On Edge Interactions and Surface Tension

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Dedicated to Bernard D. Coleman on the occasion of his sixtieth birthday

Contents

Introduction

The idea of a "line distribution of force over the edges of a body" seems to have occurred first to TOUPIN in 1962 (see p. 403 of [To1]). However, his treatment suffered from the same defect as most treatments of distributions of forces prior to 1960, namely these distributions were only implicit in formulas for resultant forces. The systems of forces giving rise to these resultants were not brought into the open. (See the remark by C. TRUESDELL in "General References" on p. 156 of [Tr].) The present paper is a first attempt to bring into the open the systems of forces that give rise to distributions of forces over edges.

The problem of how to give a precise definition of an edge interaction and TOUPIN's early work on this problem were brought to our attention by MAURIZIO VIANELLO in a discussion in Pisa in June 1987. During this discussion, the three of us came up with the germ of an idea from which the concept of regions in edge contact as described by Definition 4 of Section 3 and the corresponding Assumption I of Section 5 were developed later.

A precise concept of a "system of forces" was first proposed by one of us in 1957 [N1] and has been extensively developed since. GURTIN & WILLIAMS realized in 1967 [GW] that an analogous concept describes "systems of heat fluxes". Both kinds of systems were later subsumed under the abstract concept of an "interaction" (see Section 8 in [N2]). A definition of "interaction" that is suitable for the present context is presented in Section 4. In Sections 5 and 6 we show how "edge interactions" can be identified as special contact interactions. In Section 7 we derive the consequences of balance laws when edge interactions are present and when certain *ad hoc* assumptions are made. In Section 8 we discuss the boundary conditions that apply when edge interactions are present. In Section 9 we show how some of TOUPIN's results fit into the scheme of the present paper. Finally, we show how surface tension can be viewed as the manifestation of the presence of certain edge interactions.

Mathematicians often use terms such as "region with piecewise smooth boundary" without any further explanation (see *e.g.* axiom (S2) in [N1]), and they often give inequivalent and even obviously unintended precise definitions when challenged to make this concept precise (see Appendix II of [F]). In the context of the present paper it is crucial to have a very precise definition that does not exclude "cusped" edges. Sections 1, 2, and 3 deal with this and related issues. Our definition of "regular region" seems to be equivalent to the one given by KELLOGG in [K], p. 112. We also need to consider a somewhat more restricted concept, for which we use the term "biregular region".

What we call "edge interactions" should not be confused with external actions concentrated along curves; our theory is not related to theories of "stress-concentration" in any sense. An edge interaction between two internal parts of a given body, in the sense of the present paper, can occur only when a side of one of the parts is in "everywhere-cusped" contact with a side of the other part along a common edge of the two parts (see Definition 2 of Section 3). The additivity properties of interactions then show that if the edge is a "cusped edge" of one of the two parts, the interaction must be zero. It is for this reason that one must pay careful attention to "cusped edges", and that one can only consider what we call "biregular" contact (see Definition 4 of Section 3).

There are at least two issues which we could not settle. The first concerns the Assumptions I, II, III, and IV in Sections 5, 6, and 7. It would be desirable to derive them from other assumptions that have more transparent and natural physical interpretations. For classical surface interactions, this issue has been studied at length only recently (see [S] and [GWZ] and the literature cited there). For edge interactions, the issue is likely to be much more difficult. The second issue concerns a class of regions appropriate to represent parts of a given body (see conditions (i)–(iv) of Section 4). We do not specify explicitly such a class here, and we describe our interactions only for special situations. Even for the case of classical surface interaction, we found only recently a class (we call its members "fit regions") that we consider completely appropriate (see [NV]). The problem of finding a class that is appropriate when edge-interactions are present is likely to be extremely difficult and to require concepts from geometric measure theory that have not yet been invented.

The present paper does not consider at all the type of constitutive assumptions that should govern edge interactions. For a long time there has been evidence that non-simple materials cannot be accommodated by classical surface interactions alone and that edge interactions must necessarily occur in materials of second grade. Indeed, it seems to us that it was VIANELLO's interest in the theory of materials of second grade that motivated him to urge re-examination of TOUPIN's earlier work. We hope that the present paper will open a way to deal with these matters.

Notation and Terminology

Generally we use the notation and terminology of [FDSI]. For example, $\mathbb{P}$ denotes the set of all positive reals including zero and $\mathbb{P}^{\times} := \mathbb{P} \setminus \{0\}$ denotes the set of all strictly positive reals. The range of a mapping f is denoted by $\operatorname{Rng} f$. If A is a subset of the domain of a mapping f, then $f|_A$ denotes the restriction of f to A. If B is a set that includes the range of a mapping f, then $f|^B$ is obtained from f by changing its codomain to B. If B is a subset of a given set A, then $1_{B\subset A}$ denotes the *inclusion-mapping* from B to A, *i.e.*, the mapping whose domain is B, whose codomain is A, and whose values are given by $1_{B\subset A}(x) = x$ for all $x \in B$.

If $\mathscr{V}$ and $\mathscr{W}$ are linear spaces, then $\operatorname{Lin}(\mathscr{V}, \mathscr{W})$ denotes the set of all linear mappings from $\mathscr{V}$ to $\mathscr{W}$. The *dual space* $\mathscr{W}^*$ of $\mathscr{W}$ is defined by $\mathscr{W}^* := \operatorname{Lin}(\mathscr{W}, \mathbb{R})$.

When we use the terms "inner-product space" and "Euclidean space", it is understood that they are *genuine* in the sense of the definitions in Chapter 4 of [FDSI]. Let $\mathscr{V}$ be an inner-product space. The unit ball of $\mathscr{V}$ is denoted by $\operatorname{Ubl} \mathscr{V} := \{\boldsymbol{v} \in \mathscr{V} \mid |\boldsymbol{v}| < 1\}$ and the unit-sphere of $\mathscr{V}$ by $\operatorname{Usph} \mathscr{V} := \{\boldsymbol{v} \in \mathscr{V} \mid |\boldsymbol{v}| = 1\}$. If $\mathscr{S}$ is a subset of $\mathscr{V}$, then $\mathscr{S}^{\perp} := \{\boldsymbol{v} \in \mathscr{V} \mid \boldsymbol{v} \cdot \boldsymbol{u} = 0 \text{ for all } \boldsymbol{u} \in \mathscr{S}\}$.

We often abbreviate $\mathscr{U} := \operatorname{Usph} \mathscr{V}$ and use the notation

$$(\mathscr{U}^2)_{\perp} := \{(\boldsymbol{u}, \boldsymbol{w}) \in \mathscr{U}^2 \mid \boldsymbol{u} \cdot \boldsymbol{w} = 0\}$$

for the set of all perpendicular pairs of unit vectors. It is easily seen that $(\mathscr{U}^2)_{\perp}$ is a 3-dimensional analytic manifold imbedded in the 6-dimensional inner-product space $\mathscr{V}^2$.

The gradient of a mapping φ at a point x in the domain of φ is denoted by $\nabla_x \varphi$ if meaningful. Similarly, $\operatorname{div}_x \boldsymbol{h}$ denotes the divergence of a vector field $\boldsymbol{h}$ at a given point x of the domain of $\boldsymbol{h}$.

Let $\mathscr{D}$ be a subset of a given Euclidean space $\mathscr{E}$. The interior, closure, and boundary of $\mathscr{D}$ are denoted by $\operatorname{Int} \mathscr{D}$, $\operatorname{Clo} \mathscr{D}$, and $\operatorname{Bdy} \mathscr{D}$, respectively. We say that $\mathscr{D}$ is *regularly open* if $\mathscr{D} = \operatorname{IntClo} \mathscr{D}$.

If n and m are integers, then $n \,..\, m$ denotes the set of all integers k such that $n \leq k \leq m$. We abbreviate $n^{]} := 1 \,..\, n$, $n^{[} := 0 \,..\, (n-1)$.

To distinguish between integrals with respect to volume, surface-area, and curve-length, we write dv, da, or dl, respectively, after the integrand.

1. Regular regions, surfaces, and curves

We assume that a 3-dimensional Euclidean space $\mathscr{E}$ with translation space $\mathscr{V} := \mathscr{E} - \mathscr{E}$ is given.

Let a bounded connected C^2-manifold $\mathscr{M}$ imbedded in $\mathscr{E}$ be given (see Chapter 3 of [FDSII]). If $\dim \mathscr{M} = 3$ then $\mathscr{M}$ is just a **region**, *i.e.* a connected open set.

If $\dim \mathscr{M} = 2$, we call $\mathscr{M}$ a **surface**, and if $\dim \mathscr{M} = 1$, we call $\mathscr{M}$ a **curve**. If $\dim \mathscr{M} = 0$, then $\mathscr{M}$ is just a singleton.

We use the notation

$$\boldsymbol{E}_{\mathscr{M}} : \mathscr{M} \to \operatorname{Lin} \mathscr{V} \tag{1.1}$$

for the mapping whose value at $x \in \mathscr{M}$ is the symmetric idempotent whose range is the tangent space $\operatorname{Tan}_{\mathscr{M}}(x)$ to $\mathscr{M}$ at x, so that

$$\boldsymbol{E}_{\mathscr{M}}(x)^2 = \boldsymbol{E}_{\mathscr{M}}(x), \qquad \boldsymbol{E}_{\mathscr{M}}(x)^T = \boldsymbol{E}_{\mathscr{M}}(x) \tag{1.2}$$

and

$$\operatorname{Rng} \boldsymbol{E}_{\mathscr{M}}(x) = \operatorname{Tan}_{\mathscr{M}}(x) \quad \text{for all } x \in \mathscr{M}. \tag{1.3}$$

The **border** of $\mathscr{M}$ is defined by

$$\operatorname{Bo} \mathscr{M} := \operatorname{Clo} \mathscr{M} \setminus \mathscr{M}. \tag{1.4}$$

The border of $\mathscr{M}$ coincides with the boundary $\operatorname{Bdy} \mathscr{M}$ of $\mathscr{M}$ if $\dim \mathscr{M} = 3$, *i.e.* if $\mathscr{M}$ is a region. If $\dim \mathscr{M} < 3$, then $\operatorname{Bdy} \mathscr{M} = \operatorname{Clo} \mathscr{M}$.

Definition 1. *We say that the bounded connected C^2-manifold $\mathscr{M}$ is* **regular** *if* Clo $\mathscr{M}$ *has a finite partition $\mathfrak{P}$ with the following properties.*

(i) *Every piece of $\mathfrak{P}$ is a connected C^2-manifold.*

(ii) $\mathscr{M} \in \mathfrak{P}$.

(iii) *For every piece $\mathscr{P} \in \mathfrak{P}$, $\boldsymbol{E}_{\mathscr{P}}$ has a continuous extension to* Clo $\mathscr{P}$. *Using poetic license, we denote this extension also by*

$$\boldsymbol{E}_{\mathscr{P}} : \operatorname{Clo} \mathscr{P} \to \operatorname{Lin} \mathscr{V}. \tag{1.5}$$

(iv) *For every $\mathscr{P} \in \mathfrak{P}$ and every $\mathscr{C} \in \mathfrak{P}$ with $\mathscr{C} \subset \operatorname{Bo} \mathscr{P}$ and $\dim \mathscr{P} = \dim \mathscr{C} + 2$, there are exactly two pieces $\mathscr{S} \in \mathfrak{P}$ such that*

$$\mathscr{S} \subset \operatorname{Bo} \mathscr{P}, \qquad \mathscr{C} \subset \operatorname{Bo} \mathscr{S}. \tag{1.6}$$

We call a finite partition of Clo $\mathscr{M}$ *with the properties* (i)–(iv) a **regular partition for** $\mathscr{M}$. *For every $k \in 0 \,..\, 3$, we denote by $\mathfrak{P}_k$ the set of all manifolds in $\mathfrak{P}$ that have dimension k.*

Let a regular manifold $\mathscr{M}$ and a regular partition $\mathfrak{P}$ for $\mathscr{M}$ be given. The only piece of $\mathfrak{P}$ of dimension $m := \dim \mathscr{M}$ is $\mathscr{M}$, *i.e.*

$$\mathfrak{P}_m = \{\mathscr{M}\} \quad \text{when } m := \dim \mathscr{M}. \tag{1.7}$$

We have

$$\operatorname{Bo} \mathscr{M} = \bigcup_{k \in m[} \bigcup \mathfrak{P}_k. \tag{1.8}$$

Every piece $\mathscr{P}$ of $\mathfrak{P}$ is again a regular manifold and $\{\mathscr{S} \in \mathfrak{P} \mid \mathscr{S} \subset \operatorname{Clo} \mathscr{P}\}$ is a regular partition of $\mathscr{P}$. We define the tangent space to $\mathscr{M}$ at a point $x \in \operatorname{Bo} \mathscr{M}$ by $\operatorname{Tan}_{\mathscr{M}}(x) := \operatorname{Rng} \boldsymbol{E}_{\mathscr{M}}(x)$, using the extended $\boldsymbol{E}_{\mathscr{M}}$ of (1.5). Then (1.3) becomes valid for all $x \in \mathrm{Clo}\,\mathscr{M}$. If $\mathscr{M}$ is a **regular region** (*i.e.* if $\dim \mathscr{M} = 3$), we call the surfaces in $\mathfrak{P}_2$ **sides** of $\mathscr{M}$. If $\mathscr{M}$ is a regular region or a **regular surface** (*i.e.* if $\dim \mathscr{M} = 2$), we call the curves in $\mathfrak{P}_1$ **edges** and the singletons in $\mathfrak{P}_0$ **vertices** of $\mathscr{M}$. The only member of one of the latter is called a **vertex-point**. The condition (iv) of Definition 1 may then be restated as follows: Each edge $\mathscr{C}$ of a regular region is included in the border of exactly two sides; we call them the **sides adjacent** to $\mathscr{C}$. Each side of a regular region is a regular surface. Each vertex-point x of a regular surface belongs to the border of exactly two edges; we call them the **edges adjacent** to x. The concepts of side, edge, vertex, and vertex-point depend, of course, not only on $\mathscr{M}$ but also on the given regular partition $\mathfrak{P}$ of $\mathscr{M}$. Usually no ambiguity will arise if the appropriate partition $\mathfrak{P}$ is not explicitly described.

Remark 1. A regular manifold of dimension 1 or greater always admits infinitely many regular partitions. Indeed given such a partition, new ones can always be obtained by subdivision. Of course, there always are regular partitions with a minimum number of pieces, but there may be more than one such partition.

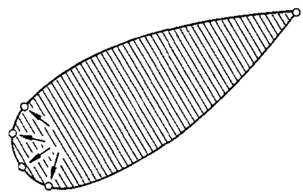

Figure 1

Figure 1 shows a regular surface with this property. A regular partition for the surface must have at least 5 pieces (the surface itself, two edges, and two vertices). There are infinitely many regular partitions with 5 pieces because one of the vertex-points can be put into infinitely many positions. □

Definition 2. Let $\mathscr{M}$ *be a regular manifold. Then the* **reduced border** $\mathrm{Rbo}\,\mathscr{M}$ of $\mathscr{M}$ *is defined to be the union of all regular manifolds of dimension* $\dim \mathscr{M} - 1$ *that are included in* $\mathrm{Bo}\,\mathscr{M}$.

Let $\mathfrak{P}$ be a regular partition of a given regular manifold $\mathscr{M}$ with $m := \dim \mathscr{M}$. Clearly, we then have

$$\bigcup \mathfrak{P}_{m-1} \subset \mathrm{Rbo}\,\mathscr{M}. \tag{1.9}$$

The example given in Remark 1 shows that it may not be possible to find a partition $\mathfrak{P}$ such that the inclusion (1.9) reduces to equality. However, one can easily show that for every $x \in \mathrm{Rbo}\,\mathscr{M}$ one can find a regular partition $\mathfrak{P}$ for $\mathscr{M}$ such that $x \in \mathscr{P}$ for some $\mathscr{P} \in \mathfrak{P}_{m-1}$.

Remark 2. Let $\mathscr{R}$ be a regular region. Then $\mathscr{R}$ is a **fit region** in the sense described in [NV]. Also, the reduced border of $\mathscr{R}$ is a subset of the reduced boundary of $\mathscr{R}$ as defined in [NV]: $\mathrm{Rbo}\,\mathscr{R} \subset \mathrm{Rby}\,\mathscr{R}$. Moreover, the set-difference $\mathrm{Rby}\,\mathscr{R} \setminus \mathrm{Rbo}\,\mathscr{R}$ is a set of area-measure zero. □

Proposition 1. *Let a regular manifold* $\mathscr{M}$ *with* $\dim \mathscr{M} \geqq 1$ *and* $x \in \operatorname{Rbo} \mathscr{M}$ *be given. Then there is exactly one* $\boldsymbol{u} \in \operatorname{Usph} \mathscr{V}$ *such that*

(i) $\boldsymbol{u} \in (\operatorname{Tan}_{\mathscr{K}}(x))^{\perp} \cap \operatorname{Tan}_{\mathscr{M}}(x)$ *for every regular manifold* $\mathscr{K}$ *of dimension* $\dim \mathscr{M} - 1$ *that contains* x *and is included in* $\operatorname{Bo} \mathscr{M}$.

