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Solitary Waves with Surface Tension I: Trajectories Homoclinic to Periodic Orbits in Four Dimensions

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1. Introduction

This paper is concerned with the existence of trajectories homoclinic to non-constant periodic orbits of the equation

$$\varepsilon^2 T^{iv}(x) + T''(x) - T(x) + T(x)^2 = 0, \quad x \in \mathbf{R}, \tag{1.1}$$

in the phase space $\mathbf{R}^4$. Motivated by results of J. T. BEALE [5] for a specific water-wave problem (see Section 1.1 below), here we consider (1.1) as prototypical of a class of Hamiltonian dynamical systems where our analysis will yield results similar to his.

In a second part, to follow later, we shall re-examine in detail the solitary water-wave problem which motivated the present investigation, and make explicit the implications of our approach for it.

1.1. A water-wave problem

Our results for equation (1.1) are strongly suggested by J. T. BEALE's elegant and powerful contribution [5] to the existence question for solitary water-waves when the Bond number (a non-negative dimensionless parameter which indicates the strength of surface tension) lies strictly between 0 and $1/3$ when the Froude number is close to 1. The Froude number is a dimensionless parameter whose value reflects the phase speed of steady waves. When $B = 0$ or $B > 1/3$, steady solitary waves whose exponential decay to zero is monotone are known to bifurcate at Froude number 1 (see [2, 3, 4, 6]). Dealing directly with the steady water-wave equations when the Bond number in the interval $(0, 1/3)$, BEALE has now shown that solitary-like waves, which settle down to small-amplitude periodic disturbances at infinity, bifurcate at Froude number 1. Though it seems likely, it has yet to be proved that the amplitudes of these periodic disturbances at infinity are non-zero. By definition, a true solitary water-wave is asymptotic to uniform horizontal flow far up and down stream.

Although it is shown in [1] that equation (1.1) has no solution which converges to 0 as $|x| \to \infty$ if $\varepsilon > 0$ is sufficiently small, it is emphasised here that the method of [1] does not extend to the water-wave problem (or, equivalently, to (1.2) below), and therefore an analogous non-existence result for the water-wave problem is unknown. The question of existence or non-existence of true solitary waves of small amplitude when $B \in (0, \frac{1}{3}]$ and the Froude number is close to 1 remains open, and, as BEALE points out in [5], a related important open problem is to decide whether the periodic oscillations to which his waves are homoclinic have non-zero amplitude.

While he does not settle the question of the existence or non-existence of true solitary waves, BEALE's work gives an important, entirely rigorous new insight into the solitary wave problem overall, when the Bond number B is in the interval $(0, \frac{1}{3}]$ and the Froude number is close to 1. He works directly with the water-wave equations using careful *a priori* estimates and Newton's method. If, however, one adopts an approach using a centre-manifold reduction (*cf.* AMICK & KIRCHGÄSSNER [2]), one find that the flow on the centre manifold is governed by a fourth-order equation of the form

$$\varepsilon^2 z^{iv} + z'' - z + z^2 + h(\varepsilon, z, z', z'', z''') = 0, \tag{1.2}$$

which has a first integral j. Here h is of higher order and $j(\varepsilon, z, z', z'', z''')$ is constant on solutions. Thus, BEALE's existence results imply analogous existence results for the equivalent problem (1.2), which, in turn, suggests that similar results for (1.1) might hold. Here we shall vindicate this remark about (1.1), and show slightly more.

The present paper concerns the existence of orbits of (1.1) which are homoclinic to non-constant periodic orbits. It complements the non-existence results of [1] and reveals the complex nature of the set of trajectories which are homoclinic to α- and ω-limit sets in a neighbourhood of the zero equilibrium: the zero equilibrium is not connected to itself by a homoclinic orbit, but uncountably many small amplitude periodic orbits are, for each $\varepsilon > 0$ sufficiently small. The structure of the set of high-frequency periodic solutions of equation (1.1) is shown in Section 2 to be that of a cylinder. This analysis plays an important part in the development which follows. We shall return to the precise form of equation (1.2) equivalent to the water-wave problem in Part II. Since our method is robust, the presence of higher-order terms in (1.2) should not impair its effectiveness. Therefore we can reasonably hope that this method for (1.1) extends to yield BEALE's existence results for equation (1.2). If so, it covers the cases with both the zero and the non-zero phase shift. To explain the significance of this remark it is necessary to give an outline of the main result.

1.2. The main results

Section 2 is a detailed treatment of the theory of τ-periodic solutions of (1.1) when $2\pi\varepsilon/\tau$ is close to 1. We prove that for each $\varepsilon \in (0, \varepsilon_0)$, where

$\varepsilon_0 > 0$ is sufficiently small, there exists a circle of even periodic orbits which contains both constant solutions $T \equiv 0$ and $T \equiv 1$. This circle comprises two semi-circles (the orbits on one being related to those of the other by translation through half a period) whose common boundary consists of the two noted equilibria, 0 and 1 (see Figure 1). This detailed analysis of the periodic case is done following a Lyapunov-Schmidt reduction.

The question addressed in Section 3 is whether any of this plethora of periodic solutions has stable and unstable manifolds which intersect non-trivially. Let $p \in [0, \frac{1}{2})$ (p stands for phase-shift). Then for every $\varepsilon > 0$ sufficiently small we show that there exists an even solution T_p^ε of (1.1) such that for any $q \in (0, 1)$,

$$\left| T_p^\varepsilon(x) - \tfrac{3}{2}\operatorname{sech}^2\left(\frac{x}{2}\right) - \varphi_\beta^\varepsilon(x - p\tau_\beta^\varepsilon)\right| \leqq \text{const.}\, \varepsilon^2 e^{-qx}, \quad x > 0,$$

where the constant is independent of ε, and $\varphi_\beta^\varepsilon$ is some τ_β^ε-periodic solution of (1.1). Notice that $\sigma(x) = (3/2)\operatorname{sech}^2(x/2)$ is an even solution of (1.1), which converges exponentially to 0 at infinity, when $\varepsilon = 0$. Here β, depending on p and ε, is a measure of the amplitude of $\varphi_\beta^\varepsilon$, and $2\pi\varepsilon/\tau_\beta^\varepsilon$ is close to 1 if ε is close to zero. Moreover, for each $N \in \mathbf{N}$ there exists a constant $C(N, p)$ such that $|\varphi_\beta^\varepsilon| \leqq C(N, p)\,\varepsilon^N$. Thus T_p^ε is an even solution of (1.1) which resembles σ in a neighbourhood of 0, and which converges exponentially to a translation of $\varphi_\beta^\varepsilon$ through p times its period as $x \to \infty$. Moreover, the persistent periodic tail has an amplitude which is $O(\varepsilon^N)$ for all N as $\varepsilon \to 0$. A detailed account of these results is given in Section 3. With this picture in mind, we can now explain a small difference between this approach and that of Beale [5] alluded to at the end of the preceding section. These results pertain to all phase shifts in $[0, \frac{1}{2})$ (see also the closing remark of Section 1.3), whereas his method seems to preclude from consideration the case $p = 0$. (Note that it is not necessary to consider all phase shifts in $[0, 1)$ because we will note later that $\varphi_\beta^\varepsilon(x - \frac{1}{2}\tau_\beta^\varepsilon) = \varphi_{-\beta}^\varepsilon(x)$. This reflects the symmetry of the circles of periodic solutions established in Section 2.)

1.3. The method

Our method is straightforward and robust. Let $\{\varphi_\beta^\varepsilon : \beta \in (-\pi, \pi]\}$ denote a parametrisation of the circle of periodic solutions whose existence is proved for $\varepsilon \in (0, \varepsilon_0)$ in Section 2. Let $p \in [0, \frac{1}{2})$. If $\varphi_\beta^\varepsilon$ has period τ_β^ε, we seek an even solution T of (1.1) such that

$$T(x) = \sigma(x) + \varphi_\beta^\varepsilon(x - p\tau_\beta^\varepsilon) + \omega(x), \quad x > 0. \tag{1.3}$$

Then ω is an even function which satisfies an equation of the form

$$\varepsilon^2\omega^{iv} + \omega'' - \omega = f(p, \beta, \varepsilon, \omega, x), \tag{1.4}$$

(the exact form is given in (3.9)). The key observation is that if $v_\varepsilon(x) = \cos k_\varepsilon x$ is a solution of $\varepsilon^2 v^{iv} + v'' - v = 0$ on $\mathbf{R}$, then a necessary condition for (1.4) to have a solution ω which decays to zero exponentially at infinity is that

$$\int_{-\infty}^{\infty} f(p, \beta, \varepsilon, \omega(x), x)\, v_\varepsilon(x)\, dx = 0. \tag{1.5}$$

The main result of Section 3.3 is that (1.5) uniquely determines β close to 0 as a function of p, ε and ω when $p \in [0, \frac{1}{2})$ and ε and ω are small enough in a sense to be specified. When this dependence is taken into account, (1.4) can be re-written

$$\varepsilon^2 \omega^{iv} + \omega'' - \omega = \hat{f}(p, \varepsilon, \omega, x). \tag{1.6}$$

We show in Section 3.4 that (1.6) is a problem for which Banach's Contraction Mapping Principle yields the existence of a unique small solution for each $p \in [0, \frac{1}{2})$ provided $\varepsilon > 0$ is sufficiently small (how small depending on p).

Finally in Section 3.5 we prove *a priori* bounds on the solutions ω of (1.4) which, when substituted into equation (1.5), yield that the functional dependence of β on ω and ε means that $|\beta| \leqq C(N, p)\, \varepsilon^N$ as $\varepsilon \to 0$ for any $N \in \mathbf{N}$. Since β is a measure of the amplitude of the periodic orbits at infinity, the homoclinic orbit whose existence is established has amplitude $3/2 + O(\varepsilon^2)$, while the periodic orbit which is its α- and ω-limit set has amplitude vanishing to all orders in ε as $\varepsilon \to 0$.

The proof here bears little superficial resemblance to that of [5]. In particular, the proof that the amplitudes of the periodic orbits are vanishing to all orders as $\varepsilon \to 0$ is proved here *a posteriori* using *a priori* estimates; in [5] it is built into the existence proof.

To complete our discussion of method, we mention a somewhat bewildering aspect of the analysis. While we deal with all phase-shifts $p \in [0, \frac{1}{2})$, the treatments of $p = 0$ and $p \in (0, \frac{1}{2})$ from an analytic viewpoint are different. The analysis when $p = 0$ is regular and straightforward. However, when $p \in (0, \frac{1}{2})$, certain singular coefficients appear in the estimates of Section 3.2, caused essentially by the analytical difficulty of extending a function such as $\cos(x - \varepsilon)$, $x \geqq \varepsilon$, as a smooth, even function C, say, on $\mathbf{R}$. It is easy to see that $\sup\{|C''(x)| : x \in (-\varepsilon, \varepsilon)\} \geqq 1/\varepsilon$ no matter how this is done. Thus non-zero phase shifts introduce inverse powers of ε which need careful analysis.

1.4. Related results

This paper was motivated by BEALE's treatment of solitary-like water-waves with periodic oscillations at infinity [5] which exist when the Froude number is close to 1. Independently, SUN [11] gave a different, less complete account of these phenomena. Later, IOOSS & KIRCHGÄSSNER [8] gave yet another approach to the same water-wave problem using normal forms for an equation similar to (1.2) which arises after centre-manifold reduction. BEALE alone has given a detailed, explicit account of the role of the phase, and an analysis of the amplitude, for the periodic oscillations which exist at infinity, but no one has so far given a proof that the periodic oscillations at infinity are non-constant.

HUNTER & SCHEURLE [7] have given an account of equation (1.1) less detailed than here. Though they predict solutions convergent to periodic solutions at infinity, their method does not seem to yield the infinity of homoclinic orbits which exist, nor other details which are features of BEALE's work, and of this paper also. In contrast to the results obtained so far for the water-wave problem, it is known that the oscillations at infinity for equation (1.1) are non-trivial because AMICK & McLEOD [1] have shown that when $\varepsilon > 0$ is sufficiently small, there are no orbits homoclinic to the constant zero solution.

Finally, IOOSS & KIRCHGÄSSNER [9] have made a fundamental new contribution to the solitary water-wave problem when the Bond number is less than 1/3. They have shown the existence of steady waves whose oscillations at infinity *decay to zero* exponentially, which bifurcate, not from Froude number 1, but from a critical Froude number which depends on the Bond number. These are true solitary water-waves whose decaying oscillations at infinity distingush them from the (until now) more familiar solitary water-waves which decay monotonically to zero at infinity when the Bond number is zero or exceeds 1/3 ([2, 3, 4, 6]).

2. Periodic Solutions

In this section a set of even, small-amplitude, high-frequency, periodic solutions of the equation

$$\varepsilon^2 T^{iv}(x) + T''(x) - T(x) + T(x)^2 = 0, \qquad x \in \mathbf{R}, \tag{2.1}$$

is described in some detail when $\varepsilon > 0$ is small. To be more precise, we are interested in small amplitude solutions of (2.1) whose period τ has $\tau/2\pi\varepsilon$ close to 1. To this end let

$$\varepsilon^2 = \delta > 0, \qquad \lambda = \frac{4\pi^2\varepsilon^2}{\tau^2} > 0, \tag{2.2}$$

$$T(x) = \delta^{-1} t(\sqrt{\lambda/\delta}\, x). \tag{2.3}$$

The t is a solution of

$$\begin{gathered} \lambda^2 t^{iv} + \lambda t'' - \delta t + t^2 = 0, \\ t(x) = t(x + 2\pi), \qquad t(x) = t(-x) \end{gathered} \tag{2.4}$$

if T is a τ-periodic even solution of (2.1), and if λ is close to 1 when $\tau/2\pi\varepsilon$ is close to 1. It will be sufficient for our purposes to find all small solutions of (2.4).

We begin by observing certain symmetries in the problem. If t satisfies (2.4), then $\tilde{t}(x) = t(x + \pi)$ defines a solution $\tilde{t}$ of (2.4), and $\hat{t}(x) = t(x) - \delta$

defines a solution of (2.4) if δ is replaced with $-\delta$. Moreover, $t \equiv 0$ and $t \equiv \delta$ are constant solutions of (2.4). We investigate the existence of other solutions of (2.4) by using the Lyapunov-Schmidt procedure, and in doing so we shall seek to reflect the symmetries just noted.

