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Quantization Effects for $-\Delta u = u(1-|u|^2)$ *in* $\mathbb{R}^2$

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1. Introduction

The study of the vortices associated with the Ginzburg-Landau energy (see [1])

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_\Omega (|u|^2-1)^2,$$

where $\Omega \subset \mathbb{R}^2$ and $u:\Omega \to \mathbb{C}$, leads in a natural way (after scaling) to the equation

$$-\Delta u_\varepsilon = u_\varepsilon(1-|u_\varepsilon|^2) \quad \text{in } \Omega_\varepsilon = \frac{1}{\varepsilon}\Omega. \tag{1.1}$$

In the first part of this paper we study the limiting situation where u satisfies

$$-\Delta u = u(1-|u|^2) \quad \text{in } \mathbb{R}^2. \tag{1.2}$$

Our main result is

Theorem 1. *Assume* $u:\mathbb{R}^2 \to \mathbb{C}$ *is a smooth function satisfying* (1.2). *Then*

$$\int_{\mathbb{R}^2} (|u|^2-1)^2 = 2\pi d^2 \tag{1.3}$$

for some integer $d = 0, 1, 2, \ldots, \infty$.

Remark 1.1. If $d < \infty$, then one can prove (see Step 1 in Section 2) that $|u(x)| \to 1$ as $|x| \to \infty$. Thus $\deg(u, S_R)$ is well-defined for R large, where S_R is a circle of radius R. We shall prove that $|\deg(u, S_R)| = d$ for R large.

Remark 1.2. If $d = 0$, the only solutions of (1.2) are constant functions. This follows easily from Theorem 1 and Liouville's Theorem.

Remark 1.3. For every integer $d = 0, 1, 2, \ldots, \infty$ there is a solution of (1.2) satisfying (1.3):

a) For $d = \infty$ we may choose a function u of the form

$$u(x_1, x_2) = Ae^{ikx_1}$$

where A and k are positive constants such that $A^2 + k^2 = 1$.

b) For every integer $0 < d < \infty$ we may find a solution of (1.2) of the form

$$u(r, \theta) = e^{id\theta} f_d(r) \tag{1.4}$$

where $f(r) = f_d(r)$ satisfies

$$\begin{aligned} -f''(r) - \frac{1}{r} f'(r) + \frac{d^2}{r^2} f = f(1 - f^2) \quad & \text{on } (0, +\infty), \\ f(0) = 0, \quad f(\infty) = 1. & \end{aligned} \tag{1.5}$$

It is well known (see, e.g., the assertion in [3]) that (1.5) admits a unique solution. Moreover, we have

$$\begin{aligned} f(r) &= 1 - \frac{d^2}{2r^2} + o\left(\frac{1}{r^2}\right) \quad \text{as } r \to \infty, \\ f'(r) &= \frac{d^3}{r^3} + o\left(\frac{1}{r^3}\right) \quad \text{as } r \to \infty. \end{aligned} \tag{1.6}$$

Multiplying (1.5) by $r^2 f'$ and integrating the result over $(0, R)$ we are led to

$$-\frac{1}{2} R^2 f'(R)^2 + \frac{d^2}{2} f^2(R) = \frac{1}{2} \int_0^R (1 - f^2)^2 \, r \, dr - \frac{R^2}{4} (1 - f^2(R))^2,$$

which implies (1.3).

In connection with this result we call attention to the very interesting

Open Problem 1: Let u be a solution of (1.2) satisfying (1.3) with $d < \infty$. Is u of the form (1.4) (up to a rotation and translation)?

Remark 1.4. Theorem 1 extends to more general nonlinearities (see Theorem 1′). For example, any solution of the equation

$$-\Delta u = u(1 - |u|^2)^3$$

satisfies

$$\int_{\mathbb{R}^2} (|u|^2 - 1)^4 = 4\pi d^2.$$

In the second part of this paper we return to (1.1) and consider a sequence u_n of solutions of

$$-\Delta u_n = u_n(1 - |u_n|^2) \quad \text{in the disc } B_{R_n} \tag{1.7}$$

with $R_n \to \infty$. Under some appropriate assumptions (see Section 4) we prove that

$$\frac{1}{2} \int_{B_{R_n}} |\nabla u_n|^2 \geqq \pi d^2 \log R_n - C. \tag{1.8}$$

Theorem 1 is used in [1] and estimate (1.8) is related to lower bounds in [1].

2. Proof of Theorem 1

Let u be a smooth function defined on $\mathbb{R}^2$, with values in $\mathbb{R}^2 \cong \mathbb{C}$, satisfying

$$-\Delta u = u(1-|u|^2) \quad \text{in } \mathbb{R}^2. \tag{2.1}$$

We assume that

$$\int_{\mathbb{R}^2} (|u|^2-1)^2 < \infty. \tag{2.2}$$

Step 1. We assert that

$$|u| \leqq 1 \quad \text{in } \mathbb{R}^2, \tag{2.3}$$

$$\nabla u \in L^\infty(\mathbb{R}^2), \tag{2.4}$$

$$|u(x)| \to 1 \quad \text{as } |x| \to \infty, \tag{2.5}$$

$$\int_{B_R} |\nabla u|^2 \leqq CR \text{ for some constant } C \text{ independent of } R. \tag{2.6}$$

Proof. From (2.1) we have

$$\Delta|u|^2 \geqq 2u\,\Delta u = 2|u|^2(|u|^2-1).$$

Set $\varphi = |u|^2-1$, so that $\varphi \in L^2(\mathbb{R}^2)$ and satisfies

$$-\Delta\varphi + 2|u|^2\varphi \leqq 0. \tag{2.7}$$

We now multiple (2.7) by $\zeta_n\varphi^+$ where $\zeta_n(x) = \zeta(x/n)$ and $0 \leqq \zeta \leqq 1$ is a function in $C_c^\infty(\mathbb{R}^2)$ such that $\zeta(x) \equiv 1$ for x near 0. Hence we obtain

$$\begin{aligned}\int_{\mathbb{R}^2} \zeta_n|\nabla\varphi^+|^2 + 2\int_{\mathbb{R}^2} |u|^2\zeta_n|\varphi^+|^2 &\leqq \frac{1}{2}\int_{\mathbb{R}^2} (\Delta\zeta_n)|\varphi^+|^2\\ &\leqq \frac{C}{n^2}\int_{\mathbb{R}^2} |\varphi^+|^2.\end{aligned}$$

As $n \to \infty$ we see that

$$\int_{\mathbb{R}^2} |\nabla\varphi^+|^2 + 2\int_{\mathbb{R}^2} |u|^2|\varphi^+|^2 \leqq 0.$$

Hence φ^+ is constant, say $\varphi^+ = c$. If $c > 0$, we would deduce that $u \equiv 0$, but this is impossible by (2.2). Hence $c = 0$, i.e., $\varphi \leqq 0$. This proves (2.3).

Going back to (2.1) and using the fact that $u \in L^\infty(\mathbb{R}^2)$ together with standard elliptic estimates we obtain (2.4).

