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Periodic Solutions to Some Problems of n-Body Type

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Abstract

We prove the existence of at least one T-periodic solution to a dynamical system of the type

$$-m_i \ddot{u}_i = \sum_{j=1, j\neq i}^{n} \nabla V_{ij}(u_i - u_j, t) \tag{1}$$

where the potentials V_{ij} are T-periodic in t and singular at the origin, $u_i \in \mathbf{R}^k$, $i = 1, \ldots, n$, and $k \geqq 3$. We also provide estimates on the H^1 norm of this solution. The proofs are based on a variant of the Ljusternik-Schnirelman method. The results here generalize to the n-body problem some results obtained by BAHRI & RABINOWITZ on the 3-body problem in [6].

1. Introduction and statement of the results

In this paper we look for periodic solutions to dynamical systems of n-body type like (1), where $V_{ij} \in \mathscr{C}^1((\mathbf{R}^k \setminus \{0\}) \times \mathbf{R}, \mathbf{R})$ are T-periodic in t and $k \geqq 3$. We consider the following assumptions on the potentials $V_{ij} (i, j = 1, \ldots, n, \ i \neq j)$:

$$V_{ij}(x, t) = V_{ji}(-x, t) \quad \forall x \in \mathbf{R}^k \setminus \{0\}, \tag{V1}$$

$$V_{ij}(x, t) \leqq 0 \quad \forall x \in \mathbf{R}^k \setminus \{0\}, \tag{V2}$$

$$\exists \rho_0 > 0, \ \exists U \in \mathscr{C}^1(\mathbf{R}^k \setminus \{0\}; \mathbf{R}) \text{ such that}$$

$$\lim_{x \to 0} U(x) = \lim_{x \to 0} V_{ij}(x, t) = +\infty, \quad -V_{i,j}(x, t) \geqq |\nabla U(x)|^2 \tag{V3}$$

$$\forall x \text{ with } 0 < |x| < \rho_0,$$

$$\exists \rho > 0, \ \exists \theta \in [0, \tfrac{\pi}{2}) \text{ such that ang } (\nabla V_{ij}(x, t), x) \leqq \theta \quad \forall x, |x| > \rho \tag{V4}$$

where $\nabla V_{ij}(x, t)$ denotes $\left(\frac{\partial}{\partial x_1} V_{ij}(x, t), \ldots, \frac{\partial}{\partial x_n} V_{ij}(x, t)\right) \in \mathbf{R}^k$, and in any Euclidean space $0 \leqq \text{ang}(x, y) \leqq \pi$ denotes the angle between x and y.

Remark. In what follows we shall assume for the sake of simplicity that $m_i = 1$. Simple variants on the estimates are needed to cover the general case.

Our main theorem is the following:

Theorem 1. *Assume that* (V1), (V2), (V3) *and* (V4) *hold. Then* (1) *has at least one T-periodic solution* $u = (u_1, \ldots, u_n)$ *such that* $u_i(t) \neq u_j(t)$, *for all* $i \neq j$ *and for all* $t \in \mathbf{R}$. *Moreover,*

$$\sum_{i=1}^{n} \int_0^T |\dot{u}_i|^2 \leqq 2c^* \leqq 2 \inf_{R>0} \left\{ \frac{2\pi^2}{T} nR^2 + T \sum_{1=i<j}^{n} \sup_{|x| \geqq 2R \sin(\pi(j-i)/n)} (-V_{ij}(x, t)) \right\}, \tag{1.1}$$

$$\frac{1}{2n} \sum_{i,j=1; i \neq i}^{n} \left(\frac{1}{T} \int_0^T u_i(t) - u_j(t) \right)^2 \leqq nR(n, \theta)^2 \left(\sqrt{\frac{c^* T}{6}} \frac{1}{\cos \theta} + \frac{\rho}{2} \right)^2 \tag{1.2}$$

where $R(n, \theta)$ *is a constant whose value is given in* (4.4).

In particular, (1.1) and (1.2) imply that there exists a constant $K(n, T) > 0$ such that

$$\| u_i - u_j \|_\infty \leqq K(n, T) (\rho + \sqrt{c^*}) (\cos \theta)^{-(n-1)}, \tag{1.3}$$

where

$$\theta = \sup_{i,j,|x| \geqq \rho} \text{ang}(\nabla V_{ij}(x, t), x)$$

and the constant c^*, as in (1.1), depends only on the interaction potentials V_{ij} and on the period.

Unfortunately, the most physically interesting potentials behave like $|x|^{-1}$ at the origin. Hence they do not satisfy the strong-force condition (V3). If we drop condition (V3) from the hypotheses of Theorem 1 and we only assume

$$V_{ij}(x, t) \to -\infty \quad \text{as } x \to 0 \ \ (\forall i \neq j), \tag{V3)*}$$

then a weaker result can be stated, namely, the existence of a *generalized* solution. Following [5], a generalized solution of (1) is an H^1 *T*-periodic function $u = (u_1, \ldots, u_n)$ that satisfies (1) in $[0, T] \setminus (\bigcup_{i,j} (u_i - u_j)^{-1}(0))$ and such that $[0, T] \cap (\bigcup_{i,j} (u_i - u_j)^{-1}(0))$ has measure zero. Hence the notion of generalized solution admits the possibility of collisions (which occur at those times t when $u_i(t) = u_j(t)$ for some i, j).

Theorem 2. *Assume* (V1), (V2), (V3)* *and* (V4) *hold. Then* (1) *has at least one T-periodic generalized solution* $u = (u_1, \ldots, u_n)$. *Moreover the same estimates* (1.1), (1.2) *and* (1.3) *hold.*

One can also prove an existence result under assumptions slightly different from (V4). Let us consider the following assumptions (for all $i \neq j$):

$$\forall M > 0 \quad \exists R > 0 \text{ such that if } |x| > R, \text{ then} \qquad \text{(V5)}$$
$$\nabla V_{ij}(x, t) \cdot x \geqq M |\nabla V_{ij}(x, t)|\, |x|^{\frac{n-3}{n-2}},$$
$$|V_{ij}(x, t)| + |\nabla V_{ij}(x, t)| \to 0 \quad \text{as } x \to +\infty \text{ uniformly in } t. \qquad \text{(V6)}$$

Then we have the following result:

Theorem 3. *Assume that* (V1), (V2), (V3), (V5) *and* (V6) *hold. Then* (1) *has at least one T-periodic solution* $u = (u_1, \ldots, u_n)$ *such that* $u_i(t) \neq u_j(t)$, *for all* $i \neq j$ *and for all* $t \in \mathbf{R}$.

From our point of view, the T-periodic solutions of (1) are critical points of the action functional

$$f(u) = \tfrac{1}{2} \int_0^T |\dot{u}|^2 - \int_0^T V(u, t) \qquad (1.4)$$

on the domain

$$\Lambda = \{u = (u_1, \ldots, u_n) \in H_T^1(\mathbf{R}; \mathbf{R}^{kn}) : u_i(t) \neq u_j(t),\ \forall t,\ \forall i \neq j\}. \qquad (1.5)$$

Here $H_T^1(\mathbf{R}, \mathbf{R}^{kn})$ is the space of H_{loc}^1 functions from $\mathbf{R}$ to $\mathbf{R}^{kn}$ with period T. V is the interaction potential

$$V(x_1, \ldots, x_n, t) = \tfrac{1}{2} \sum_{i,j=1;\, i \neq j}^{n} V_{ij}(x_i - x_j, t), \quad (x_1, \ldots, x_n) \in \mathbf{R}^{kn},\ x_i \neq x_j,\ \forall i \neq j. \qquad (1.6)$$