Let Y and X denote respectively the Banach spaces of continuous, and four-times continuously differentiable, even, 2π-periodic functions on $\mathbf{R}$ and let $G: \mathbf{R}^2 \times X \to Y$ be defined by

$$G(\lambda, \delta, t) = \lambda^2 t^{iv} + \lambda t'' - \delta t + t^2, \qquad (\lambda, \delta, t) \in \mathbf{R}^2 \times X. \tag{2.5}$$

Clearly G is infinitely differentiable. Let $K(\subset X \subset Y)$ denote span$\{1, \cos x\}$, and let X_1 and Y_1 denote respectively the closed subspaces of X and Y which are orthogonal to K in the usual L_2 inner product. If $t \in X$, then t can be written in a unique way as $t(x) = a + b\cos x + \psi(x)$, $\psi \in X_1$, and if such a function t satisfies the equation

$$G(\lambda, \delta, t) = 0 \in Y,$$

then

$$\lambda^2 \psi^{iv}(x) + \lambda \psi''(x) + (2a - \delta)\psi(x) + \tfrac{1}{2} b^2 \cos 2x + Q(\psi^2(x) + 2b\psi\cos x) = 0, \tag{2.6}$$

$$\lambda^2 b - \lambda b - \delta b + 2ab + \frac{1}{\pi}\int_{-\pi}^{\pi} \cos x \psi^2(x)\, dx + \frac{2b}{\pi}\int_{-\pi}^{\pi} \cos^2 x \psi(x)\, dx = 0, \tag{2.7}$$

$$-\delta a + a^2 + \tfrac{1}{2} b^2 + \frac{1}{2\pi}\int_{-\pi}^{\pi} \psi^2(x)\, dx = 0. \tag{2.8}$$

(Here Q denotes the projection operator on Y defined by the decomposition $Y = K \oplus Y_1$ and $Q(Y) = Y_1$.)

By the Implicit Function Theorem, equation (2.6) can be solved for ψ in a neighbourhood of $0 \in X_1$ as a smooth function of $(2a - \delta, b, \lambda)$ in a neighbourhood of $(0, 0, 1) \in \mathbf{R}^3$. This is because the linearisation of (2.6) with respect to ψ at that point is $\psi^{iv} + \psi'' = 0$, and the left-hand side defines a linear homeomorphism from X_1 onto Y_1.

Therefore, for some neighbourhood U of $(0, 0, 1)$ in $\mathbf{R}^3$, a neighbourhood V of the origin in X_1 and a smooth function $\hat{\Psi}: U \to V$ exists such that if $(a, b, \lambda, \delta) \in \mathbf{R}^4$ and $(2a - \delta, b, \lambda) \in U$, then $\hat{\Psi}(2a - \delta, b, \lambda)$ is the unique solution in V of (2.6) for this choice of (a, b, λ, δ). It is immediate, from the assertion about uniqueness and the fact that $\psi = 0$ is a solution of (2.6) when $b = 0$, that $\hat{\Psi}(2a - \delta, 0, \lambda) = 0$ for all $(2a - \delta, 0, \lambda) \in U$. Consequently,

$$\hat{\Psi}(2a - \delta, b, \lambda) = b\tilde{\Psi}(2a - \delta, b, \lambda)$$

for some smooth function $\tilde{\Psi}$. Substituting this expression into (2.6) and a further application of the Implicit Function Theorem yields that $\tilde{\Psi}(2a - \delta, 0, \lambda) = 0$, whence

$$\hat{\Psi}(2a - \delta, b, \lambda) = b^2 \Psi(2a - \delta, b, \lambda), \tag{2.9}$$

$\Psi: U \to X_1$ is smooth, and from (2.6) it follows that

$$\Psi(0,0,1) = -\tfrac{1}{24}\cos 2x. \tag{2.10}$$

Now note that if $(a, b, \lambda, \delta, \psi) \in \mathbf{R}^4 \times X_1$ is a solution of (2.6) and if we put $\hat{\psi}(x) = \psi(x+\pi)$, then $(a, -b, \lambda, \delta, \hat{\psi})$ also satisfies (2.6) and $\hat{\psi} \in X_1$. Hence, by uniqueness,

$$\Psi(2a-\delta, b, \lambda)\,(x+\pi) = \Psi(2a-\delta, -b, \lambda)\,(x). \tag{2.11}$$

Now we substitute $b^2\Psi$ for ψ in (2.7) and (2.8) to obtain the so-called bifurcation equations. They are

$$b(\lambda^2 - \lambda + 2a - \delta) + \frac{b^4}{\pi}\int_{-\pi}^{\pi} \cos x\,(\Psi(2a-\delta, b, \lambda)\,(x))^2\,dx$$
$$+ \frac{2b^3}{\pi}\int_{-\pi}^{\pi} \cos^2 x\,\Psi(2a-\delta, b, \lambda)\,(x)\,dx = 0, \tag{2.12}$$

$$a^2 - a\delta + \tfrac{1}{2}b^2 + \frac{b^4}{2\pi}\int_{-\pi}^{\pi} (\Psi(2a-\delta, b, \lambda)\,(x))^2\,dx = 0. \tag{2.13}$$

If $b = 0$, then (2.12) is satisfied and (2.13) is also satisfied if and only if $a = 0$ or $a = \delta$. Since $\hat{\Psi}(2a-\delta, 0, \lambda) = 0$, this accounts for all the constant solutions of (2.4) previously noted. For non-constant solutions $b \neq 0$. To investigate these further we may divide (2.12) by b to obtain

$$(\lambda^2 - \lambda + 2a - \delta) + \frac{b^3}{\pi}\int_{-\pi}^{\pi} \cos x\,(\Psi(2a-\delta, b, \lambda)\,(x))^2\,dx$$
$$+ \frac{2b^2}{\pi}\int_{-\pi}^{\pi} \cos^2 x\,\Psi(2a-\delta, b, \lambda)\,(x)\,dx = 0. \tag{2.14}$$

Note here that if (a, b, λ, δ) satisfies (2.13) and (2.14), so also does $(a-\delta, b, \lambda, -\delta)$, reflecting the earlier remark that if t satisfies (2.4), then so does $t - \delta$ when $-\delta$ replaces δ in the equation.

Thus all small solutions for δ in a neighbourhood of 0 are found by finding all small solutions with $\delta \geqq 0$ small. (See (2.18) below.)

Because of the identity (2.11) it follows that the left-hand sides of (2.13) and (2.14) are even functions of b (reflecting a previously noted symmetry due to translation invariance). Thus they can be written

$$a^2 - a\delta + \tfrac{1}{2}b^2 + b^4 I(2a-\delta, b^2, \lambda) = 0, \tag{2.15}$$

$$\lambda^2 - \lambda + 2a - \delta + b^2 J(2a-\delta, b^2, \lambda) = 0, \tag{2.16}$$

where

$$I(2a-\delta, b^2, \lambda) \geqq 0 \tag{2.17}$$

and I and J are smooth functions.

Before proceeding further we note that, by (2.15) and (2.17), $b \neq 0$ if and only if either $0 < a < \delta$ or $\delta < a < 0$. Recalling (2.2), and our earlier remark about symmetry in δ, we shall henceforth restrict attention to the solubility of (2.15) and (2.16) in the region

$$0 < a < \delta < \delta_0, \quad \text{for some } \delta_0 > 0 \text{ sufficiently small.} \tag{2.18}$$

Let B be used to denote b^2 in (2.15) and (2.16). We can apply the Implicit Function Theorem and solve for (B, λ) as a function of (a, δ). The result is that there exist neighbourhoods M and N of (0, 1) and (0, 0) respectively and a smooth function $\varphi: N \to M$ such that $\varphi(a, \delta) = (B(a, \delta), \lambda(a, \delta))$ gives the only solution in M of equations (2.15) and (2.16). It follows from (2.15) that $B(a, \delta) > 0$ if and only if $0 < a < \delta$ or $\delta < a < 0$. Since $B = b^2$, we need only consider the solution set $\mathbf{D} = \mathbf{D}^+ \cup \mathbf{D}^-$ where,

$$\mathbf{D}^+ = \{(a, \sqrt{B(a,\delta)}, \lambda(a, \delta), \delta) : 0 < a < \delta < \delta_0\}, \tag{2.19}$$

$$\mathbf{D}^- = \{(a, -b, \lambda, \delta) : (a, b, \lambda, \delta) \in \mathbf{D}^+\}$$

for some $\delta_0 > 0$ sufficiently small. Because of (2.15)

$$B(0, \delta) = B(\delta, \delta) = 0, \tag{2.20}$$

and because λ is close to 1 and (2.16) holds, we find that

$$\lambda(0, \delta) = \tfrac{1}{2}\left(1 + \sqrt{1 + 4\delta}\right) \quad \text{and} \quad \lambda(\delta, \delta) = \tfrac{1}{2}\left(1 + \sqrt{1 - 4\delta}\right). \tag{2.21}$$

Thus we have found the complete picture: locally all small solutions of (2.4) lie on "a cone" which, via the scalings given in (2.2) and (2.3), transforms into a "cylinder" of high-frequency solutions of (2.1). This is illustrated in Figure 1.

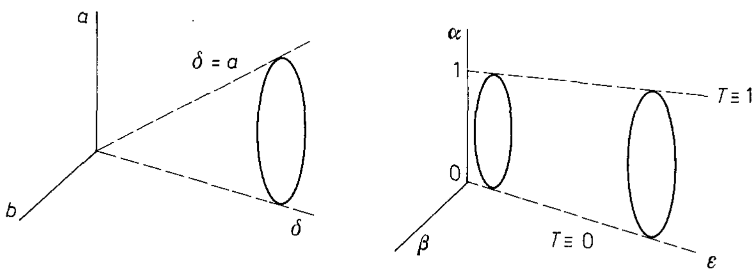

Fig. 1. In both diagrams the dotted lines denote lines of constant solutions. The one on the left pertains to equation (2.4) and that on the right to equation (2.1).

With this image of the solution set of (2.4) in mind, we consider equations (2.15) and (2.16) for a fixed $\delta \in (0, \delta_0)$. An application of the Implicit Function Theorem at $(a, b^2, \lambda) = (0, 0, \lambda(0, \delta))$ shows that (a, λ) is uniquely determined as a function of b^2. Let us write

$$(a, \lambda) = (A_\delta(b^2), \Lambda_\delta(b^2)), \quad b \in I_\delta,$$

where I_δ is an open interval about 0 in $\mathbf{R}$ and $(A_\delta(0), \Lambda_\delta(0)) = (0, \lambda(0,\delta))$. If the left-hand side of (2.15), (2.16) is considered to define a function from a neighbourhood of $(0,0,1)$ in $\mathbf{R}^3$ to $\mathbf{R}^2$, the Jacobian matrix of partial derivatives with respect to (a,λ) has determinant of the form

$$D(a,b,\lambda,\delta) = (2a-\delta)(2\lambda-1) + O(b^2) \quad \text{as } b \to 0. \tag{2.22}$$

Moreover, at a solution (2.15) and (2.17) imply the existence of $b_0 > 0$ and a constant (both independent of δ) such that

$$b^2 \leqq 2(a\delta - a^2) \leqq \text{const.}\, b^2 \quad \text{if } |b| < b_0. \tag{2.23}$$

Therefore there is a constant (Γ^2 say) independent of δ such that if (a,b,λ,δ) satisfies (2.15) and (2.16), $\delta \in (0,\delta_0)$ and $b^2 \leqq \Gamma^2\delta^2$, then

$$a \in (0, \tfrac{1}{4}\delta) \quad \text{if } a \in (0, \tfrac{1}{2}\delta). \tag{2.24}$$

In particular, Γ can be chosen such that every solution (a,b,λ,δ) of (2.15), (2.16) with $a \in (0,\frac{1}{2}\delta)$ and $b^2 \leqq \Gamma^2\delta^2$ has

$$a \in (0, \tfrac{1}{4}\delta), \quad D(a,b,\lambda,\delta) \neq 0. \tag{2.25}$$

From (2.25) and the Implicit Function Theorem it can be inferred that the domain of definition of $(A_\delta, \Lambda_\delta)$ can be extended to the interval $(0, \Gamma^2\delta^2)$ for some $\Gamma > 0$ independent of δ. We summarise this result and give estimates on the implicit functions as follows.

Lemma 2.1. *There exist $\delta_0 > 0$ and $\Gamma > 0$ such that, for $\delta \in (0,\delta_0)$, on each interval $(-\Gamma\delta, \Gamma\delta)$ are defined smooth, real-valued, even functions a_δ, λ_δ with the following properties*:

$$(a_\delta(b), b, \lambda_\delta(b), \delta) \textit{ satisfies } (2.15) \textit{ and } (2.16) \textit{ if } b \in (0, \Gamma\delta), \tag{2.26}$$

$$a_\delta(0) = 0 \quad \textit{and} \quad 0 < a_\delta(b) < \tfrac{1}{4}\delta \quad \textit{if } b \in (0,\Gamma\delta), \tag{2.27}$$

$$K_1\delta\, a_\delta(b) \leqq b^2 \leqq K_2\delta a_\delta(b), \quad b \in (-\Gamma\delta, \Gamma\delta), \tag{2.28}$$

$$\left(\frac{K_1}{\delta}\right) b \leqq \frac{da_\delta}{db}(b) \leqq \left(\frac{K_2}{\delta}\right) b, \quad b \in (0,\Gamma\delta), \tag{2.29}$$

$$\tfrac{1}{2}\left(1 + \sqrt{1 + 4\delta - 8a_\delta(b)}\right) + K_1 b^2 \leqq \lambda_\delta(b) \leqq \tfrac{1}{2}\left(1 + \sqrt{1 + 4\delta - 8a_\delta(b)}\right) + K_2 b^2,$$
$$b \in (-\Gamma\delta, \Gamma\delta), \tag{2.30}$$

$$-K_2\left(\frac{b}{\delta}\right) \leqq \frac{d\lambda_\delta}{db}(b) \leqq -K_1\left(\frac{b}{\delta}\right), \quad b \in (0,\Gamma\delta). \tag{2.31}$$

Here K_1 and K_2 are positive constants, independent of δ and b.

Proof. The existence of Γ, δ_0 and the functions a_δ and λ_δ follows from the discussion preceding the statement of the lemma and the Implicit Function

Theorem, by putting $a_\delta(b) = A(\delta, b^2)$ and $\lambda_\delta(b) = \Lambda(\delta, b^2)$. The estimates then follow by choosing δ_0 sufficiently small upon differentiating (2.15) and (2.16) with respect to b. q.e.d.