We now prove (2.5). Suppose that it were not true, so that there would be a sequence $|x_n| \to \infty$ such that $|u(x_n)| \leqq 1-\delta$ for some $\delta > 0$. Hence $|u(x)| \leqq 1-(\delta/2)$ for $x \in B(x_n, \delta/2M)$ where $M = \|\nabla u\|_{L^\infty}$. Thus, we have

$$\int_{B(x_n,\delta/2M)} (|u|^2-1)^2 \geqq \frac{\delta^2}{4}\cdot\frac{\pi\delta^2}{4M^2}. \tag{2.8}$$

On the other hand, since (2.2) holds, there is some R_0 such that

$$\int_{|x|>R_0} (|u|^2 - 1)^2 < \frac{\delta^2}{4} \cdot \frac{\pi\delta^2}{4M^2}. \tag{2.9}$$

Since $|x_n| \to \infty$, this yields a contradiction.

Finally, we prove (2.6). We multiply (2.1) by u and integrate the result over B_R:

$$\int_{B_R} |\nabla u|^2 = \int_{S_R} \frac{\partial u}{\partial \nu} u + \int_{B_R} |u|^2 (1 - |u|^2) \tag{2.10}$$

where $S_R = \partial B_R$ and ν denotes the outward normal to B_R. Note that

$$\left| \int_{S_R} \frac{\partial u}{\partial \nu} u \right| \leqq 2M\pi R, \tag{2.11}$$

$$\int_{B_R} |u|^2 (1 - |u|^2) \leqq \int_{B_R} (1 - |u|^2) \leqq \sqrt{\pi}\, R \left[\int_{B_R} (1 - |u|^2)^2 \right]^{1/2}. \tag{2.12}$$

Combining (2.2), (2.10), (2.11) and (2.12) we obtain (2.6).

Step 2. From (2.5) we deduce that

$$|u(x)| \geqq \tfrac{3}{4} \quad \text{for } |x| = R \geqq R_0. \tag{2.13}$$

Hence

$$d = \deg(u, S_R)$$

is well-defined for $R \geqq R_0$ and is independent of R. Without loss of generality we may assume that $d \geqq 0$ (the general case follows by complex conjugation). Clearly, there exists a smooth real-valued function $\psi(x)$ (which is single-valued), defined for $|x| \geqq R_0$, such that

$$u(x) = |u(x)|\, e^{i(d\theta + \psi(x))} = \rho(x)\, e^{i\varphi(x)} \tag{2.14}$$

where

$$\rho(x) = |u(x)|, \tag{2.15}$$

$$\varphi(x) = d\theta + \psi(x). \tag{2.16}$$

(*Warning*: φ is not well-defined globally as a single-valued function; however, it is well-defined and smooth, locally on the set $|x| \geqq R_0$). A basic estimate is:

Proposition 1.

$$\int_{\mathbb{R}^2 \setminus B_{R_0}} |\nabla\rho|^2 + |\nabla\psi|^2 < \infty. \tag{2.17}$$

Proof. We first express (2.1) in terms of ρ and ψ. Inserting (2.14) into (2.1) we have

$$-\Delta u = -(\Delta\rho)\, e^{i\varphi} - 2i(\nabla\rho\cdot\nabla\varphi)\, e^{i\varphi} - \rho e^{i\varphi}(i\Delta\varphi - |\nabla\varphi|^2)$$
$$= \rho e^{i\varphi}(1-\rho^2).$$

Separating the real and imaginary parts we obtain

(2.18) $$\rho\Delta\varphi + 2\nabla\rho\cdot\nabla\varphi = 0 \quad \text{for } |x| > R_0,$$

(2.19) $$-\Delta\rho + \rho|\nabla\varphi|^2 = \rho(1-\rho^2) \quad \text{for } |x| > R_0.$$

We rewrite (2.18) as

(2.20) $$\operatorname{div}(\rho^2\nabla\varphi) = 0 \quad \text{for } |x| > R_0.$$

Note that, by (2.16),

(2.21) $$\nabla\varphi = d\nabla\theta + \nabla\psi = \frac{d}{r}V + \nabla\psi$$

where $V(x)$ is the vector field in $\mathbb{R}^2\setminus\{0\}$ defined by

$$V(r\cos\theta, r\sin\theta) = (-\sin\theta, \cos\theta).$$

Combining (2.20) and (2.21) we have

(2.22) $$\operatorname{div}\left(\rho^2\left(\frac{d}{r}V + \nabla\psi\right)\right) = 0 \quad \text{for } |x| > R_0.$$

Step 1 (*in the proof of Proposition 1*). For every $R > R_0$ we have

(2.23) $$\int_{S_R} \rho^2 \frac{\partial\psi}{\partial\nu} = 0.$$

Proof. Consider the vector-field

$$D = (u \wedge u_{x_1}, u \wedge u_{x_2})$$

(which is well-defined and smooth on all of $\mathbb{R}^2$). Note that

(2.24) $$\operatorname{div} D = u \wedge \Delta u = 0$$

by (2.1). Integrating (2.24) over B_R we have

(2.25) $$\int_{S_R} D\cdot\nu = 0 \quad \forall R > 0.$$

On the other hand, a direct computation (differentiating (2.14)) shows that

$$D = \rho^2\nabla\varphi \quad \text{for } |x| > R_0.$$

The desired conclusion follows from the fact that $V\cdot\nu = 0$ on S_R.

Step 2 (*in the proof of Proposition 1*). We assert that

$$\int_{\mathbb{R}^2 \setminus B_{R_0}} |\nabla \psi|^2 < \infty .$$

Proof. The main ingredients are (2.20), (2.6), (2.23) and a method suggested by L. NIRENBERG [5] in proving a Liouville-type theorem for uniformly elliptic equations in divergence form. In the Liouville-type situation, the elliptic equation holds on *all* of $\mathbb{R}^2$; here, the equation (2.20) makes sense only on $|x| > R_0$, but we have instead the information (2.23). Set

$$\psi_R = \frac{1}{2\pi R} \int_{S_R} \psi .$$

Multiplying (2.20) by $(\psi - \psi_R)$ and integrating the result over $A_R = B_R \setminus B_{R_0}$ we obtain

$$\int_{A_R} \rho^2 \left(\frac{d}{r} V + \nabla \psi \right) \nabla \psi = \int_{S_R} \rho^2 \left(\frac{d}{r} V \cdot \nu + \frac{\partial \psi}{\partial \nu} \right) (\psi - \psi_R) - \int_{S_{R_0}} \rho^2 \left(\frac{d}{r} V \cdot \nu + \frac{\partial \psi}{\partial \nu} \right) (\psi - \psi_R) . \tag{2.26}$$

Note that $V \cdot \nu = 0$ so that, by (2.23),

$$\int_{S_{R_0}} \rho^2 \left(\frac{d}{r} V \cdot \nu + \frac{\partial \psi}{\partial \nu} \right) (\psi - \psi_R) = \int_{S_{R_0}} \rho^2 \frac{\partial \psi}{\partial \nu} \psi = C \tag{2.27}$$

where C is independent of R.