(ii) *There is a* $\delta \in \mathbb{P}^{\times}$ *such that*

$$(x + (\operatorname{Tan}_{\mathscr{M}}(x))^{\perp} + \mathbb{P}\boldsymbol{u}) \cap \mathscr{M} \cap (x + \delta \operatorname{Ubl} \mathscr{V}) = \emptyset. \tag{1.10}$$

Rather than giving a formal proof of Proposition 1, we illustrate its geometric meaning in Figure 2 for the case when $\dim \mathscr{M} = 2$. We note that $\mathscr{H} := x + (\operatorname{Tan}_{\mathscr{M}}(x))^{\perp} + \mathbb{P}\boldsymbol{u}$ is, in this case, the half-plane that touches $\mathscr{M}$ at x and is perpendicular to both the tangent-plane to $\mathscr{M}$ at x and to the edge $\mathscr{K}$ that contains x and is included in $\operatorname{Bo} \mathscr{M}$.

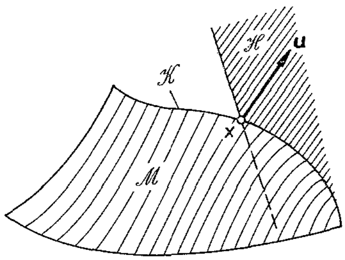

Figure 2

Definition 3. *Let* x *be a point on the reduced border of a regular manifold* $\mathscr{M}$. *We call the unit vector* $\boldsymbol{u}$ *described in Proposition 1 the* **outer unit normal** *to* $\mathscr{M}$ *at* x. *The function that associates this outer unit normal with* $x \in \operatorname{Rbo} \mathscr{M}$ *will be denoted by*

$$\boldsymbol{v}_{\mathscr{M}}: \operatorname{Rbo} \mathscr{M} \to \operatorname{Usph} \mathscr{V}. \tag{1.11}$$

We note that $\boldsymbol{v}_{\mathscr{M}}$ is continuous and that

$$\boldsymbol{E}_{\mathscr{M}}(x)\, \boldsymbol{v}_{\mathscr{M}}(x) = \boldsymbol{v}_{\mathscr{M}}(x) \quad \text{for all } x \in \operatorname{Rbo} \mathscr{M}. \tag{1.12}$$

If $\mathfrak{P}$ is a regular partition for $\mathscr{M}$ we have

$$\boldsymbol{E}_{\mathscr{S}}(x)\, \boldsymbol{v}_{\mathscr{M}}(x) = \boldsymbol{0} \quad \text{for all } x \in \mathscr{S} \tag{1.13}$$

and all $\mathscr{S} \in \mathfrak{P}_{m-1}$ when $m := \dim \mathscr{M}$.

Let a regular region $\mathscr{R}$ and a regular partition $\mathfrak{P}$ for $\mathscr{R}$ be given. Also, let $\mathscr{S}$ be a side of $\mathscr{R}$ belonging to $\mathfrak{P}$, *i.e.* $\mathscr{S} \in \mathfrak{P}_2$. It follows from condition (iii) of Definition 1, applied to $\mathscr{S}$, that $\boldsymbol{v}_{\mathscr{R}|\mathscr{S}}: \mathscr{S} \to \operatorname{Usph} \mathscr{V}$ has a continuous extension to $\operatorname{Clo} \mathscr{S}$. We denote this extension by

$$\boldsymbol{v}_{\mathscr{R};\mathscr{S}}: \operatorname{Clo} \mathscr{S} \to \operatorname{Usph} \mathscr{V}. \tag{1.14}$$

It follows from Definition 3 and condition (i) of Proposition 1 that

$$\boldsymbol{v}_{\mathscr{S}}(x) \cdot \boldsymbol{v}_{\mathscr{R};\mathscr{S}}(x) = 0 \quad \text{for all } x \in \operatorname{Rbo} \mathscr{S}. \tag{1.15}$$

Let $\mathscr{C}$ be an edge of $\mathscr{R}$ belonging to $\mathfrak{P}$, *i.e.* $\mathscr{C} \in \mathfrak{P}_1$, and let $\mathscr{S}$ and $\mathscr{S}'$ be the two sides of $\mathscr{R}$ adjacent to $\mathscr{C}$. Given $x \in \mathscr{C}$, we have $\boldsymbol{v}_{\mathscr{R};\mathscr{S}}(x) = \boldsymbol{v}_{\mathscr{R};\mathscr{S}'}(x)$ only in the exceptional case when x belongs to the reduced boundary of $\mathscr{R}$.

2. Functions on manifolds, surface divergence

Let $\mathscr{E}$ and $\mathscr{V}$ and a manifold $\mathscr{M}$ as in Section 1 be given. Also, let a finite-dimensional linear space $\mathscr{W}$ be given. If a function $\eta : \mathscr{M} \to \mathscr{W}$ is differentiable one can define, for each $x \in \mathscr{M}$, the gradient $\nabla_x \eta \in \operatorname{Lin}(\operatorname{Tan}_{\mathscr{M}}(x), \mathscr{W})$, of η at x (see, *e.g.*, (33.4) of [FDSII]).

Definition 1. *Let $\eta : \mathscr{M} \to \mathscr{W}$ be differentiable. We then define*

$$\nabla \eta : \ \mathscr{M} \to \operatorname{Lin}(\mathscr{V}, \mathscr{W})$$

by

$$\nabla \eta(x) := \nabla_x \eta := \nabla_x \eta\, \boldsymbol{E}_{\mathscr{M}}(x)|^{\operatorname{Tan}_{\mathscr{M}}(x)} \quad \text{for all } x \in \mathscr{M}. \tag{2.1}$$

We say that η is **of class C^1** *if $\nabla \eta$ is continuous. If $\mathscr{M}$ is a surface,* we call $\nabla \eta$ the **surface-gradient** of η.

We say *that a function $\eta : \operatorname{Clo} \mathscr{M} \to \mathscr{W}$ is* **of class C^1** *if $\eta|_{\mathscr{M}}$ is of class C^1 and if $\nabla(\eta|_{\mathscr{M}}) : \mathscr{M} \to \operatorname{Lin}(\mathscr{V}, \mathscr{W})$ has a continuous extension to* Clo $\mathscr{M}$, *which will then be denoted by $\nabla \eta$.*

We note that, if $\eta : \mathscr{M} \to \mathscr{W}$ is differentiable, we have

$$\operatorname{Tan}_{\mathscr{M}}(x)^{\perp} \subset \operatorname{Null} \nabla_x \eta \quad \text{for all } x \in \mathscr{M}. \tag{2.2}$$

Definition 2. *Let $\mathscr{S}$ be a surface of class C^2 and let $\boldsymbol{h} : \mathscr{S} \to \mathscr{V}$ be differentiable. The* **surface-divergence**

$$\operatorname{div}_{\mathscr{S}} \boldsymbol{h} : \ \mathscr{S} \to \mathbb{R} \tag{2.3}$$

is defined to be the value-wise trace of $\nabla \boldsymbol{h}$, i.e.,

$$(\operatorname{div}_{\mathscr{S}} \boldsymbol{h})(x) := \operatorname{tr}(\nabla \boldsymbol{h}(x)) \quad \text{for all } x \in \mathscr{S}. \tag{2.4}$$

If $\boldsymbol{H} : \mathscr{S} \to \operatorname{Lin}(\mathscr{V}, \mathscr{W})$ is differentiable then its **surface-divergence**

$$\operatorname{div}_{\mathscr{S}} \boldsymbol{H} : \ \mathscr{S} \to \mathscr{W} \tag{2.5}$$

is characterized by

$$\omega(\operatorname{div}_{\mathscr{S}} \boldsymbol{H}(x)) = \operatorname{div}_{\mathscr{S}}(\boldsymbol{H}^T \omega)(x) \quad \text{for all } x \in \mathscr{S}, \omega \in \mathscr{W}^*. \tag{2.6}$$

(Compare this with the definitions of divergence in Section 67 of [FDSI].)

Definition 3. *Let $\mathscr{S}$ be a surface of class C^2. We say that a function $\boldsymbol{h} : \mathscr{S} \to \mathscr{V}$ is* **tangential** *if $\boldsymbol{h}(x) \in \operatorname{Tan}_{\mathscr{S}}(x)$ for all $x \in \mathscr{S}$. We say that a function $\boldsymbol{H} : \mathscr{S} \to \operatorname{Lin}(\mathscr{V}, \mathscr{W})$ is* **tangential** *if*

$$\operatorname{Tan}_{\mathscr{S}}(x)^{\perp} \subset \operatorname{Null} \boldsymbol{H}(x) \quad \text{for all } x \in \mathscr{S}. \tag{2.7}$$

From use of (22.9) of [FDSI], it is evident that $\boldsymbol{H} : \mathscr{S} \to \operatorname{Lin}(\mathscr{V}, \mathscr{W})$ is tangential if and only if $\boldsymbol{H}^T \omega : \mathscr{S} \to \mathscr{V}$ is tangential for every $\omega \in \mathscr{W}^*$. If $\eta : \mathscr{S} \to \mathscr{W}$ is differentiable, then $\nabla \eta : \mathscr{S} \to \operatorname{Lin}(\mathscr{V}, \mathscr{W})$ is tangential by (2.2).

The following result is an easy consequence of the classical Stokes Theorem (a proof will be given in [FDSII]).

Surface-divergence Theorem. *Let $\mathscr{S}$ be a regular surface* (in the sense of Definition 1 of Section 1). *For every function* $\boldsymbol{h} : \mathrm{Clo}\,\mathscr{S} \to \mathscr{V}$ *that is of class* C^1 (in the sense of Definition 1) *and tangential* (in the sense of Definition 3) *we have*

$$\int_{\mathscr{S}} \mathrm{div}_{\mathscr{S}}\, \boldsymbol{h}\, da = \int_{\mathrm{Rbo}\,\mathscr{S}} (\boldsymbol{h} \cdot \boldsymbol{\nu}_{\mathscr{S}})\, dl, \tag{2.8}$$

where $\boldsymbol{\nu}_{\mathscr{S}}$ *is the outer unit normal function of* $\mathscr{S}$ (as defined by Definition 3 of Section 1).

Using (2.6), one immediately obtains the following

Corollary. *Let $\mathscr{S}$ be a regular surface and $\mathscr{W}$ a finite-dimensional linear space. For every tangential function* $\boldsymbol{H} : \mathrm{Clo}\,\mathscr{S} \to \mathrm{Lin}\,(\mathscr{V}, \mathscr{W})$ *of class* C^1 *we have*

$$\int_{\mathscr{S}} \mathrm{div}_{\mathscr{S}}\, \boldsymbol{H}\, da = \int_{\mathrm{Rbo}\,\mathscr{S}} (\boldsymbol{H}\boldsymbol{\nu}_{\mathscr{S}})\, dl. \tag{2.9}$$

3. Contact of surfaces and of regions

Definition 1. *Let* $\mathscr{M}$ and $\mathscr{M}'$ *be regular manifolds with* $\dim \mathscr{M} = \dim \mathscr{M}' \geqq 1$. *The* **contact** *of $\mathscr{M}$ and $\mathscr{M}'$ is defined by*

$$\mathrm{Ctc}\,(\mathscr{M}, \mathscr{M}') := \mathrm{Bo}\,\mathscr{M} \cap \mathrm{Bo}\,\mathscr{M}' \tag{3.1}$$

and the **reduced contact** *by*

$$\mathrm{Rtc}\,(\mathscr{M}, \mathscr{M}') := \mathrm{Rbo}\,\mathscr{M} \cap \mathrm{Rbo}\,\mathscr{M}'. \tag{3.2}$$

If $\mathscr{C}$ is a regular manifold included in $\mathrm{Ctc}\,(\mathscr{M}, \mathscr{M}')$, *we say that $\mathscr{M}$ and $\mathscr{M}'$ are* **in contact along** *$\mathscr{C}$; if* $\mathscr{C} \subset \mathrm{Rtc}\,(\mathscr{M}, \mathscr{M}')$ *we say that $\mathscr{M}$ and $\mathscr{M}'$ are* **in smooth contact** *along $\mathscr{C}$.*

Definition 2. *Let $\mathscr{S}$ and $\mathscr{S}'$ be regular surfaces. We say that $\mathscr{S}$ and $\mathscr{S}'$ are* **in cusped contact** *at a given* $x \in \mathrm{Rtc}\,(\mathscr{S}, \mathscr{S}')$ *if*

$$\boldsymbol{\nu}_{\mathscr{S}}(x) = \boldsymbol{\nu}_{\mathscr{S}'}(x). \tag{3.3}$$

We say that $\mathscr{S}$ and $\mathscr{S}'$ are **in everywhere-cusped contact** *along a given curve* $\mathscr{C} \subset \mathrm{Rtc}\,(\mathscr{S}, \mathscr{S}')$ *if* (3.3) *holds for all* $x \in \mathscr{C}$. *We say that $\mathscr{S}$ and $\mathscr{S}'$ are in* **nowhere-cusped contact** *along $\mathscr{C}$ if*

$$\boldsymbol{\nu}_{\mathscr{S}}(x) \neq \boldsymbol{\nu}_{\mathscr{S}'}(x) \tag{3.4}$$

for all $x \in \mathscr{C}$.

Every regular surface is in everywhere-cusped contact with itself along each of its edges. Of course, if $\mathscr{S}$ and $\mathscr{S}'$ are regular surfaces in contact along a curve $\mathscr{C}$, (3.3) may hold for some $x \in \mathscr{C}$ while (3.4) holds for other $x \in \mathscr{C}$. Then $\mathscr{S}$ and $\mathscr{S}'$ are neither in everywhere-cusped nor in nowhere-cusped contact along $\mathscr{C}$.

Definition 3. *We say that a subset $\mathscr{R}$ of $\mathscr{E}$ is a* **biregular region** *if it is a regular region and if there is a regular partition $\mathfrak{P}$ for $\mathscr{R}$ such that for every edge $\mathscr{C}$ in $\mathfrak{P}$, the two sides in $\mathfrak{P}$ adjacent to $\mathscr{C}$ are either in everywhere-cusped contact or in nowhere-cusped contact along $\mathscr{C}$. We call such a partition a* **biregular partition** *for $\mathscr{R}$. We say that an edge $\mathscr{C}$ in $\mathfrak{P}$ is a* **cusped edge** *if the two sides in $\mathfrak{P}$ adjacent to $\mathscr{C}$ are in everywhere-cusped contact along $\mathscr{C}$.*

Remark 1. Given a regular partition $\mathfrak{P}$ for a biregular region $\mathscr{R}$, one can always find a refinement of $\mathfrak{P}$ that is biregular partition for $\mathscr{R}$. □

Definition 4. *Let $\mathscr{R}$ and $\mathscr{R}'$ be disjoint biregular regions. We say that $\mathscr{R}$ and $\mathscr{R}'$ are in* **biregular contact** *if one can find biregular partitions $\mathfrak{P}$ and $\mathfrak{P}'$ for $\mathscr{R}$ and $\mathscr{R}'$, respectively, such that*

$$\operatorname{Ctc}(\mathscr{R},\mathscr{R}') = \bigcup(\mathfrak{P} \cap \mathfrak{P}') \tag{3.5}$$

and if for every $\mathscr{C} \in \mathfrak{P}_1 \cap \mathfrak{P}'_1$, every $\mathscr{S} \in \mathfrak{P}_2$, and every $\mathscr{S}' \in \mathfrak{P}'_2$ such that $\mathscr{C} \subset \operatorname{Rtc}(\mathscr{S},\mathscr{S}')$, the surfaces $\mathscr{S}$ and $\mathscr{S}'$ are either in everywhere-cusped or in nowhere-cusped contact along $\mathscr{C}$.

