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Variational Identities and Applications to Differential Systems

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1. Introduction

In the work of POHOŽAEV [P] and in recent work of PUCCI & SERRIN [PS], variational identities are discussed which are useful for solving various questions about elliptic differential equations and particular elliptic systems. Here we obtain a number of further applications of these identities to systems.

It was noted in [PS] that these identities are closely related to Noether's Theorem for variational problems. By working out the divergence-term arising in Noether's Theorem and making some simplifications, one obtains a general identity for variational problems. If a variational integral has a symmetry group $\boldsymbol{G}$ with the following infinitesimal generator

$$v = h^i(x)\,\frac{\partial}{\partial x^i} - a(x)\;u\,\frac{\partial}{\partial u},$$

then the conserved quantity (divergence-term) arising in Noether's Theorem is exactly the divergence form in the Pucci-Serrin identity [PS], which does not require the variational integral to have a symmetry group.

Throughout this paper we employ the summation convention.

The first idea of applying a slightly modified version of the Pucci-Serrin identity came up in the study of the following system of partial differentical equations [PV]:

$$\text{(P1)}\qquad \begin{cases} \Delta u = g(v), & (1.1)\\ \Delta v = f(u). & (1.2)\end{cases}$$

When we study radially symmetric solutions of Problem (P1) we obtain after some transformations [AP] that

$$\text{(P2)}\qquad \begin{cases} y'' = t^{-k} g(z), & (1.3)\\ z'' = t^{-k} f(y). & (1.4)\end{cases}$$

The equations (1.3) and (1.4) are closely related to the Emden-Fowler equation. Multiplying (1.3) by tz', (1.4) by ty' and integrating over $[t, \infty)$ gives after a few manipulations two functionals $H_1(t)$ and $H_2(t)$. They can be combined into a family of functionals:

$$H_\theta(t) = \theta H_1(t) + (1-\theta)\, H_2(t), \qquad \theta \in \boldsymbol{R}. \tag{1.5}$$

The family $H_\theta(t)$ enables us to prove some existence and non-existence theorems for Problem (P2) [PV]. The fact that we have a parameter θ plays an important role. In Section 2 we shall develop variational identities in which the parameter θ enters in a natural way.

Consider the following Lagrangian density:

$$L = (Du, Dv) + G(v) + F(u), \tag{1.6}$$

$$Du = \operatorname{grad} u, \qquad Dv = \operatorname{grad} v, \tag{1.7}$$

$$G'(v) = g(v), \qquad F'(u) = f(u), \tag{1.8}$$

$$f(0) = g(0) = 0, \tag{1.8a}$$

where $u, v \in C^1(\Omega)$, and $(\cdot, \cdot)$ is the inner product in $\boldsymbol{R}^N$. We want to derive an identity which is closely related to (1.5). For this a simple generalization of the Pucci-Serrin identity for systems is needed. This identity will be derived in Section 2. In Section 3 we shall show with a number of examples of variational problems how the identity of Section 2 can be used. As the examples will show, we employ identities using the infinitesimal generator of the group of dilatations, which also plays a important rôle in the concentration-compactness principle [L].

In some cases it is not easy, if not impossible, to formulate a system in terms of a variational problem. It would be nice to have some kind of machinery to handle such problems, such as

$$-\Delta u = g(u, v), \tag{1.9}$$

$$-\Delta v = f(u, v). \tag{1.10}$$

In Section 3 we shall indicate how to deal with this situation. Basically we consider modified Euler-Lagrange equations, which enable us to handle the above equations and related ones (corresponding to generalized forces in mechanics). The modified equations still give a nice integral identity. This idea was also explored in [PS].

In Section 4 an example of a perturbed system is given which is similar to the problem studied by Brezis & Nirenberg [BrN].

In this paper we do not consider higher-order variational problems, although the main ideas discussed here immediately carry over to such problems.

2. The generalized identity

In this section a simple but very useful modification of the Pucci-Serrin identity for first-order variational problems for systems is discussed in order to deal with functionals of indefinite nature. Consider the following variational problem:

$$\delta \int_\Omega L(x, \boldsymbol{u}, \boldsymbol{p})\, dx = 0, \tag{2.1}$$

where $\boldsymbol{u} = (u_k)$, $\boldsymbol{p} = (p_i^k)$, $p_i^k = \dfrac{\partial u_k}{\partial x^i}$, $k = 1, \ldots, s$, $i = 1, \ldots, N$, and Ω is a bounded domain in $\boldsymbol{R}^N$, with smooth boundary $\partial\Omega$. The corresponding Euler-Lagrange equations are

$$\operatorname{div}\left(\frac{\partial L}{\partial p_i^k}\right) - \frac{\partial L}{\partial u_k} = 0, \quad k = 1, \ldots, s. \tag{2.2}$$

Theorem 2.1. *Let $L \in C^1(\Omega \times \boldsymbol{R}^s \times \boldsymbol{R}^{N\times s})$ and let $\boldsymbol{u} = (u_1, \ldots, u_s) : \Omega \to \boldsymbol{R}^s$ be a solution of* (2.2) *with $u_k \in C^2(\Omega)$. Let a_{kl}, $h^i \in C^1(\Omega)$. Then*

$$\operatorname{div}\left(h^i L - h^j \frac{\partial u_k}{\partial x^j}\frac{\partial L}{\partial p_i^k} - a_{kl} u_l \frac{\partial L}{\partial p_i^k}\right)$$

$$= \frac{\partial h^i}{\partial x^i} L + h^i \frac{\partial L}{\partial x^i} - \left(\frac{\partial u_k}{\partial x^j}\frac{\partial h^j}{\partial x^i} + u_l \frac{\partial a_{kl}}{\partial x^i}\right)\frac{\partial L}{\partial p_i^k} - a_{kl}\left(\frac{\partial u_l}{\partial x^i}\frac{\partial L}{\partial p_i^k} + u_l \frac{\partial L}{\partial u_k}\right) \quad \text{in } \Omega. \tag{2.3}$$

Furthermore,

$$\oint_{\partial\Omega} \left(\left(h^i L - h^j \frac{\partial u_k}{\partial x^j}\frac{\partial L}{\partial p_i^k} - a_{kl} u_l \frac{\partial L}{\partial p_i^k}\right), n\right) ds$$

$$= \int_\Omega \left(\frac{\partial h^i}{\partial x^i} L + h^i \frac{\partial L}{\partial x^i} - \left(\frac{\partial u_k}{\partial x^j}\frac{\partial h^j}{\partial x^i} + u_l \frac{\partial a_{kl}}{\partial x^i}\right)\frac{\partial L}{\partial p_i^k} - a_{kl}\left(\frac{\partial u_l}{\partial x^i}\frac{\partial L}{\partial p_i^k} + u_l \frac{\partial L}{\partial u_k}\right)\right) dx, \tag{2.4}$$

where n is the outward normal on $\partial\Omega$.

Proof. Expanding the divergence term in (2.3) and using (2.2) gives the required result. The integral from is obtained by applying the Gauss Divergence Theorem.

We observe that the divergence term in Theorem 2.1 is exactly the conserved quantity in Noether's Theorem for (2.1) if the symmetry group $\boldsymbol{G}$ has the following generator (see also OLVER [O]):

$$v = h^i(x) \frac{\partial}{\partial x^i} - a_{kl}(x)\, u_l \frac{\partial}{\partial u_k}.$$

The modification of the Pucci-Serrin identity discussed in this section consists in allowing $a(x)$ to be an $s \times s$-matrix (a_{kl}). In Section 3 we shall discuss a number of applications of Theorem 2.1, in which (a_{kl}) has diagonal form.