We also observe that

$$\int_{A_R} \frac{d}{r} V \cdot \nabla \psi = \int_{A_R} \frac{d}{r} \frac{\partial \psi}{\partial \tau} = 0 . \tag{2.28}$$

Combining (2.26), (2.27) and (2.28) we are led to

$$\int_{A_R} \rho^2 |\nabla \psi|^2 \leqq \int_{S_R} \left| \frac{\partial \psi}{\partial \nu} \right| |\psi - \psi_R| + \int_{A_R} (1 - \rho^2) \frac{d}{r} |\nabla \psi| + C . \tag{2.29}$$

By the Cauchy-Schwarz inequality, we have

$$\left| \int_{S_R} \frac{\partial \psi}{\partial \nu} (\psi - \psi_R) \right| \leqq \left[\int_{S_R} \left| \frac{\partial \psi}{\partial \nu} \right|^2 \right]^{1/2} \left(\int_{S_R} |\psi - \psi_R|^2 \right)^{1/2} .$$

Recall the Poincaré inequality:

$$\int_{S_R} |\psi - \psi_R|^2 \leqq R^2 \int_{S_R} |\nabla_\tau \psi|^2. \tag{2.30}$$

[On S_1 this inequality is well known since the second eigenvalue of $-\psi''$ on S_1 is 1; we emphasize that the constant 1 is sharp and that it plays an essential role in the argument. Inequality (2.30) on S_R follows by scaling.] Therefore we obtain

$$\left| \int_{S_R} \frac{\partial \psi}{\partial \nu} (\psi - \psi_R) \right| \leqq \frac{R}{2} \int_{S_R} \left| \frac{\partial \psi}{\partial \nu} \right|^2 + \frac{R}{2} \int_{S_R} |\nabla_\tau \psi|^2 = \frac{R}{2} \int_{S_R} |\nabla \psi|^2. \tag{2.31}$$

Going back to (2.29) and using (2.31) we see that

$$\int_{A_R} \rho^2 |\nabla \psi|^2 \leqq \frac{R}{2} \int_{S_R} |\nabla \psi|^2 + \int_{A_R} (1-\rho^2) \frac{d}{r} |\nabla \psi| + C. \tag{2.32}$$

Finally, we note that

$$\int_{A_R} (1-\rho^2) \frac{d}{r} |\nabla \psi| \leqq \frac{d}{R_0} \left[\int_{A_R} (1-\rho^2)^2 \right]^{1/2} \left[\int_{A_R} |\nabla \psi|^2 \right]^{1/2}. \tag{2.33}$$

Recall (see (2.13)) that, for $|x| \geqq R_0$,

$$\rho^2(x) \geqq \alpha > \tfrac{1}{2} \tag{2.34}$$

($\alpha = \frac{9}{16}$). From (2.32), (2.33) and (2.34) we deduce that

$$\int_{A_R} |\nabla \psi|^2 \leqq \frac{R}{2\alpha} \int_{S_R} |\nabla \psi|^2 + C + C \left[\int_{A_R} |\nabla \psi|^2 \right]^{1/2}. \tag{2.35}$$

For $R \geqq R_0$ set

$$f(R) = \int_{A_R} |\nabla \psi|^2,$$

so that, by (2.35),

$$f(R) \leqq \frac{R}{2\alpha} f'(R) + C + C f(R)^{1/2}. \tag{2.36}$$

The desired conclusion of Step 2 now follows from

Lemma 1. *Any function f satisfying* (2.36) *and*

$$f(R) \leqq CR \quad \forall R \geqq R_0 \tag{2.37}$$

is bounded on $(R_0, +\infty)$.

Proof. From (2.36) it follows easily that

$$f(R) \leqq \frac{R}{\beta} f'(R) + C \tag{2.38}$$

with $\beta > 1$ by (2.34). Set

$$g(R) = f(R) - C,$$

so that

$$g(R) \leqq \frac{R}{\beta} g'(R),$$

and thus

$$(R^{-\beta} g(R))' \geqq 0.$$

We assert that

$$g(R) \leqq 0 \quad \forall R \geqq R_0.$$

Suppose not, so that $g(R_1) > 0$ for some R_1. Then

$$g(R) \geqq \left(\frac{R}{R_1}\right)^{\beta} g(R_1) \quad \forall R \geqq R_1,$$

which is impossible by (2.37).

This completes the proof of Lemma 1 and thereby Step 2.

Step 3 (*in the proof of Proposition 1*).

$$\int_{\mathbb{R}^2 \setminus B_{R_0}} |\nabla \rho|^2 < \infty. \tag{2.39}$$

Proof. We now use (2.19). First observe that

$$\int_{\mathbb{R}^2 \setminus B_{R_0}} (1 - \rho) |\nabla \varphi|^2 < \infty. \tag{2.40}$$

Indeed, we have

$$|\nabla \varphi| = \left| \frac{d}{r} V + \nabla \psi \right| \leqq \frac{d}{r} + |\nabla \psi|,$$

and thus

$$|\nabla \varphi|^2 \leqq 2 \left(\frac{d^2}{r^2} + |\nabla \psi|^2 \right).$$

Inequality (2.40) follows from Step 1 and the fact that $\frac{1}{r^2}(1 - \rho) \in L^1$ by (2.2).

Fix some smooth function η such that

$$\eta(x) = 1 \text{ for } |x| \leqq 1, \qquad \eta(x) = 0 \text{ for } |x| \geqq 2.$$

Set

$$\eta_R(x) = \eta\left(\frac{x}{R}\right).$$

Multiplying (2.19) by $(1-\rho)\eta_R$ and integrating the result over $\mathbb{R}^2 \setminus B_{R_0}$ we are led to

$$(2.41)\quad \int_{\mathbb{R}^2\setminus B_{R_0}} |\nabla\rho|^2 \eta_R \leqq -\frac{1}{2}\int_{\mathbb{R}^2\setminus B_{R_0}} \nabla(1-\rho)^2 \nabla\eta_R + \int_{\mathbb{R}^2\setminus B_{R_0}} (1-\rho)|\nabla\varphi|^2 + 2\int_{\mathbb{R}^2\setminus B_{R_0}} (1-\rho)^2 + \int_{S_{R_0}} \left|\frac{\partial\rho}{\partial\nu}\right|.$$

Note that

$$\left|\int_{\mathbb{R}^2\setminus B_{R_0}} \nabla(1-\rho)^2 \nabla\eta_R\right| = \left|\int_{\mathbb{R}^2\setminus B_{R_0}} (1-\rho)^2 \Delta\eta_R\right| \leqq \frac{C}{R^2},$$

and therefore (by (2.40) and (2.2)) the right-hand side in (2.41) remains bounded as $R \to \infty$. Passing to the limit in (2.41), as $R \to \infty$, we obtain (2.39). This completes the proof of Proposition 1.