We say that $\mathscr{R}$ and $\mathscr{R}'$ are in **simple surface contact** *along a given regular surface $\mathscr{S}$ if they are in biregular contact and if the partitions $\mathfrak{P}$ and $\mathfrak{P}'$ above can be chosen such that*

$$\mathfrak{P}_2 \cap \mathfrak{P}'_2 = \{\mathscr{S}\} \quad \textit{and} \quad \operatorname{Ctc}(\mathscr{R},\mathscr{R}') = \operatorname{Clo}\mathscr{S}. \tag{3.6}$$

We say that $\mathscr{R}$ and $\mathscr{R}'$ are in **simple edge contact** *along a given regular curve $\mathscr{C}$ if they are in biregular contact and if the partitions $\mathfrak{P}$ and $\mathfrak{P}'$ above can be chosen such that*

$$\mathfrak{P}_1 \cap \mathfrak{P}'_1 = \{\mathscr{C}\} \quad \textit{and} \quad \operatorname{Ctc}(\mathscr{R},\mathscr{R}') = \operatorname{Clo}\mathscr{C}. \tag{3.7}$$

Partitions $\mathfrak{P}$ and $\mathfrak{P}'$ that satisfy the conditions (3.5), (3.6), *or* (3.7), *as appropriate, will be called* **appropriate partitions.**

If $\mathscr{R}$ and $\mathscr{R}'$ are in biregular contact and if $\mathfrak{P}$ and $\mathfrak{P}'$ are appropriate partitions, we have

$$\bigcup(\mathfrak{P}_2 \cap \mathfrak{P}'_2) \subset \operatorname{Rtc}(\mathscr{R},\mathscr{R}'). \tag{3.8}$$

Remark 2. One can easily define a concept of **regular contact** of disjoint regular regions by a definition analogous to (and simpler than) Definition 4. At the present time however, we do not know how to deal with contact interactions when the contact is only regular and not biregular. □

Definition 5. *Let $\mathscr{R}$ be a biregular region. We say that a subset $\mathscr{P}$ of $\mathscr{R}$ is a* **section** *of $\mathscr{R}$ if*

(i) *$\mathscr{P}$ is a biregular region,*
(ii) $\operatorname{Int}(\mathscr{R} \setminus \mathscr{P})$ *is a biregular region,*
(iii) *$\mathscr{P}$ is in biregular contact with* $\operatorname{Int}(\mathscr{R} \setminus \mathscr{P})$.

Let $\mathscr{R}$ be a biregular region and let $\mathscr{P}$ be a biregular region included in $\mathscr{R}$. If $\operatorname{Clo}\mathscr{P} \subset \mathscr{R}$ then $\mathscr{P}$ is necessarily a section of $\mathscr{R}$, but if $\operatorname{Clo}\mathscr{P} \not\subset \mathscr{R}$, then $\mathscr{P}$ need not be a section of $\mathscr{R}$.

4. Contact interactions

We assume now that a biregular bounded region $\mathscr{B}$ in $\mathscr{E}$ and a collection Ω of regularly open subsets of $\mathscr{B}$ with the following properties are given:
(i) All sections of $\mathscr{B}$ (see Definition 5 of Section 3) belong to Ω.
(ii) The intersection of any two members of Ω belongs to Ω, *i.e.*

$$\mathscr{P} \cap \mathscr{Q} \in \Omega \quad \text{for all } \mathscr{P}, \mathscr{Q} \in \Omega.$$

(iii) The **join** of any two members of Ω belongs to Ω, *i.e.*

$$\mathscr{P} \vee \mathscr{Q} := \operatorname{Int} \operatorname{Clo}(\mathscr{P} \cup \mathscr{Q}) \in \Omega \quad \text{for all } \mathscr{P}, \mathscr{Q} \in \Omega. \tag{4.1}$$

(iv) The **exterior in** $\mathscr{B}$ of any member of Ω belongs to Ω, *i.e.*

$$\mathscr{P}^b := \operatorname{Int}(\mathscr{B} \setminus \mathscr{P}) \in \Omega \quad \text{for all } \mathscr{P} \in \Omega. \tag{4.2}$$

We call the members of Ω the **parts** of $\mathscr{B}$. We denote the set of all disjoint pairs of parts of $\mathscr{B}$ by

$$(\Omega^2)_{\text{dis}} := \{(\mathscr{P}, \mathscr{Q}) \in \Omega^2 \mid \mathscr{P} \cap \mathscr{Q} = \emptyset\}. \tag{4.3}$$

The set of all sections of $\mathscr{B}$ will be denoted by Ω_{sec}, so that $\Omega_{\text{sec}} \subset \Omega$ by condition (i). The elements of

$$\Omega_{\text{int}} := \{\mathscr{P} \in \Omega \mid \operatorname{Clo} \mathscr{P} \subset \mathscr{B}\} \tag{4.4}$$

will be called **interior parts** of $\mathscr{B}$. If $\mathscr{P} \in \Omega_{\text{int}}$ is a biregular region, then $\mathscr{P} \in \Omega_{\text{sec}}$ and $\operatorname{Bdy} \mathscr{P} = \operatorname{Ctc}(\mathscr{P}, \mathscr{P}^b)$.

Given any part $\mathscr{R} \in \Omega$, we put

$$\Omega_{\mathscr{R}} := \{\mathscr{Q} \in \Omega \mid \mathscr{Q} \subset \mathscr{R}\}$$

and call the members of $\Omega_{\mathscr{R}}$ the **parts of** $\mathscr{R}$. Of course, we have $\Omega_{\mathscr{B}} = \Omega$. The conditions (ii) and (iii) remain satisfied if Ω is replaced by $\Omega_{\mathscr{R}}$ or Ω_{int} and we use the notation (4.3) also when Ω is replaced by $\Omega_{\mathscr{R}}$ or by Ω_{int}.

Now let a finite-dimensional linear space $\mathscr{W}$ be given. Given $\mathscr{R} \in \Omega$ and $F: \Omega_{\mathscr{R}} \to \mathscr{W}$, we say that F is **additive** if

$$F(\mathscr{P} \vee \mathscr{Q}) = F(\mathscr{P}) + F(\mathscr{Q}) \quad \text{for all } (\mathscr{P}, \mathscr{Q}) \in (\Omega_{\mathscr{R}}{}^2)_{\text{dis}}. \tag{4.5}$$

We say that

$$I: (\Omega^2)_{\text{dis}} \to \mathscr{W}$$

is an **interaction** if, for every $\mathscr{R} \in \Omega$, both $I(\cdot, \mathscr{R}^b): \Omega_{\mathscr{R}} \to \mathscr{W}$ and $I(\mathscr{R}^b, \cdot): \Omega_{\mathscr{R}} \to \mathscr{W}$ are additive. We say that I is a **contact-interaction** if for all $(\mathscr{P}, \mathscr{Q}) \in (\Omega^2)_{\text{dis}}$

$$\operatorname{Ctc}(\mathscr{P}, \mathscr{Q}) = \emptyset \quad \Rightarrow \quad I(\mathscr{P}, \mathscr{Q}) = 0. \tag{4.6}$$

Definition 1. *We say that a given contact-interaction is* **quasi-balanced** *if there is a continuous function* $f: \operatorname{Clo} \mathscr{B} \to \mathscr{W}$ *such that*

$$I(\mathscr{P}, \mathscr{P}^b) = \int_{\mathscr{P}} f \, dv \quad \text{for all } \mathscr{P} \in \Omega_{\text{int}}. \tag{4.7}$$

Proposition 1. *If I is a quasi-balanced contact-interaction, then I is* ***skew*** *in the sense that*

$$I(\mathscr{P}, \mathscr{Q}) = -I(\mathscr{Q}, \mathscr{P}) \quad \text{for all } (\mathscr{P}, \mathscr{Q}) \in (\Omega_{\text{int}}{}^2)_{\text{dis}}. \tag{4.8}$$

The proof follows immediately from Theorem A, p. 65, of [N2] because $\mathscr{P} \mapsto I(\mathscr{P}, \mathscr{P}^b) : \Omega_{\text{int}} \to \mathscr{W}$ is additive when (4.7) holds.

Remark 1. If the region $\mathscr{B}$ is identified with a continuous body that occupies $\mathscr{B}$ in a "placement", then a contact-interaction I may be interpreted as a system of internal forces or of internal heat-transfers as explained in Sections 9 and 10 of [N2]. In the former case we have $\mathscr{W} := \mathscr{V}$ and in the latter $\mathscr{W} := \mathbb{R}$.

The assumption that I is quasi-balanced is related, in the former interpretation, to the law of balance of forces, and in the latter interpretation to the law of balance of energy. □

Intermezzo: Wedge Interactions

This short section aims to motivate Assumption I of the following section.

Let $\mathscr{D}$ be an open disc in a plane. We use the term **wedge** for an open sector of $\mathscr{D}$. We consider a collection $\Omega_{\mathscr{D}}$ generated by wedges in the following sense: The members of $\Omega_{\mathscr{D}}$ are unions of finite pairwise disjoint collections of wedges no two of which have a side in common. In particular, the wedges belong to $\Omega_{\mathscr{D}}$. Figure 1 illustrates a member of $\Omega_{\mathscr{D}}$.

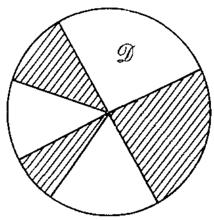

Figure 1

It is easily seen that $\Omega_{\mathscr{D}}$ is a **material universe** as described in the Appendix of [N2] if inclusion is taken as the relation "part of" and **meet, joint,** and **exterior** are defined, respectively, thus:

$$\mathscr{P} \wedge \mathscr{Q} := \mathscr{P} \cap \mathscr{Q},$$

$$\mathscr{P} \vee \mathscr{Q} := \text{Int Clo}\,(\mathscr{P} \cup \mathscr{Q}),$$

$$\mathscr{P}^e := \text{Int}\,(\mathscr{D} \setminus \mathscr{P}).$$

$\mathscr{D}$ is the **maximum** of $\Omega_{\mathscr{D}}$ and the empty set is the **minimum** of $\Omega_{\mathscr{D}}$. The collection Ω defined in Section 4 is a material universe far richer than $\Omega_{\mathscr{D}}$ as defined here.

We denote by $(\Omega_{\text{int}}{}^2)_{\text{dis}}$ the collection of all pairs of disjoint elements of $\Omega_{\mathscr{D}}$. Let $\mathscr{W}$ be a linear space as in Section 4.

Problem. *Find all interactions* $I: (\Omega_{\mathscr{D}}^2)_{\text{dis}} \to \mathscr{W}$ *such that* $I(\mathscr{P}, \mathscr{Q}) = 0$ *whenever* $\mathscr{P}$ *and* $\mathscr{Q}$ *are wedges that have no side in common.*

Solution. Let $\mathscr{U}$ be the set of all plane unit vectors. We denote by

$$(\mathscr{U}^2)_\perp := \{(\boldsymbol{u}, \boldsymbol{w}) \in \mathscr{U}^2 \mid \boldsymbol{u} \cdot \boldsymbol{w} = 0\}$$

the collection of all disjoint pairs of unit vectors. Given an interaction I as in the problem there is exactly one function

$$\gamma : (\mathscr{U}^2)_\perp \to \mathscr{W} \tag{I.1}$$

such that

$$I(\mathscr{P}, \mathscr{Q}) = \gamma(\boldsymbol{u}, \boldsymbol{w}) \tag{I.2}$$

whenever $\mathscr{P}$ and $\mathscr{Q}$ are wedges in contact along exactly one side and the unit vectors $\boldsymbol{w}$ and $\boldsymbol{u}$ are defined as follows: $\boldsymbol{w}$ is directed away from the side that $\mathscr{P}$ and $\mathscr{Q}$ have in common and $\boldsymbol{u}$ is orthogonal to $\boldsymbol{w}$ and away from $\mathscr{P}$ (see Figure 2).

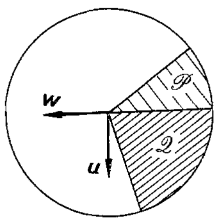

Figure 2

On the other hand, given a function γ as in (I.1), an interaction I satisfying (I.2) is uniquely determined by the biadditivity of I on the whole $(\Omega_{\mathscr{D}}^2)_{\text{dis}}$. □

Given any wedge $\mathscr{P}$, the pairs $(\boldsymbol{u}_1, \boldsymbol{w}_1)$, $(\boldsymbol{u}_2, \boldsymbol{w}_2) \in (\mathscr{U}^2)_\perp$ are determined as shown in Figure 3.

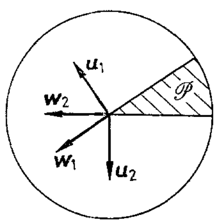

Figure 3

It follows from (I.2) and the additivity of the interaction I that

$$I(\mathscr{P}, \mathscr{P}^e) = \gamma(\boldsymbol{u}_1, \boldsymbol{w}_1) + \gamma(\boldsymbol{u}_2, \boldsymbol{w}_2). \tag{I.3}$$

Thus, by Theorem A of [N2], I is **skew** in the sense that

$$I(\mathscr{P}, \mathscr{Q}) + I(\mathscr{Q}, \mathscr{P}) = 0 \quad \text{for all } \mathscr{P}, \mathscr{Q} \in (\Omega_{\mathscr{D}}^2)_{\text{dis}}$$

if and only if

$$\gamma(\boldsymbol{u}, \boldsymbol{w}) + \gamma(-\boldsymbol{u}, \boldsymbol{w}) = 0 \quad \text{for all } (\boldsymbol{u}, \boldsymbol{w}) \in (\mathscr{U}^2)_\perp .$$

Let $\mathscr{D}' \subset \mathscr{D}$ be a given open sector of $\mathscr{D}$. We now consider the collection $\Omega_{\mathscr{D}'}$ of all members of $\Omega_{\mathscr{D}}$ that are included in $\mathscr{D}'$. $\Omega_{\mathscr{D}'}$ is a material universe itself and $(\Omega_{\mathscr{D}'}{}^2)_{\text{dis}}$ is defined in the same way as $(\Omega_{\mathscr{D}}{}^2)_{\text{dis}}$. A wedge of $\mathscr{D}'$ is **internal** if it has no side in common with $\mathscr{D}'$, it is **peripheral** otherwise.

Let $I: (\Omega_{\mathscr{D}'}^2)_{\text{dis}} \to \mathscr{W}$ be an interaction satisfying (I.2) for all pairs of *internal* wedges. Clearly, $I(\mathscr{P}, \mathscr{Q})$ cannot be computed by mere use of biadditivity if $\mathscr{P}$ or $\mathscr{Q}$ or both are peripheral. We assume that

$$I(\mathscr{P}, \mathscr{Q}) = \gamma(\boldsymbol{u}, \boldsymbol{w})$$

whenever *both* $\mathscr{P}$ and $\mathscr{Q}$ are peripheral wedges and the unit vectors $\boldsymbol{w}$ and $\boldsymbol{u}$ are defined as follows: $\boldsymbol{w}$ is directed away from the side that $\mathscr{P}$ and $\mathscr{D}'$ have in common and $\boldsymbol{u}$ is orthogonal to $\boldsymbol{w}$ and is directed away from $\mathscr{P}$ (see Figure 4).

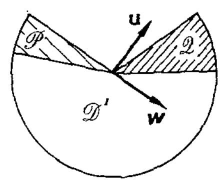

Figure 4

Thus it is easily seen that $I: (\Omega_{\mathscr{D}'}^2)_{\text{dis}} \to \mathscr{W}$ is again uniquely determined by γ.

5. Edge interactions

We assume that a biregular bounded region $\mathscr{B}$, a collection of parts of $\mathscr{B}$, and a contact-interaction $I: (\Omega^2)_{\text{dis}} \to \mathscr{W}$ are given as described in Section 4. In this section we will discuss the consequences of two special assumptions about the nature of I.

Let a section $\mathscr{P}$ of $\mathscr{B}$ be given. It easily follows from Definition 5 of Section 3 that, if $\mathfrak{P}$ is an appropriate biregular partition for $\mathscr{P}$, every piece of $\mathfrak{P}$ either is included entirely in $\mathscr{B}$ or is included entirely in Bdy $\mathscr{B}$. In the former case, we say that the piece is **internal,** in the latter, **peripheral.** Thus a given side, edge, or vertex of $\mathscr{P}$, if it belongs to $\mathfrak{P}$, is either internal or peripheral. All pieces of $\mathfrak{P}$ are internal if and only if $\mathscr{P}$ is an interior part of $\mathscr{B}$.

Definition 1. *Let $\mathscr{P}$ and $\mathscr{Q}$ be sections of $\mathscr{B}$ that are in biregular contact and let $\mathfrak{P}$ and $\mathfrak{Q}$ be appropriate biregular partitions for $\mathscr{P}$ and $\mathscr{Q}$, respectively. Also, let an edge $\mathscr{C} \in \mathfrak{P}_1 \cap \mathfrak{Q}_1$ be given, so that $\mathscr{P}$ and $\mathscr{Q}$ are in contact along $\mathscr{C}$.*

We say that $\mathscr{P}$ and $\mathscr{Q}$ are in **inessential contact** *along $\mathscr{C}$ either if $\mathscr{C}$ is a cusped edge of $\mathscr{P}$ and both sides of $\mathscr{P}$ adjacent to $\mathscr{C}$ are internal, or if $\mathscr{C}$ is a cusped edge of $\mathscr{Q}$ and both sides of $\mathscr{Q}$ adjacent to $\mathscr{C}$ are internal; otherwise we say that $\mathscr{P}$ and $\mathscr{Q}$ are in* **essential contact** *along $\mathscr{C}$.*

We define the set $\mathfrak{K}(\mathscr{C})$, consisting of sides of $\mathscr{P}$, as follows:

(i) *If* $\mathscr{P}$ *and* $\mathscr{Q}$ *are in inessential contact along* $\mathscr{C}$, *then* $\mathfrak{K}(\mathscr{C}) := \emptyset$.