If we put Theorem 2.1 in a more general setting, it is natural to consider the case in which a and h^i may also depend on $\boldsymbol{u}$:

$$v = h^i(x, \boldsymbol{u}) \frac{\partial}{\partial x^i} + a_k(x, \boldsymbol{u}) \frac{\partial}{\partial u_k}.$$

Here v is the infinitesimal generator of an 1-parameter group $\boldsymbol{G}$ of transformations. From (2.3) we can then obtain a corresponding identity in a straightforward manner. This more general situation will not be discussed in this paper.

3. Examples

In this section we shall discuss some examples to demonstrate the use of Theorem 2.1. In all cases (a_{kl}) will be of diagonal form. In these examples we shall establish a number of non-existence results.

1. *The biharmonic operator.* We consider the problem:

$$\text{(I)} \quad \begin{cases} \Delta^2 u = f(u), \quad u > 0 & \text{in } \Omega, & (3.1) \\ u = 0 & \text{on } \partial\Omega, & (3.2\text{a}) \\ \Delta u = 0 & \text{on } \partial\Omega, & (3.2\text{b}) \end{cases}$$

in which it is assumed that f satisfies the following hypotheses:

$$\text{(H1)} \quad \begin{cases} f \in C(\boldsymbol{R}), \\ f(0) = 0, f(u) > 0 \quad \text{if } u > 0. \end{cases}$$

This problem was studied in [PV]. It is similar to a problem which was studied by PUCCI & SERRIN [PS], who considered different boundary conditions:

$$u = 0, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{3.2c}$$

It is convenient to write equation (3.1) and the boundary conditions (3.2a) and (3.2b) as

$$\text{(Ia)} \quad \begin{cases} \Delta u = v, \quad u > 0 & \text{in } \Omega, & (3.3) \\ \Delta v = f(u) & \text{in } \Omega, & (3.4) \\ u = v = 0 & \text{on } \partial\Omega. & (3.5) \end{cases}$$

For this problem we have the Lagrangian

$$L = (Du, Dv) + \tfrac{1}{2} v^2 + F(u), \tag{3.6}$$

$$F'(u) = f(u), \quad f(0) = 0. \tag{3.7}$$

Theorem 3.1. *Let* $(u, v) \in (C^2(\Omega) \cap C^1(\bar{\Omega}))^2$ *be a solution of Problem* (Ia) *in which* $s = 2$, $a_k(x) = a_k = \text{const.}$, $h^i(x) = x^i$, $k = 1, \ldots, s$, $i = 1, \ldots, N$. *Then* u *and* v *satisfy the identity*

$$\int_\Omega \left((NF(u) - a_1 u f(u)) + \left(\frac{N}{2} - a_2\right) v^2 + (N - 2 - a_1 - a_2)(Du, Dv) \right) dx$$

$$= - \oint_{\partial\Omega} (Du, Dv)(n, x)\, ds. \tag{3.8}$$

Proof. When we apply Theorem 2.1 to (3.6) we immediately obtain

$$\oint_{\partial\Omega} (Du, Dv)(x, n)\, ds - \oint_{\partial\Omega} \left(\left(x^j \frac{\partial u}{\partial x^j} \frac{\partial v}{\partial x^i} + x^j \frac{\partial v}{\partial x^j} \frac{\partial u}{\partial x^i} \right), n \right) ds$$

$$= \int_\Omega \left((NF(u) - a_1 u f(u)) + \left(\frac{N}{2} - a_2\right) v^2 + (N - 2 - a_1 - a_2)(Du, Dv) \right) dx. \tag{3.9}$$

Because $u = v = 0$ on $\partial\Omega$ we can write

$$\frac{\partial u}{\partial x^i} = \frac{\partial u}{\partial n} n_i, \qquad \frac{\partial v}{\partial x^i} = \frac{\partial v}{\partial n} n_i.$$

Substituting these identities into $x^j \dfrac{\partial u}{\partial x^j} \dfrac{\partial v}{\partial x^i}$ we obtain

$$x^j \frac{\partial u}{\partial n} n_j \frac{\partial v}{\partial x^i} n_i = \frac{\partial u}{\partial n} n_i \frac{\partial v}{\partial x^i} x^j n_j = \frac{\partial u}{\partial x^i} \frac{\partial v}{\partial x^i} x^j n_j.$$

The substitution of this expression into the left-hand side of (3.9) yields

$$- \oint_{\partial\Omega} (Du, Dv)(x, n)\, ds,$$

from which (3.8) follows.

It should be noticed that this identity is similar to one derived by PUCCI & SERRIN for the biharmonic operator. If the version of the Pucci-Serrin identity for systems is applied to the equations (3.3) and (3.4), then an additional integration by parts is required in order to obtain (3.8). Here the expression of Theorem 2.1 immediately gives the required identity. In later examples we give identities for certain systems which cannot be immediately derived from the Pucci-Serrin identity for systems.

By employing the identity (3.8) we obtain the following non-existence theorem:

Theorem 3.2. *Suppose Ω and f satisfy*

(1) *Ω is star-shaped and $\partial\Omega$ is smooth,*

(2) $NF(u) - a_1 u f(u) \leqq 0, \quad u \neq 0,$

(3) $a_2 - \frac{N}{2} \geqq 0, \quad N - 2 - a_1 - a_2 \geqq 0,$

for some constants a_1 and a_2. Then Problem (Ia) *has no solution $(u, v) \in (C^2(\Omega) \cap C^1(\bar{\Omega}))^2$.*

Proof. Observe that because $u > 0$ in Ω, $f(u) > 0$ in Ω and so

$$\Delta v > 0 \quad \text{in } \Omega.$$

Because $v = 0$ on $\partial\Omega$ it follows from the Maximum Principle that

$$v < 0 \quad \text{in } \Omega.$$

Hence

$$\Delta u < 0 \quad \text{in } \Omega.$$

From the Boundary-Point Lemma we now conclude that

$$\frac{\partial u}{\partial n} < 0, \quad \frac{\partial v}{\partial n} > 0 \quad \text{on } \partial\Omega.$$

Thus

$$(Du, Dv) = \frac{\partial u}{\partial n} \frac{\partial v}{\partial n} (n, n) < 0 \quad \text{on } \partial\Omega.$$

Choosing the origin at a point about which Ω is star-shaped we conclude that

$$(Du, Dv)\,(x, n) < 0 \quad \text{on } \partial\omega \subset \partial\Omega,$$

where $\partial\omega$ has positive measure in $\partial\Omega$. Now we deduce from (3.8) that if $u > 0$, then

$$\int_\Omega \left((NF(u) - a_1 u f(u)) + \left(\frac{N}{2} - a_2\right) v^2 + (N - 2 - a_1 - a_2)\,(Du, Dv) \right) dx > 0. \tag{3.9a}$$

On the other hand, if we multiply (3.3) by v and integrate over Ω, we obtain

$$-\int_\Omega (Du, Dv)\, dx = \int_\Omega v^2\, dx > 0,$$

and so, using conditions (2) and (3) we obtain

$$\int_\Omega \left((N - 2 - a_1 - a_2)\,(Du, Dv) + (NF(u) - a_1 u f(u)) + \left(\frac{N}{2} - a_2\right) v^2 \right) dx \leqq 0. \tag{3.10}$$

The inequalities (3.9a) and (3.10) are contradictory unless $u = v = Du = Dv = 0$ in Ω. This proves the theorem.

