lambda t \cdot N_{wt} = M_{wt},\ \mathscr{L}^4(N_{wt}/z) \therefore \mathscr{L}^4$ (M_{wt}/z) are cases of 5.2. And indeed, from the premises that Ali (or my neighbour) is the mayor, and that Ali (or my neighbour) is such that it is frequently the case that *he* is sick, one can safely conclude that the mayor is such that it is frequently the case that *he* is sick.

It is left to the reader to check that similar comments apply to the construction $\mathscr{L}^6$ and its *de re* companion $\mathscr{L}^7$.

6. Exposure

6.1. Definition. If $\mathscr{A}$ is z, z is *exposed in* $\mathscr{A}$. If $\mathscr{A}$ is (I) [see 4.1] and either z is *exposed in* one of $\mathscr{X}^0, \mathscr{X}^1, \ldots \mathscr{X}, {}^m$, or $\mathscr{X}^0$ is of the form (II), and z is distinct from $x^1, \ldots, x^m$ and *exposed in* $\mathscr{Y}$, then z is *exposed in* $\mathscr{A}$.

For instance, z is exposed in $\mathscr{L}^1, \mathscr{L}^2, \mathscr{L}^4, \mathscr{L}^5$ and $\mathscr{L}^7$, but not in $\mathscr{L}^8$ or $\mathscr{L}^9$. It is readily seen that if z is exposed in $\mathscr{A}$ then z is free in $\mathscr{A}$.

For any type ξ, let Σ^ξ be the existential quantifier over ξ-objects, i.e., the total function of type $o(o\xi)$ which takes T at a ξ-class just in case the class is not empty. Where I is a o-construction and z is a ζ-variable, we shall write $(\exists z)\mathrm{I}$ for $\Sigma^\zeta[\lambda w\lambda t\lambda z\mathrm{I}]_{wt}$. Linguistic constructions of the form $\lambda w\lambda t.(\exists z)\mathscr{A}_{wt}$ will be spoken of as *existential constructions.*

The following are examples of existential constructions: $\lambda w\lambda t.(\exists z)\mathscr{L}^1_{wt}$, which constructs the proposition that at least one individual is sick, $\lambda w\lambda t.(\exists z)\mathscr{L}^2_{wt}$ and $\lambda w\lambda t.(\exists z)\mathscr{L}^4_{wt}$, both constructing the proposition that at least one individual is such that it is frequently the case that he is sick, $\lambda w\lambda t.(\exists z)\mathscr{L}^5_{wt}$ and $\lambda w\lambda t.(\exists z)\mathscr{L}^7_{wt}$ both constructing the proposition that at least one individual is such that Ali believes that he is sick, and $\lambda w\lambda t.(\exists z)\mathscr{L}^9_{wt}$, which constructs the proposition that at least one individual is such that the class of individuals equiponderous with him is void.

The following theorem follows easily from Lemma 11.6 (proven in Section 11).

6.2. Theorem. $(\mathscr{A}/\pi; \mathscr{X}/\zeta_{(l,k)}; z/\zeta; 0 \leqslant k, l \leqslant 1)$. *Let* $\mathscr{X}$ *be free for* z *in* $\mathscr{A}$ *and let* z *be exposed and* (1,1)*-hospitable in* $\mathscr{A}$. *Then*

$$\mathscr{A}(\mathscr{X}_{(k,l)}/z) \therefore \lambda w\lambda t.(\exists z)\mathscr{A}_{wt}.$$

To illustrate, let us recall that z is exposed, (0,0)-hospitable, and (1,1)-hospitable in $\mathscr{L}^1$. Hence by 6.2, both $\mathscr{L}^1(A/z) \therefore \lambda w\lambda t.(\exists z)\mathscr{L}^1_{wt}$ and $\mathscr{L}^1(M_{wt}/t) \therefore \lambda w\lambda t.(\exists z)\mathscr{L}^1_{wt}$. And indeed, the conclusion that at least one individual is sick is validly drawn from the premise that Ali is sick, as well as from the premise that the mayor is sick. In $\mathscr{L}^2$ z is exposed, (0,0)-hospitable but not (1,1)-hospitable; hence while $\mathscr{L}^2(A/z) \therefore \lambda w\lambda t.(\exists z)\mathscr{L}^2_{wt}$ is a case of 6.2, $\mathscr{L}^2(M_{wt}/z) \therefore \lambda w\lambda t.(\exists z)\mathscr{L}^2_{wt}$ is not. The proposition that there exists an individual such that it is frequently the case that he is sick, clearly follows from the premise that it is frequently the case that Ali is sick. But it does not follow from the premise that it is frequently the case that the mayor is sick: if the office of the mayor changes holders often enough, the premise may be true and the conclusion false. In $\mathscr{L}^5$, z also fails of (1,1)-hospitality; hence $\mathscr{L}^5(M_{wt}/z) \therefore \lambda w\lambda t.(\exists z)\mathscr{L}^5_{wt}$ is no instance of 6.2 either. As is easy to see, from the premise that Ali believes that the mayor is sick one cannot safely conclude that there is an individual such that Ali believes that that individual is sick; for if Ali does not known exactly who holds the mayoral office (especially if the office is in fact vacant) the premise may be true and the conclusion false. In $\mathscr{L}^9$ z is (1,1)-hospitable, but not exposed. Thus 6.2 does not endorse the inference $\mathscr{L}^9(M_{wt}/z) \therefore \lambda w\lambda t.(\exists z)\mathscr{L}^9_{wt}$. And indeed, from the premise that the class of individuals who are equiponderous with the mayor is empty it by no means follows that there is an individual such that the class of those equiponderous with *him* is empty; for if the office of the mayor is unoccupied the premise is true and the conclusion is false.

7. Languages

To tell someone that Ali is sick I must somehow draw his attention to the construction $\lambda w\lambda t.S_{wt}A$. Communication is exchange of linguistic constructions over a frame.

Most constructions (variables and constructions of rank higher than 0) over a frame are not ξ-objects over the frame for any ξ, and neither are relations between individuals and constructions. Hence if communication is to proceed *within* the frame in question, constructions must be *coded*, i.e. represented by objects over the frame. Individuals can then communicate constructions indirectly by relating themselves to the representing objects.