Step 3: Completion of the proof of Theorem 1. We assert that

$$(2.42)\quad \int_{\mathbb{R}^2} (\rho^2-1)^2 = 2\pi d^2.$$

The Pohožaev identity applied to (2.1) shows that (see, e.g., [1], proof of Theorem III.3), for every $r > 0$,

$$(2.43)\quad \int_{S_r} \left|\frac{\partial u}{\partial\nu}\right|^2 + \frac{1}{r}\int_{B_r} (|u|^2-1)^2 = \int_{S_r} \left|\frac{\partial u}{\partial\tau}\right|^2 + \frac{1}{2}\int_{S_r} (|u|^2-1)^2.$$

Set

$$E = \int_{\mathbb{R}^2} (|u|^2-1)^2, \quad E(r) = \int_{B_r} (|u|^2-1)^2.$$

Clearly $E(r) \to E$ as $r \to \infty$ and

$$(2.44)\quad \frac{1}{\log R}\int_0^R \frac{E(r)}{r}\,dr \to E \quad \text{as } R \to +\infty.$$

Integrating (2.43) for $r \in (0, R)$ we have

$$(2.45)\quad \int_{B_R} \left|\frac{\partial u}{\partial\nu}\right|^2 + \int_0^R \frac{E(r)}{r}\,dr = \int_{B_R} \left|\frac{\partial u}{\partial\tau}\right|^2 + \frac{1}{2}E(R).$$

Note that, for $r > R_0$,

$$(2.46)\quad \left|\frac{\partial u}{\partial\nu}\right|^2 = \left|\frac{\partial\rho}{\partial\nu}\right|^2 + \rho^2\left|\frac{\partial\varphi}{\partial\nu}\right|^2 \leqq |\nabla\rho|^2 + |\nabla\psi|^2,$$

and

$$\left|\frac{\partial u}{\partial \tau}\right|^2 = \left|\frac{\partial \rho}{\partial \tau}\right|^2 + \rho^2 \left|\frac{\partial \varphi}{\partial \tau}\right|^2 = \left|\frac{\partial \rho}{\partial \tau}\right|^2 + \rho^2 \left(\frac{d}{r} + \frac{\partial \psi}{\partial \tau}\right)^2. \tag{2.47}$$

Hence

$$\left|\left|\frac{\partial u}{\partial \tau}\right|^2 - \frac{d^2}{r^2}\right| \leqq |\nabla \rho|^2 + \frac{(1-\rho^2)\, d^2}{r^2} + \frac{2d}{r}|\nabla \psi| + |\nabla \psi|^2. \tag{2.48}$$

From (2.46) and Proposition 1 we deduce that

$$\int_{B_R} \left|\frac{\partial u}{\partial v}\right|^2 \leqq C \quad \text{as } R \to \infty, \tag{2.49}$$

and similarly that

$$\int_{B_R \setminus B_{R_0}} \left|\left|\frac{\partial u}{\partial \tau}\right|^2 - \frac{d^2}{r^2}\right| \leqq C(\log R)^{1/2} \quad \text{as } R \to \infty.$$

Hence

$$\frac{1}{\log R} \int_{B_R} \left|\frac{\partial u}{\partial \tau}\right|^2 \to 2\pi d^2 \quad \text{as } R \to +\infty. \tag{2.50}$$

Combining (2.44), (2.45), (2.49) and (2.50) we see that

$$E = 2\pi d^2.$$

This completes the proof of Theorem 1.

3. Some Additional Results and Open Problems

3.1. General nonlinearities

Theorem 1 extends to a large class of nonlinear equations. More precisely, let $f: \mathbb{R} \to \mathbb{R}$ be a (smooth) function satisfying

$$f(0) = 0 \quad \text{and} \quad f(t) \operatorname{sign} t \geqq 0 \quad \forall t \in \mathbb{R}, \tag{3.1}$$

$$\begin{gathered}\text{there exist constants } \gamma > 0 \text{ and } p > 0 \\ \text{such that } f(t) \sim \gamma t^p \text{ for } t > 0,\ t \text{ small},\end{gathered} \tag{3.2}$$

$$\begin{gathered}\text{there exist constants } A > 0 \text{ and } \delta > 0 \\ \text{such that } |f(t)| \geqq \delta \text{ for } t \leqq -A.\end{gathered} \tag{3.3}$$

Theorem 1′. *Assume u is a smooth function on $\mathbb{R}^2$ with values into $\mathbb{C}$ satisfying*

$$-\Delta u = u f(1 - |u|^2) \quad \text{in } \mathbb{R}^2, \tag{3.4}$$

$$\int_{\mathbb{R}^2} F(1 - |u|^2) < \infty \tag{3.5}$$

where

$$F(t) = \int_0^t f(s)\,ds$$

and f satisfies (3.1)–(3.3). *Then*

$$\int_{\mathbb{R}^2} F(1-|u|^2) = \pi d^2$$

for some integer $d = 0, 1, 2, \ldots$.

The proof of Theorem 1′ is essentially the same as the proof of Theorem 1 and is omitted.

3.2. Finite-energy solutions of (1.2)

In Theorem 1 we considered solutions of (1.2) satisfying

$$\int_{\mathbb{R}^2} (|u|^2 - 1)^2 < \infty .$$

If we consider instead solutions of (1.2) satisfying

$$\int_{\mathbb{R}^2} |\nabla u|^2 < \infty , \tag{3.6}$$

then u must be a constant function. More precisely, we have

Theorem 2 (Cazenave [2]). *Assume u satisfies* (1.2) *and* (3.6). *Then either*

$$u \equiv 0$$

or

$$u \equiv \text{Const.} = C \quad \textit{with } |C| = 1 .$$

Proof. We divide the proof into several steps.
Step 1. We have

$$|u| \leqq 1 . \tag{3.7}$$

Set $\varphi = (|u| - 1)^+$ so that $\nabla\varphi \in L^2(\mathbb{R}^2)$. By Kato's inequality (see [4]) we have

$$\Delta\varphi \geqq \frac{u}{|u|} \operatorname{sign}^+(|u| - 1)\,\Delta u = |u|\,(|u| + 1)\,\varphi \tag{3.8}$$

by (2.1). We now multiply (3.8) by $\zeta_n(x) = \zeta(x/n)$ where ζ is a fixed function, $0 \leqq \zeta \leqq 1$, $\zeta(x) = 1$ for $|x| \leqq 1$ and $\zeta(x) = 0$ for $|x| \geqq 2$. We find

$$\int_{\mathbb{R}^2} |u|^2 \varphi\zeta_n \leqq \int_{\mathbb{R}^2} |\nabla\varphi|\,|\nabla\zeta_n| \leqq \frac{C}{n} \int_{n<|x|<2n} |\nabla\varphi| \to 0$$

since $\nabla\varphi \in L^2$. Hence, we are led to

$$|u|^2 \varphi \equiv 0,$$

which implies that for every x either $|u(x)| = 0$ or $\varphi(x) = 0$. In both cases we find that $\varphi(x) = 0$ and hence $|u| \leqq 1$ in $\mathbb{R}^2$.

Step 2. Either

$$\int_{\mathbb{R}^2} |u|^2 < \infty \tag{3.9}$$

or

$$\int_{\mathbb{R}^2} (1 - |u|^2) < \infty . \tag{3.10}$$

Proof. Recall that

$$\Delta |u|^2 = 2|\nabla u|^2 + 2|u|^2 (|u|^2 - 1),$$

and thus

$$|u|^2 (1 - |u|^2) = |\nabla u|^2 - \tfrac{1}{2} \Delta |u|^2 .$$

Multiplying by ζ_n as above we deduce that

$$\int_{\mathbb{R}^2} |u|^2 (1 - |u|^2) = \int_{\mathbb{R}^2} |\nabla u|^2 < \infty . \tag{3.11}$$

We assert that

$$B = \{x \in \mathbb{R}^2; \tfrac{1}{4} \leqq |u(x)| \leqq \tfrac{3}{4}\} \text{ is bounded}.$$

This follows easily from (3.11) and the fact that $\nabla u \in L^\infty(\mathbb{R}^2)$. Suppose $B \subset B_{R_0}$. Since $\mathbb{R}^2 \setminus B_{R_0}$ is connected, we deduce that either

$$|u(x)| \leqq \tfrac{1}{4} \quad \text{on } \mathbb{R}^2 \setminus B_{R_0}$$

or

$$|u(x)| \geqq \tfrac{3}{4} \quad \text{on } \mathbb{R}^2 \setminus B_{R_0} .$$

Combining this with (3.11) we obtain the desired conclusion.