If we take

$$a_1 + a_2 = N - 2, \qquad a_2 = \frac{N}{2}, \tag{3.11}$$

we obtain

Corollary 3.3. *Suppose Ω and f satisfy*
(1) *Ω is star-shaped, $\partial\Omega$ is smooth,*
(2) $NF(u) - \dfrac{N-4}{2} uf(u) \leqq 0, \quad u \neq 0.$
Then Problem (I) *has no solution $u \in C^4(\Omega) \cap C^3(\bar{\Omega})$.*

Remark. For the function

$$f(u) = u^p, \quad p > 1,$$

we obtain non-existence by Corollary 3.3 if

$$p \geqq \frac{N+4}{N-4}.$$

Remark. Observe that because we have $a_1 = \dfrac{N-4}{2}$ and $a_2 = \dfrac{N}{2}$, the infinitesimal generator of the group of dilatations becomes

$$v = x^i \frac{\partial}{\partial x^i} - \frac{N-4}{2} u \frac{\partial}{\partial u} - \frac{N}{2} v \frac{\partial}{\partial v}.$$

This corresponds to

$$u(x) \mapsto \lambda^{-\frac{N-4}{2}} u\left(\frac{x}{\lambda}\right), \quad v(x) \to \lambda^{-\frac{N}{2}} u\left(\frac{x}{\lambda}\right), \quad \lambda \in \boldsymbol{R}^+.$$

Clearly this group of dilatations is not a symmetry group of the Lagrangian given by (3.6), when the nonlinearity $f(u)$ does not satisfy $NF(u) - \dfrac{N-4}{2} uf(u) = 0$.

2. *An indefinite variational system.* Now we consider the problem

$$\text{(II)} \quad \begin{cases} \Delta u = g(v), \quad u > 0 & \text{in } \Omega, \quad (3.12) \\ \Delta v = f(u) & \text{in } \Omega, \quad (3.13) \\ u = 0 & \text{on } \partial\Omega, \quad (3.14) \\ v = 0 & \text{on } \partial\Omega, \quad (3.15) \end{cases}$$

in which the functions f and g are assumed to satisfy the following hypotheses:

$$\text{(H2)} \quad \begin{cases} f, g \in C(\boldsymbol{R}); \\ f(u) > 0 \quad \text{if } u > 0; \quad f(u) < 0 \quad \text{if } u < 0; \quad f(0) = 0; \\ g(u) > 0 \quad \text{if } u > 0; \quad g(u) < 0 \quad \text{if } u < 0; \quad g(0) = 0. \end{cases}$$

This problem has the Lagrangian

$$L = (Du, Dv) + G(v) + F(u), \tag{3.16}$$

$$G'(v) = g(v), \quad F'(u) = f(u). \tag{3.17}$$

Using the above Lagrangian we can apply Theorem 2.1 and derive a variational identity for Problem (II). The group of transformations we use are again dilatations, although slightly different from the one we described in Section 2.

Theorem 3.4. *Let* $(u, v) \in (C^2(\Omega) \cap C^1(\bar{\Omega}))^2$ *be a solution of Problem* (II) *in which* $s = 2$, $a_k(x) = a_k = \text{const.}$, $h^i(x) = x^i$, $k = 1, \ldots, s$, $i = 1, \ldots, N$. *Then*

$$\int_\Omega ((NF(u) - a_1 uf(u)) + (NG(v) - a_2 vg(v)) + (N - 2 - a_1 - a_2)(Du, Dv))\, dx = -\oint_{\partial\Omega} (Du, Dv)(n, x)\, ds. \tag{3.18}$$

The proof of this theorem is completely similar to the proof of Theorem 3.1.

By employing this identity we obtain the following non-existence theorem:

Theorem 3.5. *Suppose that* Ω, f *and* g *satisfy*
(1) Ω *is star-shaped and* $\partial\Omega$ *is smooth,*
(2) $NF(u) - a_1 uf(u) \leqq 0, \quad u \neq 0,$
(3) $NG(v) - a_2 vg(v) \leqq 0, \quad v \neq 0,$
(4) $N - 2 - a_1 - a_2 \geqq 0,$
for some constants a_1 *and* a_2. *Then Problem* (II) *has no solution* $(u, v) \in (C^2(\Omega) \cap C^1(\bar{\Omega}))^2$.

Proof. As in the proof of Theorem 3.2 we conclude from the Maximum Principle that

$$v < 0 \quad \text{in } \Omega.$$

Again by employing the Boundary-Point Lemma we obtain

$$(Du, Dv)(x, n) < 0 \quad \text{on } \partial\omega \subset \partial\Omega.$$

This then gives

$$\int_\Omega ((NF(u) - a_1 uf(u)) + (NG(v) - a_2 vg(v)) + (N - 2 - a_1 - a_2)(Du, Dv))\, dx > 0.$$

If we multiply (3.12) by v and then integrate over Ω, we obtain

$$-\int_\Omega (Du, Dv)\, dx = \int_\Omega vg(v)\, dx > 0, \quad v \neq 0.$$

From here on the proof is similar to that of Theorem 3.2.

If we take

$$a_1 + a_2 = N - 2,$$

we obtain

Corollary 3.6. *Suppose Ω, f and g satisfy*
(1) *Ω is star-shaped and $\partial\Omega$ is smooth,*
(2) $NF(u) - a_1 uf(u) \leqq 0, \quad u \neq 0,$
(3) $NG(v) - (N-2-a_1)\, vg(u) \leqq 0, \quad v \neq 0,$
for some constant a_1. Then Problem (II) *has no solution $(u, v) \in \left(C^2(\Omega) \cap C^1(\bar{\Omega})\right)^2$.*

Remark. For the functions

$$f(u) = u^p, \quad g(v) = v|v|^{q-1}, \tag{3.19}$$

we obtain non-existence if

$$p \geqq \frac{N-a_1}{a_1}, \tag{3.20}$$

$$q \geqq \frac{2+a_1}{N-2-a_1}. \tag{3.21}$$

This result agrees with Example 2 where $q = 1$ and $a_1 = \dfrac{N-4}{2}$. Note that this particular choice of a_1 corresponds to the dilatation group used for the biharmonic operator.

Remark. Some existence and non-existence theorems were obtained by GALAKTIONOV, KURDYUMOV & SAMARSKII [GKS] in their study of steady states of parabolic systems closely related to Problem (II).

3. *The pluri-harmonic operator.* In this example we shall consider the pluri-harmonic operator as a second-order system in order to study other 'Dirichlet-like' boundary conditions different from those considered by [PS].