The generality of our considerations will not be diminished if we restrict ourselves to numerical codes, i.e., to codes in which constructions are represented by numbers. (Codes of other sorts are easily reduced to numerical codes via Gödelization.) Thus, by a *code* over frame F we shall understand a mapping of a class of numbers (τ-objects) into the class of constructions over F. If $\mathfrak{Q}$ is a code which takes number N to a construction, then N is said to be a *code number* of the construction *in* $\mathfrak{Q}$.

A typical code $\mathfrak{Q}$ is many-one: it assigns, in some cases, one and the same construction to more than one number. Two code numbers of the same construction will be called $\mathfrak{Q}$-equivalent. An arithmetical function is a *syntactic function* relative to $\mathfrak{Q}$ if it preserves $\mathfrak{Q}$-equivalence; in other words, if it takes $\mathfrak{Q}$-equivalent values at $\mathfrak{Q}$-equivalent arguments.

If N is a code number in $\mathfrak{Q}$ of a construction which is closed and constructs object X, then N is also said to be a *name of X in $\mathfrak{Q}$,* or briefly a *$\mathfrak{Q}$-name of X.*

For any type ξ, let $\mathfrak{Q}^{\xi}$ be the function which takes a number N to ξ-object X iff N is a $\mathfrak{Q}$-name of X. Although $\mathfrak{Q}$ itself is not an object over the frame, $\mathfrak{Q}^{\xi}$ is and has type $\xi\tau$. Numbers at which $\mathfrak{Q}^{\pi}$ is defined are called $\mathfrak{Q}$-sentences. A sentence is $\mathfrak{Q}$-true if $\mathfrak{Q}^{\xi}$ takes it to a true proposition.

By itself a code is not yet a communication system. In order to communicate a construction by means of its code number, a communicator must somehow single out that number, or, as we shall say, he must *display* it. As there are many different methods of displaying numbers, some incompatible with others, a particular communication system is not specified until one such method is fixed upon, i.e., until it is agreed exactly what it takes for any given individual to display any given number at any given time. Any such agreement can be represented by what will be called a *display operation,* an operation which, in any world at any time, takes each individual to the number, if any, he displays in that world at that time. A display operation is thus simply an object of type $(\tau\iota)_{\tau\omega}$ over the frame in question.

An ordered couple of the form $(\mathfrak{Q}, D)$, where $\mathfrak{Q}$ is a code and D a display operation, will be called a *language* (over the framework in question). $\mathfrak{Q}$ is the semantic component of the language; it determines which objects are meaningful in the sense of signifying constructions. D is the pragmatic component; it determines how a user of the language must relate himself to such a meaningful object in order to communicate the construction signified by it.

In the English language, code numbers are displayed by uttering or inscribing finite sequences of phonemes or graphemes, called *expressions.* For example, by saying or writing 'Muhammad Ali', any person can display a name, say U, of Muhammad Ali, and by saying or writing 'Cassius Clay', any person can display another name, say V, of Muhammad Ali. By saying or writing 'I', any person can display a name of himself; let $I^{\mathfrak{E}}$ be the function which takes every person to the name so displayed.

Modes of combining English expressions into compounds correspond to syntactic functions. For example, any person who can display a name of an individual X by uttering or inscribing expression X and a name of individual Y by uttering or inscribing expression Y, can display a code number of the identity construction $\lambda w\lambda t.X = Y$ by uttering or inscribing the concatenation of X with 'is' and Y. Let $H^{\mathfrak{E}}$ be the syntactic function which takes any two such names to the corresponding code number of the identity construction.

8. Linguistic attitudes

Relations which individuals bear to numbers qua names of a language are called *linguistic attitudes.* A typical linguistic attitude is the *assert-true* relation, the relation which an individual bears to a sentence he displays with the intention that his audience accept it as true. Another example is the *believe-true* relation; it is the relation which an individual bears to a sentence he displays to himself and considers it true.

Linguistic attitudes differ from propositional attitudes in logical type: the propositional attitudes of assertion and belief are of the type $(o\iota\pi)_{\tau\omega}$, whereas the corresponding linguistic attitudes are of the type $(o\iota\tau)_{\tau\omega}$. Linguistic attitudes also differ from their propositional counterparts in being relative to languages. There is no assert-true or believe-true relation *simpliciter*, only the assert-true relation *with respect to* a given language $(\mathfrak{Q}, D)$ — call it $A^{(\mathfrak{Q},D)}$ —, the believe-true relation *with respect to* $(\mathfrak{Q}, D)$ — call it $B^{(\mathfrak{Q},D)}$ — etc. One and the same individual may believe-true a sentence as a name in one language but not as a name in another.

It is readily seen that there is no logical connection between the belief and the believe-true relations. An individual who knows no language may believe a proposition (Fido may believe that there is meat in the fridge) without believing-true any name of that proposition in any language. On the other hand, a language speaker who does not know, or is mistaken about, which particular proposition is named by a sentence of the language, may adopt the believe-true attitude to the sentence without believing the proposition, and vice versa.

From the fact that someone takes the believe-true attitude to a sentence, it does not follow that he takes the same attitude to every other sentence which names the same proposition. The difference between propositional and linguistic belief is thus also manifested in the fact that although

$$\lambda w\lambda t.\mathscr{P} = \mathscr{Q},\ \lambda w\lambda t.B_{wt}\mathscr{X}\mathscr{P} \therefore \lambda w\lambda t.B_{wt}\mathscr{X}\mathscr{Q}$$

is a valid inference form (indeed a case of 5.2), the linguistic counterpart

$$\lambda w\lambda t.[\lambda w\lambda t.\mathfrak{Q}^{\pi}\mathscr{N}]_{wt} = [\lambda w\lambda t.\mathfrak{Q}^{\pi}\mathscr{M}]_{wt},\ \lambda w\lambda t.B_{wt}^{(\mathfrak{Q},D)}\mathscr{X}\mathscr{N} \therefore \lambda w\lambda t.B_{wt}^{(\mathfrak{Q},D)}\mathscr{X}\mathscr{M}$$

is not.

Let $\mathfrak{E}$ be the English language. Does uttering (or writing) the verb 'believe' amount to a reference, in $\mathfrak{E}$, to the propositional attitude B (as was taken for granted in Section 4) or to the linguistic attitude $B^{\mathfrak{E}}$?