Step 3: Completion of the proof of Theorem 2. The main idea is to use the Pohožaev identity. We first assert that

$$\int_{\mathbb{R}^2} (\Delta u) \left(\sum x_i \frac{\partial u}{\partial x_i} \right) \zeta_n \to 0 . \tag{3.12}$$

Indeed, a standard integration by parts yields

$$\left| \int_{\mathbb{R}^2} (\Delta u) \left(\sum x_i \frac{\partial u}{\partial x_i} \right) \zeta_n \right| \leqq C \int_{\mathbb{R}^2} |x| \, |\nabla u|^2 |\nabla \zeta_n|$$

$$\leqq C \int_{n \leqq |x| \leqq 2n} |\nabla u|^2 \to 0 \quad \text{as } n \to \infty .$$

On the other hand, using equation (1.2) together with (3.12) we are led to

$$\int_{\mathbb{R}^2} u(1 - |u|^2) \left(\sum x_i \frac{\partial u}{\partial x_i} \right) \zeta_n \to 0 . \tag{3.13}$$

If (3.9) holds, we write

$$(3.14)\quad \int_{\mathbb{R}^2} u(1-|u|^2)\left(\sum x_i \frac{\partial u}{\partial x_i}\right)\zeta_n = \int_{\mathbb{R}^2} \sum x_i \frac{\partial}{\partial x_i}\left(\tfrac{1}{2}|u|^2 - \tfrac{1}{4}|u|^4\right)\zeta_n$$

$$= -\int_{\mathbb{R}^2} \left(|u|^2 - \tfrac{1}{2}|u|^4\right)\zeta_n + o(1)$$

since

$$\int_{\mathbb{R}^2} |x|\left(\tfrac{1}{2}|u|^2 + \tfrac{1}{4}|u|^4\right)|\nabla\zeta_n| \leqq C \int_{n<|x|<2n} |u|^2 \to 0.$$

Combining (3.13), (3.14) and passing to the limit as $n \to \infty$ we obtain

$$\int_{\mathbb{R}^2} \left(|u|^2 - \tfrac{1}{2}|u|^4\right) = 0,$$

which implies that $u \equiv 0$ (since $|u| \leqq 1$).

If (3.10) holds, we write

$$(3.15)\quad \int_{\mathbb{R}^2} u(1-|u|^2)\left(\sum x_i \frac{\partial u}{\partial x_i}\right)\zeta_n = -\int_{\mathbb{R}^2} \sum x_i \left[\frac{\partial}{\partial x_i}\frac{1}{4}(|u|^2-1)^2\right]\zeta_n$$

$$= \frac{1}{2}\int_{\mathbb{R}^2} (|u|^2-1)^2\zeta_n + o(1)$$

since

$$\int_{\mathbb{R}^2} |x|\,(|u|^2-1)^2|\nabla\zeta_n| \leqq C \int_{n<|x|<2n} (1-|u|^2) \to 0.$$

In this case we conclude that

$$\int_{\mathbb{R}^2} (|u|^2-1)^2 = 0.$$

Returning to (3.11) we see that $\nabla u = 0$ and the conclusion follows.

3.3 Further open problems

Problem 2. Let u be a solution of (1.2) such that $|u| \to 1$ at infinity. Is

$$\int_{\mathbb{R}^2} (|u|^2-1)^2 < \infty\,?$$

We say that a solution u of (1.2) is a *local minimizer* if for every bounded set $\Omega \subset \mathbb{R}^2$ we have

$$\tfrac{1}{2}\int_{\Omega} |\nabla u|^2 + \tfrac{1}{4}\int_{\Omega} (|u|^2-1)^2 \leqq \tfrac{1}{2}\int_{\Omega} |\nabla v|^2 + \tfrac{1}{4}\int_{\Omega} (|v|^2-1)^2$$

for every $v \in H^1(\Omega; \mathbb{C})$ with $v = u$ on $\partial\Omega$.

Problem 3. Prove that $u = e^{i\theta} f_1(r)$ defined by (1.4), (1.5) is a local minimizer.

Problem 4. Prove that every local minimizer u must be either of the form $u = \text{Const.} = C$ with $|C| = 1$ or $u = e^{i\theta} f_1(r)$ (modulo a rotation, a translation and complex conjugation).

Problem 5. Prove (or disprove) that any solution u of (1.2), (1.3) with $d < \infty$ has a single zero. If the answer is negative, what can be said about the configuration of $[u = 0]$? Can any arbitrary finite set (with appropriate prescribed degrees) coincide with $[u = 0]$ for some u?

Problem 6. Prove (or disprove) that any solution u of (1.2), (1.3) (with $d < \infty$) such that 0 is the unique zero of u must be of the form $e^{id\theta} f_d(r)$ (modulo an isometry) as in (1.4), (1.5).

3.4. *Further results*

After our work was completed, I. SHAFRIR [6] proved that any solution u of (1.2), (1.3) with $d < \infty$ satisfies, as $|x| \to \infty$,

(i) $$1 - |u|^2 = \frac{d^2}{|x|^2} + o\left(\frac{1}{|x|^2}\right),$$

(ii) $$|\nabla |u|| = \frac{d^2}{|x|^3} + o\left(\frac{1}{|x|^3}\right),$$

(iii) $$|\Delta |u|| \leqq \frac{2d^2}{|x|^4} + o\left(\frac{1}{|x|^4}\right),$$

(iv) $$|\det(\nabla u)| = O\left(\frac{1}{|x|^4}\right).$$

I. SHAFRIR [6] has also shown that any local minimizer which is not constant must have a single zero of degree ± 1.

4. A Lower Bound for the Energy

Let u be a (smooth) map from $B_R = [|x| < R]$ into $\mathbb{C}$. We assume that

(4.1) $$|u| \leqq 1 \quad \text{in } B_R,$$

(4.2) $$|u(x)| \geqq a \qquad \forall x \in A_{R,R_0} = B_R \setminus B_{R_0},$$

(4.3) $$\frac{1}{R_0^2} \int_{B_R} (|u|^2 - 1)^2 \leqq K$$

for some positive constants a, R_0 and K.

Assumption (4.2) implies that

$$\deg(u, S_r) = d$$

is well-defined and independent of r for $R_0 < r < R$. Since u does not vanish on A_{R,R_0}, we may write locally in A_{R,R_0}

$$u = \rho e^{i\varphi},$$

and then

$$|\nabla u|^2 = |\nabla \rho|^2 + \rho^2 |\nabla \varphi|^2.$$

Our main result is:

Theorem 3. *Assume u satisfies* (4.1)–(4.3); *then*

$$\int_{B_R} |\nabla u|^2 \geqq \int_{A_{R,R_0}} \rho^2 |\nabla \varphi|^2 \geqq d^2 \left[2\pi \log\left(\frac{R}{R_0}\right) - C\right] \tag{4.4}$$

where C depends only on a and K.