(III)
$$\begin{cases} \Delta u_i = -u_{i+1}, \quad u_1 > 0 \quad i = 1, \dots, s-1 & \text{in } \Omega, \quad (3.22)\\ \Delta u_s = -f(u_1) & \text{in } \Omega, \quad (3.23)\\ \mathscr{B}_m(u_i, Du_i) = 0, \quad m, i = 1, \dots, s & \text{on } \partial\Omega. \quad (3.24)\end{cases}$$

It should be noted that we placed minus signs on the left-hand sides of the equations (3.22) and (3.23). In this way we can look for positive solutions $u_i > 0$, $i = 1, \dots, s$. This is a direct consequence of the Maximum Principle. We assume that f satisfies the following hypotheses:

(H3)
$$\begin{cases} f \in C(\boldsymbol{R}), \\ f(0) = 0, \quad f(u) > 0 \quad \text{if } u > 0. \end{cases}$$

The functionals $\mathscr{B}_m$ describe the boundary conditions. Our objective is to consider different choices of $\mathscr{B}_m$ which may occur when the pluri-harmonic operator is considered as a system. For example, the triharmonic equation

(A)
$$\Delta^3 u = -f(u)$$

allows four such possibilities, arising as Dirichlet boundary conditions for different forms of (A). Thus for (A) these boundary conditions are $u = 0$, $\frac{\partial u}{\partial n} = 0$ and $\Delta u = 0$ on $\partial \Omega$. The question of non-existence of solutions of this problem is studied in [PS].

When (A) is written as

$$\text{(B)} \qquad \begin{cases} \Delta^2 u = v, \\ \Delta v = -f(u), \end{cases}$$

the corresponding Dirichlet boundary conditions become $u = 0$, $\frac{\partial u}{\partial n} = 0$ and $v = 0$ on $\partial \Omega$. Writing (A) as

$$\text{(C)} \qquad \begin{cases} \Delta u = -v, \\ \Delta^2 v = f(u), \end{cases}$$

we have $u = 0$, $v = 0$ and $\frac{\partial v}{\partial n} = 0$ on $\partial \Omega$. Finally, if (A) is written as the system

$$\text{(D)} \qquad \begin{cases} \Delta u = -v, \\ \Delta v = -w, \\ \Delta w = -f(u), \end{cases}$$

the Dirichlet boundary conditions become $u = 0$, $v = 0$ and $w = 0$ on $\partial \Omega$. These latter conditions are the central boundary conditions in this paper. Of course, in all these problems u is taken to be positive.

In considering these four situations we shall restrict our attention to radially symmetric solutions on $\Omega = B_1$. For this case we examine System (D), and the boundary conditions which agree with the four situations. Transforming to new variables:

$$t = (N-2)^{N-2} r^{-(N-2)}, \quad k = \frac{2N-2}{N-2}, \quad N \neq 2,$$

$$y_i(t) = u_i(r), \quad u_1 = u, \quad u_2 = v, \quad u_3 = w, \quad \text{for } i = 1, \ldots, 3,$$

as in [AP, PV] we obtain

$$\text{(III a)} \qquad \begin{cases} y_i'' = -t^{-k} y_{i+1}, \quad y_1 > 0 \quad i = 1, \ldots, s-1, & (3.25) \\ y_s'' = -t^{-k} f(y_1), & (3.26) \\ y_i'(\infty) = 0, \quad i = 1, \ldots, s, & (3.27) \\ \mathscr{B}_m(y_i, y_i')(T) = 0, \quad i, m = 1, \ldots, s, & (3.28) \\ T = (N-2)^{(N-2)}. & (3.29) \end{cases}$$

For this system it is possible to give a Lagrangian

$$L = \frac{1}{2} \sum_{i=1}^{s} y_i' y_{s-i+1}' - \frac{1}{2} \sum_{i=1}^{s-1} t^{-k} y_{i+1} y_{s-i+1} - t^{-k} F(y_1) \tag{3.30}$$

in which

$$F'(y_1) = f(y_1). \tag{3.31}$$

In [AP] the case $s = 1$ is discussed in great detail. The case $s = 2$ is discussed in [PV]. Here we shall discuss the case $s = 3$ as an illustration. The Lagrangian is then

$$L = y_1' y_3' + \frac{1}{2} y_2'^2 - t^{-k} y_2 y_3 - t^{-k} F(y_1), \tag{3.32}$$

and the Euler-Lagrange equations are

(IV)
$$\begin{cases} y_1'' = -t^{-k} y_2, & (3.33) \\ y_2'' = -t^{-k} y_3, & (3.34) \\ y_3'' = -t^{-k} f(y_1). & (3.35) \end{cases}$$

The boundary conditions corresponding to (A)–(D) become

$$\text{A:}\quad y_1(T) = 0, \quad y_1'(T) = 0, \quad y_2(T) = 0, \tag{3.36}$$

$$\text{B:}\quad y_1(T) = 0, \quad y_1'(T) = 0, \quad y_3(T) = 0, \tag{3.37}$$

$$\text{C:}\quad y_1(T) = 0, \quad y_2(T) = 0, \quad y_2'(T) = 0, \tag{3.38}$$

$$\text{D:}\quad y_1(T) = 0, \quad y_2(T) = 0, \quad y_3(T) = 0. \tag{3.39}$$

Remark. We can immediately make some observations about the solution (y_1, y_2, y_3). In all cases we have by assumption that $y_1 > 0$ on (T, ∞) and $y_1(T) = 0$. In addition we have that $y_3 > 0$ and is concave on (T, ∞) in Case B, and that $y_i > 0$, $i = 1, 2, 3$ on (T, ∞) in Case D.

From Theorem 2.1 we have the following identity:

$$-I_j(y_i, y_i')(T) = -\int_T^\infty \left((1 + a_1 + a_3)\, y_1' y_3' + \left(\tfrac{1}{2} + a_2\right) y_1'^2\right) ds$$

$$+ \int_T^\infty \left((k - 1 + a_2 + a_3)\, s^{-k} y_2 y_3 + (k-1)\, s^{-k} F(y_1) + a_1 s^{-k} y_1 f(y_1)\right) ds, \tag{3.40}$$

where $j = 1, \ldots, 4$, and the boundary terms I_j are given by

$$\text{A:}\quad I_1 = -\frac{1}{2} T y_2'^2, \tag{3.41}$$

$$\text{B:}\quad I_2 = -\frac{1}{2} T y_2'^2 - a_2 y_2 y_2', \tag{3.42}$$

$$\text{C:}\quad I_3 = -T y_1' y_3' - a_3 y_3 y_1', \tag{3.43}$$

$$\text{D:}\quad I_4 = -T y_1' y_3' - \frac{1}{2} T y_2'^2. \tag{3.44}$$

The expressions (3.41)–(3.44) are evaluated by using the boundary conditions at infinity at $t = T$. Because u_1, u_2 and u_3 are assumed to be smooth, we require that $y_1(\infty)$, $y_2(\infty)$, $y_3(\infty)$ exist and be finite. For a more detailed discussion we refer to [PV].