(ii) *If* $\mathscr{P}$ *and* $\mathscr{Q}$ *are in essential contact along* $\mathscr{C}$, *then* $\mathfrak{K}(\mathscr{C}) := \{\mathscr{S} \in \mathfrak{P}_2 \mid \mathscr{S}$ *is adjacent to* $\mathscr{C}$ *and either* (*a*) *or* (b) *below holds*$\}$, *where* (a) *and* (b) *are the following conditions*:

(a) $\mathscr{S}$ *is an internal side of* $\mathscr{P}$ *that is in cusped contact with an internal side of* $\mathscr{Q}$.

(b) $\mathscr{S}$ *is a peripheral side of* $\mathscr{P}$ *that is in contact with a peripheral side of* $\mathscr{Q}$.

If we wish to emphasize the dependence of $\mathfrak{K}(\mathscr{C})$ on $\mathscr{P}$ and $\mathscr{Q}$, we replace $\mathfrak{K}$ by $\mathfrak{K}_{\mathscr{P},\mathscr{Q}}$. It is clear that $\mathfrak{K}(\mathscr{C})$ is a set of cardinality 0, 1, or 2. We illustrate the various situations that determine $\mathfrak{K}(\mathscr{C})$ in Figures 1–6 below. Each of these shows a cross-section orthogonal to $\mathscr{C}$.

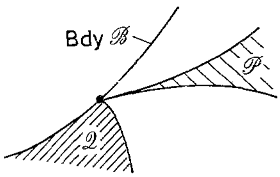

Figure 1

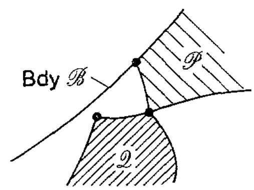

Figure 2

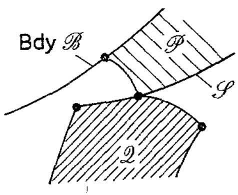

Figure 3

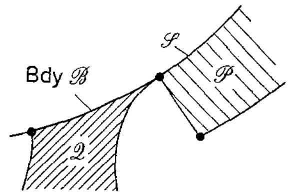

Figure 4

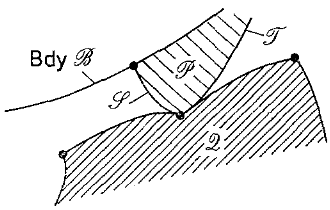

Figure 5

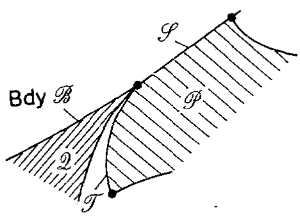

Figure 6

In Figure 1, we have $\mathfrak{K}(\mathscr{C}) = \emptyset$ because $\mathscr{P}$ and $\mathscr{Q}$ are in inessential contact along $\mathscr{C}$. The contact is essential in Figures 2–6. Specifically, we have $\mathfrak{K}(\mathscr{C}) = \emptyset$ in Figure 2, $\mathfrak{K}(\mathscr{C}) = \{\mathscr{S}\}$ in Figures 3 and 4, and $\mathfrak{K}(\mathscr{C}) = \{\mathscr{S}, \mathscr{T}\}$ in Figures 5 and 6. In Figures 3 and 5 we have $\mathscr{S} \in \mathfrak{K}(\mathscr{C})$ because condition (a) is satisfied. In Figures 4 and 6 we have $\mathscr{S} \in \mathfrak{K}(\mathscr{C})$ because condition (b) is satisfied.

If $\mathscr{P}$ and $\mathscr{Q}$ are in simple edge contact along the curve $\mathscr{C}$, then every cross-section of $\mathscr{C}$ in a small neighborhood of $\mathscr{C}$ looks like a wedge contact as described in the Intermezzo. The following assumption states, roughly, that the edge interaction along $\mathscr{C}$ is obtained by integrating wedge interactions on the cross-sections along $\mathscr{C}$.

Assumption I. *There is a function*

$$\gamma : \mathrm{Clo}\,\mathscr{B} \times (\mathscr{U}^2)_\perp \to \mathscr{W} \tag{5.1}$$

of class C^1 *such that*

$$I(\mathscr{P}, \mathscr{Q}) = \int_{\mathscr{C}} \gamma(x, \boldsymbol{v}_{\mathscr{P};\mathscr{S}}(x), \boldsymbol{v}_{\mathscr{S}}(x))\, dl_x \tag{5.2}$$

whenever $\mathscr{P}$ *and* $\mathscr{Q}$ *are disjoint sections of* $\mathscr{B}$ *in simple edge contact along the curve* $\mathscr{C}$ *and* $\mathfrak{R}(\mathscr{C}) = \{\mathscr{S}\}$ (see Definition 1 above and Definition 4 of Section 3).

Proposition 1. *We have*

$$I(\mathscr{P}, \mathscr{Q}) = \int_{\mathscr{C}} \sum_{\mathscr{S} \in \mathfrak{R}(\mathscr{C})} \gamma(x, \boldsymbol{v}_{\mathscr{P};\mathscr{S}}(x), \boldsymbol{v}_{\mathscr{S}}(x))\, dl_x \tag{5.3}$$

whenever $\mathscr{P}$ *and* $\mathscr{Q}$ *are disjoint sections of* $\mathscr{B}$ *in simple edge contact along the curve* $\mathscr{C}$.

Proof. First, we assume that $\mathscr{P}$ and $\mathscr{Q}$ are in essential contact along $\mathscr{C}$. We then have the following three cases.

(α) $\mathfrak{R}(\mathscr{C})$ *is empty.* It is easily seen that we can then construct a section $\mathscr{Q}'$ of $\mathscr{B}$, disjoint from both $\mathscr{P}$ and $\mathscr{Q}$, such that $\mathscr{P}$ is in simple edge contact along $\mathscr{C}$ with both $\mathscr{Q}'$ and $\mathscr{Q} \vee \mathscr{Q}'$ and such that

$$\mathfrak{R}_{\mathscr{P},\mathscr{Q}'}(\mathscr{C}) = \mathfrak{R}_{\mathscr{P},\mathscr{Q}\vee\mathscr{Q}'}(\mathscr{C}) = \{\mathscr{S}\}$$

for some surface $\mathscr{S} \subset \mathrm{Bo}\,\mathscr{P}$ (see Figure 7). It follows from Assumption I that

$$I(\mathscr{P}, \mathscr{Q}') = I(\mathscr{P}, \mathscr{Q} \vee \mathscr{Q}').$$

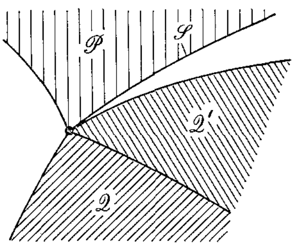

Figure 7

Using the additivity of $I(\mathscr{P}, \cdot)$, we conclude that $I(\mathscr{P}, \mathscr{Q}) = 0$, which is the same as (5.3) when $\mathfrak{R}(\mathscr{C}) = \emptyset$.

(β) $\mathfrak{R}(\mathscr{C})$ *is a singleton.* Then (5.3) simply reduces to (5.2).

(γ) $\mathfrak{R}(\mathscr{C})$ *is a doubleton,* say $\mathfrak{R}(\mathscr{C}) := \{\mathscr{T}_1, \mathscr{T}_2\}$. It is easily seen that we can then construct disjoint sections $\mathscr{Q}_1$ and $\mathscr{Q}_2$, both in simple edge contact with $\mathscr{P}$ along $\mathscr{C}$, such that $\mathscr{Q} = \mathscr{Q}_1 \vee \mathscr{Q}_2$ and $\mathfrak{R}_{\mathscr{P},\mathscr{Q}_1}(\mathscr{C}) = \{\mathscr{T}_1\}$, $\mathfrak{R}_{\mathscr{P},\mathscr{Q}_2}(\mathscr{C}) = \{\mathscr{S}_2\}$ (see Figure 8).

Using the additivity of $I(\mathscr{P}, \cdot)$, we get

$$I(\mathscr{P}, \mathscr{Q}) = I(\mathscr{P}, \mathscr{Q}_1) + I(\mathscr{P}, \mathscr{Q}_2).$$

Now $I(\mathscr{P}, \mathscr{Q}_1)$ and $I(\mathscr{P}, \mathscr{Q}_2)$ can be evaluated using Assumption I. The result is (5.3) with $\mathfrak{R}(\mathscr{C}) = \{\mathscr{T}_1, \mathscr{T}_2\}$.

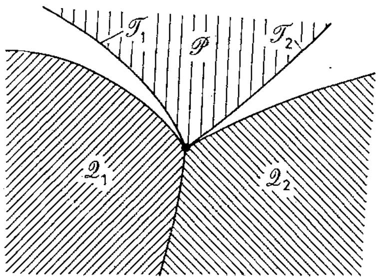

Figure 8

We now assume that $\mathcal{P}$ and $\mathcal{Q}$ are in inessential contact along $\mathcal{C}$. In particular, assume that $\mathcal{C}$ is a cusped edge of $\mathcal{P}$ and both sides of $\mathcal{P}$ adjacent to $\mathcal{C}$ are internal, but that $\mathcal{C}$ is *not* a cusped edge of $\mathcal{Q}$. We can then construct a section $\mathcal{P}'$ of $\mathcal{B}$, disjoint from both $\mathcal{P}$ and $\mathcal{Q}$, such that both $\mathcal{P}'$ and $\mathcal{P} \vee \mathcal{P}'$ are in essential simple edge contact with $\mathcal{Q}$ along $\mathcal{C}$ and $\mathfrak{K}_{\mathcal{P}\vee\mathcal{P}',\mathcal{Q}}(\mathcal{C}) = \mathfrak{K}_{\mathcal{P}',\mathcal{Q}}(\mathcal{C})$ (see Figure 9).

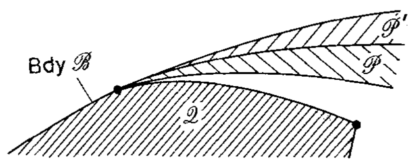

Figure 9

Using the additivity of $I(\cdot, \mathcal{Q})$, we have $I(\mathcal{P} \vee \mathcal{P}', \mathcal{Q}) = I(\mathcal{P}, \mathcal{Q}) + I(\mathcal{P}', \mathcal{Q})$. Since (5.3) can be used to determine $I(\mathcal{P} \vee \mathcal{P}', \mathcal{Q})$ and $I(\mathcal{P}', \mathcal{Q})$ and since they have the same value, it follows that $I(\mathcal{P}, \mathcal{Q}) = 0$. All other cases can be treated in a similar way. □

The following result is an easy consequence of Proposition 1 and the additivity of $I(\mathcal{P}, \cdot)$.

Proposition 2. *Let $\mathcal{P}$ and $\mathcal{Q}$ be disjoint sections of $\mathcal{B}$ in biregular contact and let* $\mathfrak{P}$ and $\mathfrak{Q}$ *be appropriate biregular partitions for $\mathcal{P}$ and $\mathcal{Q}$, respectively, so that*

$$\operatorname{Ctc}(\mathcal{P}, \mathcal{Q}) = \bigcup (\mathfrak{P} \cap \mathfrak{Q}) \tag{5.4}$$

(see Definition 4 of Section 3). *If $\mathfrak{P}$ and $\mathfrak{Q}$ have no surfaces in common*, i.e. *if* $\mathfrak{P}_2 \cap \mathfrak{Q}_2 = \emptyset$, *we have*

$$I(\mathcal{P}, \mathcal{Q}) = \sum_{\mathcal{C} \in \mathfrak{P}_1 \cap \mathfrak{Q}_1} \int_{\mathcal{C}} \sum_{\mathcal{S} \in \mathfrak{K}(\mathcal{C})} \gamma(x, \boldsymbol{v}_{\mathcal{P};\mathcal{S}}(x), \boldsymbol{v}_{\mathcal{S}}(x))\, dl_x. \tag{5.5}$$

Assumption II. *For every biregular part $\mathcal{P}$ of $\mathcal{B}$ there is a continuous function*

$$\tau_{\mathcal{P}}: \ \operatorname{Rbo} \mathcal{P} \setminus \operatorname{Bdy} \mathcal{B} \ \to \ \mathcal{W} \tag{5.6}$$

such that

$$I(\mathcal{P}, \mathcal{Q}) = \int_{\mathcal{S}} \tau_{\mathcal{P}}\, da \tag{5.7}$$

whenever (a) $\mathscr{Q}$ *is a section of* $\mathscr{B}$ *disjoint from* $\mathscr{P}$ *that is in simple surface contact with* $\mathscr{P}$ *along the regular surface* $\mathscr{S}$. (b) $\mathscr{Q}$ *has exactly one side other than* $\mathscr{S}$, *say* $\mathscr{T}$, *and* $\mathscr{T}$ *is an internal side, and* (c) *all edges of* $\mathscr{Q}$ *are cusped.*

Figure 10 illustrates a situation for which (5.7) is valid.

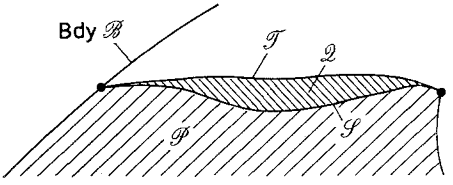

Figure 10

Proposition 3. *We have*

$$\gamma(x, \boldsymbol{u}, -\boldsymbol{w}) = -\gamma(x, \boldsymbol{u}, \boldsymbol{w}) \quad \text{for all } x \in \operatorname{Clo} \mathscr{B}, \quad (\boldsymbol{u}, \boldsymbol{w}) \in (\mathscr{U}^2)_\perp. \tag{5.8}$$

Proof. Let $x \in \mathscr{B}$ and $(\boldsymbol{u}, \boldsymbol{w}) \in (\mathscr{U}^2)_\perp$ be given. We consider a line-segment $\mathscr{C}$ included in $\mathscr{B}$ whose midpoint is x and which is perpendicular to both $\boldsymbol{u}$ and $\boldsymbol{w}$. We can easily construct five interior sections $\mathscr{P}$, $\mathscr{Q}_1$, $\mathscr{Q}_1'$, $\mathscr{Q}_2$, $\mathscr{Q}_2'$ with the following properties (see Figure 11): (i) The sections are pairwise disjoint and in biregular contact. (ii) The sections $\mathscr{Q}_1$ and $\mathscr{Q}_2$ are in simple surface contact with $\mathscr{P}$ along the two sides $\mathscr{S}_1$, $\mathscr{S}_2$ of $\mathscr{P}$ that are adjacent to $\mathscr{C}$ in some biregular partition for $\mathscr{P}$. (iii) The sections $\mathscr{Q}_1$ and $\mathscr{Q}_2$ satisfy the conditions (b) and (c) of Assumption II. (iv) The sections $\mathscr{Q}_1'$ and $\mathscr{Q}_2'$ are in simple edge contact with $\mathscr{P}$ along $\mathscr{C}$ such that $\mathfrak{K}_{\mathscr{P},\mathscr{Q}_1'}(\mathscr{C}) = \{\mathscr{S}_1\}$ and $\mathfrak{K}_{\mathscr{P},\mathscr{Q}_2'}(\mathscr{C}) = \{\mathscr{S}_2\}$. (v) $\mathscr{S} := \mathscr{S}_1 \cup \mathscr{S}_2 \cup \mathscr{C}$ is a regular surface and $\mathscr{Q} := \mathscr{Q}_1 \vee \mathscr{Q}_1' \vee \mathscr{Q}_2 \vee \mathscr{Q}_2'$ is a section that is in simple surface contact with $\mathscr{P}$ along $\mathscr{S}$ such that the conditions (b) and (c) of Assumption II are satisfied. (vi) $\mathscr{S}$ is a subset of a plane and $\boldsymbol{v}_{\mathscr{P}}|_{\mathscr{S}}$ is a constant with value $\boldsymbol{u}$. The functions $\boldsymbol{v}_{\mathscr{S}_1}|_{\mathscr{C}}$ and $\boldsymbol{v}_{\mathscr{S}_2}|_{\mathscr{C}}$ are constants with values $\boldsymbol{w}$ and $-\boldsymbol{w}$, respectively.