Let us briefly discuss the hypotheses (V1)–(V5) and describe the main difficulties which arise when dealing with the functional f. Assumption (V1) corresponds to Newton's third law of mechanics, since it implies that $-\nabla V_{ij}(u_i - u_j, t) = \nabla V_{ji}(u_j - u_i, t)$, that is, the force exerted on the j-th body by the i-th body is the opposite of the force exerted on the i-th body by the j-th body. It is a necessary condition for the variational formulation of the problem. This assumption can be dropped if we write system (1) as

$$-m_i \ddot{u}_i = \tfrac{1}{2} \sum_{i,j=1;\, i \neq j}^{n} \nabla V_{ij}(u_i - u_j, t) + \nabla V_{ji}(u_j - u_i, t).$$

There are three main difficulties with the variational approach to our problem. The first is a lack of completeness due to the fact that the domain of f is open. The strong-force condition (V3), which implies $f(u) \to +\infty$ as $u \to \partial \Lambda$, is a standard way to avoid this difficulty. The second is that f does not necessarily satisfy the Palais-Smale (PS) condition at any level, although it satisfies it on bounded sets of H. Therefore we shall use a deformation lemma that only requires a local form of the (PS) condition. A corresponding version of this deformation lemma has been introduced by Majer [14] for the case of functionals defined on a manifold M with boundary ∂M. Roughly speaking, if f has no critical points in M at the level c, then the sublevel $\{f \leqq c + \varepsilon\} \cup \partial M$ is contractible into $\{f \leqq c - \varepsilon\} \cup \partial M$ provided f, $f_{|\partial M}$ satisfy the (PS) condition and the functional gradient ∇f is not exactly opposite to the inward normal at ∂M. In our situation, let

$$g(u) = \frac{1}{2n} \sum_{i,j=1}^{n} \left(\frac{1}{T} \int_0^T u_i(t) - u_j(t) \right)^2 . \tag{1.7}$$

Then a (PS) sequence $(u_\nu)_\nu$ is precompact (up to translations of the whole systems) if and only if $g(u_\nu)$ is bounded. A natural setting in which to use this deformation lemma is to restrict f to some $M_b = \{g \leqq b\}$. We shall show how assumption (V4) ensures the above-mentioned "non-opposition" property at ∂M_b. Finally the third difficulty consists in finding a candidate for a critical level for f. To do this we shall make use of arguments of a topological nature that exploit the special structure of the loop space Λ. In particular, we shall prove the existence of at least one level c such that $\{f \leqq c - \varepsilon\} \cup \partial M$ is not deformable into $\{f \leqq c + \varepsilon\} \cup \partial M$.

In recent years, quite a large number of papers have appeared on the existence of periodic solutions to dynamical systems with singular potentials. We quote [1, 2, 5, 8, 9, 11, 12, 13, 22] and the references therein for the case of one body moving in a gravitation-like field. Concerning the n-body problem, symmetrical cases have been studied in [7, 8, 20, 23]. A symmetry constraint on the function space allows us to overcome the lack of compactness, since the restricted potential is coercive; therefore the existence of one solution can be derived from a minimization argument. The common problem of these papers is how to avoid collision solutions, when the strong-force condition (V3) is weakened.

In contrast, we shall deal with non-symmetrical potentials and we shall focus our attention on the variational method, retaining the strong force condition in order to ensure the closure of the sublevels of f. In this framework, the three-body problem ($n = 3$) has been treated by BAHRI & RABINOWITZ in [6], where the existence of *infinitely many* solutions is proved. The proof is based on the theory of critical points at infinity and makes considerable use of algebraic topology; it also gives a very detailed description of the behavior of all the diverging (PS) sequences and of the topology of the sublevel sets. The nontrivial way of generalizing this approach to any $n \geqq 3$ has been shown very recently in [20]. An advantage of our arguments is that they are based only on the above-mentioned deformation lemma and on elementary topology. The same construction together with a further topological argument yields to a proof of infinitely many solutions to (1) [16]. We do not consider here the problem for fixed energy; see [3, 15] and the references therein. For other related problems and for classical results see [4, 10, 17–19, 21].

Notation

n denotes the number of bodies with positions $(u_1, \ldots, u_n)$.
k denotes the dimension of space of the position of each body: $u_i \in \mathbf{R}^k$.
T denotes the period.

H denotes the Sobolev space of periodic functions:

$$H = \{u(t) = (u_1(t), \ldots, u_n(t)) \in H^1_{\mathrm{loc}}(\mathbf{R}; \mathbf{R}^{kn}) : u(t+T) = u(t), \forall t\}$$

endowed with the inner product

$$u \cdot v = \int_0^T (\dot{u}(t) \cdot \dot{v}(t) + u(t) \cdot v(t))\, dt.$$

Λ is the subset of all the collisionless orbits:

$$\Lambda = \{u \in H : u_i(t) \neq u_j(t), \forall t, \forall i \neq j\}.$$

We identify the subspace of h of all the constant functions with $\mathbf{R}^{kn}$:

$$\mathbf{R}^{kn} \cong \{u \in H : \dot{u}(t) = 0, \forall t\}.$$

For every function u, $[u]$ denotes its mean value:

$$[u] = \frac{1}{T} \int_0^T u(s)\, ds.$$

If $x, y \in \mathbf{R}^k$, the angle between x and y is

$$\operatorname{ang}(x, y) = \begin{cases} \arccos \dfrac{x \cdot y}{|x|\,|y|} & \text{if } |x|\,|y| \neq 0 \\ 0 & \text{otherwise}. \end{cases}$$

Recall the subadditivity of angles:

$$\operatorname{ang}(x, y) \leqq \operatorname{ang}(x, z) + \operatorname{ang}(z, y) \qquad \forall x, y, z \in \mathbf{R}^k,\ z \neq 0.$$

For any functional $f: X \to \mathbf{R}$ we denote: $\{f \leqq c\} = \{x \in X : f(x) \leqq c\}$ and use the analogous notations $\{f = c\}$ and $\{f \geqq c\}$.

2. The deformation lemma

Definition 2.1. Let X be a topological space and let $A, B \subseteq X$. A is *deformable into* B in X if there is a continuous homotopy $\eta : A \times [0, 1] \to X$ such that $\eta(\cdot, 0) = \mathrm{id}|_A$ and $\eta(A, 1) \subseteq B$.

If X is an absolute neighbourhood retract and A is closed, in the Definition 2.1 we can equivalently take homotopies $\eta : X \times [0, 1] \to X$ such that $\eta(\cdot, 0) = \mathrm{id}_{|X}$ (Homotopy Extension Property, cf. [24]).

Lemma 2.1. *Let Λ be an open subset of any Hilbert space H and let $f \in \mathscr{C}^1(\Lambda; \mathbf{R})$, $g \in \mathscr{C}^2(\Lambda; \mathbf{R})$. Assume that there are c and $\tilde{c}$ with $\tilde{c} > c$ and $b \in \mathbf{R}$ such that*

$$\lim_{\nu \to +\infty} f(x_\nu) = +\infty \quad \textit{if } x_\nu \to x_0 \in \partial\Lambda \textit{ and } g(x_\nu) \textit{ is bounded}. \tag{2.1}$$

$$\nabla g(x) \neq 0 \quad \forall x,\ g(x) = b,\ f(x) \leqq \tilde{c}. \tag{2.2}$$

$$\text{Every sequence } (x_\nu) \text{ in } \Lambda \text{ such that } f(x_\nu) \to c, \ \limsup_{\nu\to\infty} g(x_\nu) \leqq b \text{ and } \nabla f(x_\nu) \to 0 \text{ possesses a convergent subsequence.} \tag{2.3}$$

$$\text{Every sequence } (x_\nu) \text{ in } \Lambda \text{ such that } f(x_\nu) \to c, \ g(x_\nu) \to b \text{ and } \nabla f(x_\nu) - \lambda_\nu \nabla g(x_\nu) \to 0 \text{ with } \lambda_\nu \geqq 0, \text{ possesses a convergent subsequence.} \tag{2.4}$$

$$\nabla f(x) \neq \lambda \nabla g(x) \qquad \forall x, \forall \lambda > 0; \ g(x) = b, \ f(x) = c. \tag{2.5}$$

$$\forall \varepsilon \in (0, \tilde{c} - c] \text{ the set } \{f \leqq c + \varepsilon\} \cup (\{f \leqq \tilde{c}\} \cap \{g \geqq b\}) \text{ is not deformable into } \{f \leqq c - \varepsilon\} \cup (\{f \leqq \tilde{c}\} \cap \{g \geqq b\}) \text{ in } \Lambda. \tag{2.6}$$

Then f has at least one critical point $x \in \Lambda$ such that $f(x) = c$ and $g(x) \leqq b$.