In this example we take

$$a_1 = \frac{2k-5}{2}, \quad a_2 = -\frac{1}{2}, \quad a_3 = \frac{3-2k}{2}.$$

Thus if $k \in (2, 2\frac{1}{2})$, which corresponds to $N > 6$, then $a_1 < 0$, $a_2 < 0$, $a_3 < 0$. With this choice of the constants a_1, a_2 and a_3 it is possible to determine the sign of $I_j(T)$ for all values of j. The identity now becomes

$$-I_j(y_i, y_i')(T) = \int_T^\infty \left((k-1)\, s^{-k} F(y_1) - \frac{5-2k}{2}\, s^{-k} y_1 f(y_1) \right) ds.$$

Lemma 3.7. *Suppose* $k \in (2, 2\frac{1}{2})$. *If* (y_1, y_2, y_3) *is a solution of the equations* (3.33)–(3.35), *then*

$$I_j(y_i, y_i')(T) < 0 \quad \text{for } j = 1, 2, 3, 4.$$

Proof. We treat the cases $j = 1, 2, 3, 4$ in succession.
$j = 1$: In this case the result is obvious.
$j = 2$: As we already observed in the last remark, y_3 is concave, and so $y_3 > 0$ on (T, ∞) because $y_3(T) = 0$. Thus, by (3.34), y_2 is concave and $y_2'(T) > 0$ because $y_2'(\infty) = 0$. As before we conclude from (3.33) and the conditions at $t = T$ and at infinity that y_2 must have a sign change and that $y_2(T) < 0$. Because $a_2 < 0$ and $y_3'(T) > 0$, the assertion follows from (3.42).
$j = 3$: By (3.35), y_3 is a concave function, whence, because $y_3'(\infty) = 0$, there exists two possiblities: Either y_3 has one sign on (T, ∞), or it has one zero and $y_3(\infty) > 0$. If y_3 has one sign, then $y_2'(\infty) \neq 0$, by (3.34) and (3.38), so y_3 must have a zero and $y_3(T) < 0$. This means that $y_2 > 0$ on (T, ∞) and that y_1 is concave on (T, ∞). Because $y_1(T) = 0$ it follows that $y_1'(T) > 0$. By the choice of k, $a_3 < 0$, so that substitution into (3.43) yields the desired inequality.
$j = 4$: As was already noted, $y_i > 0$ on (T, ∞) for $i = 1, 2, 3$. Integration of the equations (3.33)–(3.35) over (T, ∞) yields that $y_i'(T) > 0$ for $i = 1, 2, 3$, which, when substituted in (3.44), proves the assertion.

We now can formulate the following non-existence results for Problem (IV):

Theorem 3.8. *Let* $T > 0$ *and let* f *satisfy*

$$(k-1)\, F(y_1) - \frac{5-2k}{2}\, y_1 f(y_1) \leqq 0 \quad y_1 > 0. \tag{3.45}$$

Then Problem (IV) *has no solution* $y_1, y_2, y_3 \in C^2((T, \infty)) \cap C^1([T, \infty))$ *which satisfies one of the boundary conditions A to D.*

Proof. If (3.45) is fulfilled, the right-hand side of identity (3.40) becomes non-negative while the left-hand side is negative by Lemma 3.7.

Remark. If we consider the function $f(y_1) = y_1^p$, we obtain non-existence if

$$p \geqq \frac{4k-7}{5-2k} = \frac{N+6}{N-6}, \quad N > 6.$$

Next we shall discuss a particular variant of Problem (III). We can write down the following Lagrangian:

$$L = \tfrac{1}{2} \sum_{k=1}^{s} (Du_k, Du_{s-k+1}) - \tfrac{1}{2} \sum_{k=1}^{s-1} u_{k+1} u_{s-k+1} - F(u_1) \tag{3.46}$$

$$F'(u_1) = f(u_1).$$

We are interested here in specific boundary conditions analogous to those of [PS] and [EFT], i.e., the Dirichlet boundary conditions for the system. We accordingly consider the following system of equations:

$$\text{(III b)} \qquad \begin{cases} \Delta u_k = -u_{k+1}, \quad u_k > 0 \quad k = 1, \ldots, s-1 & \text{in } \Omega, \\ \Delta u_s = -f(u_1) & \text{in } \Omega, \\ u_k = 0, \quad k = 1, \ldots, s & \text{on } \partial\Omega. \end{cases}$$

If we employ the identity (2.4) with $a_k(x) = a_k$ and $h^i(x) = x^i$, we obtain the identity

$$\int_\Omega \left((NF(u_1) - a_1 u_1 f(u_1)) + \sum_{k=1}^{s-1} \left(\frac{N}{2} - a_{k+1}\right) u_{k+1} u_{s-k+1} - \sum_{k=1}^{s} \left(\frac{N}{2} - 1 - a_k\right) (Du_k, Du_{s-k+1}) \right) dx$$

$$= \tfrac{1}{2} \oint_{\partial\Omega} \left(\sum_{k=1}^{s} (Du_k, Du_{s-k+1}) \right) (n, x)\, ds. \tag{3.47}$$

We can simplify (3.47) by imposing some restrictions on the coefficients. As before we require that the coefficients of $u_{k+1} u_{s-k+1}$ for $k = 1, \ldots, s-1$, and the coefficients of (Du_k, Du_{s-k+1}) for $k = 1, \ldots, s$ all vanish. It is easily observed that to achieve this we obtain exactly s linear equations in the variables a_k for $k = 1, \ldots, s$.

Remark. We have to point out that by the above procedure the number of independent parameters a_k reduces to zero. However, if we replace the

Lagrangian density given in (3.36) by

$$L = \tfrac{1}{2} \sum_{k=1}^{s} (Du_k, Du_{s-k+1}) - H(u_1, u_2, \ldots, u_s),$$

in which

$$\frac{\partial H}{\partial u_i} = h_i(u_1, u_2, \ldots, u_s), \quad i = 1, 2, \ldots, s,$$

we obtain a number of degrees of freedom on the parameters $a_1, \ldots, a_s$. This can best be illustrated by Example 2, where we had

$$H(u_1, u_2) = F(u_1) + G(u_2).$$

In this example one degree of freedom exists by the relation $a_1 + a_2 = N - 2$ between the parameters a_1 and a_2. This can also be done for systems of s equations.

For Problem (IIIb) we now obtain the lemma:

Lemma 3.9. *Let* $u_k \in C^2(\Omega) \cap C^1(\bar{\Omega})$ *for* $k = 1, \ldots, s$ *be a solution of Problem* (IIIb), *and let* $a_k(x) = a_k$, $h^i(x) = x^i$ *for* $k = 1, \ldots, s$, $i = 1, \ldots, N$. *Then the following identity holds*:

$$\int_{\Omega} \left(NF(u_1) - \frac{N-2s}{2} u_1 f(u_1) \right) = \tfrac{1}{2} \oint_{\partial\Omega} \left(\sum_{k=1}^{s} (Du_k, Du_{s-k+1}) \right) (n, x)\, ds. \tag{3.48}$$

Proof. We consider the cases in which s is even ($s \geqq 4$) and s is odd ($s \geqq 3$) successively. The cases $s = 1$ and $s = 2$ have been dealt with before.
s is even: If we set the coefficients occuring in (3.47) equal to zero, we obtain two sets of equations:

$$\text{(E1)} \qquad \begin{cases} N - 2 - a_1 - a_s = 0, \\ N - 2 - a_2 - s_{s-1} = 0, \\ \vdots \\ N - 2 - a_{\frac{s}{2}} - a_{1+\frac{s}{2}} = 0, \end{cases}$$

$$\text{(E2)} \qquad \begin{cases} N - a_2 - a_s = 0, \\ N - a_3 - s_{s-1} = 0, \\ \vdots \\ N - a_{\frac{s}{2}} - a_{2+\frac{s}{2}} = 0, \\ \dfrac{N}{2} - a_{1+\frac{s}{2}} = 0. \end{cases}$$

Together (E1) and (E2) yield s equations with s variables. If we subtract the first equation of (E1) from the first equation of (E2) and the second one of (E1) from the second one of (E2), etc., we obtain

$$a_{\frac{s}{2}} = \frac{N-4}{2}, \quad a_{k-1} = a_k - 2, \quad k = 2, \ldots, \frac{s}{2}.$$