Sometimes the reference is undeniably to B. When one says 'Fido believes that there is meat in the fridge', one hardly implies that Fido is adopting a *linguistic* attitude, or that he is capable of doing so. For all one says, Fido may have no language at all.

On the other hand, if Ali is afflicted with amnesia one may wish to say that although

(a) Ali believes that Ali is Ali,

it is not the case that

(b) Ali believes that Ali is Clay

or that

(c) Ali believes that he himself is Clay.

Yet the proposition that Ali is Ali is the same as that Ali is Clay. It is clearly not this proposition that Ali has forgotten. His problem rather is that he has forgotten that he is *called* 'Clay'; as a result, he does not know that the sentences displayable by him by means of the expressions 'Ali is Clay' and 'I am Clay' stand for that proposition. Thus (a), (b), and (c) must in this case be construed as reporting Ali's *linguistic* attitude to these different sentences. They must be construed as expressive of the constructions

(a*) $\lambda w \lambda t . B^{\mathfrak{C}} A\,[\lambda w \lambda t . H^{\mathfrak{C}}\, U U]_{wt}$
(b*) $\lambda w \lambda t . B^{\mathfrak{C}} A\,[\lambda w \lambda t . H^{\mathfrak{C}}\, U V]_{wt}$
(c*) $\lambda w \lambda t . B^{\mathfrak{C}} A\,[\lambda w \lambda t . H^{\mathfrak{C}}\,[\lambda w \lambda t . I^{\mathfrak{C}} A]_{wt}\, V]_{wt}$.

The propositions constructed by (a*), (b*), and (c*) are logically independent.

9. Natural deduction for partial type logic

The aim of the last three sections is to prove the lemmas appealed to in connection with Theorems 5.2 and 6.2. The present section gives an exposition of a deductive system which is then applied in Sections 10 and 11 to prove the respective Lemmas 10.6 and 11.6.

An ordered couple whose first component is a ξ-atom a and whose second component is a ξ-construction A, symbolically a : A, will be called a *match*. Valuation v is said to *satisfy* a : A if a and A v-construct one and the same object. We shall also allow for matches whose first component is missing, symbolically :A. Valuation v *satisfies* :A just in case A is v-improper. Two matches are said to be *patently incompatible* if they are of the form A^1 : A, A^2 : A, where A^1 and A^2 are distinct objects, or of the form a : A, :A. Patently incompatible matches are clearly never satisfied by one and the same valuation. If $\mathfrak{M}$ is a : A then x is *free in* $\mathfrak{M}$ just in case x is free in a or A; moreover, $\mathfrak{M}(\mathrm{x}^1, \ldots, \mathrm{x}^m/x^1 \ldots, x^m)$ is $\mathrm{a}(\mathrm{x}^1, \ldots, x^m/x^1, \ldots, x^m)$: $\mathrm{A}(\mathrm{x}^1, \ldots, \mathrm{x}^m/x^1, \ldots, x^m)$. If $\mathfrak{M}$ is :A then x is *free in* $\mathfrak{M}$ just in case x is free in A; moreover, $\mathfrak{M}(\mathrm{x}^1, \ldots, \mathrm{x}^m/x^1, \ldots, x^m)$ is :A $(\mathrm{x}^1, \ldots, \mathrm{x}^m/x^1, \ldots, x^m)$. If Φ is a class of matches, x is *free in* Φ just in case it is free in at least one member of Φ; moreover, $\Phi(\mathrm{x}^1, \ldots, \mathrm{x}^m/x^1, \ldots, x^m)$ is the class of matches of the form $\mathfrak{M}(\mathrm{x}^1, \ldots, \mathrm{x}^m/x^1, \ldots, x^m)$, where $\mathfrak{M}$ is in Φ.

A couple whose first component is a finite set Φ of matches and whose second component is a match $\mathfrak{M}$ is called a *sequent* and symbolized thus: $\Phi \to \mathfrak{M}$. We shall write $\mathfrak{M}^1, \ldots, \mathfrak{M}^m \to \mathfrak{M}$ for $\{\mathfrak{M}^1, \ldots, \mathfrak{M}^m\} \to \mathfrak{M}$. $\Phi \to \mathfrak{M}$ is *valid* if every valuation which satisfies all members of Φ also satifies $\mathfrak{M}$.

In what follows we shall state a number of validity-preserving operations on sequents, called *rules of derivation*. Rules of derivation will be stated in the following form:

$$\Phi^1 \to \mathfrak{M}^1;\ \Phi^2 \to \mathfrak{M}^2;\ \ldots;\ \Phi^k \to \mathfrak{M}^k // \Phi \to \mathfrak{M}.$$

The rule says that whenever the sequents to the left of the double slash are valid, so is the one on the right.

The definition of validity and 3.1 justify the following rules of derivation (the type distribution being as follows: f, f, g, $F/\eta\xi^1 \ldots \xi^m$; x^1, x^1, X^1/ξ^1; ...; x^m, x^m, X^m/ξ^m; y, y, Y/η):

9.1 $//\Phi \to \mathfrak{M}$ provided $\mathfrak{M}$ belongs to Φ.

9.2 $\Psi \to \mathfrak{M}//\Phi \to \mathfrak{M}$ provided Ψ is a subset of Φ.

9.3 $\Phi, \mathfrak{N} \to \mathfrak{M}; \Phi \to \mathfrak{N}//\Phi \to \mathfrak{M}$

9.4 $//\mathrm{y} : \mathrm{y}$

9.5 $\Phi \to \mathfrak{Q}^1; \Phi \to \mathfrak{Q}^2//\Phi \to \mathfrak{M}$ provided $\mathfrak{Q}^1$ and $\mathfrak{Q}^2$ are patently incompatible.

9.6 $\Phi, :\mathrm{Y} \to \mathfrak{M}; \Phi, y : \mathrm{Y}//\Phi \to \mathfrak{M}$ provided y is not free in $\Phi, \mathrm{Y}, \mathfrak{M}$.

9.7 $\Phi \to \mathrm{y} : \mathrm{F}\mathrm{X}^1 \ldots \mathrm{X}^m; \Phi, f : \mathrm{F}, x^1 : \mathrm{X}^1, \ldots, x^m : X^m \to \mathfrak{M}//\Phi \to \mathfrak{M}$ provided $f, x^1, \ldots, x^m$ are distinct and not free in $\Phi, \mathrm{F}, \mathrm{X}^1, \ldots, \mathrm{X}^m, \mathfrak{M}$.