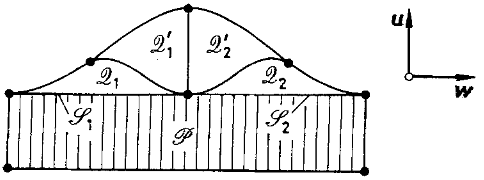

Figure 11

It follows from (i) and (v) and Assumption II that

$$I(\mathscr{P}, \mathscr{Q}_1 \vee \mathscr{Q}_1' \vee \mathscr{Q}_2 \vee \mathscr{Q}_2') = I(\mathscr{P}, \mathscr{Q}) = \int_{\mathscr{S}} \tau_{\mathscr{P}} \, da. \tag{5.9}$$

It follows from (i), (ii), (iii), and Assumption II that

$$I(\mathscr{P}, \mathscr{Q}_1) = \int_{\mathscr{S}_1} \tau_{\mathscr{P}} \, da, \quad I(\mathscr{P}, \mathscr{Q}_2) = \int_{\mathscr{S}_2} \tau_{\mathscr{P}} \, da \tag{5.10}$$

and from (i), (iv), (vi), and Assumption I that

$$I(\mathscr{P}, \mathscr{Q}_1') = \int_{\mathscr{C}} \gamma(y, \boldsymbol{v}_{\mathscr{P};\mathscr{S}_1}(y), \boldsymbol{v}_{\mathscr{S}_1}(y))\, dl_y = \int_{\mathscr{C}} \tau(y, \boldsymbol{u}, \boldsymbol{w})\, dl_y,$$
$$I(\mathscr{P}, \mathscr{Q}_2') = \int_{\mathscr{C}} \gamma(y, \boldsymbol{v}_{\mathscr{P};\mathscr{S}_2}(y), \boldsymbol{v}_{\mathscr{S}_2}(y))\, dl_y = \int_{\mathscr{C}} \gamma(y, \boldsymbol{u}, -\boldsymbol{w})\, dl_y. \tag{5.11}$$

Using the additivity of $I(\mathscr{P}, \cdot)$ and the additivity of surface integrals, we conclude from (5.9)–(5.11) that

$$0 = \int_{\mathscr{C}} (\gamma(y, \boldsymbol{u}, \boldsymbol{w}) + \gamma(y, \boldsymbol{u}, -\boldsymbol{w}))\, dl_y.$$

Since the line-segment $\mathscr{C}$ centered at x can be made arbitrarily short, it follows that (5.8) holds. If $x \in \operatorname{Bdy} \mathscr{B}$ rather than $x \in \mathscr{B}$, then (5.8) follows from the assumed continuity of γ. □

Definition 1. *Let $\mathscr{S}$ be a regular surface. A continuous function $\boldsymbol{n} : \operatorname{Clo} \mathscr{S} \to \mathscr{U}$ is called an* **orientation** *of $\mathscr{S}$ if $\boldsymbol{n}(x) \in (\operatorname{Tan}_{\mathscr{S}}(x))^{\perp}$ for all $x \in \operatorname{Clo} \mathscr{S}$. We say that $\mathscr{S}$ is* **orientable** *if admits an orientation $\boldsymbol{n}$.*

By an **oriented regular surface** *we mean a pair $(\mathscr{S}, \boldsymbol{n})$ of a regular surface $\mathscr{S}$ and an orientation $\boldsymbol{n}$ of $\mathscr{S}$.*

An orientable regular surface $\mathscr{S}$ admits exactly two orientations, and if $\boldsymbol{n}$ is one then $-\boldsymbol{n}$ is the other. Let $(\mathscr{S}, \boldsymbol{n})$ be an oriented surface. Then

$$\boldsymbol{E}_{\mathscr{S}} = 1_{\mathscr{V}} - \boldsymbol{n} \otimes \boldsymbol{n} \tag{5.12}$$

and $\boldsymbol{n}$ is of class C^1. If $\boldsymbol{n}$ is constant then $\mathscr{S}$ must be subset of a plane.

Proposition 4. *For every oriented regular surface $(\mathscr{S}, \boldsymbol{n})$ with $\mathscr{S} \subset \mathscr{B}$ there is a continuous function*

$$\sigma_{(\mathscr{S},\boldsymbol{n})} : \ \operatorname{Clo} \mathscr{S} \to \mathscr{W}$$

such that

$$\sigma_{(\mathscr{S},\boldsymbol{n})} = \tau_{\mathscr{P}}|_{\mathscr{S}} \tag{5.13}$$

whenever $\mathscr{P}$ is a biregular part of $\mathscr{B}$ with $\mathscr{S} \subset \operatorname{Rbo} \mathscr{P}$ and $\boldsymbol{n} = \boldsymbol{v}_{\mathscr{P}|\mathscr{S}}$.

Moreover, if $(\mathscr{S}, \boldsymbol{n})$ and $(\mathscr{S}', \boldsymbol{n}')$ are oriented regular surfaces such that $\mathscr{S}' \subset \mathscr{S}$ and $\boldsymbol{n}' = \boldsymbol{n}_{|\mathscr{S}'}$, then

$$\sigma_{(\mathscr{S}',\boldsymbol{n}')} = \sigma_{(\mathscr{S},\boldsymbol{n})}|_{\mathscr{S}'}. \tag{5.14}$$

The proof of Proposition 4, which will not be given here, is only slightly more complicated than the proof of the corresponding proposition for classical surface interactions (see *e.g.* Theorem 1 on p. 271 of [N1]).

Proposition 5. *Let $\mathscr{P}$ and $\mathscr{Q}$ be sections of $\mathscr{B}$ in biregular contact and let $\mathfrak{P}$ and $\mathfrak{Q}$ be appropriate biregular partitions of $\mathscr{P}$ and $\mathscr{Q}$, respectively. Then*

$$I(\mathscr{P}, \mathscr{Q}) = \sum_{\mathscr{S} \in \mathfrak{P}_2 \cap \mathfrak{Q}_2} \int_{\mathscr{S}} \sigma_{(\mathscr{S}, \boldsymbol{v}_{\mathscr{P}|\mathscr{S}})}\, da + \sum_{\mathscr{C} \in \mathfrak{P}_1 \cap \mathfrak{Q}_1} \int_{\mathscr{C}} \sum_{\mathscr{S} \in \mathfrak{R}(\mathscr{C})} \gamma(x, \boldsymbol{v}_{\mathscr{P};\mathscr{S}}(x), \boldsymbol{v}_{\mathscr{P}}(x))\, dl_x. \tag{5.15}$$

Proof. Let $\mathscr{S} \in \mathfrak{P}_2 \cap \mathfrak{Q}_2$ be given. It is easily seen that one can construct a section $\mathscr{R}_{\mathscr{S}}$ of $\mathscr{Q}$ with the following properties (see Figure 12): (i) $\mathscr{R}_{\mathscr{S}}$ is in simple surface contact with $\mathscr{P}$ along $\mathscr{S}$; (ii) $\mathscr{R}_{\mathscr{S}}$ has exactly one side other than $\mathscr{S}$, say $\mathscr{S}'$, and $\mathscr{S}'$ is included in $\mathscr{Q}$; (iii) all edges of $\mathscr{R}_{\mathscr{S}}$ are cusped. Moreover, the construction can be done in such a way that the collection $\{\mathscr{R}_{\mathscr{S}} \mid \mathscr{S} \in \mathfrak{P}_2 \cap \mathfrak{Q}_2\}$ is pairwise disjoint and $\operatorname{Ctc}(\mathscr{R}_{\mathscr{S}}, \mathscr{R}_{\mathscr{S}'}) \subset \operatorname{Bdy} \mathscr{P}$ for any two distinct $\mathscr{S}, \mathscr{S}' \in \mathfrak{P}_2 \cap \mathfrak{Q}_2$.

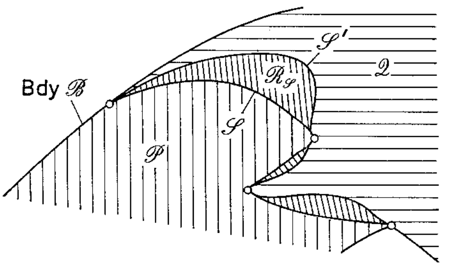

Figure 12

It is easily seen that

$$\mathscr{Q}' := \operatorname{Int}(\mathscr{Q} \setminus \bigcup \{\mathscr{R}_{\mathscr{S}} \mid \mathscr{S} \in \mathfrak{P}_2 \cap \mathfrak{Q}_2\}) \tag{5.16}$$

is a section of $\mathscr{Q}$ that has an appropriate biregular partition $\mathfrak{Q}'$ such that $\mathfrak{P}_2 \cap \mathfrak{Q}_2' = \emptyset$ and $\mathfrak{P}_1 \cap \mathfrak{Q}_1' = \mathfrak{P}_1 \cap \mathfrak{Q}_1$. The section $\mathscr{Q}'$ is in biregular contact with $\mathscr{P}$ and Proposition 2 can be used to evaluate $I(\mathscr{P}, \mathscr{Q}')$. In view of condition (iii) above, we also have $\mathfrak{K}_{\mathscr{P},\mathscr{Q}}(\mathscr{C}) = \mathfrak{K}_{\mathscr{P},\mathscr{Q}'}(\mathscr{C})$ for all $\mathscr{C} \in \mathfrak{Q}_1 \cap \mathfrak{P}_1$, so that (5.5) gives

$$I(\mathscr{P}, \mathscr{Q}') = \sum_{\mathscr{C} \in \mathfrak{P}_1 \cap \mathfrak{Q}_1} \int_{\mathscr{C}} \sum_{\mathscr{S} \in \mathfrak{K}(\mathscr{C})} \gamma(x, \boldsymbol{v}_{\mathscr{P};\mathscr{S}}(x), \boldsymbol{v}_{\mathscr{S}}(x))\, dl_x \tag{5.17}$$

when $\mathfrak{K} := \mathfrak{K}_{\mathscr{P},\mathscr{Q}}$. Using (5.16) and the additivity of $I(\mathscr{P}, \cdot)$ one easily proves using induction that

$$I(\mathscr{P}, \mathscr{Q}) = I(\mathscr{P}, \mathscr{Q}') + \sum_{\mathscr{S} \in \mathfrak{P}_2 \cap \mathfrak{Q}_2} I(\mathscr{P}, \mathscr{R}_{\mathscr{S}}). \tag{5.18}$$

Now, in view of the conditions (i), (ii), and (iii) above. Assumption II can be used to evaluate $I(\mathscr{P}, \mathscr{R}_{\mathscr{S}})$. Using also Proposition 4, we get

$$I(\mathscr{P}, \mathscr{R}_{\mathscr{S}}) = \int_{\mathscr{S}} \sigma_{(\mathscr{S}, \boldsymbol{v}_{\mathscr{P}}|_{\mathscr{S}})}\, da. \tag{5.19}$$

Combining (5.17), (5.18), and (5.19), we get the desired result (5.15). □

Proposition 6 shows that the values of the interaction I for sections in biregular contact are uniquely determined by the function γ and the functions $\sigma_{(\mathscr{S},\boldsymbol{n})}$. The problem of how to express the values of I in more general situations is open.

If $\gamma = 0$ we say that the interaction I is a **pure surface interaction.** If $\sigma_{(\mathscr{S},\boldsymbol{n})} = 0$ for all oriented regular surfaces $(\mathscr{S}, \boldsymbol{n})$ we say that I is a **pure edge interaction.** Every interaction satisfying Assumptions I and II is then the value-wise sum of a pure surface interaction and a pure edge interaction.

Remark 1. Suppose that a function

$$\gamma : \mathrm{Clo}\,\mathscr{B} \times (\mathscr{U}^2)_\perp \to \mathscr{W}$$

of class C^1 satisfying $\gamma(x, \boldsymbol{u}, -\boldsymbol{w}) = -\gamma(x, \boldsymbol{u}, \boldsymbol{w})$ for all $(x, \boldsymbol{u}, \boldsymbol{w}) \in \mathrm{Clo}\,\mathscr{B} \times (\mathscr{U}^2)_\perp$ has been prescribed. Also suppose that, for each oriented regular surface $(\mathscr{S}, \boldsymbol{n})$ with $\mathscr{S} \subset \mathscr{B}$, a continuous function $\sigma_{(\mathscr{S},\boldsymbol{n})} : \mathrm{Clo}\,\mathscr{S} \to \mathscr{W}$ has been *prescribed* in such a way that (5.14) holds. We can then *define* $I(\mathscr{P}, \mathscr{Q})$ by (5.15) for all disjoint pairs $(\mathscr{P}, \mathscr{Q})$ of sections of $\mathscr{B}$ in biregular contact. One can then prove that I is *conditionally biadditive* in the sense that

$$I(\mathscr{P}, \mathscr{Q}_1 \vee \mathscr{Q}_2) = I(\mathscr{P}, \mathscr{Q}_1) + I(\mathscr{P}, \mathscr{Q}_2)$$

and

$$I(\mathscr{Q}_1 \vee \mathscr{Q}_2, \mathscr{P}) = I(\mathscr{Q}_1, \mathscr{P}) + I(\mathscr{Q}_2, \mathscr{P})$$

hold provided that: (i) $\mathscr{P}$, $\mathscr{Q}_1$ and $\mathscr{Q}_2$ are pairwise disjoint sections of $\mathscr{B}$; (ii) $\mathscr{Q}_1 \vee \mathscr{Q}_2$ is a section of $\mathscr{B}$; (iii) $\mathscr{P}$ is in biregular contact with $\mathscr{Q}_1$, $\mathscr{Q}_2$, and $\mathscr{Q}_1 \vee \mathscr{Q}_2$. □

6. Resultant actions

We assume that a region $\mathscr{B}$ and a contact interaction I as described in Section 5 are given. In particular, we assume that the Assumptions I and II are satisfied.

The following result describes the *resultant* action on a given section of $\mathscr{B}$, *i.e.* the action $I(\mathscr{P}, \mathscr{P}^b)$ on a section $\mathscr{P}$ of $\mathscr{B}$ by the exterior $\mathscr{P}^b$ of $\mathscr{P}$ in $\mathscr{B}$, provided that the condition (6.1) below is valid. We will see in Section 7 that (6.1) is automatically satisfied if I is quasi-balanced in the sense of Definition 1 of Section 4.

Proposition 1. *Assume that the function* γ *of* Assumption I *satisfies*

$$\gamma(x, -\boldsymbol{u}, \boldsymbol{w}) = -\gamma(x, \boldsymbol{u}, \boldsymbol{w}) \quad \textit{for all} \quad x \in \mathrm{Clo}\,\mathscr{B}, \quad (\boldsymbol{u}, \boldsymbol{w}) \in (\mathscr{U}^2)_\perp. \tag{6.1}$$

Let $\mathscr{P}$ *be a section of* $\mathscr{B}$ *and let* $\mathfrak{P}$ *be an appropriate biregular partition for* $\mathscr{P}$. *Then*

$$I(\mathscr{P}, \mathscr{P}^b) = \sum_{\mathscr{S} \in \mathfrak{P}_2^{\mathrm{int}}} \Big(\int_{\mathscr{S}} \sigma_{(\mathscr{S}, \boldsymbol{v}_{\mathscr{P}}|_{\mathscr{S}})}\, da + \int_{\mathrm{Rbo}\,\mathscr{S}} \gamma(x, \boldsymbol{v}_{\mathscr{P};\mathscr{S}}(x), \boldsymbol{v}_{\mathscr{S}}(x))\, dl_x \Big)$$
$$+ \sum_{\mathscr{S} \in \mathfrak{P}_2^{\mathrm{per}}} \int_{\mathrm{Rbo}\,\mathscr{S} \cap \mathrm{Bdy}\,\mathscr{P}^b} \gamma(x, \boldsymbol{v}_{\mathscr{P};\mathscr{S}}(x), \boldsymbol{v}_{\mathscr{S}}(x))\, dl_x \tag{6.2}$$

where $\mathfrak{P}_2^{\mathrm{int}}$ *is the set of all internal sides of* $\mathscr{P}$ *and* $\mathfrak{P}_2^{\mathrm{per}}$ *is the set of all peripheral sides of* $\mathscr{P}$.

Proof. It is easily seen that we may choose an appropriate biregular partition $\mathfrak{Q}$ for $\mathscr{P}^b$ such that

$$\mathfrak{P}_2 \cap \mathfrak{Q}_2 = \mathfrak{P}_2^{\mathrm{int}}. \tag{6.3}$$

i.e. such that the internal sides of $\mathscr{P}$ are exactly the sides that $\mathscr{P}$ and $\mathscr{P}^b$ have in common.