We finish this section with an interpretation of this as a result about equation (2.1) using (2.2) and (2.3). We introduce some convenient notation. For $\beta \in (-\Gamma, \Gamma)$ let

$$\begin{gathered} \varepsilon = \sqrt{\delta}, \quad \mu_\varepsilon(\beta) = \varepsilon^{-1}\sqrt{\lambda_{\varepsilon^2}(\varepsilon^2\beta)}, \quad \alpha_\varepsilon(\beta) = \varepsilon^{-2}a_{\varepsilon^2}(\varepsilon^2\beta), \\ \Psi^\varepsilon_\beta(x) = \Psi(2\varepsilon^2\alpha_\varepsilon(\beta) - \varepsilon^2, \ \varepsilon^2\beta, \ \varepsilon^2\mu^2_\varepsilon(\beta))\,(x), \end{gathered} \tag{2.32}$$

$$\varphi^\varepsilon_\beta(x) = \alpha_\varepsilon(\beta) + \beta\cos(\mu_\varepsilon(\beta)\,x) + \varepsilon^2\beta^2\Psi^\varepsilon_\beta(\mu_\varepsilon(\beta)\,x). \tag{2.33}$$

With this notation we have the following result.

Theorem 2.2. *If* $\varepsilon \in (0, \varepsilon_0)$, *then*

$$C_\varepsilon = \{\varphi^\varepsilon_\beta : -\Gamma < \beta < \Gamma\}$$

is a smooth curve of solutions of (2.1) *such that*

$$\varphi^\varepsilon_\beta \text{ has period } \frac{2\pi}{\mu_\varepsilon(\beta)} = \tau^\varepsilon_\beta, \text{ say},$$

and the following estimates hold:

$$0 \leqq \alpha_\varepsilon(\beta) \leqq \tfrac{1}{4}, \qquad \beta \in (-\Gamma, \Gamma), \tag{2.34}$$

$$K_1\alpha_\varepsilon(\beta) \leqq \beta^2 \leqq K_2\alpha_\varepsilon(\beta), \qquad \beta \in (-\Gamma, \Gamma), \tag{2.35}$$

$$K_1\beta \leqq \frac{d\alpha_\varepsilon}{d\beta}(\beta) \leqq K_2\beta, \qquad \beta \in (0, \Gamma), \tag{2.36}$$

$$\begin{aligned} &\frac{1}{2\varepsilon^2}\{1 + \sqrt{1 + 4\varepsilon^2 - 8\varepsilon^2\alpha_\varepsilon(\beta)}\} + K_1\varepsilon^2\beta^2 \leqq \mu_\varepsilon(\beta)^2 \\ &\qquad \leqq \frac{1}{2\varepsilon^2}\{1 + \sqrt{1 + 4\varepsilon^2 - 8\varepsilon^2\alpha_\varepsilon(\beta)}\} + K_2\varepsilon^2\beta^2, \quad \beta \in (-\Gamma, \Gamma), \end{aligned} \tag{2.37}$$

$$-K_2\varepsilon\beta \leqq \frac{d\mu_\varepsilon}{d\beta}(\beta) \leqq -K_1\varepsilon\beta, \qquad \beta \in (0, \Gamma). \tag{2.38}$$

Here K_1 *and* K_2 *are positive constants independent of* β *and* ε.

3. Homoclinic Orbits

3.1. Introductory remarks

Recall that the equation $\sigma'' - \sigma + \sigma^2 = 0$ on $\mathbf{R}$ is satisfied by the function

$$\sigma(x) = \tfrac{3}{2}\operatorname{sech}^2\frac{x}{2}. \tag{3.1}$$

Thus σ is a solution of equation (1.1) with $\varepsilon = 0$. Moreover, for each $\varepsilon \in (0, \varepsilon_0)$ and $\beta \in (-\Gamma, \Gamma)$ it is proved in Section 2 that there exists a solution $\varphi_\beta^\varepsilon$ which is periodic of period τ_β^ε where $\tau_\beta^\varepsilon = 2\pi\varepsilon(1 + O(\varepsilon^2))$ as $\varepsilon \to 0$ uniformly for $\beta \in (-\Gamma, \Gamma)$, by (2.37). For any $p \in [0, \frac{1}{2})$ let

$$\varphi_{\beta,p}^\varepsilon(x) = \varphi_\beta^\varepsilon(x - p\tau_\beta^\varepsilon), \quad x \in \mathbf{R}. \tag{3.2}$$

The purpose of this section is to prove that for each $p \in [0, \frac{1}{2})$ there exists a solution T of (1.1) for all ε sufficiently small (how small depending on p) such that T is even and

$$|T(x) - \varphi_{\beta,p}^\varepsilon(x)| \to 0 \text{ exponentially as } x \to \infty,$$

where β is a function of ε and p. Recall that, in the notation of (2.33), for any $p \in [0, \frac{1}{2})$

$$\varphi_{\beta,p}^\varepsilon(x) = \alpha_\varepsilon(\beta) + \beta\cos(\mu_\varepsilon(\beta)\,x - 2\pi p) + \varepsilon^2\beta^2\Psi_\beta^\varepsilon(\mu_\varepsilon(\beta)\,x - 2\pi p). \tag{3.3}$$

Note that μ_ε and α_ε are even functions of β. It should be noted that there is no loss of generality in restricting p to the interval $[0, \frac{1}{2})$ since by (2.11), (2.32) and (2.33) $\varphi_{\beta,\frac{1}{2}}^\varepsilon = \varphi_{-\beta}^\varepsilon$. For technical reasons the analysis in the case $p = 0$ and $p \in (0, \frac{1}{2})$ is different, though some notation is common to both.

Since $\tau_\beta^\varepsilon = 2\pi\varepsilon(1 + O(\varepsilon^2))$ as $\varepsilon \to 0$ uniformly for $\beta \in (-\Gamma, \Gamma)$ it is possible to choose $\varepsilon_0 > 0$ such that $\tau_\beta^\varepsilon \in [\pi\varepsilon, 3\pi\varepsilon]$ for all $\varepsilon \in (0, \varepsilon_0)$ and $\beta \in (-\Gamma, \Gamma)$. Let $\eta: \mathbf{R} \to \mathbf{R}$ be a C^∞-function such that

$$\eta(x) = 0 \text{ if } x \leqq \tfrac{\pi}{2}, \quad \eta(x) = 1 \text{ if } x \geqq \pi$$

and such that $|\eta^{(k)}(x)| \leqq M$, $k \in \{0, 1, 2, 3, 4\}$, $x \in \mathbf{R}$. (Here, as elsewhere, $f^{(k)}$ is a convenient notation for the kth derivative of f.) Now for all $\varepsilon \in (0, \varepsilon_0)$ and $p \in (0, \frac{1}{2})$ let

$$\eta_p^\varepsilon(x) = \eta\left(\frac{x}{p\varepsilon}\right),$$

so that

$$(\eta_p^\varepsilon)^{(k)}(0) = (\eta_p^\varepsilon)^{(k)}(p\tau_\beta^\varepsilon) = 0, \quad k \in \mathbf{N}, \quad \eta_p^\varepsilon(p\tau_\beta^\varepsilon) = 1, \quad \eta_p^\varepsilon(0) = 0, \tag{3.4}$$

$$(\eta_p^\varepsilon)^{(k)} \leqq M(p\varepsilon)^{-k}, \quad k \in \{0, 1, 2, 3, 4\}. \tag{3.5}$$

Now let

$$\Phi_{\beta,0}^\varepsilon(x) = \varphi_\beta^\varepsilon(x), \quad x \in \mathbf{R}, \tag{3.6}$$

and for $p \in (0, \frac{1}{2})$ let $\Phi_{\beta,p}^\varepsilon$ be an even function on $\mathbf{R}$ given by

$$\Phi_{\beta,p}^\varepsilon(x) = \eta_p^\varepsilon(x)\,\varphi_{\beta,p}^\varepsilon(x), \quad x \geqq 0. \tag{3.7}$$

Now we seek a solution T of (1.1) in the form

$$T(x) = \sigma(x) + \Phi_{\beta,p}^\varepsilon(x) + \omega(x) \tag{3.8}$$

where ω is an even function which tends to zero exponentially at infinity. Such a function T satisfies (1.1) if and only if ω satisfies

$$\varepsilon^2\omega^{iv} + \omega'' - \omega + 2\sigma\omega = -\varepsilon^2\sigma^{iv} - 2\sigma\Phi^{\varepsilon}_{\beta,p} - 2\Phi^{\varepsilon}_{\beta,p}\omega - \omega^2 - V^{\varepsilon}_{\beta,p}, \quad (3.9)$$

where

$$V^{\varepsilon}_{\beta,0}(x) = 0, \quad x \in \mathbf{R}, \quad (3.10)$$

$$V^{\varepsilon}_{\beta,p} = \varepsilon^2(\Phi^{\varepsilon}_{\beta,p})^{iv} + (\Phi^{\varepsilon}_{\beta,p})'' - \Phi^{\varepsilon}_{\beta,p} + (\Phi^{\varepsilon}_{\beta,p})^2. \quad (3.11)$$

The remainder of this paper focuses on the question of existence for a solution of equation (3.9) which decays to zero exponentially at infinity.

3.2. *Function spaces and invertible operators*

In this section we define some function spaces and fashion some results on invertibility of operators suitable for the analysis to follow.

Let $q \in (0, 1)$ be fixed throughout, and let Z denote the Banach space of continuous even functions f on $\mathbf{R}$ with

$$\|f\| = \sup\{e^{q|x|}|f(x)| : x \in \mathbf{R}\} < \infty.$$

Note that integration on (x, ∞) gives $\|f^{(k)}\| \leqq q^{k-n}\|f^{(n)}\|$, $0 \leqq k \leqq n$. Let

$$Z_n = \{f \in Z : f^{(k)} \in Z, \ 0 \leqq k \leqq n, \ k \in \mathbf{N} \cup \{0\}\}$$

with

$$\|f\|_n = \sum_{k=0}^{n} \|f^{(k)}\|.$$

The usual space of r-th power "Lebesgue integrable functions" on $\mathbf{R}$ will be denoted by $L_r(\mathbf{R})$ and its norm will be denoted $|\cdot|_r$, $1 \leqq r \leqq \infty$. Our purpose is to study the operator on the left-hand side of (3.9). For convenience let

$$\mathscr{A}_\varepsilon u = \varepsilon^2 u^{iv} + u'' - u + 2\sigma u.$$

We need some further notation. Suppose that

$$k_\varepsilon = \varepsilon^{-1}\sqrt{\{(1 + \sqrt{(1 + 4\varepsilon^2)})/2\}} \quad \text{so that} \quad \varepsilon^2 k_\varepsilon^4 - k_\varepsilon^2 - 1 = 0. \quad (3.12)$$

Then $v_\varepsilon(x) = \cos k_\varepsilon x$ is the bounded solution of the equation

$$\varepsilon^2 v^{iv} + v'' - v = 0; \quad (3.13)$$

it is unique (up to translations and scalar multiplications). Let $K_\varepsilon = \varepsilon k_\varepsilon$ and note that $K_\varepsilon \to 1$ as $\varepsilon \to 0$. Now define a differential operator D_ε by

$$D_\varepsilon u = u'' - K_\varepsilon^{-2}u + 2K_\varepsilon^{-2}\sigma u, \quad \varepsilon \in [0, \varepsilon_0)$$

where K_0 is defined to be 1. The following results are given with applications to equation (3.9) in mind.

Lemma 3.1. *The number* $\varepsilon_0 > 0$ *can be chosen such that*
(a) D_ε *defines a linear homeomorphism from* Z_2 *onto* Z *such that the norm of* D_ε^{-1} *is bounded independent of* $\varepsilon \in [0, \varepsilon_0)$.
(b) *If* f *is a continuous even function with compact support in* $(-a, a)$, $a < 1$,

$$\left| \int_{-a}^{a} f(x)\, dx \right| = c, \qquad D_0 u = f \text{ and } u \in Z,$$

then there are constants independent of a, c *and* ε *such that*

$$\|u\| \leqq \text{const. } (c + a^2 |f|_\infty)$$

$$|u'(x)| \leqq \text{const. } (c + a^2 |f|_\infty)\, |\sigma''(x)|, \qquad |x| \geqq a,$$

$$|u'(x)| \leqq \text{const. } (1 + a)\, |f|_\infty |x|, \qquad |x| \leqq a,$$

and if $D_\varepsilon u_\varepsilon = f$, *then* $\|u - u_\varepsilon\|_2 \leqq \text{const. } \varepsilon^2 (c + a^2 |f|_\infty)$.
(c) *If* $D_\varepsilon u = v_\varepsilon \sigma$, *then* $u = -k_\varepsilon^{-2} \sigma v_\varepsilon + \varepsilon w_\varepsilon$ *where* $\|w_\varepsilon\|_2$ *is bounded independently of* $\varepsilon \in [0, \varepsilon_0)$.

Proof. Since $K_\varepsilon \to 1$ as $\varepsilon \to 0$, it is enough, by continuity, to establish (a) for D_0. Part (a) in this case is already proved in [2], Lemma 13, and [10].
(b) Suppose $D_0 u = f$. Then there is a constant such that

$$|u|_\infty + |u'|_\infty \leqq |u|_1 + |u'|_1 + |u''|_1 \leqq \text{const.} |f|_1 \leqq \text{const.}\, a |f|_\infty. \tag{3.14}$$

(See [10, p. 276].) Hence

$$|u'(a)| = \left| \int_0^a u''(x)\, dx \right| = \left| \int_0^a \{u(x) - 2\sigma(x)\, u(x) + f(x)\} dx \right|$$

$$\leqq \text{const. } (a^2 |f|_\infty + c). \tag{3.15}$$

Now $D_0 \sigma' = 0$ on $[a, \infty)$ because $\sigma'' - \sigma + \sigma^2 = 0$ on $\mathbf{R}$, and $D_0 u = 0$ on $[a, \infty)$ because $\operatorname{supp}(f) \subset (-a, a)$. Hence $u = \alpha \sigma'$ on $[a, \infty)$. (Otherwise u and σ' would be linearly independent solutions of $D_0 w = 0$ on $[a, \infty)$, which would imply that all solutions of $D_0 w = 0$ on $[a, \infty)$ tend to 0 at infinity. If $x_0 > a$ is such that $\sigma(x_0) = \frac{1}{4}$, and w_0 is the solution on $[x_0, \infty)$ of the initial-value problem $D_0 w = 0$ with $w(0) = w'(0) = 1$, then clearly $w_0(x) \to \infty$ as $x \to \infty$. This would then be a contradiction.) From (3.15) and the fact that $a \in (0, 1)$ we deduce that

$$|\alpha| = |u'(a)/\sigma''(a)| \leqq \text{const. } (a^2 |f|_\infty + c), \tag{3.16}$$

whence

$$|u'(x)| \leqq \text{const. } (a^2 |f|_\infty + c)\, |\sigma''(x)|, \qquad |x| \geqq a. \tag{3.17}$$

Moreover, for $x \in [0, a]$,

$$|u'(x)| = \left| \int_0^x u''(x)\, dx \right| \leqq \text{const. } (1 + a)\, |f|_\infty x \tag{3.18}$$

by (3.14). Also

$$|u(x)| \leqq \text{const.}\,(a^2|f|_\infty + c)\,|\sigma'(x)|, \qquad x \geqq a, \tag{3.19}$$

and for $x \in [0, a]$

$$|u(x)| \leqq |u(a)| + \int_x^a |u'(x)|\,dx \leqq |u(a)| + \text{const.}\,(1+a)\,|f|_\infty\,(a^2 - x^2)$$

$$\leqq \text{const.}\,(c + a^2|f|_\infty), \tag{3.20}$$

by (3.18). Combining (3.19) and (3.20) completes the proof of the estimates of $D_0^{-1}f$ in (b). If $D_\varepsilon u_\varepsilon = f$, then $D_\varepsilon(u_\varepsilon - u) = (K_\varepsilon^{-2} - 1)\,(u - 2\sigma u)$. Hence

$\|u_\varepsilon - u\|_2 \leqq \text{const.}\,\varepsilon^2\|u\|$, from the definition of K_ε and part (a),

$\leqq \text{const.}\,\varepsilon^2(c + a^2|f|_\infty)$, by the preceding estimate.