9.8 $\Phi \to \mathrm{y} : \mathrm{F}\mathrm{X}^1 \ldots \mathrm{X}^m; \Phi \to \mathrm{x}^1 : \mathrm{X}^1; \ldots; \Phi \to \mathrm{x}^m : \mathrm{X}^m//\Phi \to \mathrm{y} : \mathrm{F}\mathrm{x}^1 \ldots \mathrm{x}^m$.

9.9 $\Phi \to \mathrm{y} : \mathrm{F}\mathrm{x}^1 \ldots \mathrm{x}^m; \Phi \to \mathrm{x}^1 : \mathrm{X}^1; \ldots; \Phi \to \mathrm{x}^m : \mathrm{X}^m//\Phi \to \mathrm{y} : \mathrm{F}\mathrm{X}^1 \ldots \mathrm{X}^m$.

9.10 $\Phi, y : \mathrm{f}x^1 \ldots x^m \to y : \mathrm{g}x^1 \ldots x^m; \Phi \to y : \mathrm{g}x^1 \ldots x^m \to y : \mathrm{f}x^1 \ldots x^m//$ $//\Phi \to \mathrm{f} : \mathrm{g}$ provided $x^1, \ldots, x^m$, and y are distinct and not free in Φ, f, g.

9.11 $\Phi, f : \lambda x^1 \ldots x^m \mathrm{Y} \to \mathfrak{M}//\Phi \to \mathfrak{M}$ provided f is not free in Φ, Y, and $\mathfrak{M}$.

9.12 $\Phi \to \mathrm{y} : [\lambda x^1 \ldots x^m \mathrm{Y}]\mathrm{X}^1 \ldots \mathrm{X}^m//\Phi \to \mathrm{y} : \mathrm{Y}(\mathrm{X}^1, \ldots, \mathrm{X}^m/x^1, \ldots, x^m)$ provided for $1 \leqslant s \leqslant m$, X^s is free for x^s in Y.

9.13 $\Phi \to \mathrm{x}^1 : \mathrm{X}^1; \ldots; \Phi \to \mathrm{x}^m : \mathrm{X}^m; \Phi \to \mathrm{y} : \mathrm{Y}(\mathrm{X}^1, \ldots, \mathrm{X}^m/x^1, \ldots, x^m)//$ $\Phi \to \mathrm{y} : [\lambda x^1 \ldots x^m \mathrm{Y}]\mathrm{X}^1 \ldots \mathrm{X}^m$ provided for $1 \leqslant s \leqslant m$, X^s is free for x^s in Y.

Where R is a class of rules of derivation and H a class of sequents, we shall write $\mathrm{H}/_R\mathfrak{S}$ to say that the sequent $\mathfrak{S}$ is derivable from members of H by means of members of R. Moreover, we shall write $\mathfrak{S}^1, \ldots, \mathfrak{S}^m/_R\mathfrak{S}$ for $\{\mathfrak{S}^1, \ldots, \mathfrak{S}^m\}/_R\mathfrak{S}$.

In the rest of this section '/' shall stand for $/_{\{9.1-9.13\}}$. It is easy to show that

9.14 $\Phi \to \mathfrak{M}/\Phi(\mathrm{x}^1, \ldots, \mathrm{x}^m/x^1, \ldots, x^m) \to \mathfrak{M}(\mathrm{x}^1, \ldots, \mathrm{x}^m/x^1, \ldots, x^m)$ provided $\mathrm{x}^1, \ldots, \mathrm{x}^m$ are free for $x^1, \ldots, x^m$, respectively, in $\mathfrak{M}$ and in every member of Φ.

In what follows, by $\Phi\{\mathrm{A} \Leftrightarrow_{\mathrm{x}} \mathrm{B}\}$ we shall understand the couple of sequents $\Phi, \mathrm{x} : \mathrm{A} \to \mathrm{x} : \mathrm{B}$ and $\Phi, \mathrm{x} : \mathrm{B} \to \mathrm{x} : \mathrm{A}$, and by $\Phi\{\mathrm{A} \Leftrightarrow \mathrm{B}\}$ we shall understand $\Phi\{\mathrm{A} \Leftrightarrow_x \mathrm{B}\}$, where x is the first variable of appropriate type which does not occur in Φ, A or B. The following are simple derivability results listed for future reference[6].

9.15 $/\Phi\{\mathrm{Y} \Leftrightarrow \mathrm{Y}\}$

9.16 $\Phi\{\mathrm{Y}^1 \Leftrightarrow \mathrm{Y}^2\}; \Phi\{\mathrm{Y}^2 \Leftrightarrow \mathrm{Y}^3\}/\Phi\{\mathrm{Y}^1 \Leftrightarrow \mathrm{Y}^3\}$

9.17 $\Phi\{\mathrm{F} \Leftrightarrow \mathrm{G}\}; \Phi\{\mathrm{X}^1 \Leftrightarrow \mathrm{Y}^1\}; \ldots; \Phi\{\mathrm{X}^m \Leftrightarrow \mathrm{Y}^m\}/\Phi\{\mathrm{F}\mathrm{X}^1 \ldots \mathrm{X}^m \Leftrightarrow \mathrm{G}\mathrm{Y}^1 \ldots \ldots \mathrm{Y}^m\}$

[6] For proofs, see [6].

9.18 $/\Phi\{[\lambda x^1 \ldots x^m \mathrm{Y}]\mathrm{x}^1 \ldots \mathrm{x}^m \Leftrightarrow \mathrm{Y}(\mathrm{x}^1, \ldots, \mathrm{x}^m / x^1, \ldots, x^m)\}$ provided for $1 \leqslant s \leqslant m$ x^s is free for x^s in Y.

9.19 $\Phi\{\mathrm{X} \Leftrightarrow \mathrm{Y}\} / \Phi\{\lambda x^1 \ldots x^m \mathrm{X} \Leftrightarrow \lambda x^1 \ldots x^m \mathrm{Y}\}$ provided none of $x^1, \ldots, x^m$ is free in Φ.