Solving these gives

$$a_1 = a_2 - 2 = \ldots = a_{\frac{s}{2}} - 2\left(\frac{s}{2} - 1\right) = \frac{N}{2} - 2 - s + 2 = \frac{N}{2} - s.$$

s is odd: Just as for even values of s we obtain two sets of equations from (3.47):

$$\text{(E3)} \qquad \begin{cases} N - a_2 - \ a_s \quad = 0, \\ N - a_3 - \ s_{s-1} \quad = 0, \\ \vdots \\ N - a_{\frac{s+1}{2}} - a_{1+\frac{s+1}{2}} = 0, \end{cases}$$

$$\text{(E4)} \qquad \begin{cases} 2 - \ N + \ a_1 + \ a_s \quad = 0, \\ 2 - \ N + \ a_2 + \ s_{s-1} \quad = 0, \\ \vdots \\ 2 - \ N + \ a_{\frac{s-1}{2}} + a_{1+\frac{s+1}{2}} = 0, \\ \frac{N}{2} - 1 - \ a_{\frac{s+1}{2}} \quad = 0. \end{cases}$$

We immediately have

$$a_{\frac{s+1}{2}} = \frac{N-2}{2}.$$

If we add the first equation of (E3) to the first equation of (E4) and the second equation of (E3) to the second equation of (E4), etc., we obtain

$$a_{k-1} = a_k - 2, \quad k = 2, \ldots, \frac{s+1}{2}.$$

Direct verification yields

$$a_1 = a_{\frac{s+1}{2}} - 2\left(\frac{s+1}{2} - 1\right) = \frac{N}{2} - 1 - s - 1 + 2 = \frac{N}{2} - s.$$

This outcome is similar to the one in the even case. The lemma is proved by substituting the value obtained for a_1 in identity (3.47).

By employing Lemma 3.9 we are able to derive the following nonexistence theorem for Problem (IIIb).

Theorem 3.10. *Suppose Ω and f satisfy*
(1) *Ω is star-shaped,*
(2) $NF(u_1) - \frac{N-2s}{2} u_1 f(u_1) \leqq 0, \quad u_1 \neq 0.$
Then Problem (IIIb) *has no solution $u_k \in C^2(\Omega) \cap C^1(\bar{\Omega})$ for $k = 1, \ldots, s$.*

Proof. From the equations of Problem (IIIb) we immediately deduce that

$$\Delta u_k < 0 \quad \text{in } \Omega \quad \text{for } k = 1, \ldots, s.$$

From the Boundary-Point Lemma we conclude that

$$\frac{\partial u_k}{\partial n} < 0 \quad \text{on } \partial\Omega, \quad k = 1, \ldots, s.$$

Hence $(Du_k, Du_{s-k+1}) = \frac{\partial u_k}{\partial n} \frac{\partial u_{s-k+1}}{\partial n} (n, n) > 0$ on $\partial\Omega$ for all k. Because Ω is star-shaped this gives

$$\tfrac{1}{2} \oint_{\partial\Omega} \left(\sum_{k=1}^{s} (Du_k, Du_{s-k+1}) \right) (n, x)\, ds > 0.$$

Therefore

$$\int_{\Omega} \left(NF(u_1) - \frac{N-2s}{2} u_1 f(u_1) \right) dx > 0. \tag{3.49}$$

On the other hand, by employing condition (2) we obtain

$$\int_{\Omega} \left(NF(u_1) - \frac{N-2s}{2} u_1 f(u_1) \right) dx \leqq 0. \tag{3.50}$$

Since (3.49) and (3.50) are contradictory, the theorem is proved.

Remark. Consider the function

$$f(u_1) = u_1^p.$$

We obtain a non-existence theorem if $p \geqq \frac{N+2s}{N-2s}$. This is the same result as that derived in [PS] for the case of Problem (IIIc) with Dirichlet boundary conditions.

Theorem 3.10 yields a non-existence theorem for the function

$$f(u_1) = \lambda u_1^q + u_1^{\frac{N+2s}{N-2s}}$$

if

$$\lambda \leqq 0 \quad \text{and} \quad q < \frac{N+2s}{N-2s}. \tag{3.51}$$

To see this observe that condition (2) becomes

$$\lambda \left(\frac{N-2s}{2} - \frac{N}{q+1} \right) u_1^{q+1} \leqq 0, \qquad u_1 \geqq 0.$$

Thus, if $\lambda < 0$ and $q < \frac{N+2s}{N-2s}$, we arrive at a contradiction. The case $\lambda = 0$ is discussed in the Remark above.

Remark. Following [PS] and [EFJ] we can write Problem (III b) as

$$\text{(III c)} \qquad \begin{cases} (-\Delta)^s u - \lambda u^q - u^{\frac{N+2s}{N-2s}} = 0 & \text{in } \Omega, \\ (-\Delta)^k u > 0, \quad k = 0, \ldots, s-1 & \text{in } \Omega, \\ u = \Delta u = \ldots = (\Delta)^{s-1} u = 0 & \text{on } \partial\Omega. \end{cases}$$

4. Linearizations around solutions. In this example we shall derive an identity for a system consisting of the elliptic equation $\Delta u + \lambda u + u^p = 0$ and its linearization

$$\text{(L1)} \qquad \begin{cases} \Delta u + \lambda u + u^p = 0, \quad u > 0 & \text{in } \Omega, & (3.52) \\ \Delta w + \lambda w + p u^{p-1} w = 0 & \text{in } \Omega, & (3.53) \\ u = 0, \ w = 0 & \text{on } \partial\Omega. & (3.54) \end{cases}$$

This system arises in the study of uniqueness and stability of solutions of (3.52).

For System (L1) it is possible to give the Lagrangian

$$L = (Du, Dw) - \lambda uw - u^p w. \tag{3.55}$$

With this information we can apply Theorem 2.1, which leads to the following lemma:

Lemma 3.11. *Let* $(u, w) \in (C^2(\Omega) \cap C^1(\bar{\Omega}))^2$ *be a solution of Problem* (L1), *and let* $a_k(x) = a_k = \text{const.}$, $h^i(x) = x^i$ *for* $k = 1, 2$, $i = 1, \ldots, N$. *Then*

$$\int_\Omega \left((a_1 p + a_2 - N) u^p w + \lambda (a_1 + a_2 - N) uw + (N - 2 - a_1 - a_2)(Du, Dw) \right) dx = - \oint_{\partial\Omega} (Du, Dw)(n, x)\, ds. \tag{3.56}$$

Proof. Apply Theorem 2.1, using expression (3.55).

With the choice of parameters

$$a_1 = \frac{2}{p-1}, \quad a_2 = \frac{(N-2)\,p - N}{p-1} \tag{3.57}$$

in (3.56) we obtain the identity

$$\oint_{\partial\Omega} (Du, Dw)\,(n, x)\,ds = 2\lambda \int_{\Omega} uw\,dx. \tag{3.58}$$

Identity (3.58) was recently also obtained by S. C. LIN [Li] using a different method, in connection with the study of uniqueness of solutions of equation (3.52) when $\lambda = 0$ and Ω is a convex domain in $\boldsymbol{R}^2$.