(c) Note that

$$D_\varepsilon(-k_\varepsilon^{-2}\sigma v_\varepsilon) = \sigma v_\varepsilon - \varepsilon r_\varepsilon$$

where

$$r_\varepsilon(x) = \varepsilon^{-1}\{k_\varepsilon^{-2}\sigma''(x)\,v_\varepsilon(x) - 2k_\varepsilon^{-1}\sigma'(x)\sin k_\varepsilon x - k_\varepsilon^{-2}K_\varepsilon^{-2}\,\sigma(x)\,v_\varepsilon(x)$$
$$+ 2k_\varepsilon^{-2}K_\varepsilon^{-2}\sigma^2(x)\,v_\varepsilon(x)\}.$$

Now $\|r_\varepsilon\|$ is bounded independently of ε since $\varepsilon k_\varepsilon \to 1$ as $\varepsilon \to 0$. Thus part (c) follows by part (a). q.e.d.

The following result lies at the heart of the analysis. The operator $\mathscr{A}_\varepsilon$ is defined before (3.12).

Theorem 3.2. *Suppose that $u \in Z_4$ and $\mathscr{A}_\varepsilon u = f$. Then there exists a constant independent of ε such that*

$$\varepsilon\|u'\| + \|u\| \leqq \text{const.}\,\|D_\varepsilon^{-1}f\|_1 \leqq \text{const.}\,\|f\|. \tag{3.21}$$

Moreover, if $f = g + h$ where h has compact support in $(-a, a)$ and $\left|\int_{-a}^{a} h(x)\,dx\right| = c$, then

$$\varepsilon\|u'\| + \|u\| \leqq \text{const.}\,(c + a^2|h|_\infty + \|g\|). \tag{3.22}$$

Proof. Let

$$v(x) = \varepsilon^2 u''(x) + K_\varepsilon^2 u(x) - 2\varepsilon^2 K_\varepsilon^{-2}\sigma(x)\,u(x). \tag{3.23}$$

Then because $\mathscr{A}_\varepsilon u = f$ and $\sigma'' = \sigma - \sigma^2$, it follows that

$$D_\varepsilon v = \varepsilon^2\{-4K_\varepsilon^{-2}\sigma' u' - 2K_\varepsilon^{-4}(K_\varepsilon^2 - 1 + (2 - K_\varepsilon^2)\,\sigma)\,\sigma u\} + f$$

$$= \varepsilon^2\{w_1 + w_2\} + f, \quad \text{say}, \tag{3.24}$$

where $\|w_1\| \leqq \text{const.}\|u'\|$ and $\|w_2\| \leqq \text{const.}\|u\|$. A substitution from (3.23) now gives

$$\varepsilon^2 u'' + K_\varepsilon^2 u = \varepsilon^2 D_\varepsilon^{-1}(w_1 + w_2) + 2\varepsilon^2 K_\varepsilon^{-2}\sigma u + D_\varepsilon^{-1} f.$$

Now a multiplication by u' followed by integration over (x, ∞) for $x \geqq 0$ gives

$$\varepsilon^2 u'(x)^2 + K_\varepsilon^2 u(x)^2 = \left| 2\int_x^\infty \{\varepsilon^2 D_\varepsilon^{-1}(w_1 + w_2) + 2\varepsilon^2 K_\varepsilon^{-2}\sigma u + D_\varepsilon^{-1} f\} u'\, dy \right|$$

$$\leqq \text{const.}\left(\varepsilon^2 \left\{ e^{-2qx}\|u\|\,\|u'\| + \left|\int_x^\infty (D_\varepsilon^{-1} w_1)\, u'\, dy\right|\right\} + \left|\int_x^\infty (D_\varepsilon^{-1} f)\, u'\right|\right)$$

$$\leqq \text{const.}\left(\varepsilon^2 e^{-2qx}\{\|u\|\,\|u'\| + \|D_\varepsilon^{-1} w_1\|\,\|u\|\} + \int_x^\infty |(D_\varepsilon^{-1} w_1)' u|\, dy\right.$$

$$\left. + e^{-2qx}\|D_\varepsilon^{-1} f\|\,\|u\| + \int_x^\infty |(D_\varepsilon^{-1} f)' u|\, dy\right)$$

$$\leqq \text{const.}\, e^{-2qx}\{\varepsilon^2\|u\|\,\|u'\| + \|u\|\,\|D_\varepsilon^{-1} f\|_1\}. \tag{3.25}$$

A similar estimate holds for $x \leqq 0$. Now multiply both sides by $e^{2q|x|}$ and take the supremum of each term on the left-hand side and add to get

$$\varepsilon^2\|u'\|^2 + K_\varepsilon^2\|u\|^2 \leqq \text{const.}\{\varepsilon^2\|u\|\,\|u'\| + \|u\|\,\|D_\varepsilon^{-1} f\|_1\}$$

$$\leqq \text{const.}\{(\varepsilon^3\|u'\|^2 + \varepsilon\|u\|^2) + \|u\|\,\|D_\varepsilon^{-1} f\|_1\},$$

by Young's Inequality. Hence for all $\varepsilon > 0$ sufficiently small, there exists a constant such that

$$\varepsilon^2\|u'\|^2 + \|u\|^2 \leqq \text{const.}\,\|u\|\,\|D_\varepsilon^{-1} f\|_1,$$

from which (3.21) follows. (Recall that $\|D_\varepsilon^{-1} f\|_1 \leqq \text{const.}\|f\|$ by Lemma 3.1.)

To obtain (3.22) we return to (3.25), with $f = g + h$ to get

$$\varepsilon^2 u'(x)^2 + K_\varepsilon^2 u(x)^2 \leqq \text{const.}\,(\varepsilon^2 e^{-2qx}\|u\|\,\|u'\| + \|u\|\,\|g\| + e^{-2qx}\|D_\varepsilon^{-1} h\|\,\|u\|$$

$$+ \left(\int_x^\infty |(D_\varepsilon^{-1} h)' u|\, dy + \varepsilon^2(c + a^2|h|_\infty)\,\|u\|\right)$$

$$\leqq \text{const.}\,(e^{-2qx}\{\varepsilon^2\|u\|\,\|u'\| + \|u\|\,(\|g\| + c + a^2|h|_\infty)\})$$

since, by Lemma 3.1(b), $\|D_\varepsilon^{-1}h - D_0^{-1}h\|_2 \leqq \text{const.}\, \varepsilon^2(c + a^2|h|_\infty)$ and

$$\int_x^\infty |(D_0^{-1}h)'u|\,dy \leqq \text{const.}\ (c + a^2|h|_\infty) \int_x^\infty |u\sigma''|\,dy$$

$$\leqq \text{const.}\, e^{-2qx}(c + a^2|h|_\infty)\,\|u\|, \quad x \geqq a, \tag{3.26}$$

and for $x \in (0, a)$

$$\int_x^\infty |(D_0^{-1}h)'u|\,dy = \left(\int_x^a + \int_a^\infty\right) |(D_0^{-1}h)'u|\,dy$$

$$\leqq \text{const.}\left(\int_x^a ((1+a)|h|_\infty y)\,u(y)\,dy + e^{-2qa}(c + a^2|h|_\infty)\|u\|\right)$$

$$\leqq \text{const.}\,(a^2(1+a)|h|_\infty + c + a^2|h|_\infty)\,\|u\|\,e^{-2qx}$$

$$\leqq \text{const.}\ (c + a^2|h|_\infty)\,\|u\|\,e^{-2qx}.$$

Hence

$$\varepsilon^2\|u'\|^2 + \|u\|^2 \leqq \text{const.}\ (\varepsilon^2\|u\|\,\|u'\| + \|u\|\,(\|g\| + c + a^2|h|_\infty)),$$

and (3.22) now follows as in an earlier calculation. q.e.d.

3.3. A necessary condition for existence

If equation (3.9) is to have a solution $\omega \in Z$, then

$$\int_{-\infty}^\infty \{\varepsilon^2 k_\varepsilon^4 \sigma v_\varepsilon + 2\sigma\, v_\varepsilon \Phi_{\beta,p}^\varepsilon + 2\sigma v_\varepsilon \omega + 2\Phi_{\beta,p}^\varepsilon \omega v_\varepsilon + \omega^2 v_\varepsilon + v_\varepsilon V_{\beta,p}^\varepsilon\}\,dx = 0 \tag{3.27}$$

because v_ε satisfies (3.13). The purpose of this section is to show how (3.27) implies that β is a function of ω. To see this we need some estimates based on (2.38), (3.3), (3.6), (3.7), (3.10) and (3.11).

If $p \in (0, \frac{1}{2})$, $\varepsilon \in (0, \varepsilon_0)$ and $\beta \in (-\Gamma, \Gamma)$, then

$$V_{\beta,p}^\varepsilon \text{ is even}, \quad \operatorname{supp}(V_{\beta,p}^\varepsilon) \subset (-\tfrac{3}{2}\pi p\varepsilon, \tfrac{3}{2}\pi p\varepsilon), \tag{3.28}$$

$$\left|\int_{-\infty}^\infty V_{\beta,p}^\varepsilon(x)\,dx\right| = \left|\int_{-p\tau_\beta^\varepsilon}^{p\tau_\beta^\varepsilon} V_{\beta,p}^\varepsilon(x)\,dx\right| = \left|\int_{-p\tau_\beta^\varepsilon}^{p\tau_\beta^\varepsilon} \{\Phi_{\beta,p}^\varepsilon(x) - (\Phi_{\beta,p}^\varepsilon(x))^2\}dx\right|$$

$$\leqq \left|\int_{-3\pi\varepsilon p/2}^{3\pi\varepsilon p/2} \{\Phi_{\beta,p}^\varepsilon(x) - (\Phi_{\beta,p}^\varepsilon(x))^2\}dx\right| \leqq \text{const.}\,\varepsilon p\beta, \tag{3.29}$$

$$|V_{\beta,p}^\varepsilon|_\infty \leqq \text{const.}\left(\frac{\beta}{\varepsilon^2 p^4}\right), \tag{3.30}$$

and

$$\begin{aligned}
\int_{-\infty}^{\infty} \nu_\varepsilon V^\varepsilon_{\beta,p}\,dx &= 2\int_0^{p\tau^\varepsilon_\beta} \nu_\varepsilon V^\varepsilon_{\beta,p}\,dx \\
&= -2[\varepsilon^2\{(\Phi^\varepsilon_{\beta,p})''\nu'_\varepsilon + \Phi^\varepsilon_{\beta,p}\nu'''_\varepsilon\} + \Phi^\varepsilon_{\beta,p}\nu'_\varepsilon]_0^{p\tau^\varepsilon_\beta} + 2\int_0^{p\tau^\varepsilon_\beta} \nu_\varepsilon(\Phi^\varepsilon_{\beta,p})^2\,dx \\
&= 2\sin(k_\varepsilon p\tau^\varepsilon_\beta)\,[\varepsilon^2 k_\varepsilon(\Phi^\varepsilon_{\beta,p})'' - \varepsilon^2 k_\varepsilon^3\Phi^\varepsilon_{\beta,p} + k_\varepsilon\Phi^\varepsilon_{\beta,p}](p\tau^\varepsilon_\beta) \\
&\quad + 2\int_0^{p\tau^\varepsilon_\beta} \nu_\varepsilon(\Phi^\varepsilon_{\beta,p})^2\,dx \\
&= 2k_\varepsilon\sin(k_\varepsilon p\tau^\varepsilon_\beta)\,[\varepsilon^2(\varphi^\varepsilon_\beta)'' - (K^2_\varepsilon - 1)\,\varphi^\varepsilon_\beta]\,(0) + 2\int_0^{p\tau^\varepsilon_\beta} \nu_\varepsilon(\Phi^\varepsilon_{\beta,p})^2\,dx \\
&= 2k_\varepsilon\sin(k_\varepsilon p\tau^\varepsilon_\beta)\,\{-\varepsilon^2\beta(\mu_\varepsilon(\beta))^2 - \varepsilon^2(\alpha_\varepsilon(\beta)+\beta) + O(\varepsilon^2\beta^2(\mu_\varepsilon(\beta))^2) \\
&\quad + O(\varepsilon^4\beta)\} + 2\int_0^{p\tau^\varepsilon_\beta} \nu_\varepsilon(\Phi^\varepsilon_{\beta,p})^2\,dx \qquad \text{(by (2.33))} \\
&= 2k_\varepsilon\sin(k_\varepsilon p\tau^\varepsilon_\beta)\,\{-(1+2\varepsilon^2)\beta + O(\beta^2) + O(\varepsilon^4\beta)\} + O(\varepsilon p\beta^2) \\
&= 2k_\varepsilon\,(\sin 2\pi p + O(p\varepsilon^2\beta^2))\,\{-(1+2\varepsilon^2)\beta + O(\beta^2) + O(\varepsilon^4\beta)\} \\
&\quad + O(\varepsilon p\beta^2) \\
&= 2k_\varepsilon\sin(2\pi p)\,(-(1+2\varepsilon^2)\beta + O(\beta^2) + O(\varepsilon^4\beta)) + O(\varepsilon p\beta^2) \\
&\quad \text{as } \varepsilon\to 0. \qquad (3.31)
\end{aligned}$$