Let $\mathscr{C} \in \mathfrak{P}_1 \cap \mathfrak{Q}_1$, *i.e.* an edge that $\mathscr{P}$ and $\mathscr{P}_b$ have in common, be given, and let $\mathscr{S}_1, \mathscr{S}_2 \in \mathfrak{P}_2$ be the two sides of $\mathscr{P}$ adjacent to $\mathscr{C}$. There are two cases: (i) $\mathscr{C}$ is *not* an internal cusped edge of $\mathscr{P}$. Then $\mathscr{C}$ cannot be an internal cusped edge of $\mathscr{P}^b$ either and $\mathscr{P}$ and $\mathscr{P}^b$ are in essential contact along $\mathscr{C}$ (see Definition 1 of Section 5). In this case, the set $\mathfrak{K}(\mathscr{C})$ is easily seen to be given by $\mathfrak{K}(\mathscr{C}) = \{\mathscr{S}_1, \mathscr{S}_2\}$ so that

$$\int_{\mathscr{C}} \gamma(x, \boldsymbol{v}_{\mathscr{P};\mathscr{S}_1}(x), \boldsymbol{v}_{\mathscr{S}_1}(x))\, dl_x + \int_{\mathscr{C}} \gamma(x, \boldsymbol{v}_{\mathscr{P};\mathscr{S}_2}(x), \gamma_{\mathscr{S}_2}(x))\, dl_x$$
$$= \int_{\mathscr{C}} \sum_{\mathscr{S}\in\mathfrak{K}(\mathscr{C})} \gamma(x, \boldsymbol{v}_{\mathscr{P};\mathscr{S}}(x), \boldsymbol{v}_{\mathscr{S}}(x))\, dl_x. \tag{6.4}$$

(ii) $\mathscr{C}$ is an internal cusped edge of $\mathscr{P}$. Then $\mathscr{P}$ and $\mathscr{P}^b$ are in inessential contact along $\mathscr{C}$ and hence $\mathfrak{K}(\mathscr{C}) = \emptyset$. Hence the right side of (6.4) is zero. We also have $\boldsymbol{v}_{\mathscr{P};\mathscr{S}_1}(x) = -\boldsymbol{v}_{\mathscr{P};\mathscr{S}_2}(x)$ and $\boldsymbol{v}_{\mathscr{S}_1}(x) = \boldsymbol{v}_{\mathscr{S}_2}(x)$ for all $x \in \mathscr{C}$. Hence, by the assumption (6.1), the left side of (6.4) is also zero.

We conclude that (6.4) is valid no matter what $\mathscr{C} \in \mathfrak{P}_1 \cap \mathfrak{Q}_1$ is. If $\mathscr{C}$ is an internal edge of $\mathscr{P}$, then $\mathscr{S}_1, \mathscr{S}_2 \in \mathfrak{P}_2^{\text{int}}$. If $\mathscr{C}$ is a peripheral edge of $\mathscr{P}$, then one of $\mathscr{S}_1$ and $\mathscr{S}_2$, say $\mathscr{S}_1$, is internal while the other, $\mathscr{S}_2$, is peripheral and $\mathscr{C} \subset$ Rbo $\mathscr{S}_2 \cap$ Bdy $\mathscr{P}^b$.

Since the edges of every $\mathscr{S} \in \mathfrak{P}_2^{\text{int}}$ and the edges included in Bdy $\mathscr{P}^b$ of every $\mathscr{S} \in P_2^{\text{per}}$ all belong to $\mathfrak{P}_1 \cap \mathfrak{Q}_1$, it follows from (6.3) and (6.4) that (5.15) reduces to (6.2) when $\mathscr{Q} := \mathscr{P}^b$. □

Remark 1. If a given edge $\mathscr{C} \in \mathfrak{P}_1$ is included in the reduced boundary Rby $\mathscr{P}$ of $\mathscr{P}$, then the contributions from $\mathscr{C}$ to the line integrals in (6.2) corresponding to the two sides adjacent to $\mathscr{C}$ cancel by Proposition 3 of Section 5. Hence the right side of (6.2) does not change when $\mathfrak{P}$ is replaced by a refinement, as it shouldn't. □

We now add an assumption which is consistent with (5.8), but does not follow from it.

Assumption III. *There is a function*

$$\boldsymbol{G}: \ \text{Clo}\,\mathscr{B} \times \mathscr{U} \to \text{Lin}\,(\mathscr{V}, \mathscr{W}) \tag{6.5}$$

of class C^1 *such that*

$$\gamma(x, \boldsymbol{u}, \boldsymbol{w}) = \boldsymbol{G}(x, \boldsymbol{u})\, \boldsymbol{w} \quad \textit{for all } x \in \text{Clo}\,\mathscr{B}, \quad (\boldsymbol{u}, \boldsymbol{w}) \in (\mathscr{U}^2)_\perp. \tag{6.6}$$

The function $\boldsymbol{G}$ is not uniquely determined by γ. Indeed, if $\boldsymbol{G}'$ is a C^1-function of the type (6.5) that satisfies (6.6), then

$$\boldsymbol{G}(x, \boldsymbol{u}) := \boldsymbol{G}'(x, \boldsymbol{u})\, (\mathbf{1}_{\mathscr{V}} - (\boldsymbol{u} \otimes \boldsymbol{u})) \quad \textit{for all } x \in \text{Clo}\,\mathscr{B}, \ \ \boldsymbol{u} \in \mathscr{U} \tag{6.7}$$

defines another one. The function $\boldsymbol{G}$ defined by (6.7) satisfies, in addition to (6.6), the following *normalization condition*:

$$\boldsymbol{u} \in \text{Null}\,\boldsymbol{G}(x, \boldsymbol{u}) \quad \text{for all } x \in \text{Clo}\,\mathscr{B}, \quad \boldsymbol{u} \in \mathscr{U}, \tag{6.8}$$

It is easily seen that γ determines uniquely a C^1-function G that satisfies both (6.6) and (6.8), and from now on we will assume that G denotes this function.

Remark 2. One can prove that Assumption III is implied by the following

Assumption III′. *There is a* $k \in \mathbb{P}^\times$ *such that for every biregular oriented surface* $(\mathcal{S}, \boldsymbol{n})$ *with* $\mathcal{S} \subset \mathcal{B}$ *we have*

$$\left| \int_{\mathrm{Rbo}\,\mathcal{S}} \gamma(x, \boldsymbol{n}(x), \boldsymbol{v}_{\mathcal{S}}(x)\, dl_x \right| \leqq k \,\mathrm{area}\,(\mathcal{S}). \tag{6.9}$$

Unfortunately, we do not know of any convincing physical reasons that one might use to justify Assumption III′ or Assumption III. □

Now let an oriented regular surface $(\mathcal{S}, \boldsymbol{n})$ with $\mathcal{S} \subset \mathrm{Clo}\,\mathcal{B}$ be given. Then

$$G(1_{\mathcal{S}\subset\mathcal{E}}, \boldsymbol{n}) = (x \mapsto G\,(x, \boldsymbol{n}(x)): \ \mathrm{Clo}\,\mathcal{S} \to \mathrm{Lin}\,(\mathcal{V}, \mathcal{W}) \tag{6.10}$$

is a mapping of Class C^1 in the sense of Definition 1 of Section 2. We use the abbreviation

$$\varrho_{(\mathcal{S},\boldsymbol{n})} := \mathrm{div}_{\mathcal{S}}\,(G(1_{\mathcal{S}\subset\mathcal{E}}, \boldsymbol{n})): \ \mathcal{S} \to \mathcal{W} \tag{6.11}$$

for the surface-divergence of $G(1_{\mathcal{S}\subset\mathcal{E}}, \boldsymbol{n})$ as defined in Definition 2 of Section 2. Using (5.12) and the fact that

$$\mathrm{Tan}_{\mathcal{S}}(x) = \{\boldsymbol{n}(x)\}^\perp \quad \text{for all } x \in \mathrm{Clo}\,\mathcal{S}, \tag{6.12}$$

one can use the chain rule and the definitions of Section 2 to show that $\varrho_{(\mathcal{S},\boldsymbol{n})}$ is determined by the following explicit formula:

$$\begin{aligned} \omega\varrho_{(\mathcal{S},\boldsymbol{n})}(x) &= \omega\, \mathrm{div}_x\,(G(\cdot, \boldsymbol{n}(x))) - \boldsymbol{n}(x)\cdot\nabla_x(\omega G(\cdot, \boldsymbol{n}(x)))\,\boldsymbol{n}(x) \\ &\quad + \mathrm{tr}\,(\nabla_{\boldsymbol{n}(x)}(\omega G(x, \cdot))\,\nabla_x \boldsymbol{n}|^{\{\boldsymbol{n}(x)\}^\perp}) \quad \text{for all } x \in \mathrm{Clo}\,\mathcal{S}, \quad \omega \in \mathcal{W}^*. \end{aligned} \tag{6.13}$$

In the case when $\mathcal{W} = \mathbb{R}$ and $\mathrm{Lin}\,(\mathcal{V}, \mathbb{R}) = \mathcal{V}^* \cong \mathcal{V}$, (6.13) reduces to

$$\begin{aligned} \varrho_{(\mathcal{S},\boldsymbol{n})}(x) &= \mathrm{div}_x\,(G(\cdot, \boldsymbol{n}(x))) - \boldsymbol{n}(x)\cdot\nabla_x G(\cdot, \boldsymbol{n}(x))\,\boldsymbol{n}(x) \\ &\quad + \mathrm{tr}(\nabla_{\boldsymbol{n}(x)}\, G(x, \cdot)\,\nabla_x \boldsymbol{n}|^{\{\boldsymbol{n}(x)\}^\perp}) \quad \text{for all } x \in \mathrm{Clo}\,\mathcal{S}. \end{aligned} \tag{6.14}$$

The normalization condition (6.8) ensures that $G(1_{\mathcal{S}\subset\mathcal{E}}, \boldsymbol{n})$ is a tangential mapping in the sense of Definition 3 of Section 2. Hence we can apply the Corollary to surface-divergence Theorem stated in Section 2 to obtain the following

Proposition 2. *Let* $\mathcal{P}$ *be a section of* $\mathcal{B}$ *and let* $\mathfrak{P}$ *be an appropriate biregular partition for* $\mathcal{P}$. *For every internal side* $\mathcal{S} \in \mathfrak{P}_2^{\mathrm{int}}$ *of* $\mathcal{P}$ *we then have*

$$\begin{aligned} &\int_{\mathcal{S}} \sigma_{(\mathcal{S},\boldsymbol{v}_{\mathcal{P}}|_{\mathcal{S}})}\, da + \int_{\mathrm{Rbo}\,\mathcal{S}} \gamma(x, \boldsymbol{v}_{\mathcal{P};\mathcal{S}}(x), \boldsymbol{v}_{\mathcal{S}}(x))\, dl_x \\ &\quad = \int_{\mathcal{S}} (\sigma_{(\mathcal{S},\boldsymbol{v}_{\mathcal{P}}|_{\mathcal{S}})} + \varrho_{(\mathcal{S},\boldsymbol{v}_{\mathcal{P}}|_{\mathcal{S}})})\, da, \end{aligned} \tag{6.15}$$

where ϱ *is characterized by* (6.13).

Since $\mathcal{S} \in \mathfrak{P}^2_{\text{int}}$ implies $\mathcal{S} \subset \mathcal{B}$, one needs functions of the form $\varrho_{(\mathcal{S},\boldsymbol{n})}$ only for surfaces $\mathcal{S}$ included in $\mathcal{B}$ when applying Proposition 2. Nevertheless, it is important to note, for use in Section 8, that $\varrho_{(\mathcal{S},\boldsymbol{n})}$, as given by (6.11) or (6.13), is meaningful even when $\mathcal{S}$ is a regular surface included in the *boundary* of $\mathcal{B}$.

7. Quasi-balanced interactions

We assume again that a region $\mathcal{B}$ and a contact interaction I as described in Section 5 are given. We assume that Assumption III of Section 6 as well as the Assumptions I and II of Section 5 are satisfied. Finally, we assume that the interaction I is quasi-balanced in the sense of Definition 1 of Section 4. The continuous function $f\colon \operatorname{Clo}\mathcal{B} \to \mathcal{W}$ for which (4.7) holds is clearly uniquely determined by I.

For each oriented biregular surface $(\mathcal{S}, \boldsymbol{n})$ with $\mathcal{S} \subset \mathcal{B}$ we now define

$$\varphi_{(\mathcal{S},\boldsymbol{n})} := \sigma_{(\mathcal{S},\boldsymbol{n})} + \varrho_{(\mathcal{S},\boldsymbol{n})}\colon \ \mathcal{S} \to \mathcal{W}, \tag{7.1}$$

where $\sigma_{(\mathcal{S},\boldsymbol{n})}$ is characterized by Proposition 4 of Section 5 and $\varrho_{(\mathcal{S},\boldsymbol{n})}$ is given by (6.11) or (6.13).

It follows from Proposition 4 of Section 5 and from (6.13) that the functions $\varphi_{(\mathcal{S},\boldsymbol{n})}$ defined by (7.1) satisfy the following two conditions:

(A) For every oriented biregular surface $(\mathcal{S}, \boldsymbol{n})$ with $\mathcal{S} \subset \mathcal{B}$, $\varphi_{(\mathcal{S},\boldsymbol{n})}$ is continuous.

(B) If $(\mathcal{S}, \boldsymbol{n})$ and $(\mathcal{S}', \boldsymbol{n}')$ are oriented biregular surfaces such that $\mathcal{S}' \subset \mathcal{S} \subset \mathcal{B}$ and $\boldsymbol{n}' = \boldsymbol{n}|_{\mathcal{S}'}$, then

$$\varphi_{(\mathcal{S}',\boldsymbol{n}')} = \varphi_{(\mathcal{S},\boldsymbol{n})}|_{\mathcal{S}'}. \tag{7.2}$$

It follows from Proposition 1 of Section 4, applied to the case when $\mathcal{P}$ and $\mathcal{Q}$ are suitably chosen biregular interior parts of $\mathcal{B}$ in simple edge contact, that the function γ of Assumption I of Section 5 satisfies $\gamma(x, -\boldsymbol{u}, \boldsymbol{w}) = -\gamma(x, \boldsymbol{u}, \boldsymbol{w})$ for all $x \in \mathcal{B}$ and all $(\boldsymbol{u}, \boldsymbol{w}) \in (\mathcal{U}^2)_\perp$. In view of the assumed continuity of γ on its entire domain $\operatorname{Clo}\mathcal{B} \times (\mathcal{U}^2)_\perp$, we conclude that

$$\gamma(\cdot, -\boldsymbol{u}, \boldsymbol{w}) = -\gamma(\cdot, \boldsymbol{u}, \boldsymbol{w}) \quad \text{for all } (\boldsymbol{u}, \boldsymbol{w}) \in (\mathcal{U}^2)_\perp \tag{7.3}$$

i.e. that the assumption (6.1) of Proposition 1 of Section 6 is automatically satisfied. Using this fact and the fact that an interior section $\mathcal{P}$ does not have any peripheral sides, we conclude from (4.7) and Propositions 1 and 2 of Section 6 that the functions $\varphi_{(\mathcal{S},\boldsymbol{n})}$ also satisfy the following third condition:

(C) There is a continuous function $f\colon \operatorname{Clo}\mathcal{B} \to \mathcal{W}$ such that, for every biregular part $\mathcal{P} \in \Omega_{\text{int}}$ and every biregular partition $\mathfrak{P}$ for $\mathcal{P}$, we have

$$\sum_{\mathcal{S} \in \mathfrak{P}_2} \int_{\mathcal{S}} \varphi_{(\mathcal{S},\boldsymbol{n})}\, da = \int_{\mathcal{P}} f\, dv. \tag{7.4}$$

Now let $x \in \mathscr{B}$ and $\boldsymbol{u} \in \mathscr{U}$ be given. Then the plane section $(x + \{\boldsymbol{u}\}^{\perp}) \cap \mathscr{B}$ includes biregular plane surfaces $\mathscr{T}$ that contain x. For each such surface $\mathscr{T}$, the constant $\boldsymbol{u}$ is an orientation of $\mathscr{T}$ and hence $\varphi_{(\mathscr{T},\boldsymbol{u})}$, $\sigma_{(\mathscr{T},\boldsymbol{u})}$, and $\varrho_{(\mathscr{T},\boldsymbol{u})}$ are all meaningful. It follows from condition (B) that

$$\delta(x, \boldsymbol{u}) := \varphi_{(\mathscr{T},\boldsymbol{u})}(x) \tag{7.5}$$

depends only on x and $\boldsymbol{u}$ and not on what particular plane surface $\mathscr{T}$ has been selected. Similar statements apply to

$$\alpha(x, \boldsymbol{u}) := \sigma_{(\mathscr{T},\boldsymbol{u})}(x) \tag{7.6}$$

and

$$\beta(x, \boldsymbol{u}) := \varrho_{(\mathscr{T},\boldsymbol{u})}(x) \tag{7.7}$$

because of Proposition 4 of Section 5 and (6.13).

We now add one more assumption.