The proof of this lemma easily follows from [14, Chap. III, Propositions 7, 8]. A sketch of it is given in Section 7.

In order to prove Theorem 1 we shall apply Lemma 2.1 in the following situation: H and Λ are defined in (2) and (4) and

$$f(u) = \frac{1}{2} \int_0^T \sum_{i=1}^n |\dot{u}_i|^2 - \int_0^T \sum_{1=i\leqq j}^n V_{ij}(u_i - u_j, t), \tag{2.7}$$

$$g(u) = \frac{1}{2n} \sum_{i,j=1}^n \left(\frac{1}{T} \int_0^T u_i(t) - u_j(t) \right)^2. \tag{2.8}$$

Conditions (2.1), (2.2), (2.3) and (2.4) will be proved to hold true for every c and $b > 0$ in Section 3; their proofs are quite standard. The proofs of the remaining conditions require a deeper discussion. In Section 4 we shall show that (2.5) holds whenever b is sufficiently large, depending on c, $\tilde{c}$. In Section 5 we shall finally characterize c and $\tilde{c}$ in a variational way. The estimates (1.1) and (1.2) in the statement of Theorem 1 will then follow from the characterization of the critical level c and the associated value b.

Remark. Since f is invariant under translations of the whole system, its natural domain is the subset $\Lambda/\mathbf{R}^k$ of the quotient space $H/\mathbf{R}^k$ (which is isomorphic to $H_0 =: \{u \in H: \Sigma_i[u_i] = 0\}$). In the quotient space f really satisfies the (PS) condition in every sublevel of g, while it does not in Λ, of course, owing to the non-compact action of the group of the translations. Hence we shall apply the deformation lemma in $\Lambda/\mathbf{R}^k$; nevertheless, with a slight abuse of notation, we shall denote with the same symbols both the functions of H and the equivalence classes of $H/\mathbf{R}^k$. This identification is admissible since all the operations appearing in this paper are compatible with the equivalence relation. In particular, it is straightforward to check that condition (2.6) f and g on Λ if and only if (2.6) holds for the induced maps $\bar{f}$ and $\bar{g}$ on $\Lambda/\mathbf{R}^k$.

3. Compactness properties of f. Proofs of (2.1), (2.2), (2.3) and (2.4)

In this section f and g are as in (2.7) and (2.8) and the V_{ij}'s satisfy assumptions (V1), (V2) and (V3).

Proposition 3.1. *For every $c > 0$ there is $\delta(c) > 0$ such that*

$$\text{if } f(u) \leqq c, \text{ then } \min_{t \in \mathbf{R}} |u_i(t) - u_j(t)| \geqq \delta(c). \tag{3.1}$$

Proof. See, for example, [5] or [11]. □

Proposition 3.2.

$$\text{If } g(u) > 0, \text{ then } \nabla g(u) \neq 0. \tag{3.2}$$

Proof. Since g is quadratic, we have $\nabla g(u) \cdot u = 2g(u) > 0$. □

Remark. Actually, $\|\nabla g(u)\|$ is bounded away from zero in any set $\{g \geqq b, f \leqq c\}$, $b > 0$. Indeed, let us denote by P the orthogonal projection onto H_0; we have

$$\|Pu\| \leqq C\big(g(u) + \|\dot{u}\|_2^2\big)^{1/2} \leqq C\big(g(u) + 2f(u)\big)^{1/2}.$$

Thus

$$2g(u) = \nabla g(u) \cdot u = \nabla g(u) \cdot Pu \leqq \|\nabla g(u)\| \, \|Pu\| \leqq C\big(g(u) + 2f(u)\big)^{1/2} \|\nabla g(u)\|.$$

It follows that $\|\nabla g(u)\| \geqq 2b/C(b + 2c)^{1/2}$ for $u \in \{g \geqq b, f \leqq c\}$.

Proposition 3.3. *For every $c, b \in \mathbf{R}$, every sequence $\{u_\nu\}_\nu \subset \Lambda$ such that $f(u_\nu) \to c$, $\nabla f(u_\nu) \to 0$ and $g(u_\nu) \leqq b$ possesses a converging subsequence.*

Proof. First, $f(u_\nu) \to c$ and $g(u_\nu) \leqq b$ imply the H^1-boundness of the sequence (up to translations of the whole system) and therefore the existence of a subsequence converging in the weak topology of H^1 and in the uniform topology to some $u \in H$. From (3.1) it follows that $u \in \Lambda$. Hence $\nabla V(u_\nu, t) \cdot (u - u_\nu)$ converges uniformly to zero. Since $\nabla f(u_\nu) \to 0$ and $u - u_\nu$ is H^1-bounded, we have

$$\|\dot{u}\|_2^2 - \lim_{\nu \to \infty} \|\dot{u}_\nu\|_2^2 = \lim_{\nu \to \infty} \int_0^T \dot{u}_\nu \cdot (\dot{u} - \dot{u}_\nu)\, dt$$

$$= \lim_{\nu \to \infty} \left\{ \nabla f(u_\nu) \cdot (u - u_\nu) + \int_0^T \nabla V(u_\nu, t) \cdot (u - u_\nu)\, dt \right\} = 0.$$

Therefore u_ν converges to u strongly in H^1. □

Proposition 3.4. *Given $c \in \mathbf{R}$ and $b > 0$, every sequence $\{u_\nu\} \subset \Lambda$ such that $f(u_\nu) \to c$, $g(u_\nu) \to b$ and $\nabla f(u_\nu) - \lambda_\nu \nabla g(u_\nu) \to 0$, with $\lambda_\nu \geqq 0$, possesses a converging subsequence.*

Proof. The proof follows the proof of Proposition 3.3, since ∇g is compact and that $\limsup_\nu |\lambda_\nu| = \limsup_\nu \dfrac{|\nabla f(u_\nu)|}{|\nabla g(u_\nu)|} < = \infty$. □

4. An a priori estimate for a nonlinear eigenvalue problem. Proof of (2.5)

Proposition 4.1. *Assume that* (V1), (V2), (V3) *and* (V4) *hold. For every* $c > 0$ *there exists* $b = b(c) > 0$ *such that the nonlinear eigenvalue problem*

$$\nabla f(u) = \lambda \nabla g(u), \quad \lambda > 0 \tag{4.1}$$

has no solution u *with*

$$f(u) \leqq c, \quad g(u) > b. \tag{4.2}$$

More precisely, $b(c)$ *can be taken to be*

$$b(c) = nR(n, \theta)^2 \left(\sqrt{\frac{cT}{6}} \frac{1}{\cos \theta} + \frac{\rho}{2} \right)^2 \tag{4.3}$$

where

$$R(n, \theta) = \begin{cases} n - 1 & \text{if } \theta = 0, \\ \dfrac{c(\theta)^{n-1} - 1}{c(\theta) - 1} & \text{if } \theta > 0, \end{cases} \tag{4.4}$$

$$c(\theta) = \frac{1}{2} \left(1 + \frac{1}{\cos \theta} \right). \tag{4.5}$$

In order to prove the proposition we need some preliminaries. To begin with, let $\rho \geqq 0$ and $\theta \in [0, \frac{\pi}{2}[$ be fixed real numbers. We shall be concerned with finite families of closed balls $\mathscr{B} = \{B(x_i, r_i)\}_{i \leqq n}$ satisfying the following metric property:

$$|x - y| > \rho, \ \operatorname{ang}(x - y, x_i - x_j) < \tfrac{\pi}{2} - \theta \quad \forall\, i \neq j, \ \forall\, x \in \bar{B}(x_i, r_i), \ \forall\, y \in \bar{B}(x_j, r_j). \tag{4.6}$$