Therefore if $p\in(0,\frac{1}{2})$, Γ and ε_0 can be chosen to be independent of p such that

$$\int_{-\infty}^{\infty} \nu_\varepsilon V^\varepsilon_{\beta,p}\,dx \leqq -k_\varepsilon\beta\sin(2\pi p), \qquad \varepsilon\in(0,\varepsilon_0), \tag{3.32}$$

uniformly for $\beta\in(-\Gamma,\Gamma)$. Similar calculations yield that for $p\in(0,\frac{1}{2})$, $\varepsilon\in(0,\varepsilon_0)$ and $\beta\in(-\Gamma,\Gamma)$,

$$\left|\frac{\partial}{\partial\beta}\left(\int_{-\infty}^{\infty} V^\varepsilon_{\beta,p}(x)\,dx\right)\right| \leqq \text{const.}\ \varepsilon p, \qquad \left|\frac{\partial}{\partial\beta}V^\varepsilon_{\beta,p}\right|_\infty \leqq \text{const.}\ \varepsilon^{-2}p^{-4}, \tag{3.33}$$

$$\frac{\partial}{\partial\beta}\int_{-\infty}^{\infty} \nu_\varepsilon(x)\,V^\varepsilon_{\beta,p}(x)\,dx \leqq -k_\varepsilon\sin(2\pi p). \tag{3.34}$$

It is useful to note at this stage that for any $p \in [0, \frac{1}{2})$ (taking $\eta_0^\varepsilon \equiv 1$)

$$
\begin{aligned}
\frac{\partial}{\partial\beta}\left(\int_{-\infty}^{\infty} \sigma\, \Phi_{\beta,p}^\varepsilon \nu_\varepsilon\, dx\right) &= 2\frac{\partial}{\partial\beta}\left(\int_0^\infty \sigma(x)\, \eta_p^\varepsilon(x)\, \varphi_\beta^\varepsilon\left(x - \frac{2\pi p}{\mu_\varepsilon(\beta)}\right) \nu_\varepsilon(x)\, dx\right) \\
&= 2\frac{\partial}{\partial\beta}\left(\int_0^\infty \sigma(x)\, \eta_p^\varepsilon(x)\, \{\alpha_\varepsilon(\beta) + \beta\cos(\mu_\varepsilon(\beta)x - 2\pi p)\right. \\
&\qquad \left. + \varepsilon^2\beta^2 \Psi_\beta^\varepsilon(\mu_\varepsilon(\beta)x - 2\pi p)\}\, \nu_\varepsilon(x)\, dx\right) \\
&= 2\int_0^\infty \sigma(x)\, \eta_p^\varepsilon(x)\, \{\alpha_\varepsilon'(\beta) + \cos(\mu_\varepsilon(\beta)x - 2\pi p) \\
&\qquad + 2\varepsilon^2\beta \Psi_\beta^\varepsilon(\mu_\varepsilon(\beta)x - 2\pi p)\}\, \nu_\varepsilon(x)\, dx \\
&\quad + 2\int_0^\infty x\beta\mu_\varepsilon'(\beta)\, \sigma(x)\, \eta_p^\varepsilon(x)\{-\sin(\mu_\varepsilon(\beta)x - 2\pi p) \\
&\qquad + \varepsilon^2\beta(\Psi_\beta^\varepsilon)'\,(\mu_\varepsilon(\beta)x - 2\pi p)\, \nu_\varepsilon(x)\}\, dx \\
&\quad + 2\int_0^\infty \sigma(x)\, \eta_p^\varepsilon(x)\, \varepsilon^2\beta^2\left(\frac{\partial}{\partial\beta}(\Psi_\beta^\varepsilon)\right)(\mu_\varepsilon(\beta)x - 2\pi p)\, \nu_\varepsilon(x)\, dx \\
&= 2\alpha_\varepsilon'(\beta)\int_0^\infty \sigma(x)\, \eta_p^\varepsilon(x)\, \nu_\varepsilon(x)\, dx \\
&\quad + 2\int_0^\infty \sigma(x)\, \eta_p^\varepsilon(x)\cos(\mu_\varepsilon(\beta)x - 2\pi p)\, \nu_\varepsilon(x)\, dx \\
&\quad + O(\varepsilon\beta^2) + O(\varepsilon^2\beta) + O(\varepsilon^2\beta^2) \\
&= \alpha_\varepsilon'(\beta)\int_{-\infty}^\infty \sigma(x)\, \nu_\varepsilon(x)\, dx \\
&\quad + 2\int_0^\infty \sigma(x)\, \eta_p^\varepsilon(x)\cos(\mu_\varepsilon(\beta)x - 2\pi p)\, \nu_\varepsilon(x)\, dx \\
&\quad + O(\varepsilon\beta).
\end{aligned}
$$

Now $\alpha'(\beta) = O(\beta)$ as $\beta \to 0$ by (2.36). Hence by the Dominated Convergence Theorem, and by (3.38) and (3.40) below, it follows that if $p = 0$, then

$$\frac{\partial}{\partial\beta}\left(\int_{-\infty}^{\infty} \sigma\, \Phi^{\varepsilon}_{\beta,p} v_{\varepsilon}\, dx\right) \to 3 \quad \text{as } \varepsilon \to 0. \tag{3.35}$$

Theorem 3.3. *Let $p \in [0, \frac{1}{2})$ be given. Then there exist positive numbers ε_p and ρ_p such that if $\varepsilon \in (0, \varepsilon_p)$ and $\omega \in Z$ with $\|\omega\| \leqq \rho_p$, then there exists a unique $\beta \in (-\Gamma, \Gamma)$ for which (3.27) holds. Moreover, the dependence of β on $\omega \in Z$ is smooth and there is a constant (depending on p) such that*

$$|\beta(0)| \leqq \text{const.}\left(\varepsilon^{-3} \operatorname{cosech} \frac{\pi}{\varepsilon}\right), \tag{3.36}$$

$$\|\beta(\omega)\| \leqq \text{const.}\,(\beta(0) + \|\omega\|) \quad \textit{if} \quad \|\omega\| \leqq \rho_p. \tag{3.37}$$

Proof. We begin by observing that for any $k > 0$ a contour integration gives

$$\int_{-\infty}^{\infty} \cos kx \sigma(x)\, dx = \frac{3}{2} \int_{-\infty}^{\infty} \cos kx \operatorname{sech}^2\left(\frac{x}{2}\right) dx = 6\pi k \operatorname{cosech}(k\pi). \tag{3.38}$$

Hence if $k_1, k_2 \geqq 0$, then

$$\int_{-\infty}^{\infty} \cos(k_1 x) \cos(k_2 x)\, \sigma(x)\, dx$$

$$= 3\pi\{(k_1 + k_2) \operatorname{cosech}((k_1 + k_2)\pi) + (k_1 - k_2) \operatorname{cosech}((k_1 - k_2)\pi)\}. \tag{3.39}$$

Therefore by (2.37),

$$\int_{-\infty}^{\infty} v_{\varepsilon}(x) \cos(\mu_{\varepsilon}(\beta) x)\, \sigma(x)\, dx \to 3 \quad \text{as } \varepsilon \to 0 \tag{3.40}$$

uniformly for $\beta \in (-\Gamma, \Gamma)$. Now by using (3.38) we can rewrite (3.27) as

$$6\pi k_{\varepsilon}^5 \varepsilon^2 \operatorname{cosech}(k_{\varepsilon}\pi) + \int_{-\infty}^{\infty} v_{\varepsilon} \omega\{\omega + 2\sigma + 2\Phi^{\varepsilon}_{\beta,p}\} dx + \int_{-\infty}^{\infty} (2\sigma\, \Phi^{\varepsilon}_{\beta,p} + V^{\varepsilon}_{\beta,p})\, v_{\varepsilon} dx = 0. \tag{3.41}$$

Now $\int_{-\infty}^{\infty} 2\sigma v_{\varepsilon}\, \Phi^{\varepsilon}_{\beta,p}\, dx = \beta W_1(\beta, p, \varepsilon)$, where W_1 is bounded for $\beta \in (-\Gamma, \Gamma)$, $p \in [0, \frac{1}{2})$, $\varepsilon \in (0, \varepsilon_0)$, by (3.3) and (3.7). Also

$$\left| 6\pi k_{\varepsilon}^5 \varepsilon^2 \operatorname{cosech}(k_{\varepsilon}\pi) + \int_{-\infty}^{\infty} v_{\varepsilon} \omega\{\omega + 2\sigma + 2\Phi^{\varepsilon}_{\beta,p}\} dx \right|$$

$$\leqq \text{const.}\left[\varepsilon^{-3} \operatorname{cosech}\left(\frac{\pi}{\varepsilon}\right) + \|\omega\|\right] \tag{3.42}$$

for all $p \in [0, \frac{1}{2})$. If $p \in (0, \frac{1}{2})$, then (3.32) and the preceding two observations yield that ε_p and ρ_p can be chosen such that the existence of a unique $\beta \in (-\Gamma, \Gamma)$ for each $\varepsilon \in (0, \varepsilon_p)$ and $\omega \in Z$ with $\|\omega\| \leqq \rho_p$ follows by the Mean-Value Theorem for functions of one real variable.

Now suppose that $p = 0$, so that $V^{\varepsilon}_{\beta,p} \equiv 0$ and $\Phi^{\varepsilon}_{\beta,p} = \varphi^{\varepsilon}_{\beta}$. Therefore in this case

$$\int_{-\infty}^{\infty} 2\sigma v_\varepsilon \varphi^{\varepsilon}_{\beta,p}\, dx = 2 \int_{-\infty}^{\infty} \sigma v_\varepsilon \{\alpha_\varepsilon(\beta) + \beta \cos(\mu_\varepsilon(\beta) x) + \varepsilon^2 \beta^2 \Psi^{\varepsilon}_{\beta}(\mu_\varepsilon(\beta) x)\}\, dx$$

$$= 2\alpha_\varepsilon(\beta) \int_{-\infty}^{\infty} \sigma v_\varepsilon\, dx + 2\beta \int_{-\infty}^{\infty} \sigma(x)\, v_\varepsilon(x) \cos(\mu_\varepsilon(\beta) x)\, dx$$

$$+ 2\varepsilon^2 \beta^2 \int_{-\infty}^{\infty} \sigma(x)\, v_\varepsilon(x)\, \Psi^{\varepsilon}_{\beta}(\mu_\varepsilon(\beta) x)\, dx$$

$$= 2\beta \left\{ 6\pi \frac{\alpha_\varepsilon(\beta)}{\beta} k_\varepsilon \operatorname{cosech}(k_\varepsilon \pi) + \int_{-\infty}^{\infty} \sigma(x)\, v_\varepsilon(x) \cos(\mu_\varepsilon(\beta) x)\, dx \right.$$

$$\left. + \varepsilon^2 \beta \int_{-\infty}^{\infty} \sigma(x)\, v_\varepsilon(x)\, \Psi^{\varepsilon}_{\beta}(\mu_\varepsilon(\beta) x)\, dx \right\},$$

so that the last term in (3.41) converges to 6β as $\varepsilon \to 0$ by (3.40). The existence of ε_0 and ρ_0 such that (3.41) has a unique solution $\beta \in (-\Gamma, \Gamma)$ when $p = 0$, $\varepsilon \in (0, \varepsilon_0)$ and $\|\omega\| \leqq \rho_0$ is again immediate from the Mean-Value Theorem. Now for all $p \in [0, \frac{1}{2})$ the estimate for $|\beta(0)|$ follows from (3.41), (3.42), and the smooth dependence of β on $\omega \in Z$ follows by the Implicit Function Theorem. This completes the proof of the theorem. q.e.d.

Let us denote the functional dependence of β on ω in this theorem by writing

$$\beta = \beta^{\varepsilon}_{p}(\omega), \quad \varepsilon \in (0, \varepsilon_p), \ \|\omega\| \leqq \rho_p. \tag{3.43}$$

We have already noted that

$$|\beta^{\varepsilon}_{p}(\omega)| \leqq \text{const.} \left[\varepsilon^{-3} \operatorname{cosech}\left(\frac{\pi}{\varepsilon}\right) + \|\omega\| \right], \tag{3.44}$$

and from (3.27) we can immediately calculate the Fréchet derivative of β^{ε}_{p} by the formula

$$d\beta^{\varepsilon}_{p}[\hat{\omega}]\, \omega = - \frac{\int_{-\infty}^{\infty} 2\omega \left(\Phi^{\varepsilon}_{\beta^{\varepsilon}_{p}(\hat{\omega}),p} + \hat{\omega} + \sigma\right) v_\varepsilon\, dx}{\frac{\partial}{\partial \beta} \left\{ \int_{-\infty}^{\infty} 2\sigma v_\varepsilon\, \Phi^{\varepsilon}_{\beta,p} + 2\hat{\omega} v_\varepsilon\, \Phi^{\varepsilon}_{\beta,p} + v_\varepsilon V^{\varepsilon}_{\beta,p}\, dx \right\} \Big|_{\beta = \beta^{\varepsilon}_{p}(\hat{\omega})}}. \tag{3.45}$$

So if $p \in (0, \frac{1}{2})$, there exists a constant such that

$$\| d\beta_p^\varepsilon[\hat{\omega}] \| \leqq \text{const.}\, \varepsilon, \quad \| \hat{\omega} \| \leqq \rho_p, \tag{3.46}$$

because of (3.34), since all the other terms in (3.45) are bounded.