Proof. As in Section 2 we write on A_{R,R_0}

$$u(x) = |u(x)|\, e^{i(d\theta + \psi(x))} = \rho e^{i\varphi(x)}$$

where ψ is smooth and single-valued. We have

$$\nabla \varphi = \frac{d}{r} V + \nabla \psi$$

where $V(x)$ is the vector field in $\mathbb{R}^2 \setminus \{0\}$ defined by

$$V(r\cos\theta,\ r\sin\theta) = (-\sin\theta,\ \cos\theta).$$

Thus,

$$|\nabla \varphi|^2 = \frac{d^2}{r^2} + \frac{2d}{r}\frac{\partial \psi}{\partial \tau} + |\nabla \psi|^2, \tag{4.5}$$

where $\dfrac{\partial \psi}{\partial \tau} = V \cdot \nabla \psi$ is the derivative in the direction tangential to S_r. We write

$$\int_{A_{R,R_0}} \rho^2 |\nabla \varphi|^2 = \int_{A_{R,R_0}} \frac{\rho^2 d^2}{r^2} + 2 \int_{A_{R,R_0}} \frac{\rho^2 d}{r} \frac{\partial \psi}{\partial \tau} + \int_{A_{R,R_0}} \rho^2 |\nabla \psi|^2 \tag{4.6}$$

$$= I_1 + I_2 + I_3,$$

and we evaluate each integral separately. First we have

$$I_1 = \int_{A_{R,R_0}} \frac{d^2}{r^2} - \int_{A_{R,R_0}} (1-\rho^2)\frac{d^2}{r^2} = 2\pi d^2 \log \frac{R}{R_0} - \int_{A_{R,R_0}} (1-\rho^2)\frac{d^2}{r^2}. \tag{4.7}$$

From the Cauchy-Schwarz inequality and (4.3) we obtain

$$\int_{A_{R,R_0}} (1-\rho^2)\frac{1}{r^2} \leqq (\pi K)^{1/2} \tag{4.8}$$

and thus

$$I_1 \geqq 2\pi\, d^2 \log\frac{R}{R_0} - (\pi K)^{1/2} d^2. \tag{4.9}$$

Next, we use the fact that

$$\int_{S_r} \frac{\partial\psi}{\partial\tau} = 0 \qquad \forall\, r \in (R_0, R)$$

and we write

$$I_2 = 2 \int_{A_{R,R_0}} (\rho^2 - 1)\frac{d}{r}\frac{\partial\psi}{\partial\tau}.$$

Hence, we find by the Cauchy-Schwarz inequality and (4.3) that

$$|I_2| \leqq 2\, dK^{1/2}\left[\int_{A_{R,R_0}} |\nabla\psi|^2\right]^{1/2} \leqq 2\frac{d^2K}{a^2} + \frac{a^2}{2}\int_{A_{R,R_0}} |\nabla\psi|^2. \tag{4.10}$$

Finally, we have by (4.2) that

$$I_3 \geqq a^2 \int_{A_{R,R_0}} |\nabla\psi|^2. \tag{4.11}$$

Combining (4.9), (4.10) and (4.11) we are led to

$$\int_{A_{R,R_0}} \rho^2|\nabla\varphi|^2 \geqq 2\pi\, d^2 \log\frac{R}{R_0} - d^2\left[(\pi K)^{1/2} + \frac{2K}{a^2}\right], \tag{4.12}$$

which is the desired conclusion.

Remark 4.1. The above argument in fact gives more than (4.4), namely,

$$\int_{B_R} |\nabla u|^2 \geqq d^2\left[2\pi \log\left(\frac{R}{R_0}\right) - C\right] + \int_{B_R} |\nabla\rho|^2 + \frac{a^2}{2}\int_{A_{R,R_0}} |\nabla\psi|^2. \tag{4.13}$$

We now turn to a more general setting where there are several holes of radius R_0 in B_R. More precisely, let $a_1, a_2, \ldots, a_m$ be points in B_R such that

$$|a_j| \leqq \frac{R}{2} \qquad \forall j, \tag{4.14}$$

$$|a_j - a_k| \geqq 4R_0 \qquad \forall j, k, \quad j \neq k. \tag{4.15}$$

Set

$$\Omega = B_R \setminus \bigcup_{j=1}^{m} B(a_j, R_0)$$

with

$$R_0 \leqq \frac{R}{4}. \tag{4.16}$$

Let u be a (smooth) map from Ω into $\mathbb{C}$. We assume that

$$0 < a \leqq |u| \leqq 1 \quad \text{in } \Omega, \tag{4.17}$$

$$\frac{1}{R_0^2} \int_\Omega (|u|^2 - 1)^2 \leqq K \tag{4.18}$$

for some constants a and K.

Assumption (4.17) implies that

$$\deg(u, \partial B(a_j, R_0)) = d_j$$

is well-defined. We consider the map

$$u_0(z) = \left(\frac{z-a_1}{|z-a_1|}\right)^{d_1} \left(\frac{z-a_2}{|z-a_2|}\right)^{d_2} \cdots \left(\frac{z-a_m}{|z-a_m|}\right)^{d_m}.$$

Our main result is:

Theorem 4. *Assume that* (4.14)–(4.18) *hold. Then*

$$\int_\Omega |\nabla u|^2 \geqq \int_\Omega |\nabla u_0|^2 - C\|d\|^2 m^2 \tag{4.19}$$

where $\|d\| = \sum_j |d_j|$ *and* C *depends only on* a *and* K.

The proof relies on the following simple

Lemma 3. *Given a function* ψ *defined in* $B_{2R_0} \setminus B_{R_0}$, *there is an extension* $\bar{\psi}$ *of* ψ *defined in* B_{2R_0} *such that*

$$\int_{B_{2R_0}} |\nabla \bar{\psi}|^2 \leqq C \int_{B_{2R_0} \setminus B_{R_0}} |\nabla \psi|^2 \tag{4.20}$$

where C *is some universal constant.*

Proof. By scaling we may always assume that $R_0 = 1$ and by adding a constant to ψ we may also assume that

$$\int_{B_2 \setminus B_1} \psi = 0.$$

Poincaré's inequality implies that

$$\int_{B_2 \setminus B_1} |\psi|^2 \leqq C \int_{B_2 \setminus B_1} |\nabla \psi|^2.$$

We may then extend ψ inside B_1 by a standard reflection and cut-off technique.

Proof of Theorem 4. Set $\rho = |u|$. We may write, locally in Ω (but not globally in Ω),

$$u = \rho e^{i\varphi}$$

and then

$$|\nabla u|^2 = |\nabla \rho|^2 + \rho^2 |\nabla \varphi|^2. \tag{4.21}$$

Similarly, we may write, locally in Ω,

$$u_0 = e^{i\varphi_0}$$

with $|\nabla u_0| = |\nabla \varphi_0|$ and

$$\nabla \varphi_0(z) = \sum_j \frac{d_j V_j(z)}{|z - a_j|} \tag{4.22}$$

where $V_j(z)$ is the unit vector tangent to the circle of radius $|z - a_j|$ centered at a_j:

$$V_j(z) = \left(-\frac{y - a_j}{|z - a_j|}, \frac{x - a_j}{|z - a_j|} \right). \tag{4.23}$$

It is convenient to introduce the function ψ *globally* defined on Ω by

$$u = \rho u_0 e^{i\psi}. \tag{4.24}$$

Thus, we have

$$|\nabla u|^2 = |\nabla \rho|^2 + \rho^2 |\nabla \varphi_0 + \nabla \psi|^2, \tag{4.25}$$

and consequently

$$\int_\Omega |\nabla u|^2 \geqq \int_\Omega |\nabla \rho|^2 + \int_\Omega |\nabla u_0|^2 + \int_\Omega a^2 |\nabla \psi|^2 - X \tag{4.26}$$

with

$$X = \int_\Omega (1 - \rho^2) |\nabla u_0|^2 + \int_\Omega 2(1 - \rho^2)\, \nabla \varphi_0 \cdot \nabla \psi - \int_\Omega 2 \nabla \varphi_0 \cdot \nabla \psi.$$

We write $X = X_1 + X_2 + X_3$ and estimate each term separately.