With a little elementary geometry (4.6) can easily be shown to be equivalent to

$$\forall\, i \neq j \quad |x_i - x_j| > (r_i + r_j + \rho) \vee \frac{r_i + r_j}{\cos \theta}. \tag{4.6$'$}$$

Now, given n balls of radius r in $\mathbf{R}^k$, we wish to cover them with another family of balls, which satisfy property (4.6) and whose radii are as small as possible. We have the following result:

Lemma 4.1 (Covering Lemma). *Let a family* $\mathscr{B} = \{\bar{B}(x_i, r)\}_{i \leqq n}$ *of* n *balls of* $\mathbf{R}^k$ *be given. Then* (i) *there is another family of balls* $\mathscr{B}' = \{\bar{B}(x'_i, r'_i)\}_{i \leqq n'}$ *and a surjective map* $\sigma: \{1, \ldots, n\} \to \{1, \ldots, n'\}$ *such that, for every* $i \leqq n$, $\bar{B}(x_i, r) \subseteq \bar{B}(x'_{\sigma(i)}, r'_{\sigma(i)})$; $\mathscr{B}'$ *satisfies* (4.6) *and, for every* $i \leqq n'$,

$$r'_i \leqq R(n, \theta) \left(\frac{r}{\cos \theta} + \frac{\rho}{2} \right) + r. \tag{4.7}$$

(ii) *If in addition*

$$\frac{1}{2n}\sum_{i,j=1}^{n}|x_i-x_j|^2 > nR(n,\theta)^2\left(\frac{r}{\cos\theta}+\frac{\rho}{2}\right)^2,$$

then $n' \geqq 2$.

Proof. (i) We construct such a cover $\mathscr{B}'$ as the last term of a recursively defined finite sequence of covers:

$$\mathscr{B}_\nu = \{B_{i,\nu}\}_{1\leqq i\leqq n-\nu}, \quad B_{i,\nu} =: \bar{B}(x_{i,\nu}, r_{i,\nu}), \quad 0 \leqq \nu \leqq \nu^*.$$

Moreover, for technical reasons it is also convenient to define numbers $\mu(i,\nu)$ for all the indices (i,ν). Set $\mathscr{B}_0 = \mathscr{B}$ and $\mu(i,0) = 1$ for all $i \leqq n$. Suppose that the $\mathscr{B}_j$ and the $\mu(i,j)$ have been defined for all (i,j) with $1 \leqq j \leqq \nu$, $1 \leqq i \leqq n-j$. Then if $\mathscr{B}_\nu$ satisfies property (4.6), stop at $\nu^* =: \nu$ and take $\mathscr{B}' =: \mathscr{B}_\nu$. If it does not, then by (4.6)′ there exist two indices $i_\nu < j_\nu$ such that

$$|x_{i_\nu,\nu} - x_{j_\nu,\nu}| \leqq (r_{i_\nu,\nu} + r_{j_\nu,\nu} + \rho) \vee \frac{r_{i_\nu,\nu} + r_{j_\nu,\nu}}{\cos\theta}$$

$$\leqq \frac{r_{i_\nu,\nu} + r_{j_\nu,\nu}}{\cos\theta} + \rho.$$

Therefore $B_{i_\nu,\nu} \cup B_{j_\nu,\nu}$ is contained in a suitable ball $B_{\nu+1} =: \bar{B}(x_{\nu+1}, r_{\nu+1})$, where

$$r_{\nu+1} =: \frac{1}{2}\left(1+\frac{1}{\cos\theta}\right)(r_{i_\nu,\nu} + r_{j_\nu,\nu}) + \frac{\rho}{2}. \tag{4.8}$$

We then define $\mathscr{B}_{\nu+1}$ and the numbers $\mu(i,\nu+1)$ by

$$B_{i,\nu} = \begin{cases} B_{i,\nu} & \text{for } 1 \leqq i < j_\nu,\ i \neq i_\nu, \\ B_{\nu+1} & \text{for } i = i_\nu, \\ B_{i+1,\nu} & \text{for } j_\nu \leqq i \leqq n-\nu-1, \end{cases}$$

$$\mu(i,\nu+1) = \begin{cases} \mu(i,\nu) & \text{for } 1 \leqq i < j_\nu,\ i \neq i_\nu, \\ \mu(i_\nu,\mu_\nu) + \mu(j_\nu,\nu) & \text{for } i = i_\nu, \\ \mu(i+1,\nu) & \text{for } j_\nu \leqq i \leqq n-\nu-1. \end{cases} \tag{4.9}$$

Of course, this procedure stops at most at the n-th stage, because in that case $\mathscr{B}_n$ is a singleton and certainly satisfies property (4.6). Notice also that

$$\sum_{i=1}^{n-\nu}\mu(i,\nu) = n \quad \text{for every } \nu \text{ with } 0 \leqq \nu \leqq \nu^*.$$

Now let us consider the generic ball $B_{i,\nu}$ and estimate its radius $r_{i,\nu}$. We assert that if $\mu(i,\nu) = m$, then

$$r_{i,\nu} \leqq R(m,\theta)\left(\frac{r}{\cos\theta}+\frac{\rho}{2}\right) + r; \tag{4.10}$$

this implies in particular (4.7), as $m \leqq n$. Let us show (4.10) by induction on m. If $m = 1$, then $r_{i,\nu} = r$ and the assertion is obviously true. Suppose it holds for all $m' < m \leqq n$, and let us prove it for m. By definitions (4.9), (4.5), (4.8) there exist $h < \nu$ and $1 \leqq i_h < j_h \leqq n - h$ such that

$$\begin{aligned} B_{i,\nu} &= B_{i_h,h+1} \supset B_{i_h,h} \cup B_{j_h,h}, \\ r_{i,\nu} &= r_{i_h,h+1} = c(\theta)(r_{i_h,h} + r_{j_h,h}) + \frac{\rho}{2}, \\ \mu(i,\nu) &= \mu(i_h, h+1) = \mu(i_h, h) + \mu(j_h, h). \end{aligned} \tag{4.11}$$

Letting $\mu(i_h, h) = m'$ and $\mu(j_h, h) = m''$, we have from (4.11) and from the inductive hypothesis that

$$\begin{aligned} r_{i,\nu} &= c(\theta)(r_{i_h,h} + r_{j_h,h}) + \frac{\rho}{2} \\ &\leqq c(\theta)\left[(R(m',\theta) + R(m'',\theta))\left(\frac{r}{\cos\theta} + \frac{\rho}{2}\right) + 2r\right] + \frac{\rho}{2} \\ &= [c(\theta)(R(m',\theta) + R(m'',\theta)) + 1]\left(\frac{r}{\cos\theta} + \frac{\rho}{2}\right) + r. \end{aligned}$$

Now it is easily seen that $R(m,\theta)$ satisfies

$$c(\theta)(R(m',\theta) + R(m'',\theta)) + 1 \leqq R(m,\theta) \quad \text{whenever } m' + m'' = m.$$

Hence

$$r_{i,\nu} \leqq R(m,\theta)\left(\frac{r}{\cos\theta} + \frac{\rho}{2}\right) + r,$$

and we are done.