If $p = 0$, then $V_{\beta,p}^\varepsilon \equiv 0$ and the denominator in (3.45) is

$$2 \int_{-\infty}^{\infty} \sigma v_\varepsilon \frac{\partial}{\partial \beta} (\Phi_{\beta,p}^\varepsilon)\, dx + 2 \int_{-\infty}^{\infty} \hat{\omega} v_\varepsilon \frac{\partial}{\partial \beta} (\Phi_{\beta,p}^\varepsilon)\, dx \geqq 1$$

by (3.35) for all $\varepsilon \in (0, \varepsilon_0)$ and $\| \hat{\omega} \| \leqq \rho_0$, provided ε_0 and ρ_0 are chosen sufficiently small. Therefore we conclude that when $p = 0$ there exists a constant such that

$$\| d\beta_0^\varepsilon[\hat{\omega}] \| \leqq \text{const.} \quad \text{if } \varepsilon \in (0, \varepsilon_0),\ \| \hat{\omega} \| \leqq \rho_0. \tag{3.47}$$

We record the formula

$$d\beta_p^\varepsilon[0]\, \omega = - \frac{\int_{-\infty}^{\infty} 2\omega \left(\Phi_{\beta_p^\varepsilon(0),p}^\varepsilon + \sigma \right) v_\varepsilon\, dx}{\frac{\partial}{\partial \beta} \left\{ \int_{-\infty}^{\infty} (2\sigma \Phi_{\beta,p}^\varepsilon + V_{\beta,p}^\varepsilon)\, v_\varepsilon\, dx \right\} \Big|_{\beta = \beta_p^\varepsilon(0)}}. \tag{3.48}$$

If $p \in [0, \frac{1}{2})$, $\beta \in (-\Gamma, \Gamma)$, $\varepsilon \in (0, \varepsilon_p)$ and if $\omega \in Z$ is a solution of (3.9) with $\| \omega \| \leqq \rho_p$, then $\beta = \beta_p^\varepsilon(\omega)$. Thus (3.9) is equivalent to the problem

$$\mathscr{A}_\varepsilon \omega = G_p^\varepsilon(\omega) \tag{3.49}$$

where

$$G_p^\varepsilon(\omega) = -\varepsilon^2 \sigma^{iv} - 2\sigma\, \Phi_{\beta_p^\varepsilon(\omega),p}^\varepsilon - 2\omega\, \Phi_{\beta_p^\varepsilon(\omega),p} - \omega^2 - V_{\beta_p^\varepsilon(\omega),p}. \tag{3.50}$$

Note that

$$G_p^\varepsilon(0) = -\varepsilon^2 \sigma^{iv} - 2\sigma\, \Phi_{\beta_p^\varepsilon(0)}^\varepsilon - V_{\beta_p^\varepsilon(0),p}^\varepsilon$$

and hence that for some constant depending on p

$$\| G_p^\varepsilon(0) \| \leqq \text{const.} \| \varepsilon^2 \| \tag{3.51}$$

by (3.30) and (3.36). Note also that the Fréchet derivative of G_p^ε at $\hat{\omega}$ is given by the formula

$$dG_p^\varepsilon[\hat{\omega}]\, \omega = -d\beta_p^\varepsilon[\hat{\omega}]\, \omega \left\{ 2(\sigma + \hat{\omega}) \left(\frac{\partial}{\partial \beta} \Phi_{\beta,p}^\varepsilon \right) \Big|_{\beta = \beta_p^\varepsilon(\hat{\omega})} + \left(\frac{\partial}{\partial \beta} V_{\beta,p}^\varepsilon \right) \Big|_{\beta = \beta_p^\varepsilon(\hat{\omega})} \right\}$$

$$-2 \left(\Phi_{\beta_p^\varepsilon(\hat{\omega}),p}^\varepsilon + \hat{\omega} \right) \omega. \tag{3.52}$$

If $p \in (0, \frac{1}{2})$, then (3.52) gives that

$$dG_p^\varepsilon[\hat{\omega}]\, w = L_p^\varepsilon[\hat{\omega}]\, \omega - (d\beta_p^\varepsilon[\hat{\omega}]\, \omega) \left(\frac{\partial}{\partial \beta} V_{\beta,p}^\varepsilon \right) \Big|_{\beta = \beta_p^\varepsilon(\hat{\omega})}$$

where $\|L_p^\varepsilon[\hat\omega]\|$ is bounded for $\|\hat\omega\| \leqq \rho_p$, uniformly as $\varepsilon \to 0$. If $p = 0$, then

$$\|dG_0^\varepsilon[\hat\omega]\| \text{ is bounded for } \|\hat\omega\| \leqq \rho_0$$

uniformly as $\varepsilon \to 0$. Note the formula

$$dG_p^\varepsilon[0]\,\omega = -2\omega\,\Phi^\varepsilon_{\beta_p^\varepsilon(0),p} - (d\beta_p^\varepsilon[0]\,\omega)\left\{\frac{\partial}{\partial\beta}\,(2\sigma\,\Phi^\varepsilon_{\beta,p} + V^\varepsilon_{\beta,p})|_{\beta=\beta_p^\varepsilon(0)}\right\}. \quad (3.53)$$

In order that some estimates later are easy to obtain we write the formula (3.45) using abbreviated notation as

$$d\beta_p^\varepsilon[\hat\omega]\,\omega = -\frac{l[\hat\omega]\,\omega}{\dfrac{\partial}{\partial\beta}\,(D(\beta,\hat\omega))|_{\beta=\beta_p^\varepsilon(\hat\omega)}} = -\frac{\hat l(\omega)}{\hat D(\hat\omega)}$$

where the dependence of the formula on ε and p has been temporarily suppressed. With this notation a calculation based on (3.45) yields a formula for the second Fréchet derivative of β_p^ε, namely,

$$\begin{aligned} &d^2\beta_p^\varepsilon[\hat\omega]\,(v,w) \\ &= -2\,(\hat D(\hat\omega))^{-1}\left\{\int_{-\infty}^{\infty} v_\varepsilon v w\,dx\right\} \\ &\quad + 2\,(\hat D(\hat\omega))^{-2}\left\{\hat l(w)\int_{-\infty}^{\infty} v_\varepsilon v\,\frac{\partial}{\partial\beta}\,(\Phi^\varepsilon_{\beta,p})|_{\beta_p^\varepsilon(\hat\omega)}\,dx \right.\\ &\qquad\qquad \left. + \hat l(v)\int_{-\infty}^{\infty} v_\varepsilon w\,\frac{\partial}{\partial\beta}\,(\Phi^\varepsilon_{\beta,p})|_{\beta_p^\varepsilon(\hat\omega)}\,dx\right\} \\ &\quad - (\hat D(\hat\omega))^{-3}\left\{\hat l(v)\,\hat l(w)\int_{-\infty}^{\infty}\frac{\partial^2}{\partial\beta^2}[(2v_\varepsilon\hat\omega + 2\sigma v_\varepsilon)\,\Phi^\varepsilon_{\beta,p} + v_\varepsilon V^\varepsilon_{\beta,p}]_{\beta_p^\varepsilon(\hat\omega)}\,dx\right\}. \end{aligned}$$

When $p \in (0,\frac{1}{2})$, it follows that $|\hat D(\hat\omega)| \geqq \text{const.}\,\varepsilon^{-1}$ as $\varepsilon \to 0$, and so

$$d^2\beta_p^\varepsilon[\hat\omega]\,(v,w) = M(v,w) - \hat D(\hat\omega)^{-3}\,\hat l(v)\,\hat l(w)\int_{-\infty}^{\infty}\left(\frac{\partial^2}{\partial\beta^2}\,V^\varepsilon_{\beta,p}\right)\bigg|_{\beta=\beta_p^\varepsilon(\hat\omega)} dx \quad (3.54)$$

where $|M(v,w)| \leqq \text{const.}\,\varepsilon\|v\|\,\|w\|$ for $p \in (0,\frac{1}{2})$.

When $p = 0$, it is clear that for $\|\hat{\omega}\| \leqq \rho_0$,

$$d^2\beta_0^\varepsilon[\hat{\omega}] \text{ is bounded as } \varepsilon \to 0. \tag{3.55}$$

3.4. The existence theorem

Suppose that $p \in [0, \frac{1}{2})$, $\varepsilon \in (0, \varepsilon_p)$, and let $\mathscr{B}_p^\varepsilon : Z_4 \to Z$ be defined by

$$\mathscr{B}_p^\varepsilon u = \mathscr{A}_\varepsilon u - dG_p^\varepsilon[0]\, u, \quad u \in Z_4. \tag{3.56}$$

The following result is the key to the existence of homoclinic orbits discussed in the Introduction. Let

$$Z^\varepsilon = \left\{ u \in Z : \int_{-\infty}^{\infty} v_\varepsilon u \, dx = 0 \right\}. \tag{3.57}$$

Theorem 3.4. *Let $p \in [0, \frac{1}{2})$. Then $\varepsilon_p > 0$ can be chosen sufficiently small that $\mathscr{B}_p^\varepsilon$ is a linear homeomorphism from Z_4 onto Z^ε. There exists a constant (depending on p but independent of ε) such that*

$$\varepsilon \|u'\| + \|u\| \leqq \text{const.} \|\mathscr{B}_p^\varepsilon u\|. \tag{3.58}$$

Moreover, if $h \in Z^\varepsilon$ is such that

$$\operatorname{supp}(h) \subset (-a, a) \subset (-1, 1) \quad \text{and} \quad \left| \int_{-\infty}^{\infty} h(x)\, dx \right| = c,$$

and $\mathscr{B}_p^\varepsilon u = h$, then

$$\varepsilon \|u'\| + \|u\| \leqq \text{const.}\, (c + a^2 |h|_\infty). \tag{3.59}$$

Proof. Suppose that $u \in Z_4$ and $\mathscr{B}_p^\varepsilon u = f$. It is immediate from (3.13), (3.48), (3.53) and (3.56) that $f = Z^\varepsilon$. So $\mathscr{B}_p^\varepsilon : Z_4 \to Z^\varepsilon$. Next we show it is injective and establish (3.59).

Suppose that $u \in Z_4$ and $\mathscr{B}_p^\varepsilon u = f$. Then

$$\begin{aligned} \mathscr{A}_\varepsilon u &= dG_p^\varepsilon[0]\, u + f \\ &= -2u\, \Phi^\varepsilon_{\beta_p^\varepsilon(0),p} - (d\beta_p^\varepsilon[0]\, u) \frac{\partial}{\partial \beta} (2\sigma\, \Phi^\varepsilon_{\beta,p}) \bigg|_{\beta = \beta_p^\varepsilon(0)} \\ &\quad - (d\beta_p^\varepsilon[0]\, u) \frac{\partial}{\partial \beta} (V^\varepsilon_{\beta,p}) \bigg|_{\beta = \beta_p^\varepsilon(0)} + f \\ &= f_1 + f_2 + f_3 + f, \end{aligned}$$

say. By (3.44),

$$\|f_1\| \leqq \text{const.}\, \beta_p^\varepsilon(0) \|u\| \leqq \text{const.} \left[\varepsilon^{-3} \operatorname{cosech}^2 \left(\frac{\pi}{\varepsilon} \right) \right] \|u\|, \quad p \in [0, \tfrac{1}{2}). \tag{3.60}$$

When $p = 0$, it follows that $f_3 \equiv 0$ and

$$\left\| (d\beta_0^\varepsilon[0]\,u)\,\frac{\partial}{\partial\beta}\,(2\sigma\,\Phi_{\beta,0}^\varepsilon)\right|_{\beta=\beta_0^\varepsilon(0)} - \left(\tfrac{2}{3}\int_{-\infty}^{\infty} u\sigma v_\varepsilon\right)\sigma\,v_\varepsilon\right\| \leqq M(\varepsilon)\,\|u\|$$

where $M(\varepsilon) \to 0$ as $\varepsilon \to 0$ by (3.40), (3.48), (3.3) with $p = 0$, and (3.6). Hence

$$\|D_\varepsilon^{-1}(f_2)\|_1 \leqq \tfrac{2}{3}\left|\int_{-\infty}^{\infty} u\sigma v_\varepsilon\,dx\right| \|D_\varepsilon^{-1}(\sigma v_\varepsilon)\|_1 + M(\varepsilon)\,\|D_\varepsilon^{-1}\|\,\|u\|$$

$$\leqq \text{const.}\,(\varepsilon + M(\varepsilon))\,\|u\| \tag{3.61}$$

by Lemma 3.1(c). Thus when $p = 0$, Theorem 3.2, (3.60) and (3.61) together give

$$\varepsilon\|u'\| + \|u\| \leqq \text{const.}\,\|D_\varepsilon^{-1}(f_1 + f_2 + f_3 + f)\|_1$$

$$\leqq \text{const.}\left\{\left[\varepsilon^{-3}\operatorname{cosech}\left(\frac{\pi}{\varepsilon}\right) + \varepsilon + M(\varepsilon)\right]\|u\| + \|f\|\right\}.$$

Thus for $\varepsilon > 0$ sufficiently small,

$$\varepsilon\|u'\| + \|u\| \leqq \text{const.}\,\|f\|.$$

This gives the required inequality from which the injectivity of $\mathscr{B}_0^\varepsilon$ follows.

Now suppose that $p = (0, \frac{1}{2})$. Then (3.46) gives that

$$\|f_2\| \leqq \text{const.}\,\varepsilon\|u\|. \tag{3.62}$$

Now (3.28), (3.29), (3.30), (3.33) and (3.46) show that $\operatorname{supp}(f_3) \subset (-\frac{3}{2}\varepsilon\pi, \frac{3}{2}\varepsilon\pi)$, $|f_3|_\infty \leqq \text{const.}\,\varepsilon^{-1}$ and $|\int_{-\infty}^{+\infty} f_3\,dx| \leqq \text{const.}\,\varepsilon^2$. Thus (3.22) gives that

$$\varepsilon\|u'\| + \|u\| \leqq \text{const.}\left[\varepsilon^{-3}\operatorname{cosech}\left(\frac{\pi}{\varepsilon}\right) + \varepsilon\right]\|u\| + \|f\|, \tag{3.63}$$

for $\varepsilon > 0$ sufficiently small. In this case, too, the inequality and the injectivity have been established. The proof of (3.59) is immediate from Theorem 3.2.

Now we prove surjectivity from Z_4 onto Z^ε. Suppose that $f \in Z^\varepsilon$. Then we need to prove that there exists $\omega \in Z_4$ with

$$\varepsilon^2\omega^{iv} + \omega'' - \omega = -2\sigma\,\omega + dG_p^\varepsilon[0]\,\omega + f. \tag{3.64}$$

This can be rewritten as a system of equations

$$\omega'' - K_\varepsilon^{-2}\omega = r, \tag{3.65}$$

$$\varepsilon^2 r'' + K_\varepsilon^2 r = -2\sigma\,\omega + dG_p^\varepsilon[0]\,\omega + f. \tag{3.66}$$

If $\omega \in Z$, then the right-hand side of (3.66) is in Z^ε because $f \in Z^\varepsilon$ and $dG_p^\varepsilon[0]$ is defined via (3.48) and (3.53). Hence for any $\omega \in Z$ equation (3.66) can be solved for $r \in Z_2$ by the formula

$$r(x) = \frac{-1}{\varepsilon K_\varepsilon} \int_x^\infty \sin\left(k_\varepsilon(x-y)\right)\{-2\sigma\omega + dG_p^\varepsilon[0]\,\omega + f\}\,dy = B_p^\varepsilon\omega + \hat{f},$$

where $\hat{f} \in Z_2$ is arbitrary, since $f \in Z^\varepsilon$ is arbitrary, and $B_p^\varepsilon : Z \to Z$ is compact. Hence (3.65) and (3.66) are equivalent to

$$\omega'' - K_\varepsilon^{-2}\omega = B_p^\varepsilon\omega + \hat{f}. \tag{3.67}$$

The left-hand side of (3.67) has a bounded inverse from Z to Z_2 and so (3.67) is equivalent to

$$\omega = C_p^\varepsilon\omega + \hat{f}$$

where $C_p^\varepsilon : Z \to Z$ is compact. The injectivity of $\mathscr{B}_p^\varepsilon$ ensures that C_p^ε does not have 1 as an eigenvalue. The surjectivity now follows from the Fredholm Alternative for compact perturbations of the identity. q.e.d.