Estimate of X_1. We have

$$|\nabla u_0| \leqq \sum_j \frac{|d_j|}{|z - a_j|} \leqq \|d\| \sum_j \frac{1}{|z - a_j|}, \tag{4.27}$$

so that

$$\|\nabla u_0\|_4 \leqq \|d\| \sum_j \left\| \frac{1}{z - a_j} \right\|_4 \leqq \|d\|\, m \left(\frac{\pi}{R_0^2} \right)^{1/4}. \tag{4.28}$$

Hence, by the Cauchy-Schwarz inequality and by (4.28) and (4.18) we obtain

$$|X_1| \leqq K^{1/2} \|d\|^2 m^2 \pi^{1/2}. \tag{4.29}$$

Estimate of X_2. From (4.22) we have

$$|\nabla \varphi_0| \leqq \frac{m \|d\|}{R_0}, \tag{4.30}$$

and thus, by the Cauchy-Schwarz inequality and (4.30), we find

(4.31) $$|X_2| \leqq 2 \int_\Omega (1-\rho^2)\, |\nabla\varphi_0|\, |\nabla\psi| \leqq 2K^{1/2} m \|d\|\, \|\nabla\psi\|_2 .$$

Estimate of X_3. We have

$$\int_\Omega \nabla\varphi_0 \cdot \nabla\psi = \sum_j d_j \int_\Omega \frac{V_j \cdot \nabla\psi}{|z-a_j|}.$$

We extend ψ inside each disc $B(a_j, R_0)$ using Lemma 3 and we write, for each j,

(4.32) $$\int_\Omega \frac{V_j \cdot \nabla\psi}{|z-a_j|} = \int_{B_R \setminus B(a_j,R_0)} \frac{V_j \cdot \nabla\overline{\psi}}{|z-a_j|} - \sum_{k \neq j} \int_{B(a_k,R_0)} \frac{V_j \cdot \nabla\overline{\psi}}{|z-a_j|}.$$

Note that for $k \neq j$,

(4.33) $$\left| \int_{B(a_k,R_0)} \frac{V_j \cdot \nabla\overline{\psi}}{|z-a_j|} \right| \leqq \frac{1}{R_0} \int_{B(a_k,R_0)} |\nabla\overline{\psi}|$$

and thus, by the Cauchy-Schwarz inequality and Lemma 3,

(4.34) $$\left| \sum_{k \neq j} \int_{B(a_k,R_0)} \frac{V_j \cdot \nabla\overline{\psi}}{|z-a_j|} \right| \leqq C(m-1) \|\nabla\psi\|_2$$

for some universal constant C.

Finally, we observe that

$$\int_{S_r(a_j)} V_j \cdot \nabla\overline{\psi} = \int_{S_r(a_j)} \frac{\partial\overline{\psi}}{\partial\tau} = 0$$

for every $r \in (0, R - |a_j|)$. It follows that, with $\rho_j = R - |a_j|$, we have

$$\left| \int_{B_R \setminus B(a_j,R_0)} \frac{V_j \cdot \nabla\overline{\psi}}{|z-a_j|} \right| = \left| \int_{B_R \setminus B(a_j,\rho_j)} \frac{V_j \cdot \nabla\overline{\psi}}{|z-a_j|} \right| \leqq \frac{1}{\rho_j} \int_{B_R \setminus B(a_j,\rho_j)} |\nabla\psi|$$
$$\leqq \frac{1}{\rho_j} \|\nabla\overline{\psi}\|_2 \, (\pi R^2 - \pi\rho_j^2)^{1/2}.$$

Hence we obtain

(4.35) $$\left| \int_{B_R \setminus B(a_j,R_0)} \frac{V_j \cdot \nabla\overline{\psi}}{|z-a_j|} \right| \leqq C \|\nabla\psi\|_2 .$$

Combining (4.32), (4.34) and (4.35) we are led to

$$|X_3| \leqq Cm\|d\| \, \|\nabla\psi\|_2 . \tag{4.36}$$

Combining (4.29), (4.31) and (4.36) we find

$$\begin{aligned} |X| &\leqq CK^{1/2}\|d\|^2 m^2 + \|d\| \, m \|\nabla\psi\|_2 \, (2K^{1/2} + C) \\ &\leqq \frac{1}{2} a^2 \|\nabla\psi\|_2^2 + \frac{\|d\|^2 m^2}{a^2} (4K + C) . \end{aligned}$$

Returning to (4.26) we obtain

$$\int_\Omega |\nabla u|^2 \geqq \int_\Omega |\nabla \rho|^2 + \int_\Omega |\nabla u_0|^2 + \frac{a^2}{2} \int_\Omega |\nabla\psi|^2 - \frac{\|d\|^2 m^2}{a^2} (4K + C)$$

where C is some universal constant.

Remark 4.2. We emphasize that the above proof gives a stronger conclusion than that stated in Theorem 4, namely,

$$\int_\Omega |\nabla u|^2 \geqq \int_\Omega |\nabla \rho|^2 + \int_\Omega |\nabla u_0|^2 + \frac{a^2}{2} \int_\Omega |\nabla\psi|^2 - C\|d\|^2 m^2 \tag{4.37}$$

where C depends only on a and K.

Finally, we give an estimate for $\int_\Omega |\nabla u_0|^2$ which is convenient to use in conjunction with Theorem 4.

Theorem 5. *Assume* (4.14), (4.15) *and* (4.16) *hold. Then*

$$\int_\Omega |\nabla u_0|^2 = \left(\sum_j d_j^2\right) \log \frac{R}{R_0} - \sum_{k \neq l} d_k d_l \log \frac{|a_k - a_l|}{R} + O(m^2\|d\|^2) , \tag{4.38}$$

where $O(m^2\|d\|^2)$ *stands for a quantity* X *such that* $|X| \leqq Cm^2\|d\|^2$ *and where* C *is some universal constant.*