An interesting question arises how to handle the problem if we want to consider two different eigenvalues instead of one in Problem (L1). For this we have to state an additional property for variational identities which was also mentioned in [PS] (page 685). For the problem with two distinct eigenvalues we consider the modified Euler-Lagrange equations:

$$\operatorname{div}\left(\frac{\partial L}{\partial p_i^k}\right) - \frac{\partial L}{\partial u_k} = G_k, \quad k = 1, \ldots, s. \tag{2.2a}$$

Here G_k is some function of u, w, Du, Dw. We choose $G_1 = (\lambda - \mu)\,w$ and $G_2 = 0$. The equations (2.2a) then become

$$\text{(L2)} \qquad \begin{cases} \Delta u + \lambda u + u^p = 0, \quad u > 0 & \text{in } \Omega, & (3.59) \\ \Delta w + \mu w + p u^{p-1} w = 0 & \text{in } \Omega, & (3.60) \\ u = w = 0 & \text{on } \partial\Omega. & (3.61) \end{cases}$$

If we want to obtain variational identities for the *quasi*-variational equations (2.2a), we have to add the following expression to the right-hand sides of (2.3) and (2.4):

$$-\left(h^j \frac{\partial u_k}{\partial x^j} + a_{kl} u_l\right) G_k. \tag{3.62}$$

If we apply this modification, we obtain the following identity for Problem (L2):

$$-\oint_{\partial\Omega} (Du, Dw)\,(n, x)\,ds = -\frac{2(p\lambda - \mu)}{p-1} \int_{\Omega} uw\,dx - (\lambda - \mu) \int_{\Omega} w(x, Du)\,dx. \tag{3.63}$$

The identities (3.58) and (3.63) were derived in collaboration with HULSHOF & WEISSLER [HW], who considered them in the study of Problems (L1) and (L2). For systems with a variational structure the linearized system can be described together with the original system in one Lagrangian similar to the above example.

Remark. If we recall the non-variational system (1.9) and (1.10) from Section 1, we can give a method using the modified Euler-Lagrange equations to derive variational identities. One can proceed as follows. Set

$$\int_0^u f(s, v)\, ds = F(u, v), \qquad \int_0^v g(u, s)\, ds = G(u, v),$$

and consider the Lagrangian

$$L = (Du, Dv) - F(u, v) - G(u, v).$$

The modified Euler-Lagrange equations (2.2a) in which we set $G_1 = G_u(u, v)$ and $G_2 = F_v(u, v)$ (2.2a) yields

$$-\Delta u = g(u, v), \tag{1.9}$$

$$-\Delta v = f(u, v). \tag{1.10}$$

Using the Lagrangian L and the expression (3.62) as an additional term in (2.3) and (2.4) we can derive identities similar to the above example.

4. A multiple eigenvalue problem for systems

In this section we shall study an eigenvalue problem which is related to the Dirichlet problem for the equation $-\Delta u = \lambda u + u^{\frac{N+2}{N-2}}$ studied by Brezis & Nirenberg [BrN]. Here we perturb the linear eigenvalue problem

$$\text{(L)} \qquad \begin{cases} -\Delta u = \lambda_2 v & \text{in } \Omega, & (4.1)\\ -\Delta v = \lambda_1 u & \text{in } \Omega, & (4.2)\\ u = 0, \; v = 0 & \text{on } \partial\Omega, & (4.3) \end{cases}$$

in which Ω is a bounded domain in $\boldsymbol{R}^N$, $N > 4$, with two terms of critical growth. For Problem (L) we can prove the following lemma, in which μ_1 is the principal eigenvalue of the Laplacian with Dirichlet boundary conditions and ϕ_1 is the corresponding (positive) eigenfunction.

Lemma 4.1. *Problem* (L) *has a positive solution if and only if*

$$\lambda_1 > 0, \qquad \lambda_2 > 0, \qquad \lambda_1\lambda_2 = \mu_1^2.$$

The solution is given (*modulo a constant*) *by* $u = \frac{1}{\sqrt{\lambda_1}}\phi_1$, $v = \frac{1}{\sqrt{\lambda_2}}\phi_1$.

Proof. Multiply (4.1) by v and (4.2) by u, and integrate over Ω. This gives

$$\int_\Omega (Du, Dv)\, dx = \lambda_2 \int_\Omega v^2\, dx,$$

$$\int_\Omega (Du, Dv)\, dx = \lambda_1 \int_\Omega u^2\, dx.$$

Multiplying these identities we obtain

$$\left(\int_\Omega (Du, Dv)\,dx\right)^2 = \lambda_1\lambda_2 \int_\Omega v^2\,dx \int_\Omega u^2\,dx, \tag{4.4}$$

which yields for non-trivial solutions the condition

$$\lambda_1\lambda_2 > 0.$$

Thus there are two possibilities:
Case 1: $\lambda_1 < 0$, $\lambda_2 < 0$. Define

$$u^* = \sqrt{-\lambda_1}\; u, \tag{4.5}$$

$$v^* = -\sqrt{-\lambda_2}\; v. \tag{4.6}$$

Employing the equations (4.1) and (4.2) we obtain the following system:

$$-\Delta u^* = \sqrt{\lambda_1\lambda_2}\; v^* \quad \text{in } \Omega, \tag{4.7}$$

$$-\Delta v^* = \sqrt{\lambda_1\lambda_2}\; u^* \quad \text{in } \Omega, \tag{4.8}$$

$$u^* = 0,\ v^* = 0 \qquad \text{on } \partial\Omega. \tag{4.9}$$

Adding (4.7) and (4.8) we obtain

$$(\mathrm{L}') \qquad \begin{cases} \Delta(u^* + v^*) = \sqrt{\lambda_1\lambda_2}\;(u^* + v^*) & \text{in } \Omega, \\ u^* + v^* = 0 & \text{on } \partial\Omega, \end{cases}$$

and subtracting (4.7) from (4.8) we obtain

$$(\mathrm{L}'') \qquad \begin{cases} \Delta(u^* - v^*) = \sqrt{\lambda_1\lambda_2}\;(u^* - v^*) & \text{in } \Omega, \\ u^* - v^* = 0 & \text{on } \partial\Omega. \end{cases}$$

Multiplication of the differential equation of (L″) by $u^* - v^*$ followed by an integration over Ω yields

$$-\int_\Omega |D(u^* - v^*)|^2\,dx = \sqrt{\lambda_1\lambda_2}\int_\Omega (u^* - v^*)^2\,dx,$$

which proves that $u^* = v^*$ in $\bar{\Omega}$. Remembering the properties of the eigenvalue problem for the Laplacian, we find that the only solution to Problem (L′) with one sign (+ or −) is the first eigenfunction ϕ_1 with the eigenvalue μ_1. Thus $u^* = C\phi_1$, where $C \neq 0$ is some constant, and $\sqrt{\lambda_1\lambda_2} = \mu_1$. Since $u^*v^* = u^{*2} > 0$ it follows from (4.5) and (4.6) that $uv < 0$ so that u and v cannot both be positive. This excludes Case 1.
Case 2: $\lambda_1 > 0$, $\lambda_2 > 0$. Define

$$u^* = \sqrt{\lambda_1}\; u, \tag{4.10}$$

$$v^* = \sqrt{\lambda_2}\; v. \tag{4.11}$$

As in Case 1 we obtain

$$\text{(L')} \qquad \begin{cases} -\Delta(u^* + v^*) = \sqrt{\lambda_1\lambda_2}\ (u^* + v^*) & \text{in } \Omega, \\ u^* + v^* = 0 & \text{on } \partial\Omega, \end{cases}$$

$$\text{(L'')} \qquad \begin{cases} \Delta(u^* - v^*) = \sqrt{\lambda_1\lambda_2}\ (u^* - v^*) & \text{in } \Omega, \\ u^* - v^* = 0 & \text{on } \partial\Omega. \end{cases}$$

Again we obtain $u^* = v^*$ in $\bar{\Omega}$. This immediately yields the solution for Problem (L):

$$u = C\frac{1}{\sqrt{\lambda_1}}\phi_1, \qquad v = C\frac{1}{\sqrt{\lambda_2}}\phi_1, \qquad \mu_1 = \sqrt{\lambda_1\lambda_2}\ ,$$

where C is some nonzero constant.