9.20 $/\Phi\{[[\lambda x \lambda y \mathrm{Y}]x]y \Leftrightarrow \mathrm{Y}\}$

9.21 $\Phi \rightarrow z : \mathrm{Z} / \Phi\{\mathrm{Y} \Leftrightarrow \mathrm{Y}(\mathrm{Z}/z)\}$ provided Z is free for z in Y.

10. Hospitality revisited

10.1. THEOREM. $(\mathscr{A}/\beta_{(h,g)};\ \mathscr{Z}/\zeta_{(l,k)};\ \mathscr{Z}'/\zeta_{(l',k')};\ z/\zeta;\ 0 \leqslant g, h, k, l, k', l' \leqslant 1)$. *Let both* $\mathscr{Z}$ *and* $\mathscr{Z}'$ *be free for* z *in* $\mathscr{A}$, $(k, l) \leqslant (g, h)$, *and* $(k', l') \leqslant (g, h)$. *If* z *is both* (k, l)- *and* (k', l')-*hospitable in* $\mathscr{A}$, *then*

$$(\mathfrak{S}) \qquad \Phi\{\mathscr{A}(\mathscr{Z}_{(k,l)}/z)_{(g,h)} \Leftrightarrow \mathscr{A}(\mathscr{Z}'_{(k',l')}/z)_{(g,h)}\}$$

is derivable from

$$(\mathfrak{S}^1) \qquad \Phi\{\mathscr{Z}_{(k,l)} \Leftrightarrow \mathscr{Z}'_{(k',l')}\}.$$

PROOF by induction on the rank r of $\mathscr{A}$. First consider the case $r = 0$. If $\mathscr{A}$ is z then z is not (k, l)-hospitable in $\mathscr{A}$; if $\mathscr{A}$ is other than z then $\mathfrak{S}$ is derivable by 9.15. Assume (as an induction hypothesis) that the Theorem holds for any $\mathscr{A}$ whose rank does not exceed r. Now consider an $\mathscr{A}$ of rank $r+1$ in which z is (k, l)- and (k', l')-hospitable. If z is not free in $\mathscr{A}$, $\mathfrak{S}$ is derivable by 9.15. Assume therefore that z is free in $\mathscr{A}$. Let w^+ and t^+ be variables which are distinct from and of the same type as w and t respectively and which do not occur in Φ, $\mathscr{A}$, $\mathscr{Z}$, $\mathscr{Z}'$. Let w' be w^+ or w according as g is 0 or 1. Then

$$(\mathfrak{S}^2) \qquad \Phi(w'/w)\{\mathscr{Z}_{(k,l)} \Leftrightarrow \mathscr{Z}'_{(k',l')}\}$$

is derivable from $\mathfrak{S}^1$. For if $g = 0$ then $k = k' = 0$ and $\mathfrak{S}^2$ is derivable from $\mathfrak{S}^1$ by 9.14; otherwise $\mathfrak{S}^2$ is the same as $\mathfrak{S}^1$. Let t' be t^+ or t according as h is 0 or 1, and let Φ' be $\Phi(w', t'/w, t)$. An analogous argument shows that

$$(\mathfrak{S}^3) \qquad \Phi'\{\mathscr{Z}_{(k,l)} \Leftrightarrow \mathscr{Z}'_{(k',l')}\}$$

is derivable from $\mathfrak{S}^2$.

Case 1: $\mathscr{A}$ is of the form (I) (see 4.1). If $0 \leqslant s \leqslant m$, then

$$(\mathfrak{S}^4_{1,s}) \qquad \Phi'\{\mathscr{X}^s(\mathscr{Z}_{(k,l)}/z)_{(i^s,j^s)} \Leftrightarrow \mathscr{X}^s(\mathscr{Z}'_{(k',l')}/z)_{(i^s,j^s)}\}$$

is derivable from $\mathfrak{S}^3$. For if z is not free in $\mathscr{X}^s$, $\mathfrak{S}^4_{1,s}$ is derivable by 9.15. Otherwise, since z is (k, l)- and (k', l')-hospitable in $\mathscr{A}$, either $\mathscr{X}^s$ is z and $i_s = j_s = 0$ or $\mathscr{X}^s$ is both (k, l)- and (k', l')-hospitable, $(k, l) \leqslant (i^s, j^s)$, and $(k', l') \leqslant (i^s, j^s)$. In the former case, $\mathfrak{S}^4_{1,s}$ is the same as $\mathfrak{S}^3$; in the latter case $\mathfrak{S}^4_{1,s}$ is derivable from $\mathfrak{S}^3$ by the induction hypothesis.

Case 2: $\mathscr{A}$ is of the form (II). As z is free in $\mathscr{A}$ and both $\mathscr{Z}$ and $\mathscr{Z}'$ are free for z in $\mathscr{A}$, none of $x^1, \ldots, x^m$ is free in $\mathscr{Z}$ or $\mathscr{Z}'$. Let $y^1, \ldots, y^m$ be distinct variables which are of the same respective types as $x^1, \ldots, x^m$ and do not occur in $\Phi, \mathscr{A}, \mathscr{Z}, \mathscr{Z}', w, t$. By 9.14

$$(\mathfrak{S}_2^4) \qquad \Phi'(y^1, \ldots, y^m/x^1, \ldots, x^m)\{\mathscr{Z}_{(k,l)} \Leftrightarrow \mathscr{Z}'_{(k',l')}\}$$

is derivable from $\mathfrak{S}^3$. Now the sequent

$$(\mathfrak{S}_2^5) \qquad \Phi'(y^1, \ldots, y^m/x^1, \ldots, x^m)\{\mathscr{Y}(\mathscr{Z}_{(k,l)}/z)_{(i,j)} \Leftrightarrow \mathscr{Y}(\mathscr{Z}'_{(k',l')}/z)_{(i,j)}\}$$

is derivable from $\mathfrak{S}_2^4$. For, since z is free in $\mathscr{A}$ and z is both (k, l)- and (k', l')-hospitable in $\mathscr{A}$, either $\mathscr{Y}$ is z and $i = j = 0$ or z is (k, l)- and (k', l')-hospitable in $\mathscr{Y}$, $(k, l) \leqslant (i, j)$, and $(k', l') \leqslant (i, j)$. In the former case $\mathfrak{S}_2^5$ is the same as $\mathfrak{S}_2^4$; in the latter case S_2^5 is derivable from $\mathfrak{S}_2^4$ by the induction hypothesis. The sequent

$$(\mathfrak{S}_2^6) \qquad \Phi^1(y^1, \ldots, y^m/x^1, \ldots, x^m)\{\lambda x^1 \ldots x^m \mathscr{Y}(\mathscr{Z}_{(k,l)}/z)_{(i,j)} \Leftrightarrow \lambda x^1 \ldots \ldots x^m \mathscr{Y}(\mathscr{Z}'_{(k',l')}/z)_{(i,j)}\}$$

is derivable from $\mathfrak{S}_2^5$ by 9.19.