(ii) We have

$$\frac{1}{2n}\sum_{i,j}|x_i - x_j|^2 = \sum_{i=1}^{n}\left|x_i - \frac{1}{n}\sum_{j=1}^{n}x_j\right|^2 = \min_{y\in\mathbf{R}^k}\sum_{i=1}^{n}|x_i - y|^2.$$

If $\mathscr{B}'$ were a singleton $\{B'(x', r')\}$, then

$$\frac{1}{2n}\sum_{i,j}|x_i - x_j|^2 \leqq \sum_{i=1}^{n}|x_i - x'|^2 \leqq n(r' - r)^2 \leqq nR(n,\theta)^2\left(\frac{r}{\cos\theta} + \frac{\rho}{2}\right)^2. \quad \square$$

Proof of Proposition 4.1. Let $u \in \Lambda$, with $f(u) \leqq c$, $g(u) > b$. We shall exhibit a $v \in H$ such that

$$\nabla f(u) \cdot v \leqq 0 \quad \nabla g(u) \cdot v > 0, \tag{4.12}$$

which proves that u does not satisfy (4.1) for any $\lambda > 0$. Since

$$\|u_i - [u_i]\|_\infty \leqq \left(\frac{T}{12}\right)^{1/2}\|\dot{u}_i\|_2 \leqq \left(\frac{T}{12}\right)^{1/2}(2c)^{1/2} = \left(\frac{T\cdot c}{6}\right)^{1/2}, \tag{4.13}$$

we have $u_i(t) \in \bar{B}(x_i, r)$ for every $i \leqq n$ and every $t \in \mathbf{R}$, with $r = (T \cdot c/6)^{1/2}$ and $x_i = [u_i]$. Applying the covering lemma to the family $\mathscr{B} = \{\bar{B}(x_i, r)\}_{i \leqq n}$, we get the cover $\mathscr{B}' = \{\bar{B}(x_i', r_i')\}_{i \leqq n'}$ and the map σ. Moreover, $n' \geqq 2$, since we have supposed that

$$\frac{1}{2n} \sum_{i,j} |x_i - x_j|^2 = g(u) > b. \tag{4.14}$$

For $i = 1, \ldots, n$, we define

$$v_i = x'_{\sigma(i)} \in \mathbf{R}^k. \tag{4.15}$$

Then

$$\nabla f(u) \cdot v = -\sum_{i,j} \int_0^T \nabla V_{ij}(u_i - u_j, t) \cdot (x'_{\sigma(i)} - x'_{\sigma(j)})\, dt.$$

Notice that the only indices (i, j) which contribute to the sum are those for which $\sigma(i) \neq \sigma(j)$; in that case, since from (4.13) we have

$$u_i(t) \in \bar{B}(x_i, r), \quad u_j(t) \in \bar{B}(x_j, r) \quad \forall t \in \mathbf{R},$$

we get by (4.6) that

$$|u_i - u_j| > \rho, \quad \operatorname{ang}(u_i - u_j, x'_{\sigma(i)} - x'_{\sigma(j)}) < \tfrac{\pi}{2} - \theta \quad \forall t \in \mathbf{R}$$

where for simplicity, $u_i = u_i(t)$, $u_j = u_j(t)$. From the first of these inequalities and from the hypothesis (V4) on V_{ij} we also have

$$\operatorname{ang}(\nabla V_{ij}(u_i - u_j, t),\ u_i - u_j) \leqq \theta$$

so that, by the subadditivity property of angles,

$$\operatorname{ang}(\nabla V_{ij}(u_i - u_j, t),\ x'_{\sigma(i)} - x'_{\sigma(j)}) \leqq \tfrac{\pi}{2},$$

and we infer that

$$\nabla V_{ij}(u_i - u_j, t) \cdot (x'_{\sigma(i)} - x'_{\sigma(j)}) \geqq 0.$$

Thus $\nabla f(u) \cdot v \leqq 0$.

In a similar way, we have

$$\nabla g(u) \cdot v = 2 \cdot \frac{1}{2n} \sum_{i,j} [u_i - u_j] \cdot (x'_{\sigma(i)} - x'_{\sigma(j)}) > 0$$

for $[u_i - u_j] \cdot (x'_{\sigma(i)} - x'_{\sigma(j)}) > 0$ whenever $\sigma(i) \neq \sigma(j)$. (Notice that here we have used the fact that $n' \geqq 2$). □

5. Homotopies in Λ. Proofs of (2.6) and the main theorem

Proposition 5.1. *There is $c^* \in \mathbf{R}$ such that with every pair $\tilde{c}$, b with $\tilde{c} > c^*$ and $b > n(n-1)^2 \tilde{c} T/6$ there is associated $c_b < \tilde{c}$ such that*

$$\begin{gathered} \{f \leqq c_b + \varepsilon\} \cup (\{f \leqq \tilde{c}\} \cap \{g \geqq b\}) \\ \text{is not deformable into } \{f \leqq c_b - \varepsilon\} \cup (\{f \leqq \tilde{c}\} \cap \{g \geqq b\}) \text{ in } \Lambda \end{gathered} \tag{5.1}$$

for every $\varepsilon < \tilde{c} - c_b$.

We start with some definitions: Consider the set of all the deformations of Λ in H into the subspace of constant functions $\mathbf{R}^{kn}$:

$$\mathscr{H}=\{h\in\mathscr{C}(\Lambda\times[0,1];H):h(\cdot,0)=\mathrm{id},\ h(\Lambda,1)\subset\mathbf{R}^{kn}\}. \tag{5.2}$$

For fixed $h=(h_1,\ldots,h_n)\in\mathscr{H}$ and $u=(u_1,\ldots,u_n)\in\Lambda$, we define a relation $r_{h,u}$ in the set of the indices $I=\{1,\ldots,n\}$ by

$$\begin{gathered} i\,\mathrm{r}_{h,u}j \Leftrightarrow \exists\,\lambda_{ij}\in[0,1],\ \exists\,t_{ij}\in S^1 \\ \text{such that } h_i(u_1,\ldots,u_n,\lambda_{ij})(t_{ij})=h_j(u_1,\ldots,u_n,\lambda_{ij})(t_{ij}). \end{gathered} \tag{5.3}$$

It is immediate to check that $\mathrm{r}_{h,u}$ is reflexive and symmetric. We denote by $\mathrm{R}_{h,u}$ the smallest equivalence relation containing $\mathrm{r}_{h,u}$:

$$i\,\mathrm{R}_{h,u}j \Leftrightarrow \exists\, i_1,\ldots,i_k : i\,\mathrm{r}_{h,u}i_1,\ \mathrm{r}_{h,u}i_2,\ldots,i_k\mathrm{r}_{h,u}j.$$

Definition 5.1. Let A be a closed subset of Λ. We say that A is *admissible* if

$$\begin{gathered} \forall\, h=(h_1,\ldots,h_n)\in\mathscr{H}\quad \exists\, u=(u_1,\ldots,u_n)\in A \text{ such that} \\ i\mathrm{R}_{h,u}j,\ \forall\, i,j\in I. \end{gathered} \tag{5.4}$$

Remark 5.1. If A has the following property, then it is admissible:

$$\begin{gathered} \forall\, h=(h_1,\ldots,h_n)\in\mathscr{H}\quad \exists\, u=(u_1,\ldots,u_n)\in A,\ \exists\, i_0\in\{1,\ldots,n\}:\forall\, i\neq i_0 \\ \exists\,\lambda_{i,i_0}\in[0,1],\quad \exists\, t_{i,i_0}\in S^1: h_i(u_1,\ldots,u_n,\lambda_{i,i_0})(t_{i,i_0})=h_{i_0}(u_1,\ldots,u_n,\lambda_{i,i_0})(t_{i,i_0}). \end{gathered} \tag{5.5}$$

In view of Definition 2.1, the following property follows from Definition 5.1 (the proof will be given later in this section).

Proposition 5.2. *If A and B are closed subsets of Λ and A is deformable into B in Λ, then, if A is admissible, so is B.*

We define

$$\mathscr{A}=\{A\subseteqq\Lambda \text{ closed}: A \text{ is admissible}\} \tag{5.6}$$

and c^* as

$$c^*=\inf_{A\in\mathscr{A}}\sup_A f. \tag{5.7}$$

In the usual Ljusternik-Schnirelman theory (with the full (PS) condition) we are allowed to say that c^* is a critical point for f. Here the variational characterization of the critical level will be a little more complicated. In order to complete it we need some preliminary results.