Assumption IV. *For each* $\boldsymbol{u} \in \mathscr{U}$, *the function* $\alpha(\cdot, \boldsymbol{u}) : \mathscr{B} \to \mathscr{W}$ *given by* (7.6) *is continuous.*

It follows from (6.13) and (7.7) that

$$\omega\beta(x, \boldsymbol{u}) = \omega \operatorname{div}_x (\boldsymbol{G}(\cdot, \boldsymbol{u})) - \boldsymbol{u} \cdot \nabla_x(\omega \boldsymbol{G}(\cdot, \boldsymbol{u}))\, \boldsymbol{u} \tag{7.8}$$

for all $x \in \mathscr{B}, \boldsymbol{u} \in \mathscr{U}$, and $\omega \in \mathscr{W}^*$. Since the function $\boldsymbol{G}$ of Assumption III was assumed to be of class C^1, it follows from (7.8) that the function $\beta(\cdot, \boldsymbol{u})$: $\mathscr{B} \to \mathscr{W}$ is continuous. Hence, since $\delta = \alpha + \beta$ by (7.1) and (7.5)–(7.7), we conclude that the functions $\varphi_{(\mathscr{S},\boldsymbol{u})}$ also satisfy the following fourth condition:

(D) For every $\boldsymbol{u} \in \mathscr{U}$, the function $\delta(\cdot, \boldsymbol{u}) : \mathscr{B} \to \mathscr{W}$ given by (7.5) in terms of $\varphi_{(\mathscr{T},\boldsymbol{u})}$ is continuous.

The main result of [N3] is that the conditions (A)–(D) imply the existence of a continuous function

$$\boldsymbol{F} : \mathscr{B} \to \operatorname{Lin}(\mathscr{V}, \mathscr{W}) \tag{7.9}$$

such that

$$\boldsymbol{F}(x)\, \boldsymbol{n}(x) = \varphi_{(\mathscr{S},\boldsymbol{n})}(x) = \sigma_{(\mathscr{S},\boldsymbol{n})}(x) + \varrho_{(\mathscr{S},\boldsymbol{n})}(x) \quad \text{for all } x \in \mathscr{S} \tag{7.10}$$

for all oriented biregular surfaces $(\mathscr{S}, \boldsymbol{n})$ with $\mathscr{S} \subset \mathscr{B}$.

Remark 1. If (D) is replaced by a stronger condition that need not be satisfied by the functions $\varphi_{(\mathscr{S},\boldsymbol{n})}$ here (namely that $\varphi_{(\mathscr{S},\boldsymbol{n})}$ is uniformly bounded with respect to $\mathscr{S}$), the conclusion above was already obtained by one of us in 1957 (see [N1]). If one assumes that the values of $\tau_{\mathscr{P}}(x)$ of the functions $\tau_{\mathscr{P}}$ of Assumption II of Section 5 depend on x and $\mathscr{P}$ only through x, $\boldsymbol{v}_{\mathscr{P}}(x)$, and the tangential gradient of $\boldsymbol{v}_{\mathscr{P}}$ at x and if one also assumes that this dependence is of class C^1, then the conclusion can also be obtained from the recent result of FOSDICK & VIRGA [FV]. □

It follows from (7.3) that, under the present hypotheses, the function $\boldsymbol{G}$ of Assumption III of Section 6 must satisfy

$$\boldsymbol{G}(\cdot, -\boldsymbol{u}) = -\boldsymbol{G}(\cdot, \boldsymbol{u}) \quad \text{for all } \boldsymbol{u} \in \mathscr{U}. \tag{7.11}$$

Putting the results above together with Proposition 5 of Section 5, we obtain the following conclusion

Theorem I. *Let I be a quasi-balanced contact interaction for which Assumptions I-IV are satisfied. Then there is a function*

$$\boldsymbol{G}: \ \mathrm{Clo}\,\mathscr{B} \times \mathrm{Usph}\,\mathscr{V} \to \mathrm{Lin}\,(\mathscr{V}, \mathscr{W}) \tag{7.12}$$

of class C^1 *satisfying* (7.11) *and a continuous function*

$$\boldsymbol{F}: \ \mathscr{B} \to \mathrm{Lin}\,(\mathscr{V}, \mathscr{W}) \tag{7.13}$$

such that, for any two disjoint sections $\mathscr{P}$ and $\mathscr{Q}$ of $\mathscr{B}$ in biregular contact, we have

$$I(\mathscr{P}, \mathscr{Q}) = \sum_{\mathscr{S} \in \mathfrak{P}_2 \cap \mathfrak{Q}_2} \int_{\mathscr{S}} (\boldsymbol{F}\boldsymbol{v}_{\mathscr{P}} - \mathrm{div}_{\mathscr{S}}\, \boldsymbol{G}(1_{\mathscr{S} \subset \mathscr{E}}, \boldsymbol{v}_{\mathscr{P}}|_{\mathscr{S}}))\, da$$
$$+ \sum_{\mathscr{C} \in \mathfrak{P}_1 \cap \mathfrak{Q}_1} \int_{\mathscr{C}} \Bigl(\sum_{\mathscr{S} \in \mathfrak{K}(\mathscr{C})} \boldsymbol{G}\,(x, \boldsymbol{v}_{\mathscr{P};\mathscr{S}}(x))\, \boldsymbol{v}_{\mathscr{S}}(x) \Bigr)\, dl_x, \tag{7.14}$$

whenever $\mathfrak{P}$ and $\mathfrak{Q}$ *are appropriate biregular partitions of $\mathscr{P}$ and $\mathscr{Q}$, and when $\mathfrak{K}(\mathscr{C})$ for each $\mathscr{C} \in \mathfrak{P}_1 \cap \mathfrak{Q}_1$, is defined according to Definition 1 of Section 5.*

Remark 2. If an interaction I satisfies the hypotheses of Theorem I, it cannot be a pure edge interaction as defined at the end of Section 5. To see this, assume that I is a pure edge interaction and that $x \in \mathscr{B}$ is given. Then, by (7.1) and (7.10),

$$0 = \sigma_{(\mathscr{S},\boldsymbol{n})}(x) = \boldsymbol{F}(x)\, \boldsymbol{n}(x) - \varrho_{(\mathscr{S},\boldsymbol{n})}(x) \tag{7.15}$$

for all oriented surfaces $(\mathscr{S}, \boldsymbol{n})$ with $x \in \mathscr{S}$ and $\mathscr{S} \subset \mathscr{B}$. If (7.15) holds, then $\varrho_{(\mathscr{S},\boldsymbol{n})}(x)$ can depend on $(\mathscr{S}, \boldsymbol{n})$ only through the value $\boldsymbol{n}(x)$. But (6.13) shows that this can be the case only when $\boldsymbol{G}(x, \cdot)$ has a zero gradient, *i.e.* only when $\boldsymbol{G}(x, \boldsymbol{u})$ does not depend on $\boldsymbol{u} \in \mathscr{U}$. In view of (7.11) it follows that $\boldsymbol{G}(x, \cdot) = 0$. Since $x \in \mathscr{B}$ was arbitrary, $\boldsymbol{G}$ must be zero. □

The following result follows from Propositions 1 and 2 of Section 6 and from (7.10) and (6.6).

Theorem II. *Let I be a quasi-balanced interaction for which Assumptions I–IV are satisfied and let $\boldsymbol{G}$ and $\boldsymbol{F}$ be the functions described in Theorem I. For every section $\mathscr{P}$ of $\mathscr{B}$ and every appropriate biregular partition $\mathfrak{P}$ for $\mathscr{P}$ we then have*

$$I(\mathscr{P}, \mathscr{P}^b) = \sum_{\mathscr{S} \in \mathfrak{P}_2^{\mathrm{int}}} \int_{\mathscr{S}} \boldsymbol{F}\boldsymbol{v}_{\mathscr{P}}\, dv + \sum_{\mathscr{S} \in \mathfrak{P}_2^{\mathrm{per}}} \int_{\mathrm{Rbo}\mathscr{S} \cap \mathrm{Bdy}\mathscr{P}^b} \boldsymbol{G}(x, \boldsymbol{v}_{\mathscr{P};\mathscr{S}}(x))\, \boldsymbol{v}_{\mathscr{S}}(x)\, dl_x. \tag{7.16}$$

In particular, if $\mathscr{P}$ is an interior biregular part of $\mathscr{P}$ we have

$$I(\mathscr{P}, \mathscr{P}^b) = \int_{\mathrm{Rbo}\mathscr{P}} \boldsymbol{F}\boldsymbol{v}_{\mathscr{P}}\, dv = \int_{\mathscr{P}} f\, dv. \tag{7.17}$$

If the function $\boldsymbol{F}: \mathscr{B} \to \operatorname{Lin}(\mathscr{V}, \mathscr{W})$ is of class C^1, it follows from the divergence theorem and the fact that (7.17) holds for all biregular $\mathscr{P} \in \Omega_{\text{int}}$ that

$$f = \operatorname{div} \boldsymbol{F}. \tag{7.18}$$

8. Boundary conditions

We continue to assume that an interaction I satisfying the hypotheses of Theorems I and II of the preceding section is given. We have seen in the preceding section that the field equation (7.18) is the same as it would be if there are no edge interactions, *i.e.* if $\boldsymbol{G} = 0$. The presence of edge interaction becomes manifest only when suitable boundary conditions are imposed. Physically, such boundary conditions describe the action of the environment on the boundary of the body $\mathscr{B}$ under consideration. We consider only the case when the action of the environment is a pure surface action. *i.e.* when there are no edge interactions between $\mathscr{B}$ and the environment. Thus, we strengthen our assumption that I be quasi-balanced as follows:

Assumption V. *There are continuous functions* $f: \operatorname{Clo} \mathscr{B} \to \mathscr{W}$ *and* $h: \operatorname{Rbo} \mathscr{B} \to \mathscr{W}$ *such that*

$$I(\mathscr{P}, \mathscr{P}^b) = \int_{\mathscr{P}} f\, dv + \int_{\operatorname{Clo}\mathscr{P} \cap \operatorname{Rby}\mathscr{B}} h\, da \tag{8.1}$$

for all parts $\mathscr{P} \in \Omega$.

The function h describes the surface action of the environment on $\mathscr{B}$. Proposition 1 of Section 4 can now be strengthened as follows:

Proposition 1. *If Assumption V is valid, then I is skew in the sense that*

$$I(\mathscr{P}, \mathscr{Q}) = -I(\mathscr{Q}, \mathscr{P}) \quad \textit{for all } \mathscr{P}, \mathscr{Q} \in (\Omega^2)_{\text{dis}}. \tag{8.2}$$

The proof follows again from Theorem A, p. 76, of [N2] because $(\mathscr{P} \mapsto I(\mathscr{P}, \mathscr{P}^b)) : \Omega \to \mathscr{W}$ is additive when (8.1) holds.

Theorem III. *Let I be an interaction for which Assumptions I–V are valid and let* $\boldsymbol{G}$ *be the function determined by I as described in Theorem I. If* $\mathfrak{P}$ *is a biregular partition for* $\mathscr{B}$ *and* $\mathscr{C} \in \mathfrak{P}_1$ *is an edge of* $\mathscr{B}$, *we have*

$$\boldsymbol{G}(x, \boldsymbol{v}_{\mathscr{B};\mathscr{S}_1}(x))\, \boldsymbol{v}_{\mathscr{S}_1}(x) + \boldsymbol{G}(x, \boldsymbol{v}_{\mathscr{B};\mathscr{S}_2}(x))\, \boldsymbol{v}_{\mathscr{S}_2}(x) = 0 \quad \textit{for all } x \in \mathscr{C}, \tag{8.3}$$

when $\mathscr{S}_1, \mathscr{S}_2 \in \mathfrak{P}_2$ *are the two sides adjacent to the edge* $\mathscr{C}$.

Proof. If $\mathscr{C}$ is a cusped edge of $\mathscr{B}$, we have $\boldsymbol{v}_{\mathscr{S}_1} = \boldsymbol{v}_{\mathscr{S}_2}$ and $\boldsymbol{v}_{\mathscr{B};\mathscr{S}_1} = -\boldsymbol{v}_{\mathscr{B};\mathscr{S}_2}$ and (8.3) is valid because of (7.11). Assume, then, that $\mathscr{C}$ is not a cusped edge of $\mathscr{B}$ and let $x \in \mathscr{C}$ be given. For every curve $\mathscr{C}'$ with $x \in \mathscr{C}'$ and $\mathscr{C}' \subset \mathscr{C}$, we can construct disjoint sections $\mathscr{P}_1$ and $\mathscr{P}_2$ of $\mathscr{B}$ with the following properties: (i) $\mathscr{P}_1$ and $\mathscr{P}_2$ are in simple edge contact along $\mathscr{C}'$, (ii) one of the sides of $\mathscr{P}_1$ adjacent to $\mathscr{C}'$, say $\mathscr{S}'_1$, is included in $\mathscr{S}_1$, (iii) one of the sides of $\mathscr{P}_2$ adjacent to $\mathscr{C}'$, say $\mathscr{S}'_2$, is included in $\mathscr{S}_2$, and (iv) the internal sides of $\mathscr{P}_1$ and $\mathscr{P}_2$ that are adjacent to $\mathscr{C}'$ are not in cusped contact. Figure 1 illustrates this situation by showing a cross-section orthogonal to $\mathscr{C}$. It follows from Definition 1 of Section 5 that

$$\mathfrak{K}_{\mathscr{P}_1,\mathscr{P}_2}(\mathscr{C}') = \{\mathscr{S}'_1\}, \qquad \mathfrak{K}_{\mathscr{P}_2,\mathscr{P}_1}(\mathscr{C}') = \{\mathscr{S}'_2\}.$$

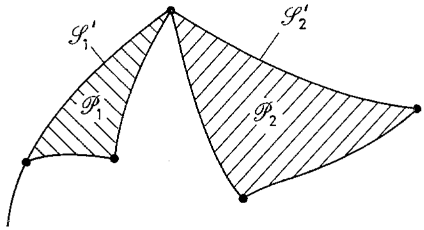

Figure 1

Since $I(\mathscr{P}_1, \mathscr{P}_2) = -I(\mathscr{P}_2, \mathscr{P}_1)$ by Proposition 1 above, it follows from Assumption I of Section 5 that

$$\int_{\mathscr{C}'} \gamma(z, \boldsymbol{v}_{\mathscr{B};\mathscr{S}_1}(z), \boldsymbol{v}_{\mathscr{S}_1}(z))\, dl_z = -\int_{\mathscr{C}'} \gamma(z, \boldsymbol{v}_{\mathscr{B};\mathscr{S}_2}(z), \boldsymbol{v}_{\mathscr{S}_2}(z))\, dl_z \tag{8.4}$$

because $\boldsymbol{v}_{\mathscr{B};\mathscr{S}_1}|_{\mathscr{C}'} = \boldsymbol{v}_{\mathscr{P}_1;\mathscr{S}'_1}|_{\mathscr{C}'}$, $\boldsymbol{v}_{\mathscr{B};\mathscr{S}_2}|_{\mathscr{C}'} = \boldsymbol{v}_{\mathscr{P}_2;\mathscr{S}'_2}|_{\mathscr{C}'}$, $\boldsymbol{v}_{\mathscr{S}'_1}|_{\mathscr{C}'} = \boldsymbol{v}_{\mathscr{S}_1}|_{\mathscr{C}'}$ and $\boldsymbol{v}_{\mathscr{S}'_2}|_{\mathscr{C}'} = \boldsymbol{v}_{\mathscr{S}_2}|_{\mathscr{C}'}$. Since the curve $\mathscr{C}'$ satisfying $x \in \mathscr{C}'$ and $\mathscr{C}' \subset \mathscr{C}$ was arbitrary, it follows from (8.4) and the continuity of γ that

$$\gamma(x, \boldsymbol{v}_{\mathscr{B};\mathscr{S}_1}(x), \boldsymbol{v}_{\mathscr{S}_1}(x)) = -\gamma(x, \boldsymbol{v}_{\mathscr{B};\mathscr{S}_2}(x), \boldsymbol{v}_{\mathscr{S}_2}(x)).$$

In view of Assumption III, (6.6), the desired result (8.3) follows. □

If $\mathscr{C}$ is not a genuine edge, *i.e.* if $\boldsymbol{v}_{\mathscr{S}_1}(x) = -\boldsymbol{v}_{\mathscr{S}_2}(x)$ for all $x \in \mathscr{C}$, then $\boldsymbol{v}_{\mathscr{B};\mathscr{S}_1}(x) = \boldsymbol{v}_{\mathscr{B};\mathscr{S}_2}(x) = \boldsymbol{v}_{\mathscr{B}}(x)$ for all $x \in \mathscr{C}$ and hence (8.3) is trivially satisfied.

Theorem IV. *Let I be an interaction for which Assumptions I–V are satisfied and which therefore determines the functions **F** and **G** as described in Theorem I. For all regular surfaces $\mathscr{S}$ included in the boundary of the body $\mathscr{B}$ we then have*

$$\lim_{y \to x} (\boldsymbol{F}(y)\, \boldsymbol{v}_{\mathscr{B}}(x)) = \varrho_{(\mathscr{S}, \boldsymbol{v}_{\mathscr{B}}|_{\mathscr{S}})}(x) - h(x) \quad \textit{for all } x \in \mathscr{S}, \tag{8.5}$$

where $\varrho_{(\mathscr{S}, \boldsymbol{v}_{\mathscr{B}}|_{\mathscr{S}})}$ is given by (6.11) *or* (6.13).