Now we turn the proof of existence of solutions. For each $p \in [0, \frac{1}{2})$ and $\varepsilon \in (0, \varepsilon_p)$ let

$$G_p^\varepsilon(u) = G_p^\varepsilon(0) + dG_p^\varepsilon[0]\,u + R_p^\varepsilon(u), \quad \|u\| \leqq \rho_p. \tag{3.68}$$

Then equation (3.9), or equivalently (3.49), has the form

$$\mathscr{B}_p^\varepsilon(u) = G_p^\varepsilon(0) + R_p^\varepsilon(u), \quad \|u\| \leqq \rho_p. \tag{3.69}$$

It is clear from (3.68) and (3.48) that $G_p^\varepsilon(0) + R_p^\varepsilon(u) \in Z^\varepsilon$ for all $u \in Z$, and hence (3.69) can be reformulated as a fixed-point problem

$$u = (\mathscr{B}_p^\varepsilon)^{-1}\left(G_p^\varepsilon(0) + R_p^\varepsilon(u)\right), \tag{3.70}$$

to which Banach's Contraction Mapping Principle applies.

The case when $p = 0$ is the more straightforward; we deal with it first. In this case, because of (3.55) there exists $d_0 > 0$ such that if $\delta \in (0, \rho_0)$ and if $\|u_1\|, \|u_2\| \leqq \delta$, then

$$\|R_p^\varepsilon(u_1) - R_p^\varepsilon(u_2)\| \leqq d_0\delta\|u_1 - u_2\|.$$

By Theorem 3.4 and (3.51),

$$\|(\mathscr{B}_p^\varepsilon)^{-1}\left(G_p^\varepsilon(0) + R_p^\varepsilon(u)\right)\| \leqq \text{const.}\,(\varepsilon^2 + d_0\|u\|^2)$$

$$\leqq \text{const.}\,(\varepsilon^2 + \|u\|^2) = C_0(\varepsilon^2 + \|u\|^2),$$

say. Hence if $\|u\| \leqq 2C_0\varepsilon_0^2$ and $\varepsilon \in (0, \varepsilon_0)$, then

$$\|(\mathscr{B}_p^\varepsilon)^{-1}\left(G_p^\varepsilon(0) + R_p^\varepsilon(u)\right)\| \leqq C_0\varepsilon_0^2(1 + 4C_0^2\varepsilon_0^2) \leqq 2C_0\varepsilon_0^2 \tag{3.71}$$

for $\varepsilon_0 > 0$ sufficiently small. Similarly

$$\|(\mathscr{B}_p^\varepsilon)^{-1}(R_p^\varepsilon(u_1) - R_p^\varepsilon(u_2)\| \leqq \text{const.}\, d_0\delta\|u_1 - u_2\|$$

where $\|u_1\|$, $\|u_2\| \leqq \delta$. Choose $\rho_0 \leqq 2C_0\varepsilon_0^2$ such that (const. $d_0\rho_0$) $\leqq 1$. Then if $\varepsilon_0 > 0$ is sufficiently small for each $\varepsilon \in (0, \varepsilon_0)$, equation (3.70) has a unique solution in the ball of radius ρ_0 about the origin in Z when $p = 0$, by Banach's Contraction Mapping Principle. Clearly there exists a constant such that $\|\omega\| \leqq \text{const.}\, \varepsilon^2$.

The argument for $p \in (0, \frac{1}{2})$ is complicated by the formula (3.54) for the second derivative of β_p^ε in a neighbourhood of 0 in Z. Let

$$\hat{Q}(v, w) = \hat{D}(\hat{\omega})^{-3}\, \hat{l}(v)\, \hat{l}(w) \int_{-\infty}^{\infty} \left(\frac{\partial^2}{\partial\beta^2} V_{\beta,p}^\varepsilon\right)\bigg|_{\beta=\beta_p^\varepsilon(\hat{w})} dx.$$

A calculation based on (2.38), (3.3), (3.7) and (3.29) gives

$$\left|\int_{-\infty}^{\infty} \left(\frac{\partial^2}{\partial\beta^2} V_{\beta,p}^\varepsilon\right) dx\right| \leqq \text{const.}\, \varepsilon.$$

Since

$$|\hat{D}(\hat{\omega})^{-3}| \leqq \text{const.}\, \varepsilon^3 \quad \text{when } p \in (0, \tfrac{1}{2}),$$

it follows that

$$\|\hat{Q}(v, w)\| \leqq \text{const.}\, \varepsilon^4 \|v\|\,\|w\|, \tag{3.72}$$

$$\|d^2\beta_p^\varepsilon[\hat{\omega}]\,(v, w)\| \leqq \text{const.}\, \varepsilon\|v\|\,\|w\| \quad (\text{by } (3.54)). \tag{3.73}$$

Also

$$\frac{\partial^2}{\partial\beta^2}(V_{\beta,p}^\varepsilon) \leqq \frac{\text{const.}}{\varepsilon^2}.$$

When $p \in (0, \frac{1}{2})$, equations (3.52) and (3.54) yield that

$$\begin{aligned} d^2G_p^\varepsilon[\hat{\omega}]\,(v, w) &= \hat{Q}(v, w) - (d^2\beta_p^\varepsilon[\hat{\omega}]\,(v, w)) \left(\frac{\partial}{\partial\beta} V_{\beta,p}^\varepsilon\right)\bigg|_{\beta=\beta_p^\varepsilon(\hat{\omega})} \\ &\quad - (d\beta_p^\varepsilon[\hat{\omega}]\,v)\,(d\beta_p^\varepsilon[\hat{\omega}]\,w) \left(\frac{\partial^2}{\partial\beta^2} V_{\beta,p}^\varepsilon\right)\bigg|_{\beta=\beta_p^\varepsilon(\hat{\omega})}, \\ &= \tilde{Q}(v, w) + \tilde{R}(v, w) \end{aligned} \tag{3.74}$$

where $\|\tilde{Q}(v, w)\| \leqq \text{const.}\|v\|\,\|w\|$ as $\varepsilon \to 0$ and

$$\text{supp}\,(\tilde{R}(v, w)) \subset (-\tfrac{3}{2}\pi\varepsilon, \tfrac{3}{2}\pi\varepsilon),$$

$$|\tilde{R}(v, w)|_\infty \leqq \text{const.}\, \|v\|\,\|w\|\,\varepsilon^{-1},$$

$$\left|\int_{-\infty}^{\infty} \tilde{R}(v, w)\,(x)\, dx\right| \leqq \text{const.}\, \varepsilon^3.$$

Now $d^2G_p^\varepsilon[\hat{w}]\,(v,w) \in Z^\varepsilon$ since $G_p^\varepsilon(w) \in Z^\varepsilon$ for all $z \in w$, and so if $\|\hat{w}\| \leqq \rho_p$, then

$$\|(\mathscr{B}_p^\varepsilon)^{-1}\, d^2G_p^\varepsilon[\hat{w}]\,(v,w)\| \leqq \text{const.}\ (\varepsilon^3+\varepsilon)\,\|u\|\,\|v\|.$$

by (3.59). Along with (3.73) this estimate shows that the remainder term $(\mathscr{B}_p^\varepsilon)^{-1}R_p^\varepsilon(u)$ in (3.68) is uniformly small as $\varepsilon \to 0$ and has a Lipschitz constant which tends to 0 as $\varepsilon \to 0$. With this observation, the Contraction Mapping Principle may be applied as in the case $p = 0$ to (3.70) to give the main result of this section, which is the following theorem.

Theorem 3.5. *Let $p \in [0, \frac{1}{2})$. Then there exist $\varepsilon_p > 0$, $\rho_p > 0$ and constants depending on p but independent of $\varepsilon \in (0, \varepsilon_p)$ with the following properties.*

(a) *If $\varepsilon \in (0, \varepsilon_p)$, then equation* (3.9) *has a unique solution $\omega_\varepsilon \in Z$ with $\|\omega_\varepsilon\| \leqq \rho_p$.*

(b) $\|\omega_\varepsilon\| \leqq \text{const.}\ \varepsilon^2$.

(c) *If $\beta_p^\varepsilon(\omega_\varepsilon)$ is defined by Theorem 3.3, then for each N there is a constant such that*

$$|\beta_p^\varepsilon(\omega_\varepsilon)| \leqq \text{const.}\ \varepsilon^N \quad \textit{as } \varepsilon \to 0.$$

Proof. Parts (a) and (b) are established in the discussion preceding the statement of this result. Part (c) is the substance of the next section of the paper.

3.5. The asymptotics of β as $\varepsilon \to 0$

In this section we complete the proof of the result announced in the Introduction by showing that in the solution of (1.1), whose existence is proved in Section 3.4, $|\beta| \leqq C(N)\,\varepsilon^N$ as $\varepsilon \to 0$ for all $N \in \mathbf{N}$ where the constant $C(N)$ is independent of ε.

To do this we need an interpolation inequality and some *a priori* estimates. The first is based on a well-known inequality of LANDAU.

Lemma 3.6. *Suppose that $u \in Z_2$. Then*

$$\|u'\|^2 \leqq 4\|u\|\,\|u''\|.$$

Proof. If $s, t > 0$, then

$$u(t+s) = u(t) + su'(t) + \int_0^s (s-w)\,u''(t+w)\,dw,$$

and so

$$|su'(t)| \leqq |u(t)| + |u(t+s)| + \left|\int_0^s (s-w)\,u''(t+w)\,dw\right|$$

$$\leqq e^{-qt}\left\{2\|u\| + \|u''\|\int_0^s (s-w)\,e^{-qw}\,dw\right\}$$

$$\leqq e^{-qt}\left\{2\|u\| + \left(\frac{s}{q} + \frac{e^{-qs}-1}{q^2}\right)\|u''\|\right\} \leqq e^{-qt}\left\{2\|u\| + \frac{s^2}{2}\|u''\|\right\}.$$

Since u is even, we infer that for all $s > 0$,

$$\|u'\| \leqq \frac{2}{s}\|u\| + \frac{s}{2}\|u''\|.$$

In particular, if $s = 2(\|u\|/\|u''\|)^{1/2}$, then

$$\|u'\|^2 \leqq 4\|u\|\,\|u''\|. \quad \text{q.e.d.}$$

In proving the next result we shall need the estimate in Theorem 3.2, along with the following trivial observation.

Lemma 3.7. *Suppose that $f \in Z_2$ and $D_\varepsilon g = f''$, $g \in Z_4$. Then*

$$\|g\|_1 = \|D_\varepsilon^{-1}(f'')\|_1 \leqq \text{const.}\,\|f\|_1 \leqq \text{const.}\,\|f'\|,$$

where the constant is independent of ε.

Proof. If $D_\varepsilon g = f''$, then

$$D_\varepsilon(g - f) = K_\varepsilon^{-2}f - 2K_\varepsilon^{-2}\sigma f,$$

whence

$$g - f = D_\varepsilon^{-1}(K_\varepsilon^{-2}f - 2K_\varepsilon^{-2}\sigma f).$$

Hence $\|g\|_1 \leqq \|f\|_1 + \text{const.}\,\|D_\varepsilon^{-1}f\|_1 \leqq \text{const.}\,\|f\|_1 \leqq \text{const.}\|f'\|$ by Lemma 3.1. q.e.d.

The key observation in this section is an *a priori* bound for solutions of (3.9). Recall that for $k \in \mathbf{N} \cup \{0\}$,

$$|(\Phi_{\beta,p}^\varepsilon)^{(k)}| \leqq \text{const.}\,\frac{\beta}{\varepsilon^k}, \qquad (V_{\beta,0}^\varepsilon)^{(k)} \equiv 0,$$

$$|(V_{\beta,p}^\varepsilon)^{(k)}|_\infty \leqq \text{const.}\,\frac{\beta}{\varepsilon^{k+2}}, \qquad \operatorname{supp}(V_{\beta,p}^{(k)}) \subset (-3\pi\varepsilon, 3\pi\varepsilon), \qquad p \in (0, \tfrac{1}{2}).$$

Here and elsewhere in this section the constants depend on p and k, and are independent of ε and β.

Theorem 3.8. *Suppose that $p \in [0, \frac{1}{2})$. Then ε_p and ρ_p can be chosen sufficiently small that if $\omega \in Z_4$ is a solution of (3.9) with $\|\omega\| \leqq \rho_p$ and $\varepsilon \in (0, \varepsilon_p)$, then $\omega \in Z_N$ for all $N \in \mathbf{N}$ and there exists a constant $c(p, N)$ such that*

$$\|\omega^{(2N-1)}\| + \|\omega^{(2N)}\| \leqq c(p,N)\left(\varepsilon + \frac{|\beta|}{\varepsilon^{2N-1}} + \frac{p|\beta|}{\varepsilon^{2N}}\right), \qquad N \in \mathbf{N}, \tag{3.75}$$

$$\|\omega\| \leqq c(p,0)\,(\varepsilon^2 + |\beta|). \tag{3.76}$$

Proof. The first step is to observe from Theorem 3.3 that ε_p and $\rho_p > 0$ can be chosen so that $|\beta| = |\beta_p^\varepsilon(\omega)|$ is smaller than any preassigned positive value if $\varepsilon \in (0, \varepsilon_p)$ and $\|\omega\| \leqq \rho_p$. Equation (3.9) can be rewritten as

$$\mathscr{A}_\varepsilon \omega = -\varepsilon^2 \sigma^{(4)} - 2\omega\, \Phi^\varepsilon_{\beta,p} - \omega^2 - 2\sigma\, \Phi^\varepsilon_{\beta,p} - V^\varepsilon_{\beta,p}. \tag{3.77}$$

Therefore by (3.28), (3.29), (3.30) and the fact that $V^\varepsilon_{\beta,p} \equiv 0$ when $p = 0$ we find that

$$\varepsilon\|\omega'\| + \|\omega\| \leqq \text{const.}\,(\varepsilon^2 + |\beta|\,\|\omega\| + \|\omega\|^2 + |\beta| + p|\beta|),$$

whence

$$\varepsilon\|\omega'\| + \|\omega\| \leqq \text{const.}\,(\varepsilon^2 + |\beta|). \tag{3.78}$$

This gives (3.76). Now differentiate (3.77) twice to obtain

$$\mathscr{A}_\varepsilon \omega'' = -4\sigma'\omega' - 2\sigma''\omega - \varepsilon^2\sigma^{(6)} - \{2\omega\, \Phi^\varepsilon_{\beta,p} + \omega^2 + 2\sigma\, \Phi^\varepsilon_{\beta,p}\}'' - (V^\varepsilon_{\beta,p})'',$$

whence, by Theorem 3.2 and Lemma 3.7,

$$\begin{aligned}
\|\omega''\| &\leqq \text{const.}\left\{\|\omega\| + \|\omega'\| + \varepsilon^2 + \|\{2\omega\, \Phi^\varepsilon_{\beta,p} + \omega^2 + 2\sigma\, \Phi^\varepsilon_{\beta,p}\}'\| + \frac{p|\beta|}{\varepsilon^2}\right\} \\
&\leqq \text{const.}\left\{\|\omega\| + \|\omega'\| + \varepsilon^2 + \frac{|\beta|}{\varepsilon}\|\omega\| + |\beta|\,\|\omega'\| + \|\omega\|\|\omega'\| + \frac{|\beta|}{\varepsilon} + \frac{p|\beta|}{\varepsilon^2}\right\} \\
&\leqq \text{const.}\left\{\|\omega\| + \|\omega'\| + \varepsilon^2 + \frac{|\beta|}{\varepsilon}\|\omega\| + \frac{|\beta|}{\varepsilon} + \frac{p|\beta|}{\varepsilon^2}\right\} \\
&\leqq \text{const.}\left(\varepsilon + \frac{|\beta|}{\varepsilon} + \frac{p|\beta|}{\varepsilon^2}\right).
\end{aligned}$$

Along with (3.78) this gives

$$\|\omega'\| + \|\omega''\| \leqq \text{const.}\left(\varepsilon + \frac{|\beta|}{\varepsilon} + \frac{p|\beta|}{\varepsilon^2}\right),$$

which means that (3.75) holds with $N = 1$.