Proposition 5.3. $\mathscr{A}\neq\emptyset$. *Moreover,*

$$c^*\leqq\inf_{R>0}\left\{\frac{2\pi^2}{T}nR^2+T\sum_{1=i<j}\sup_{|x|\geqq 2R\sin\pi(j-i)/n}(-V_{ij}(x,t))\right\}. \tag{5.8}$$

Proposition 5.4. *For every $\tilde{c}\in\mathbf{R}$ there is $b^*=b^*(\tilde{c})$ such that*

$$(\{f\leqq\tilde{c}\}\cap\{g\geqq b\})\notin\mathscr{A}\quad \textit{for every } b>b^*. \tag{5.9}$$

Moreover, $b^*(\tilde{c})$ can be taken to be

$$b^*(\tilde{c}) = n(n-1)^2 \frac{\tilde{c}T}{6}. \tag{5.10}$$

Postponing the proof of these two facts, let us turn to the proofs of Proposition 5.1 and Theorem 1.

Proof of Proposition 5.1. Let us fix any $\tilde{c} > c^*$ and $b > b^*(\tilde{c}) = n(n-1)^2 \tilde{c}T/6$ and define

$$c_b = \inf\{\gamma : \{f \leqq \gamma\} \cup (\{f \leqq \tilde{c}\} \cap \{g \geqq b\}) \in \mathscr{A}\}. \tag{5.11}$$

Notice that c_b is well defined provided $\mathscr{A} \neq \emptyset$, $c^* < +\infty$ (cf. Proposition 5.3) and $(\{f \leqq \tilde{c}\} \cap \{g \geqq b\}) \notin \mathscr{A}$, which occurs, by Proposition 5.4, whenever $b > b^*(\tilde{c})$.

Since the class $\mathscr{A}$ is monotone we have

$$c_b \leqq c^* < \tilde{c} \quad \text{for every } b > b^*(\tilde{c}); \tag{5.12}$$

moreover, for every $\varepsilon > 0$,

$$\{f \leqq c_b + \varepsilon\} \cup (\{f \leqq \tilde{c}\} \cap \{g \geqq b\}) \in \mathscr{A}, \tag{5.13}$$

while

$$\{f \leqq c_b - \varepsilon\} \cup (\{f \leqq \tilde{c}\} \cap \{g \geqq b\}) \notin \mathscr{A}. \tag{5.14}$$

Proposition 5.2 then ensures that $\{f \leqq c_b + \varepsilon\} \cup (\{f \leqq \tilde{c}\} \cap \{g \geqq b\})$ cannot be deformable into $\{f \leqq c_b - \varepsilon\} \cup (\{f \leqq \tilde{c}\} \cap \{g \geqq b\})$ in Λ. □

Proof of Theorem 1. Let us fix any $\tilde{c} > c^*$ and $\bar{b} > b(\tilde{c})$, $b(\tilde{c})$ being defined in (4.3). Let c_b be as in (5.11). Since $c_{\bar{b}} \leqq \tilde{c}$ by (5.12) and since $b(\cdot)$ is an increasing function, we have that $\bar{b} \geqq b(\tilde{c}) \geqq b(c_{\bar{b}})$. Hence, by Proposition 4.1, condition (2.5) of Lemma 2.1 is fulfilled for $c = c_{\bar{b}}$ and $b = \bar{b}$.

Moreover, by their definitions (4.3) and (5.10), $b(\cdot) \geqq b^*(\cdot)$, so that $\bar{b} > b(\tilde{c}) \geqq b^*(\tilde{c})$; thus it follows from Proposition 5.1 that assumption (2.6) is also satisfied for these values of $\tilde{c}$, c and b.

This discussion, together with the results of Section 3, allows the application of Lemma 2.1. Therefore a critical point $u_{\bar{b}}$ is found such that $f(u_{\bar{b}}) = c_{\bar{b}} \leqq c^* < \tilde{c}$ and $g(u_{\bar{b}}) \leqq \bar{b}$.

Of course we wish to minimize the estimates on the levels of f and g of this solution. Letting $\tilde{c}$ converge to c^* and $\bar{b}$ converge to $b(c^*)$ and using the compactness property of (2.3), we obtain the existence of one critical point u at level $c = c_{b(c^*)} \leqq c^*$ such that $g(u) \leqq b(c^*)$. The estimates (1.1) and (1.3) just follow from the estimate on c^* of Proposition 5.3 and from the definition of $b(c^*)$ (4.3). □

We end this section with the proofs of Propositions 5.2, 5.3 and 5.4.

Proof of Proposition 5.2. Let $h \in \mathscr{H}$; according to Definition 5.1 we have to find a $w \in B$ such that $r_{h,w}$ generates the total equivalence relation on I, i.e., $i\mathrm{R}_{h,w}j$ for all $i, j \in I$. By Definition 2.1 and the remark below there exists a continuous homotopy $\eta : \Lambda \times [0, 1] \to \Lambda$ such that $\eta(\cdot, 0) = \mathrm{id}_{|\Lambda}$ and $\eta(A, 1) \subseteq B$. Consider the juxtaposition of η and h:

$$h \cdot \eta(u, \lambda) := \begin{cases} \eta(u, 2\lambda) & \text{if } 0 \leqq \lambda \leqq \frac{1}{2}, \\ h(\eta(u, 1), 2\lambda - 1) & \text{if } \frac{1}{2} \leqq \lambda \leqq 1. \end{cases}$$

Then, clearly, $h \cdot \eta \in \mathscr{H}$; so, since $A \in \mathscr{A}$, there exists $u \in A$ such that $\mathrm{r}_{h \cdot \eta, u}$ generates the total equivalence relation on I. But it is readily seen that $\mathrm{r}_{h \cdot \eta, u} = r_{h, \eta(u, 1)}$, so that we finish the proof by taking $w = \eta(u, 1) \in B$. □

Proof of Proposition 5.3. We are going to exhibit a particular closed set having the property (5.5) and therefore the property (5.4). Let us fix an orthonormal system $(e_1, e_2, \dots, e_k)$ of $\mathbf{R}^k$: Define

$$S^{k-2} = \{x \in \mathbf{R}^k : |x| = 1,\ x \cdot e_1 = 0\}.$$

For simplicity of notation we are going to assume that the period T is equal to 1. Indeed the subset Λ in the space of T-periodic functions is clearly homeomorphic, by scaling of the period, to the corresponding subset Λ in the space of 1-periodic functions.

Define the continuous map

$$U:\ (S^{k-2})^{n-1} \to H,\ U(x_2, \dots, x_n) = (u_1(x_2, \dots, x_n), \dots, u_n(x_2, \dots, x_n))$$

by

$$u_1(x_2, \dots, x_n)(t) = e_2(1 - \cos 2\pi)\, t + e_1 \sin 2\pi t,$$

$$u_i(x_2, \dots, x_n) = x_i \left(1 - \cos 2\pi \left(t + \frac{i-1}{n}\right)\right)$$
$$+ e_1 \sin 2\pi \left(t + \frac{i-1}{n}\right), \quad i = 2, \dots, n.$$

We set $A^* = U((S^{k-2})^{n-1})$ and we assert that (5.5) is fulfilled for $i_0 = 1$. It is easily seen that $A^* \subset \Lambda$. Now let us fix any $h \in \mathscr{H}$ and define $\psi = (\psi_2, \dots, \psi_n) : (S^{k-2} \times B^2)^{n-1} \to \mathbf{R}^{k(n-1)}$ by

$$\psi_i(x_2, \dots, x_n, \rho_2 e^{2\pi i t_2}, \dots, \rho_n e^{2\pi i t_n})$$
$$= h_i(U(x_1, \dots, x_n), 1 - \rho_i)(t_i) - h_1(U(x_1, \dots, x_n), 1 - \rho_i)(t_i), \quad i = 2, \dots, n.$$