Now let Q be such that $\mathscr{A}$ is $\lambda w \lambda t$Q. We shall show that the sequent

$$(\mathfrak{S}^7) \qquad \Phi'\{\mathrm{Q}(\mathscr{Z}_{(k,l)}/z) \Leftrightarrow \mathrm{Q}(\mathscr{Z}'_{(k',l')}/z)\}$$

is derivable from $\mathfrak{S}^1$. In Case 1, $\mathfrak{S}^7$ is derivable by 9.17 from $\mathfrak{S}_{1,0}^4 - \mathfrak{S}_{1,m}^4$, which, as we have seen, are derivable from $\mathfrak{S}^1$. In Case 2, $\mathfrak{S}^7$ is derivable by 9.14 from $\mathfrak{S}_2^6$, which, as we have seen, is derivable from $\mathfrak{S}^1$. Now sequent

$$(\mathfrak{S}^8) \qquad \Phi'\{[\lambda t\mathrm{Q}(\mathscr{Z}_{(k,l)}/z)]_{(0,h)} \Leftrightarrow [\lambda t\mathrm{Q}(\mathscr{Z}'_{(k',l')}/z)]_{(0,h)}\}$$

is derivable from $\mathfrak{S}^7$. For if $h = 0$ then t is not free in Φ', hence $\mathfrak{S}^8$ is derivable from $\mathfrak{S}^7$ by 9.19. If, on the other hand, $h = 1$ then the sequents $\{[\lambda t\mathrm{Q}.(\mathscr{Z}_{(k,l)}/z)]_{(0,h)} \Leftrightarrow \mathrm{Q}(\mathscr{Z}_{(k,l)}/z)\}$ and $\{[\lambda t\mathrm{Q}(\mathscr{Z}'_{(k',l')}/z)]_{(0,h)} \Leftrightarrow \mathrm{Q}(\mathscr{Z}'_{(k',l')}/z)\}$ are derivable by 9.18, hence $\mathfrak{S}^8$ is derivable from $\mathfrak{S}^7$ by 9.16. Now from $\mathfrak{S}^8$ we can derive

$$(\mathfrak{S}^9) \qquad \Phi'\{[\lambda w\lambda t\mathrm{Q}(\mathscr{Z}_{(k,l)}/z)]_{(g,h)} \Leftrightarrow [\lambda w\lambda t\mathrm{Q}(\mathscr{Z}'_{(k',l')}/z)]_{(g,h)}\}.$$

For if $g = 0$ then w is not free in Φ' and, since the type of $\mathscr{A}$ is of the form $\xi_{\tau\omega}$, $h = 0$; hence $\mathfrak{S}^9$ is derivable from $\mathfrak{S}^8$ by 9.19. If, on the other hand, $g = 1$ then the sequents $\{[\lambda w\lambda t\mathrm{Q}(\mathscr{Z}_{(k,l)}/z)]_{(g,0)} \Leftrightarrow \lambda t\mathrm{Q}(\mathscr{Z}_{(k,l)}/z)\}$ and $\{[\lambda w\lambda t\mathrm{Q}(\mathscr{Z}'_{(k',l')}/z)]_{(g,0)} \Leftrightarrow \lambda t\mathrm{Q}(\mathscr{Z}'_{(k',l')}/z)\}$ are derivable by 9.18 and the sequent $\{t \Leftrightarrow t\}$ by 9.15. Hence by 9.17, the sequents $\{[\lambda w\lambda t\mathrm{Q}(\mathscr{Z}_{(k,l)}/z)]_{(g,h)} \Leftrightarrow [\lambda t\mathrm{Q}(\mathscr{Z}_{(k,l)}/z)]_{(0,h)}\}$ and $\{[\lambda w\lambda t\mathrm{Q}(\mathscr{Z}'_{(k',l')}/z)]_{(g,h)} \Leftrightarrow [\lambda t\mathrm{Q}(\mathscr{Z}'_{(k',l')}/z)]_{(0,h)}\}$ are also derivable. Consequently, $\mathfrak{S}^9$ is derivable from $\mathfrak{S}^8$ by 9.16. But $\mathfrak{S}$ is derivable from $\mathfrak{S}^9$ by 9.14. □

The following are three rules of derivation involving identity:

10.2 $\Phi, i : \mathrm{x} = \mathrm{y} \to \mathfrak{M} // \Phi \to \mathfrak{M}$ provided i is not free in $\Phi, \mathrm{x}, \mathrm{y}, \mathfrak{M}$.

10.3 $\Phi \to \mathrm{x} : \mathrm{X} // \Phi \to \mathsf{T} : \mathrm{x} = \mathrm{X}$

10.4 $\Phi \to \mathsf{T} : \mathrm{x} = \mathrm{X} // \Phi \to \mathrm{x} : \mathrm{X}$.

Writing $/$ for $/_{\{9.1,\ldots,.13,10.2,\ldots,10.4\}}$ it is easy to show that

10.5 $\Phi \to \mathsf{T} : \mathrm{X} = \mathrm{Y} / \Phi \{\mathrm{X} \Leftrightarrow \mathrm{Y}\}$.