Remark. The proof of Lemma 4.1 gives insight into the complete spectrum of Problem (L). Clearly it depends strongly on the spectrum of the Laplacian. In the (λ_1, λ_2)-plane we obtain curves of eigenvalues:

$$\lambda_1\lambda_2 = \mu_i^2, \qquad i \in N,$$

where μ_i are the eigenvalues of the Laplacian. The positive branch of $\lambda_1\lambda_2 = \mu_1^2$ corresponds to the positive solutions of Problem (L).

Remark. From Theorem 2.1 we have the following identity for Problem (L):

$$\lambda_1\left(a_1 - \frac{N}{2}\right)\int_\Omega u^2\,dx + \lambda_2\left(\frac{N-4}{2} - a_1\right)\int_\Omega v^2\,dx = -\oint_{\partial\Omega}(Du, Dv)(x, n)\,ds. \tag{4.12}$$

Choosing a_1 so that either the first or the second term on the left-hand side of (4.12) vanishes we derive

$$\lambda_1 = \frac{1}{2}\frac{\oint_{\partial\Omega}(Du, Dv)(x, n)\,ds}{\int_\Omega u^2\,dx}, \qquad \lambda_2 = \frac{1}{2}\frac{\oint_{\partial\Omega}(Du, Dv)(x, n)\,ds}{\int_\Omega v^2\,dx}.$$

After normalising u and v so that $\|u\| = 1$ and $\|v\| = 1$ in $L^2(\Omega)$ we obtain

$$\sqrt{\lambda_1\lambda_2} = \frac{1}{2}\oint_{\partial\Omega}(Du, Dv)(n, x)\,ds. \tag{4.13}$$

A similar relation was found by F. Rellich [R] for the Laplacian.

Now we formulate the problem which we are interested in:

$$\text{(P1)} \qquad \begin{cases} -\Delta u = \mu v + v^q, \quad u > 0 & \text{in } \Omega, & (4.14) \\ -\Delta v = \lambda u + u^p, \quad v > 0 & \text{in } \Omega, & (4.15) \\ u = v = 0 & \text{on } \partial\Omega. & (4.16) \end{cases}$$

Here (p, q) is a pair of critical exponents as derived in Example 2 of Section 3:

$$p = \frac{N-\gamma}{\gamma}, \quad q = \frac{2+\gamma}{N-2-\gamma}, \tag{4.17}$$

where we choose

$$\gamma \in \left(\frac{N-4}{2}, \frac{N}{2}\right), \tag{4.18}$$

so that $1 < p < \frac{N+4}{N-4}$, and $1 < q < \frac{N+4}{N-4}$. For Problem (P1) we now state the following theorem:

Theorem 4.2. *Let* (λ, μ) *be in one of the areas*
(1) $\lambda \leqq 0, \quad \mu \leqq 0$,
(2) $\lambda\mu \geqq \mu_1^2$.
Then Problem (P1) *has no solution.*

Proof. To prove part (1) we shall derive an identity which can be formed by using the Lagrangian for the variational problem:

$$L = (Du, Dv) - \frac{\lambda}{2} u^2 - \frac{1}{p+1} u^{p+1} - \frac{\mu}{2} v^2 - \frac{1}{q+1} v^{q+1}.$$

If we apply Theorem 2.1 with $a_k(x) = \text{const}$ and $h^i(x) = x$ we obtain the identity

$$\lambda \left(a_1 - \frac{N}{2}\right) \int_\Omega u^2 \, dx + \mu \left(a_2 - \frac{N}{2}\right) \int_\Omega v^2 \, dx$$
$$+ \left(a_1 - \frac{N}{p+1}\right) \int_\Omega u^{p+1} \, dx + \left(a_2 - \frac{N}{q+1}\right) \int_\Omega v^{q+1} \, dx$$
$$+ (N - 2 - a_1 - a_2) \int_\Omega (Du, Dv) \, dx = - \oint_{\partial\Omega} (Du, Dv)(x, n) \, ds. \tag{4.19}$$

(1) For a_1 and a_2 we make the following choices:

$$a_1 = \gamma, \quad a_2 = N - 2 - \gamma, \quad \text{so } N - 2 - a_1 - a_2 = 0.$$

The third, fourth, and fifth terms in the identity (4.19) vanish and we obtain

$$\lambda \left(\gamma - \frac{N}{2}\right) \int_\Omega u^2 \, dx + \mu \left(\frac{N-4}{2} - \gamma\right) \int_\Omega v^2 \, dx = - \oint_{\partial\Omega} (Du, Dv)(x, n) \, ds. \tag{4.20}$$

Note that

$$\gamma - \frac{N}{2} < 0, \quad \frac{N-4}{2} - \gamma < 0.$$

The right-hand side of (4.20) is non-positive. However, if $\lambda < 0$ and $\mu < 0$, the left-hand side is strictly positive. Hence for these values of λ and μ we have a contradiction. Actually if either λ or μ is zero, (4.20) still gives a contradiction. When λ and μ are both equal to zero, we obtain non-existence from Example 2 of Section 3.

(2) To prove this part we need Lemma 4.1. We shall denote by (ϕ, ψ) the positive solution of Problem (L). Using the equations (4.14) and (4.15) we deduce that

$$\lambda_2 \int_\Omega u\psi = -\int_\Omega u\Delta\phi = -\int_\Omega \phi\Delta u = \mu \int_\Omega v\phi + \int_\Omega v^q\phi > \mu \int_\Omega v\phi, \tag{4.21}$$

$$\lambda_1 \int_\Omega v\phi = -\int_\Omega v\Delta\psi = -\int_\Omega \psi\Delta v = \lambda \int_\Omega u\psi + \int_\Omega u^p\psi > \lambda \int_\Omega u\psi. \tag{4.22}$$

Thus

$$\lambda_2 \int_\Omega u\psi > \mu \int_\Omega v\phi, \tag{4.23}$$

$$\lambda_1 \int_\Omega v\phi > \lambda \int_\Omega u\psi. \tag{4.24}$$

Combining (4.23) and (4.24) and using the fact that λ, μ, λ_1, λ_2 are all positive we obtain

$$\lambda_1\lambda_2 \int_\Omega u\psi > \lambda\mu \int_\Omega u\psi. \tag{4.25}$$

Because $\int_\Omega u\psi > 0$ this gives

$$\mu_1^2 = \lambda_1\lambda_2 > \lambda\mu.$$

Here we used Lemma 4.1. This now completes the prove of the theorem.

Remark. After this paper was completed I learned from E. Mitidieri about his paper [Mi], in which he obtained results similar to those derived in Section 3, by a different method.

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