Keeping in mind the dependence of ψ on the homotopy h, proving (5.5) becomes equivalent to proving that

$$0 \in \psi(S^{k-2} \times B^2)^{n-1}.$$

We first observe that ψ is continuous, since by definition, $h \in \mathscr{H}$ implies that $h(u, 1) = \text{const}$. We are going to prove that ψ has topological degree ± 1 with respect to the value 0. To do this we define the continuous map

$$\tilde{\psi}: (S^{k-2} \times B^2)^{n-1} \to \mathbf{R}^{k(n-1)}$$

by

$$\tilde{\psi}_i(x_2, \ldots, x_n, \rho_2 e^{2\pi i t_2}, \ldots, \rho_n e^{2\pi i t_n})$$
$$= \begin{cases} u_i(x_2, \ldots, x_n)(t_i) - 2(\rho_i - \frac{1}{2})\, u_1(t) + 2(1 - \rho_i)\, e_2 & \text{if } \frac{1}{2} \leqq \rho_i \leqq 1, \\ u_i(x_2, \ldots, x_n)(t_i) - 8(\frac{1}{2} - \rho_i)\, e_2 - e_2 & \text{if } \frac{1}{4} \leqq \rho_i \leqq \frac{1}{2}, \\ 4\rho_i u_i(x_2, \ldots, x_n)(t_i) - 3e_2 & \text{if } 0 \leqq \rho_i \leqq \frac{1}{4}. \end{cases}$$

In the same way that ψ corresponds to the homotopy h, the map $\tilde{\psi}$ corresponds to some $\tilde{h} \in \mathscr{H}$. Therefore $\tilde{h}$ is defined as the juxtaposition of three homotopies. Notice that the first, defined for $\frac{1}{2} \leqq \rho \leqq 1$, makes the first body collapse to its mean value, e_2, without any collision. The second, defined for $\frac{1}{4} \leqq \rho \leqq \frac{1}{2}$, is a translation along the direction $-e_2$ of all the bodies except u_1; it unlinks the orbit of the first body from the orbits of all the other bodies. The third, defined for $0 \leqq \rho \leqq \frac{1}{4}$, contracts the system $(u_2, \ldots, u_n)$ to a constant without any collision with the first body. We remark that collisions with the first body can occur only if $\frac{1}{4} \leqq \rho \leqq \frac{1}{2}$.

We have

$$\tilde{\psi}_i(x_2, \ldots, x_n, \rho_2 e^{2\pi i t_2}, \ldots, \rho_n e^{2\pi i t_n}) = \psi_i(x_2, \ldots, x_n, \rho_2 e^{2\pi i t_2}, \ldots, \rho_n e^{2\pi i t_n})$$

whenever $\rho_i = 1$. Thus the homotopy $\lambda\tilde{\psi} + (1 - \lambda)\,\psi$ has no zeros on the boundary of the domain $(S^{k-2} \times B^2)^{n-1}$; indeed, on the boundary, for at least one index i, we have $\rho_i = 1$, so that by definition, $\lambda\tilde{\psi}_i + (1 - \lambda)\,\psi_i = \psi_i \neq 0$. By well-known properties of the topological degree we then obtain that ψ and $\tilde{\psi}$ have the same topological degree with respect to the value 0. Now, by a few elementary computations, we can see that each regular value $z = (z_2, \ldots, z_n) \in \mathbf{R}^{k(n-1)}$ in a sufficiently small neighborhood of the origin has only one counterimage for $\tilde{\psi}$. We then conclude that $\deg\big((S^{k-2} \times B^2)^{n-1}, \tilde{\psi}, 0\big) = \pm 1$.

In order to prove (5.8), we are going to evalute the supremum of the functional f over all the homotheties of A^*, since, by Proposition 5.2, if $A^* \in \mathscr{A}$, then $RA^* \in \mathscr{A}$, for every positive R. It is easily seen that, by the definition of A^*,

$$|u_i(t) - u_j(t)| \geqq 2R \sin \frac{\pi(j-i)}{n} \qquad \forall\, u = (u_1, \ldots, u_n) \in RA^*,\ \forall\, i < j,$$

so that

$$c^* \leqq \inf_{R>0} \sup_{RA^*} f \leqq \inf_{R>0} \left\{ \frac{2\pi^2}{T} nR^2 + T \sum_{1=i<j}^{n} \sup_{|x| \geqq 2R \sin(\pi(j-i)/n)} (-V_{ij}(x, t)) \right\}. \qquad \square$$

Proof of Proposition 5.4. Let $u \in \Lambda$ with $f(u) \leqq \tilde{c}$ and $g(u) > b^*(\tilde{c})$. As we have already pointed out in the proof of Position 4.1, $u_i(t)$ belongs to the ball $B\big([u_i], \sqrt{T\tilde{c}/6}\big)$ for every t. According to the notations of the covering

Lemma 4.1, we set $x_i = [u_i]$, $r = \sqrt{T\tilde{c}/6}$, $\theta = 0$ and $\rho = 0$. Then the assumption $g(u) > b^*(\tilde{c})$ corresponds to

$$\frac{1}{2n} \sum_{i,j=1}^{n} |x_i - x_j|^2 > nR(n, \theta)^2 \left(\frac{r}{\cos\theta} + \frac{\rho}{2}\right)^2 = n(n-1)^2 \frac{\tilde{c}T}{6}.$$

Now we apply the covering Lemma 4.1, obtaining the existence of a family of balls $\mathscr{B}' = \{B(x_i', r_i')\}_{i \leqq n'}$ and a surjective map $\sigma: \{1, \ldots, n\} \to \{1, \ldots, n'\}$ such that

$$B(x_i, r) \subseteq B(x'_{\sigma(i)}, r'_{\sigma(i)})$$

and (4.6) holds, that is, since ρ, $\theta = 0$,

$$B(x'_{\sigma(i)}, r'_{\sigma(i)}) \cap B(x'_{\sigma(j)}, r'_{\sigma(j)}) = \emptyset, \quad \text{if } \sigma(i) \neq \sigma(j).$$

Moreover, $n' \geqq 2$, so that there really exist i and j such that $\sigma(i) \neq \sigma(j)$.

Now we consider the continuous homotopy

$$h(u, \lambda) = (1 - \lambda) u + \lambda[u].$$

Of course, since $h_i(u, \lambda)(t) \in B(x_i, r)$, for every $\lambda \in [0, 1]$ and for every t, we have

$$h_i(u, \lambda)(t) - h_j(u, \lambda)(t) = 0 \Rightarrow \sigma(i) = \sigma(j).$$

Thus, for every $u \in \{f \leqq \tilde{c}\} \cup \{g \geqq b\}$, the equivalence relation $R_{u,h}$ considered in Definition 5.1 is contained in the equivalence relation $i \simeq j \Leftrightarrow \sigma(i) = \sigma(j)$ which has $n' \geqq 2$ equivalence classes. Hence (5.4) is violated and $\{f \leqq \tilde{c}\} \cup \{g \geqq b\}$ is not admissible. □

6. Further results and comments

Proof of Theorem 2. One simply considers approximating problems satisfying the strong-force condition (V3), taking, for example $V_{ij}^{\varepsilon} = V_{ij} - \varepsilon|x|^{-2}$. From Theorem 1 one obtains solutions u_ε of the approximating problems, which are uniformly bounded in H^1 due to estimates (1.1) and (1.3). These have a weak H^1 limit point $u \in \bar{\Lambda}$, which is in C^2 and satisfies (1) outside a closed set of measure zero, i.e., $[0, T] \setminus (\bigcup_{i,j} (u_i - u_j)^{-1}(0))$. □

Now we are going to prove some corollaries and variants of Theorem 1. Our first result shows that assumption (V4) can be relaxed, simply by using estimate (1.3) and a truncation argument; as a counterpart, we lose the extra information given by estimate (1.2). Let us consider the following assumption:

$$\begin{gathered} \forall M > 0 \quad \exists R > 0 \text{ such that if } |x| > R, \\ \text{then } \nabla V_{ij}(x, t) \cdot x \geqq M |\nabla V_{ij}(x, t)| \, |x|^{\frac{n-2}{n-1}} \quad \forall i \neq j. \end{gathered} \tag{V7}$$

Theorem 6.1. *Assume* (V1), (V2), (V3) *and* (V7) *hold. Then* (1) *has at least one T-periodic solution* $u = (u_1, \ldots, u_n)$ *such that* $u_i(t) \neq u_j(t)$, *for all* $i \neq j$ *and for all* $t \in \mathbf{R}$.