10.6 Lemma. *On the hypothesis of Theorem* 5.2,

$$/\mathsf{T} : [\lambda w \lambda t . \mathscr{Z}_{(k,l)} = \mathscr{Z}'_{(k',l')}]_{wt}, \mathsf{T} : \mathscr{A}(\mathscr{Z}_{(k,l)}/z)_{wt} \to \mathsf{T} : \mathscr{A}(\mathscr{Z}'_{(k',l')}/z)_{wt}.$$

Proof. Let $\mathfrak{M}$ be $\mathsf{T} : [\lambda w \lambda t . \mathscr{Z}_{(k,l)} = \mathscr{Z}'_{(k',l')}]_{wt}$. By 9.20, $/\mathfrak{M} \to \mathsf{T} : \mathscr{Z}_{(k,} = \mathscr{Z}'_{(k',l')}$; Hence by 10.5, $/\mathfrak{M}\{\mathscr{Z}_{(k,l)} \Leftrightarrow \mathscr{Z}'_{(k',l')}\}$. Thus by 10.1, $/\mathfrak{M}\{\mathscr{A}(\mathscr{Z}_{(k,l)}/z)_{wt} \Leftrightarrow \mathscr{A}(\mathscr{Z}'_{(k',l')}/z)_{wt}\}$, whence 10.6 follows by 9.4. □

11. Exposure revisited

11.1 Theorem. $(\mathscr{A}/\beta_{\tau\omega}; \mathscr{Z}/\zeta_{(l,k)}; z, x/\zeta; \mathrm{b}/\beta; 0 \leqslant k, l \leqslant 1)$. *Let* $\mathscr{Z}$ *be free for* z *in* $\mathscr{A}$ *and let* x *not occur in* $\Phi, \mathscr{A}, \mathscr{Z}, w, t, \mathfrak{M}$. *If* z *is exposed and* (1,1)-*hospitable in* $\mathscr{A}$ *then*

($\mathfrak{S}$) $\Phi \to \mathfrak{M}$

is derivable from

($\mathfrak{S}^1$) $\Phi \to \mathrm{b} : \mathscr{A}(\mathscr{Z}_{(k,l)}/z)_{wt}$ *and* ($\mathfrak{S}^2$) $\Phi, x : \mathscr{Z}_{(k,l)} \to \mathfrak{M}$.

Proof by induction on the rank r of $\mathscr{A}$. First consider the case $r = 0$. If $\mathscr{A}$ is z, z is not (1,1)-hospitable in $\mathscr{A}$; otherwise z is not exposed in $\mathscr{A}$. Assume (as an induction hypothesis) that the Theorem holds for any $\mathscr{A}$ whose rank does not exceed r. Now consider an $\mathscr{A}$ of rank $r+1$ in which z is exposed and (1,1)-hospitable. By 6.1, $\mathscr{A}$ is of the form (I), hence by 9.20 and 9.3,

($\mathfrak{S}^3$) $\Phi \to \mathrm{b} : \mathscr{X}^0(\mathscr{Z}_{(k,l)}/z)_{(i^0,j^0)} \mathscr{X}^1(\mathscr{Z}_{(k,l)}/z)_{(i^1,j^1)} \ldots \mathscr{X}^m(\mathscr{Z}_{(k,l)}/z)_{(i^m,j^m)}$

is derivable from $\mathfrak{S}^1$; moreover, one of the following two cases obtains.

Case 1: z is exposed in one of $\mathscr{X}^0, \mathscr{X}^1, \ldots, \mathscr{X}^m$, say in $\mathscr{X}^s$. Let u be a variable of type ξ^s which is not free in $\Phi, \mathscr{X}^s, \mathscr{Z}, w, t, \mathfrak{M}$. We shall show that

($\mathfrak{S}_1^4$) $\Phi, u : \mathscr{X}^s(\mathscr{Z}_{(k,l)}/z)_{(i^s,j^s)} \to \mathfrak{M}$

is derivable from $\mathfrak{S}^2$. By the definition of (1,1)-hospitality, either $\mathscr{X}^s$ is z and $i^s = j^s = 0$, or z is (1,1)-hospitable and $i^s = j^s = 1$. In the former case $\mathfrak{S}_1^4$ is derivable from $\mathfrak{S}^2$ by 9.14; in the latter case it is derivable from $\mathfrak{S}^2$ and the derivable sequent $\Phi, u : \mathscr{X}^s(\mathscr{Z}_{(k,l)}/z)_{wt} \to u : \mathscr{X}^s(\mathscr{Z}_{(k,l)}/z)_{wt}$ by the induction hypothesis. Thus in either case, $\mathfrak{S}_1^4$ is derivable from $\mathfrak{S}^2$. But $\mathfrak{S}$ is derivable from $\mathfrak{S}^3$ and $\mathfrak{S}_1^4$ by 9.7.

Case 2: $\mathscr{X}^0$ is of the form (II) and z is distinct from $x^1, \ldots, x^m$ and exposed in $\mathscr{Y}$. Then z is free in $\mathscr{Y}$ and consequently (since z is (1,1)-hospitable in $\mathscr{A}$) z is (1,1)-hospitable in $\mathscr{X}^0$ and $i^0 = j^0 = 1$; moreover (since $\mathscr{Z}$ is free for z in $\mathscr{A}$), none of $x^1, \ldots, x^m$ is free in $\mathscr{Z}$. Let $y^1, \ldots, y^m$ be variables of the same respective types as $x^1, \ldots, x^m$ and not occur in Φ, $\mathscr{A}$, $\mathscr{Z}$, w, t, $\mathfrak{M}$. Furthermore, let Φ', $\mathfrak{M}'$, $\mathscr{X}^{1\prime}, \ldots$, and $\mathscr{X}^{m\prime}$ be $\Phi(y^1, \ldots, y^m/x^1, \ldots, x^m)$, $\mathfrak{M}(y^1, \ldots, y^m/x^1, \ldots, x^m)$, $\mathscr{X}^1(y^1, \ldots, y^m/x^1, \ldots, x^m), \ldots$, and $\mathscr{X}^m(y^1, \ldots, y^m/x^1, \ldots, x^m)$ respectively. By 9.14,

($\mathfrak{S}_2^4$) $\quad \Phi', x: \mathscr{Z}_{(k,l)} \to \mathfrak{M}'$

is derivable from $\mathfrak{S}^2$ and

($\mathfrak{S}_2^5$) $\quad \Phi' \to \mathrm{b}: \mathscr{X}^0(\mathscr{Z}_{(k,l)}/z)_{(i^0,j^0)} \mathscr{X}^{1\prime}(\mathscr{Z}_{(k,l)}/z)_{(i^1,j^1)} \ldots \mathscr{X}^{m\prime}(\mathscr{Z}_{(k,l)}/z)_{(i^1,j^1)}$

is derivable from $\mathfrak{S}^3$. By 9.20, $\{\mathscr{X}^0(\mathscr{Z}_{(k,l)}/z)_{(i^0,j^0)} \Leftrightarrow \lambda x^1 \ldots x^m \mathscr{Y}(\mathscr{Z}_{(k,l)}/z)_{(i,j)}\}$ is derivable, hence by 9.15, 9.17, and 9.3,