Proof. Let a regular surface $\mathscr{S}$ included in Bdy $\mathscr{B}$ and $x \in \mathscr{S}$ be given. It is not hard to construct a section $\mathscr{P}$ of $\mathscr{B}$ with an appropriate biregular partition $\mathfrak{P} := \{\mathscr{P}, \mathscr{T}, \mathscr{T}', \mathscr{C}\}$ for $\mathscr{P}$ such that (i) $\mathscr{T}$ is an external side of $\mathscr{P}$ with $x \in \mathscr{T}$ and Clo $\mathscr{T} \subset \mathscr{S}$. (ii) $\mathscr{T}'$ is an internal side of $\mathscr{P}$, and (iii) $\mathscr{C} = \text{Bo}\, \mathscr{T} = \text{Bo}\, \mathscr{T}'$ is the only edge of $\mathscr{P}$ and $\mathscr{C}$ is a cusped edge (see Figure 2). Furthermore, $\mathscr{P}$ can be constructed in such a way that $\mathscr{P}$ is included in an arbitrarily small neighborhood of x and the values of $\boldsymbol{v}_{\mathscr{P}}|_{\mathscr{T}'}$ differ arbitrarily little from $-\boldsymbol{v}_{\mathscr{B}}(x)$. Since $\mathfrak{P}_2^{\text{int}} = \{\mathscr{T}'\}$, $\mathfrak{P}_2^{\text{per}} = \{\mathscr{T}\}$ and Rbo $\mathscr{T} \subset \text{Bdy}\, \mathscr{P}^b$, we see that the formula (7.16) of Theorem II gives

$$I(\mathscr{P}, \mathscr{P}^b) = \int_{\mathscr{T}'} \boldsymbol{F} \boldsymbol{v}_{\mathscr{P}}\, da + \int_{\text{Rbo}\mathscr{T}} \boldsymbol{G}(z, \boldsymbol{v}_{\mathscr{P};\mathscr{T}}(z))\, \boldsymbol{v}_{\mathscr{S}}(z)\, dl_z. \tag{8.6}$$

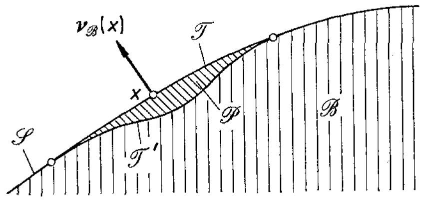

Figure 2

Using the corollary to the surface-divergence theorem stated in Section 2, the notation (6.11), and the fact that $\boldsymbol{v}_{\mathscr{P};\mathscr{T}} = \boldsymbol{v}_{\mathscr{B}}|_{\text{Clo}\,\mathscr{T}}$, it follows from (8.6) that

$$I(\mathscr{P}, \mathscr{P}^b) = \int_{\mathscr{T}'} \boldsymbol{F}\, \boldsymbol{v}_{\mathscr{P}}\, da + \int_{\mathscr{T}} \varrho_{(\mathscr{T}, \boldsymbol{v}_{\mathscr{B}}|_{\mathscr{T}})}\, da. \tag{8.7}$$

On the other hand, since $\text{Clo}\,\mathscr{P} \cap \text{Rby}\,\mathscr{B} = \text{Clo}\,\mathscr{T}$, it follows from (8.1) that

$$I(\mathscr{P}, \mathscr{P}^b) = \int_{\mathscr{P}} f\, dv + \int_{\mathscr{T}} h\, da$$

and hence, by (8.7), that

$$\int_{\mathscr{T}'} \boldsymbol{F} \boldsymbol{v}_{\mathscr{P}}\, da = \int_{\mathscr{T}} (h - \varrho_{(\mathscr{T}, \boldsymbol{v}_{\mathscr{B}}|_{\mathscr{T}})})\, da + \int_{\mathscr{P}} f\, dv. \tag{8.8}$$

Since $\boldsymbol{v}_{\mathscr{P}}|_{\mathscr{T}'}$ differs arbitrarily little from $-\boldsymbol{v}_{\mathscr{B}}(x)$, we obtain the desired result (8.5) by dividing (8.8) by the area of $\mathscr{T}$ and then taking the limit as $\mathscr{P}$ shrinks to $\{x\}$. □

9. Examples, surface tension

As a special example, we now consider the case when the function $\boldsymbol{G}$ of Assumption III, Section 6 can be expressed in the form (6.7) when $\boldsymbol{G}'(x, \boldsymbol{u})$ depends *linearly* on $\boldsymbol{u}$. More precisely, we assume that there is a function

$$\boldsymbol{A}:\ \text{Clo}\,\mathscr{B} \to \text{Lin}\,(\mathscr{V}, \text{Lin}\,(\mathscr{V}, \mathscr{W})) \tag{9.1}$$

of class C^1 such that $\boldsymbol{G}:\ \text{Clo}\,\mathscr{B} \times \mathscr{U} \to \text{Lin}\,(\mathscr{V}, \mathscr{W})$ is given by

$$\boldsymbol{G}(x, \boldsymbol{u}) = (\boldsymbol{A}(x)\, \boldsymbol{u})\, (\boldsymbol{1}_{\mathscr{V}} - \boldsymbol{u} \otimes \boldsymbol{u}) \quad \text{for all } x \in \text{Clo}\,\mathscr{B},\quad \boldsymbol{u} \in \mathscr{U}. \tag{9.2}$$

Let an oriented regular surface $(\mathscr{S}, \boldsymbol{n})$ with $\mathscr{S} \subset \text{Clo}\,\mathscr{B}$ be given. If we substitute (9.2) into (6.11), we obtain

$$\varrho_{(\mathscr{S}, \boldsymbol{n})} = \text{div}_{\mathscr{S}}\, ((\boldsymbol{A}\,|_{\mathscr{S}}\, \boldsymbol{n})\, (\boldsymbol{1}_{\mathscr{V}} - \boldsymbol{n} \otimes \boldsymbol{n})). \tag{9.3}$$

In view of (2.6), (9.3) is equivalent to

$$\omega \varrho_{(\mathscr{S}, \boldsymbol{n})} = \text{div}_{\mathscr{S}}\, ((\boldsymbol{1}_{\mathscr{V}} - \boldsymbol{n} \otimes \boldsymbol{n})\, (\omega \boldsymbol{A}\,|_{\mathscr{S}}\, \boldsymbol{n})) \quad \text{for all } \omega \in \mathscr{W}^*, \tag{9.4}$$

where $\omega \boldsymbol{A} \in \text{Lin}\,(\mathscr{V}, \mathscr{V}^*) \cong \text{Lin}\,\mathscr{V}$ is characterized by

$$((\omega \boldsymbol{A})\, \boldsymbol{u}) \cdot \boldsymbol{v} = \omega((\boldsymbol{A}\boldsymbol{u})\, \boldsymbol{v}) \quad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in \mathscr{V}, \omega \in \mathscr{W}^*. \tag{9.5}$$

Now let $\omega \in \mathscr{W}^*$ be given and put

$$\boldsymbol{B} := \omega \boldsymbol{A}\,|_{\mathscr{S}}:\ \mathscr{S} \to \text{Lin}\,\mathscr{V} \tag{9.6}$$

so that (9.4) gives

$$\omega \varrho_{(\mathscr{S},n)} = \operatorname{div}_{\mathscr{S}} (Bn - (n \cdot Bn)\, n). \tag{9.7}$$

In order to evaluate the right side of (9.7) we observe that the surface divergence satisfies rules that are analogous to the rules stated in Section 67 of [FDSI] for the ordinary divergence. In these rules, ordinary gradients must be replaced by surface-gradients defined according to Definition 1 of Section 2. The analogue of Proposition 2 of Section 67 gives

$$\operatorname{div}_{\mathscr{S}} (Bn) = n \cdot \operatorname{div}_{\mathscr{S}} (B^T) + \operatorname{tr} (B^T \nabla n). \tag{9.8}$$

The analogue of Proposition 1 of Section 67 gives

$$\operatorname{div}_{\mathscr{S}} ((n \cdot Bn)\, n) = (n \cdot Bn) \operatorname{div}_{\mathscr{S}} n + \nabla(n \cdot Bn) \cdot n. \tag{9.9}$$

Now, if $\eta : \mathscr{S} \to \mathbb{R}$ is of class C^1, then $\nabla\eta$ is tangential (see Definition 3 of Section 2) by (2.2), which means that $(\nabla\eta) \cdot n = 0$. Hence the second term on the right side of (9.9) is zero. Therefore, using (9.9), (9.8), and (9.6), we see that (9.7) yields

$$\omega \varrho_{(\mathscr{S},n)} = n \cdot \operatorname{div}_{\mathscr{S}} (B^T) + \operatorname{tr} (B^T \nabla n) - (n \cdot Bn) \operatorname{div}_{\mathscr{S}} n. \tag{9.10}$$

Given $x \in \mathscr{S}$, the lineon $\bar{L} = -\nabla_x n$ has the following significance (see Chapter 3 of [FDSII]): $\bar{L}$ is symmetric, one of its spectral values is zero and the other two are the principal curvatures of $\mathscr{S}$ at x. The mean curvature of $\mathscr{S}$ at x is given by

$$\mathrm{H}_{\mathscr{S}}(x) := \tfrac{1}{2} \operatorname{tr} \bar{L} = -\tfrac{1}{2} \operatorname{tr} \nabla_x n = -\tfrac{1}{2} (\operatorname{div}_{\mathscr{S}} n)(x) \tag{9.11}$$

($\bar{L}$ is closely related to what are often called the "Weingarten map" and the "Second Fundamental Form"). Using (9.11), (9.6), and the fact that ∇n has values in $\operatorname{Sym} \mathscr{V}$, we see that (9.10) becomes

$$\omega \varrho_{(\mathscr{S},n)} = n \cdot \operatorname{div}_{\mathscr{S}} ((\omega A)^T|_{\mathscr{S}}) + \operatorname{tr} (\nabla n (\omega A|_{\mathscr{S}})) + 2\mathrm{H}_{\mathscr{S}} (n \cdot (\omega A|_{\mathscr{S}})\, n), \tag{9.12}$$

valid for all $\omega \in \mathscr{W}^*$. The use of (9.12) leads to a more explicit form for the boundary condition (8.5).

If we substitute (9.2) into (8.3) and observe that $v_{\mathscr{S}_1}(x) \cdot v_{\mathscr{B};\mathscr{S}_1}(x) = 0 = v_{\mathscr{S}_2}(x) \cdot v_{\mathscr{B};\mathscr{S}_2}(x)$ for all $x \in \mathscr{C}$ we see that the boundary-edge condition (8.3) becomes

$$(A(x)\, v_{\mathscr{B};\mathscr{S}_1}(x))\, v_{\mathscr{S}_1}(x) + (A(x)\, v_{\mathscr{B};\mathscr{S}_2}(x))\, v_{\mathscr{S}_2}(x)) = 0 \quad \text{for all } x \in \mathscr{C} \tag{9.13}$$

when G has the form (9.2).

Remark 1. If we substitute (9.12) into the boundary condition (8.5), we obtain a result that is consistent with one of boundary conditions that Toupin obtained in his treatment of non-simple elastic materials (see equations (10.9) or (10.14) on p. 102 or [To2] or equation (7.9) on p. 402 of [To1]). The boundary-edge condition (9.13) is also consistent with a condition found by Toupin (see (7.11) on p. 402 of [To1]). □

We now specialize further. We assume that the space $\mathscr{W}$ coincides with $\mathscr{V}$ and that the function A of (9.1) is defined by

$$A(x)\, u := (k(x) \cdot u)\, 1_{\mathscr{V}} \quad \text{for all } x \in \operatorname{Clo} \mathscr{B},\, u \in \mathscr{V} \tag{9.14}$$

in terms of a given function

$$\boldsymbol{k} : \mathrm{Clo}\,\mathscr{B} \to \mathscr{V} \tag{9.15}$$

of class C^1. Let $\boldsymbol{w} \in \mathscr{V}$ be given, so that $\boldsymbol{w}\cdot \in \mathscr{V}^*$ (see the explanation of the identification $\mathscr{V} \cong \mathscr{V}^*$ in Section 41, p. 134, of [FDSI]). We then have

$$\boldsymbol{w} \cdot \boldsymbol{A} = \boldsymbol{w} \otimes \boldsymbol{k}, \qquad (\boldsymbol{w} \cdot \boldsymbol{A})^T = \boldsymbol{k} \otimes \boldsymbol{w}. \tag{9.16}$$

Substitution of (9.16) and $\omega := \boldsymbol{w}\cdot$ into (9.10) gives

$$\boldsymbol{w}\cdot \varrho_{(\mathscr{S},\boldsymbol{n})} = \boldsymbol{n} \cdot \mathrm{div}_{\mathscr{S}} (\boldsymbol{k}|_{\mathscr{S}} \otimes \boldsymbol{w}) + \mathrm{tr}\,(\nabla \boldsymbol{n}(\boldsymbol{w} \otimes \boldsymbol{k}|_{\mathscr{S}})) + 2\mathrm{H}_{\mathscr{S}}(\boldsymbol{k}|_{\mathscr{S}} \cdot \boldsymbol{n})\,(\boldsymbol{n} \cdot \boldsymbol{w}).$$

Using analogues of the rules (67.10) and (66.9) of [FDSI] we obtain

$$\boldsymbol{w}\cdot \varrho_{(\mathscr{S},\boldsymbol{n})} = \nabla(\boldsymbol{k}|_{\mathscr{S}} \cdot \boldsymbol{n}) \cdot \boldsymbol{w} + 2\mathrm{H}_{\mathscr{S}}(\boldsymbol{k}|_{\mathscr{S}} \cdot \boldsymbol{n})\,(\boldsymbol{n} \cdot \boldsymbol{w}).$$

Since this equation is valid for all $\boldsymbol{w} \in \mathscr{V}$, we conclude that

$$\varrho_{(\mathscr{S},\boldsymbol{n})} = \nabla \sigma_{\mathscr{S}} + 2\mathrm{H}_{\mathscr{S}}\,\sigma_{\mathscr{S}}\boldsymbol{n} \qquad \text{where } \sigma_{\mathscr{S}} := \boldsymbol{k}|_{\mathscr{S}} \cdot \boldsymbol{n}. \tag{9.17}$$

If (9.14) and hence (9.17) is valid and if the function $\boldsymbol{F} : \mathscr{B} \to \mathrm{Lin}\,\mathscr{V}$ has a continuous extension (also denoted by $\boldsymbol{F}$) to Clo $\mathscr{B}$, then the boundary condition (8.5) becomes

$$\boldsymbol{F}\boldsymbol{v}_{\mathscr{B}}|_{\mathscr{S}} = \nabla \sigma_{\mathscr{S}} + 2\mathrm{H}_{\mathscr{S}}\sigma_{\mathscr{S}}\boldsymbol{v}_{\mathscr{B}|\mathscr{S}} - \boldsymbol{h}|_{\mathscr{S}}. \tag{9.18}$$

We now assume, in addition, that $\boldsymbol{F}$ has the form $\boldsymbol{F} := -p\mathbf{1}_{\mathscr{V}}$ for a given

$$p : \mathrm{Clo}\,\mathscr{B} \to \mathbb{R} \tag{9.19}$$

and that $\boldsymbol{h} = -p_0\boldsymbol{v}_{\mathscr{B}}$ for a given

$$p_0 : \mathrm{Rbo}\,\mathscr{B} \to \mathbb{R}. \tag{9.20}$$

In this case, (19.8) reduces to

$$(p_0 - p)\,\boldsymbol{v}_{\mathscr{B}}|_{\mathscr{S}} = \nabla \sigma_{\mathscr{S}} + 2\mathrm{H}_{\mathscr{S}}\sigma_{\mathscr{S}}\boldsymbol{v}_{\mathscr{B}}|_{\mathscr{S}}. \tag{9.21}$$

Since $\nabla \sigma_{\mathscr{S}}$ is tangential by (2.2), it follows from (9.21) that $\nabla \sigma_{\mathscr{S}} = 0$ and

$$p_0 - p = 2\mathrm{H}_{\mathscr{S}}\sigma_{\mathscr{S}}, \qquad \sigma_{\mathscr{S}} = \text{constant}. \tag{9.22}$$

Remark 2. (9.22) is the classical formula for surface tension in fluids. It has been derived here *without* assuming that the interface between different fluids behaves like a material skin. The classical derivations of (9.22) are based on the assumption that Bdy $\mathscr{B}$ is a "material surface" in some sense. A rigorous modern treatment of such material surfaces is given in [GM], which also contains the derivation of a formula similar to (9.18). □

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