Proof. With obvious notations we may write

$$u_0 = e^{i\Sigma_j d_j \theta_j} ,$$

so that

$$|\nabla u_0| = |\Sigma \, d_j \nabla \theta_j| .$$

On the other hand, the functions θ_j and $\log|x - a_j|$ are harmonic conjugates and thus

$$|\nabla u_0| = |\nabla \, (\Sigma \, d_j \log|x - a_j|)| .$$

Set

$$v_j(x) = \log|x - a_j|, \quad v = \sum_j d_j v_j .$$

We have

$$\int_\Omega |\nabla v|^2 = \int_{\partial B_R} \frac{\partial v}{\partial \nu} v - \sum_j \int_{\partial B(a_j, R_0)} \frac{\partial v}{\partial \nu} v \tag{4.39}$$

where ν denotes the outward normal to B_R and to $B(a_j, R_0)$. On ∂B_R we have

$$\frac{R}{2} \leqq |x - a_j| \leqq 2R$$

and hence

$$\big|\log|x - a_j| - \log R\big| \leqq C,$$

so that

$$|v - (\textstyle\sum d_j) \log R| \leqq C\|d\| m. \tag{4.40}$$

On the other hand we have

$$\Delta \log|x - a_j| = 2\pi\, \delta_{a_j}$$

and therefore

$$\Delta v = 2\pi \sum d_j \delta_{a_j},$$

so that

$$\int_{\partial B_R} \frac{\partial v}{\partial \nu} = 2\pi \sum d_j. \tag{4.41}$$

Combining (4.40) and (4.41) we see that

$$\left| \int_{\partial B_R} \frac{\partial v}{\partial \nu} v - 2\pi (\textstyle\sum d_j)^2 \log R \right| \leqq C\|d\| m \int_{\partial B_R} \left| \frac{\partial v}{\partial \nu} \right|.$$

Finally, we observe that on ∂B_R,

$$\frac{\partial}{\partial \nu} \log|x - a_j| = \frac{x - a_j}{|x - a_j|^2} \cdot \frac{x}{R}$$

and consequently

$$\left| \frac{\partial}{\partial \nu} \log|x - a_j| \right| \leqq \frac{2}{R}.$$

Hence

$$\left| \frac{\partial v}{\partial \nu} \right| \leqq \frac{2}{R} \|d\| m, \qquad \int_{\partial B_R} \left| \frac{\partial v}{\partial \nu} \right| \leqq 4\pi \|d\| m.$$

Thus, we are led to

$$\int_{\partial B_R} \frac{\partial v}{\partial \nu} v = 2\pi (\textstyle\sum d_j)^2 \log R + O(\|d\|^2 m^2). \tag{4.42}$$

Next, we have to evaluate

$$\sum_{j,k,l} \int_{\partial B(a_j, R_0)} d_k d_l \frac{\partial v_k}{\partial \nu} v_l.$$

It is convenient to distinguish several cases:

$$\text{Case 1: } j = l, \quad k \neq l,$$
$$\text{Case 2: } j = l, \quad k = l,$$
$$\text{Case 3: } j \neq l, \quad j = k,$$
$$\text{Case 4: } j \neq l, \quad j \neq k.$$

Case 1: $j = l$, $k \neq l$. We have

$$\int_{\partial B(a_j, R_0)} \frac{\partial v_k}{\partial \nu} v_l = \log R_0 \int_{\partial B(a_j, R_0)} \frac{\partial v_k}{\partial \nu} = 0 \tag{4.43}$$

since v_k is harmonic on $B(a_j, R_0)$.

Case 2: $j = l$, $k = l$. We have

$$\int_{\partial B(a_j, R_0)} \frac{\partial v_k}{\partial \nu} v_l = \log R_0 \int_{\partial B(a_j, R_0)} \frac{\partial v_j}{\partial \nu} = 2\pi \log R_0 \tag{4.44}$$

since $\Delta v_j = 2\pi\, \delta_{a_j}$ on $B(a_j, R_0)$.

Case 3: $j \neq l$, $j = k$. We have

$$\begin{aligned} \int_{\partial B(a_j, R_0)} \frac{\partial v_k}{\partial \nu} v_l &= \frac{1}{R_0} \int_{\partial B(a_j, R_0)} v_l \\ &= 2\pi v_l(a_j) = 2\pi \log |a_j - a_l| \end{aligned} \tag{4.45}$$

since v_l is harmonic on $B(a_j, R_0)$.

Case 4: $j \neq l$, $j \neq k$. On $\partial B(a_j, R_0)$ we have

$$|v_l(x) - v_l(a_j)| = \left| \log \frac{|x - a_l|}{|a_j - a_l|} \right| .$$

But

$$|a_j - a_l| - R_0 \leqq |x - a_l| \leqq |a_j - a_l| + R_0$$

so that, by (4.15),

$$\frac{3}{4} \leqq \frac{|x - a_l|}{|a_j - a_l|} \leqq \frac{5}{4}$$

and thus

$$|v_l(x) - v_l(a_j)| \leqq C \quad \text{on } \partial B(a_j, R_0) .$$

Hence

$$\int_{\partial B(a_j, R_0)} \frac{\partial v_k}{\partial \nu} v_l = \int_{\partial B(a_j, R_0)} \frac{\partial v_k}{\partial \nu} (v_l - v_l(a_j))$$

and consequently

$$\left| \int_{\partial B(a_j,R_0)} \frac{\partial v_k}{\partial \nu} v_l \right| \leqq 2\pi C R_0 \left\| \frac{\partial v_k}{\partial \nu} \right\|_{L^\infty(\partial B(a_j,R_0))}.$$

On the other hand, on $\partial B(a_j, R_0)$, we have

$$\left| \frac{\partial v_k}{\partial \nu} \right| \leqq \frac{1}{|x-a_k|} \leqq \frac{1}{|a_j-a_k|-R_0} \leqq \frac{1}{3R_0}$$

and therefore we have

$$\left| \int_{\partial B(a_j,R_0)} \frac{\partial v_k}{\partial \nu} v_l \right| \leqq C. \tag{4.46}$$

Combining all the cases we see that

$$\sum_{j,k,l} \int_{\partial B(a_j,R_0)} d_k d_l \frac{\partial v_k}{\partial \nu} v_l = 2\pi \left(\sum_j d_j^2 \right) \log R_0 + 2\pi \sum_{k \neq l} d_k d_l \log |a_k - a_l| + O(m^2 \|d\|^2). \tag{4.47}$$

Combining (4.39), (4.42) and (4.47) we obtain

$$\int_\Omega |\nabla v|^2 = 2\pi \left(\sum d_j \right)^2 \log R - 2\pi \left(\sum d_j^2 \right) \log R_0 - 2\pi \sum_{k \neq l} d_k d_l \log |a_k - a_l| + O(m^2 \|d\|^2)$$

and this yields the desired conclusion.

Remark 4.3. In the special case where $d_j \geqq 0$ for all j, we deduce from (4.14) and Theorem 5 that

$$\int_\Omega |\nabla u_0|^2 \geqq \left(\sum_j d_j^2 \right) \log \frac{R}{R_0} + O(m^2 \|d\|^2).$$

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References

1. F. BETHUEL, H. BREZIS & F. HÉLEIN, *Ginzburg-Landau Vortices*, Birkhäuser (to appear).
2. TH. CAZENAVE, personal communication.
3. P. HAGAN, Spiral waves in reaction diffusion equations, *SIAM J. Applied Math.* **42** (1982), p. 762–786.

4. T. KATO, Schrödinger operators with singular potentials, *Israel J. Math.* **13** (1972), p. 135–148.
5. L. NIRENBERG, personal communication.
6. L. SHAFRIR, Remarks on solutions of $-\Delta u = u(1-|u|^2)$ in $\mathbb{R}^2$, *C. R. Acad. Sci. Paris* (to appear).

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