($\mathfrak{S}_2^6$) $\quad \Phi' \to \mathrm{b}: [\lambda x^1 \ldots x^m \mathscr{Y}(\mathscr{Z}_{(k,l)}/z)_{(i,j)}] \mathscr{X}^{1\prime}(\mathscr{Z}_{(k,l)}/z)_{(i^1,j^1)} \ldots \mathscr{X}^{m\prime}(\mathscr{Z}_{(k,l)}/z)_{(i^m,j^m)}$

is derivable from $\mathfrak{S}_2^5$. Let Ψ be the sequence $x^1: \mathscr{X}^{1\prime}(\mathscr{Z}_{(k,l)}/z)_{(i^1,j^1)}, \ldots, x^m: \mathscr{X}^{m\prime}(\mathscr{Z}_{(k,l)}/z)_{(i^m,j^m)}$. By 9.1 and 9.8,

($\mathfrak{S}_2^7$) $\quad \Phi', \Psi \to \mathrm{b}: [\lambda x^1 \ldots x^m \mathscr{Y}(\mathscr{Z}_{(k,l)}/z)_{(i,j)}] x^1 \ldots x^m$

is derivable from $\mathfrak{S}_2^6$, whence

($\mathfrak{S}_2^8$) $\quad \Phi', \Psi \to \mathrm{b}: \mathscr{Y}(\mathscr{Z}_{(k,l)}/z)_{(i,j)}$

is derivable by 9.12. Now the sequent

($\mathfrak{S}_2^9$) $\quad \Phi', \Psi \to \mathfrak{M}'$

is derivable from $\mathfrak{S}_2^8$ and $\mathfrak{S}_2^4$. For, as z is (1,1)-hospitable in $\mathscr{X}^0$, either $\mathscr{Y}$ is z and $i = j = 0$, or z is (1,1)-hospitable in $\mathscr{Y}$ and $i = j = 1$. In the former case, $\mathfrak{S}_2^9$ is derivable from $\mathfrak{S}_2^8$ and $\mathfrak{S}_2^4$ by 9.14 and 9.3; in the latter case by the induction hypothesis. Now

($\mathfrak{S}_2^{10}$) $\quad \Phi' \to \mathfrak{M}'$

is derivable from $\mathfrak{S}_2^6$ and $\mathfrak{S}_2^9$ by 9.7, and $\mathfrak{S}$ is derivable from $\mathfrak{S}_2^{10}$ by 9.14. □

Σ^ξ obeys the following rules of derivation:

11.2 $\quad \Phi, i: \Sigma^\xi \mathrm{c} \to \mathfrak{M} // \Phi \to \mathfrak{M}$
11.3 $\quad \Phi \to \mathsf{T}: \mathrm{CX} // \Phi \to \mathsf{T}: \Sigma^\xi \mathrm{C}$
11.4 $\quad \Phi \to \mathsf{T}: \Sigma^\xi \mathrm{C}; \Phi, \mathsf{T}: Cx \to \mathfrak{M} // \Phi \to \mathfrak{M}$ provided x is not free in Φ, C, $\mathfrak{M}$.

Writing $/$ for $/_{\{9.1,\ldots,9.13,10.2,\ldots,10.4,11.2,\ldots,11.4\}}$, it is easy to show that, in view of 9.13, 9.20, 9.17 and 11.3, we have

11.5 $\quad \Phi \to \mathrm{x}: \mathrm{X}; \Phi \to \mathsf{T}: \mathrm{I}(\mathrm{X}/z) / \Phi \to \mathsf{T}: (\exists z)\mathrm{I}$ provided X is free for z in I.

11.6. LEMMA. *On the hypothesis of Theorem* 6.2,

$$/\mathsf{T} : \mathscr{A}(\mathscr{Z}_{(k,l)}/z)_{wt} \rightarrow \mathsf{T} : (\exists z)\mathscr{A}_{wt}.$$

PROOF. Let x be a ζ-variable which does not occur in $\mathscr{A}, \mathscr{Z}, w, t$. By 10.1 and the (1,1)-hospitality of z in $\mathscr{A}$, $x : \mathscr{Z}_{(k,l)}\{\mathscr{A}(\mathscr{Z}_{(k,l)}/z)_{wt} \Leftrightarrow \mathscr{A}(x/z)_{wt}\}$ is derivable, and consequently, $\mathsf{T} : \mathscr{A}(\mathscr{Z}_{(k,l)}/z)_{wt}, x : \mathscr{Z}_{(k,l)} \rightarrow \mathsf{T} : \mathscr{A}(x/z)$ is derivable. By 11.5, $\mathsf{T} : \mathscr{A}(\mathscr{Z}_{(k,l)}/z)_{wt}, x : \mathscr{Z}_{(k,l)} \rightarrow \mathsf{T} : (\exists z)\mathscr{A}_{wt}$ is then derivable, whence 11.6 follows by 11.1 and the exposure of z in $\mathscr{A}$.

References

[1] G. CURRIE and G. ODDIE, *Changing numbers*, ***Theoria*** 56 (1980), pp. 148–64.

[2] P. MATERNA, *Question-like and non-question-like imperative sentences*, ***Linguistics and Philosophy*** 4 (1981), pp. 393–404.

[3] P. TICHÝ, *Two kinds of intensional logic*, ***Epistemologia*** 1 (1978), pp. 143–64.

[4] P. TICHÝ, *De dicto and de re*, ***Philosophia*** 8 (1978), pp. 1–16.

[5] P. TICHÝ, *Existence and God*, ***Journal of Philosophy*** 76 (1979), pp. 403–20.

[6] P. TICHÝ, *Foundations of partial type theory*, ***Reports on Mathematical Logic*** 14 (1981), pp. 57–72.

[7] P. TICHÝ, *Einzeldinge als Amtsinhaber; Vorbetrachtungen zur transparenten intensionalen Logik*, ***Zeitschrift für Semiotik***, forthcoming.

UNIVERSITY OF OTAGO
DUNEDIN, NEW ZEALAND

Received June 10, 1984
Final version received April 24, 1985