Now the proof is by induction on N. Suppose that for some $M \in \mathbf{N}$, $M \geqq 2$, inequality (3.75) holds for all $N \in \mathbf{N}$ with $N \leqq M - 1$. We shall infer that it holds for M, and the result will follow by mathematical induction. Differentiation of (3.77) $2M$ times gives

$$\begin{aligned}
\mathscr{A}_\varepsilon \omega^{(2M)} = &-\sum_{r=0}^{2M-1} ({}^{(2M)}C_r)\, \sigma^{(2M-r)}\, \omega^{(r)} - \varepsilon^2 \sigma^{(2M+4)} \\
&- \{(2\omega\, \Phi^\varepsilon_{\beta,p} + \omega^2 + 2\sigma\, \Phi^\varepsilon_{\beta,p})^{(2M-2)}\}'' - (V^\varepsilon_{\beta,p})^{(2M)},
\end{aligned}$$

whence, by Theorem 3.2 and Lemma 3.7,

$$\begin{aligned}
\|\omega^{(2M)}\| \leqq \text{const.}\Bigg\{\left(\sum_{r=0}^{2M-1}\|\omega^{(r)}\|\right) &+ \varepsilon^2 + \|(2\omega\, \Phi^\varepsilon_{\beta,p} + \omega^2 + 2\sigma\, \Phi^\varepsilon_{\beta,p})^{(2M-1)}\| \\
&+ \varepsilon^2\,|(V^\varepsilon_{\beta,p})^{2M}|_\infty\Bigg\} \leqq
\end{aligned}$$

$$\leqq \text{const.}\left\{\left(\|\omega^{(2M-1)}\| + \|\omega\| + \sum_{r=1}^{2M-2}\|\omega^{(r)}\|\right) + \varepsilon^2\right.$$

$$\left. + \sum_{r=1}^{2M-2}\left(\frac{|\beta|}{\varepsilon^{2M-1-r}}\right)\|\omega^{(r)}\| + \sum_{r=1}^{2M-2}\|\omega^{(r)}\|\|\omega^{(2M-1-r)}\| + \frac{|\beta|}{\varepsilon^{2M-1}} + \frac{p|\beta|}{\varepsilon^{2M}}\right\}$$

$$\leqq \text{const.}\left\{\|\omega^{(2M-1)}\| + \sum_{j=1}^{M-1}(\|\omega^{(2j-1)}\| + \|\omega^{(2j)}\|) + \varepsilon^2\right.$$

$$+ \sum_{j=1}^{M-1}\left[\left(\frac{|\beta|}{\varepsilon^{2M-2j}}\right)\|\omega^{(2j-1)}\| + \left(\frac{|\beta|}{\varepsilon^{2M-1-2j}}\right)\|\omega^{(2j)}\|\right]$$

$$+ \sum_{j=1}^{M-1}\|\omega^{(2j-1)}\|\|\omega^{(2M-2j)}\| + \|\omega^{(2j)}\|\|\omega^{(2M-2j-1)}\|$$

$$\left. + \frac{|\beta|}{\varepsilon^{2M-1}} + \frac{p|\beta|}{\varepsilon^{2M}}\right\}.$$

Now since (3.75) holds for all $N \in \mathbf{N}$ with $N \leqq M - 1$, this gives

$$\|\omega^{(2M)}\| \leqq \text{const.}\left\{\|\omega^{(2M-1)}\| + \varepsilon + \frac{|\beta|}{\varepsilon^{2M-1}} + \frac{p|\beta|}{\varepsilon^{2M}}\right\}. \tag{3.79}$$

Hence, by Lemma 3.6,

$$\|\omega^{(2M-1)}\|^2 \leqq 4\|\omega^{(2M-2)}\|\|\omega^{(2M)}\|$$

$$\leqq \text{const.}\|\omega^{(2M-2)}\|\left\{\|\omega^{(2M-1)}\| + \varepsilon + \frac{|\beta|}{\varepsilon^{2M-1}} + \frac{p|\beta|}{\varepsilon^{2M}}\right\},$$

from which it follows (by the theory of roots of quadratic equations) that

$$\|\omega^{(2M-1)}\|^2 \leqq \text{const.}\left\{\|\omega^{(2M-2)}\|^2 + \|\omega^{(2M-2)}\|\left(\varepsilon + \frac{|\beta|}{\varepsilon^{2M-1}} + \frac{p|\beta|}{\varepsilon^{2M}}\right)\right\}.$$

Hence, since (3.75) holds for $N = M - 1$, we find that

$$\|\omega^{(2M-1)}\| \leqq \text{const.}\left(\varepsilon + \frac{|\beta|}{\varepsilon^{2M-1}} + \frac{p|\beta|}{\varepsilon^{2M}}\right).$$

A substitution of this into (3.79) yields that (3.75) holds when $N = M$. The result now follows by mathematical induction. q.e.d.

This result and Lemma 3.6 show that for any $N \in \mathbf{N}$,

$$\|\omega^{(2N+1)}\|^2 \leqq 4\|\omega^{(2N)}\|\|\omega^{(2N+2)}\|$$

$$\leqq \text{const.}\left(\varepsilon + \frac{|\beta|}{\varepsilon^{2N-1}} + \frac{p|\beta|}{\varepsilon^{2N}}\right)\left(\varepsilon + \frac{|\beta|}{\varepsilon^{2N+1}} + \frac{p|\beta|}{\varepsilon^{2N+2}}\right)$$

$$= \text{const.}\left(\varepsilon^2 + \frac{|\beta|}{\varepsilon^{2N}} + \frac{p|\beta|}{\varepsilon^{2N+1}} + \frac{|\beta|}{\varepsilon^{2N-2}} + \frac{|\beta|^2}{\varepsilon^{4N}} + \frac{p|\beta|^2}{\varepsilon^{4N+1}}\right.$$

$$\left. + \frac{p|\beta|}{\varepsilon^{2N-1}} + \frac{p|\beta|^2}{\varepsilon^{4N+1}} + \frac{p^2|\beta|^2}{\varepsilon^{4N+2}}\right)$$

$$\leqq \text{const.}\left(\varepsilon^2 + \frac{|\beta|}{\varepsilon^{2N}} + \frac{|\beta|^2}{\varepsilon^{4N}} + \frac{p|\beta|}{\varepsilon^{2N+1}} + \frac{p^2|\beta|^2}{\varepsilon^{4N+2}}\right)$$

$$\leqq \text{const.}\left(\varepsilon^2 + 1 + \frac{|\beta|^2}{\varepsilon^{4N}} + \frac{p^2|\beta|^2}{\varepsilon^{4N+2}}\right)$$

by the Cauchy-Schwarz Inequality. Hence

$$\|\omega^{(2N+1)}\| \leqq \text{const.}\left(1 + \varepsilon + \frac{|\beta|}{\varepsilon^{2N}} + \frac{p|\beta|}{\varepsilon^{2N+1}}\right). \tag{3.80}$$

Combining (3.75) and (3.80) we find that for any $r \in \mathbf{N}$, $r \geqq 2$,

$$\|\omega^{(r)}\| \leqq \text{const.}\left(1 + \frac{|\beta|}{\varepsilon^{r-1}} + \frac{p|\beta|}{\varepsilon^{r}}\right). \tag{3.81}$$

As usual the constant here depends on r and p, but is independent of ε, ω and β.

The main result of this section is now almost immediate from (3.40) and (3.81):

Theorem 3.8. *If T is a solution of* (1.1) *in the form* (3.8) (*whose existence for $p \in [0, \frac{1}{2})$, $\varepsilon \in (0, \varepsilon_p)$ with $\|\omega\| \leqq \rho_p$ was proved in the preceding section*), *then for $N \in \mathbf{N}$ there exists a constant depending on N and p such that*

$$|\Phi^{\varepsilon}_{\beta,p}|_\infty \leqq \text{const.}\, \varepsilon^N \quad \textit{as } \varepsilon \to 0.$$

Proof. We have seen in the proof of Theorem 3.3 that equation (3.41), which determines $\beta^{\varepsilon}_p(\omega)$, is of the form

$$6\pi k_\varepsilon^5 \varepsilon^2 \operatorname{cosech}(k_\varepsilon \pi) + \int_{-\infty}^{\infty} v_\varepsilon \omega(\omega + 2\sigma)\, dx + 2\int_{-\infty}^{\infty} \omega\, \Phi^{\varepsilon}_{\beta,p}\, dx$$

$$+ \beta\{R(\beta,\varepsilon,p) + pS(\beta,\varepsilon,p)\} = 0 \tag{3.82}$$

where $R(\beta,\varepsilon,0) \to 6$ and $|S(\beta,\varepsilon,p)| \geqq \text{const.}(1/\varepsilon)$ as $\varepsilon \to 0$, if $p \in (0,\frac{1}{2})$. Also, for each N there exists $C(N)$ such that

$$6\pi k_\varepsilon^5 \varepsilon^2 \operatorname{cosech}(k_\varepsilon \pi) \leqq C(N)\, \varepsilon^N \quad \text{as } \varepsilon \to 0. \tag{3.83}$$

Now $v_\varepsilon(x) = \cos(\mu_\varepsilon(\beta)x)$ and so $2N$ integrations by parts gives

$$\left|\int_{-\infty}^{\infty} v_\varepsilon \omega(\omega + 2\sigma)\,dx\right| \leqq (\mu_\varepsilon(\beta))^{-2N}\left|\int_{-\infty}^{\infty}\{\omega^2 + 2\sigma\omega\}^{(2N)}\,dx\right|$$

$$\leqq \text{const.}\,\varepsilon^{2N}\left\{\sum_{r=0}^{2N}\|\omega^{(r)}\|\,(\|\omega^{(2N-r)}\| + 1)\right\}$$

$$\leqq \text{const.}\,\varepsilon^{2N}\left\{\|\omega\|\|\omega^{(2N)}\| + \|\omega'\|\|\omega^{(2N-1)}\| + \sum_{r=2}^{2N}\|\omega^{(r)}\|\right.$$

$$\left. + \sum_{r=2}^{2N-2}\|\omega^{(r)}\|\|\omega^{(2N-r)}\|\right\}$$

$$\leqq \text{const.}\,\varepsilon^{2N}\left\{(\varepsilon^2 + \beta)\left(1 + \frac{|\beta|}{\varepsilon^{2N-1}} + \frac{p|\beta|}{\varepsilon^{2N}}\right)\right.$$

$$+ \left(\varepsilon + \frac{\beta}{\varepsilon}\right)\left(1 + \frac{|\beta|}{\varepsilon^{2N-2}} + \frac{p|\beta|}{\varepsilon^{2N-1}}\right)$$

$$+ \sum_{r=2}^{2N}\left(1 + \frac{|\beta|}{\varepsilon^{r-1}} + \frac{p|\beta|}{\varepsilon^{r}}\right)$$

$$\left. + \sum_{r=2}^{2N-2}\left(1 + \frac{|\beta|}{\varepsilon^{r-1}} + \frac{p|\beta|}{\varepsilon^{r}}\right)\left(1 + \frac{|\beta|}{\varepsilon^{2N-r-1}} + \frac{p|\beta|}{\varepsilon^{2N-r}}\right)\right\}$$

$$\leqq \text{const.}\,\varepsilon^{2N}\left\{1 + \frac{|\beta|}{\varepsilon^{2N-1}} + \frac{p|\beta|}{\varepsilon^{2N}}\right\}$$

$$\leqq \text{const.}\{\varepsilon^{2N} + \varepsilon|\beta| + p|\beta|\}. \tag{3.84}$$

A substitution of (3.83) and (3.84) in (3.82) gives the existence of an absolute constant A and a constant K independent of ε and β but depending on N and p such that

$$|\beta(R(\beta,\varepsilon,p) + pS(\beta,\varepsilon,p))| \leqq K(N,p)\,(\varepsilon^N + \varepsilon|\beta| + p|\beta|) + A(\varepsilon^2 + \beta)\beta.$$

Hence ε_p and $\rho_p > 0$ can be chosen sufficiently small that

$$|\beta| = |\beta_p^\varepsilon(\omega)| \leqq \text{const.}\,\varepsilon^N$$

if $\|\omega\| \leqq \rho_p$ and $\varepsilon \in (0, \varepsilon_0)$, since $R(\beta,\varepsilon,0) \to 6$ as $\varepsilon \to 0$, $S(\beta,\varepsilon,p) \geqq$ const. $(1/\varepsilon)$ as $\varepsilon \to 0$ if $p \in (0, \frac{1}{2})$, and

$$|\beta| = |\beta_p^\varepsilon(\omega)| \leqq \text{const.}\,(\varepsilon^{-3}\operatorname{cosech}(\pi/\varepsilon) + \rho_p)$$

by Theorem 3.3. This completes the proof of the theorem. q.e.d.

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