Proof. Let $c^* = c^*(V_{ij})$ as in (1.1) and let $K(n, T)$ be the constant defined in (1.3). Choose $\rho \geqq 1$ so that the inequality in (V7) holds for $M = 1$; choose $\rho_1 \geqq \rho$ so that the same holds for

$$M_1 =: [2K(n, T)(\rho + \sqrt{c^*})]^{1/(n-1)}. \tag{6.1}$$

Finally define

$$R =: M_1^{n-1} \rho_1. \tag{6.2}$$

From our choices we obtain

$$\nabla V_{ij}(x, t) \cdot \frac{x}{|x|} \geqq |\nabla V_{ij}(x, t)| \, \rho_1^{-1/(n-1)} \quad \text{if } \rho \leqq |x| \leqq \rho_1, \tag{6.3}$$

and using (6.2) we also obtain

$$\begin{aligned} \nabla V_{ij}(x, t) \cdot \frac{x}{|x|} &\geqq M_1 |\nabla V_{ij}(x, t)| \, R^{-1/(n-1)} \\ &= |\nabla V_{ij}(x, t)| \, \rho_1^{-1/(n-1)}, \quad \text{if } \rho_1 \leqq |x| \leqq R. \end{aligned} \tag{6.4}$$

Therefore,

$$\cos \operatorname{ang}(\nabla V_{ij}(x, t), x) \geqq \rho_1^{-1/(n-1)} \quad \text{if } \rho \leqq |x| \leqq R \tag{6.5}$$

(when $\nabla V_{ij}(x, t) = 0$, this is true by definition of angle and by the fact that $\rho_1 \geqq 1$). Now let Φ be a cut-off function satisfying $0 \leqq \Phi \leqq 1$; $\Phi' \leqq 0$; $\Phi(t) = 1$ if $t \leqq \frac{1}{2}$, $\Phi(t) = 0$ if $t \geqq 1$, and define the truncated potentials

$$\bar{V}_{ij}(x, t) = V_{ij}(x, t) \, \Phi\left(\frac{|x|}{R}\right). \tag{6.6}$$

Hence

$$\nabla \bar{V}_{ij}(x, t) = \Phi\left(\frac{|x|}{R}\right) \nabla V_{ij}(x, t) + \frac{1}{R} V_{ij}(x, t) \, \Phi'\left(\frac{|x|}{R}\right) \frac{x}{|x|},$$

and, since $V_{ij} \Phi' \geqq 0$, we obtain

$$\cos \operatorname{ang}(\nabla \bar{V}_{ij}(x, t), x) \geqq \cos \operatorname{ang}(\nabla V_{ij}(x, t), x).$$

This, together with (6.5) and $\nabla \bar{V}_{ij}(x, t) = 0$ for $|x| \geqq R$, yields condition (V4) for the $\bar{V}_{ij}$'s:

$$\cos \operatorname{ang}(\nabla \bar{V}_{ij}(x, t), x) \geqq \rho_1^{-1/(n-1)} \quad \text{if } |x| \geqq \rho.$$

Now Theorem 1 applies to the potentials $\bar{V}_{ij}$ and we obtain a solution u of the analog of system (1) for the $\bar{V}_{ij}$'s, with the estimate (1.3):

$$\|u_i - u_j\|_\infty \leqq K(n, T)(\rho + \sqrt{c^*}) \rho_1$$

(here $c^*((\bar{V}_{ij})) \leqq (c^*(V_{ij}))$ because $-\bar{V}_{ij} \leqq -V_{ij}$). Hence (6.1) and (6.2) yield

$$\|u_i - u_j\|_\infty \leqq \tfrac{1}{2} M_1^{n-1} \rho_1 = \tfrac{1}{2} R;$$

and we conclude that u is actually a solution of (1), since it takes values in the set where $\bar{V}_{ij}$ and V_{ij} coincide. □

Another variant of Theorem 1 can be obtained by assuming that the V_{ij}'s and their gradients vanish at infinity. This kind of requirement is quite natural in the physical setting of the n-body problem. We assume that

$$|V_{ij}(x,t)| + |\nabla V_{ij}(x,t)| \to 0 \quad \text{as } |x| \to +\infty \text{ uniformly in } t,\ \forall\, i \neq j. \tag{V6}$$

Theorem 6.2. *Assume that* (V1), (V2), (V3), (V4) *and* (V6) *hold. Then the same conclusion as in Theorem 1 holds. Moreover, u satisfies the further estimate*

$$\sum_{i,j=1;\, i\neq j}^{n} \left(\frac{1}{T}\int_0^T u_i(t) - u_j(t)\right)^2 \leqq 4(n-1)^2 R(n-1,\theta)^2 \left(\frac{\sqrt{\frac{c^*T}{6}} + \bar{\rho}}{\cos\theta} + \frac{\rho}{2}\right)^2 \tag{6.7}$$

where the constant $\bar{\rho}$ depends only on the rate of growth of the V_{ij}'s at zero and on the rate of decay of the $|V_{ij}| + |\nabla V_{ij}|$'s at infinity and not on the angle θ.

In order to prove Theorem 6.2 we need some preliminary results and a slight variant of Proposition 4.1. In what follows we shall assume that (V1), (V2), (V3), (V4) and (V6) hold.

Proposition 6.1. *For every $\tilde{c} > 0$ there is $\varepsilon(\tilde{c}) > 0$ such that if $f(u) \leqq \tilde{c}$ and $\|\dot{u}\|_2 \leqq \varepsilon(\tilde{c})$, then*

$$|[u_i] - [u_j]| > 2\sqrt{\frac{T}{12}}\|\dot{u}\|_2, \quad \forall\, i \neq j. \tag{6.8}$$

Proof. Let us assume that, for some pair of indices i and j,

$$|[u_i] - [u_j]| \leqq 2\sqrt{\frac{T}{12}}\|\dot{u}\|_2.$$

Then

$$\|u_i - u_j\|_\infty \leqq 4\sqrt{\frac{T}{12}}\|\dot{u}\|_2;$$

on the other hand,

$$\tilde{c} \geqq f(u) \geqq \int_0^T -V_{ij}(u_i - u_j) \geqq T \inf_{t\in\mathbf{R},\, 0<|x|\leqq 4\sqrt{\frac{T}{12}}\|\dot{u}\|_2} (-V_{ij}(x,t)).$$

Since, from (V3), $\lim_{x\to 0} - V_{ij}(x,t) = +\infty$ uniformly in t, we obtain the conclusion. □

Remark 6.1. The constant $\varepsilon(\tilde{c})$ only depends on the rate of growth of $-V_{ij}$ as x tends to 0. We can then conclude that, if $V_{ij} = \bar{V}_{ij}$ in some neighborhood of the origin, then the corresponding constant $\varepsilon(\tilde{c})$ could be the same for the V_{ij}'s and the $\bar{V}_{ij}$'s.

Proposition 6.2.

$$(\{g \geqq b^*(\bar{c})\} \cup \{\|\dot{u}\|_2 \leqq \varepsilon(\bar{c})\}) \cap \{f \leqq \bar{c}\} \notin \mathscr{A} \tag{6.9}$$

when $b^*(\bar{c}) = n(n-1)\,\bar{c}T/6$.

Proof. Let us consider the homotopy h introduced in the proof of Proposition 5.4:

$$h(u, \lambda) = (1-\lambda)\,u + \lambda\,[u].$$

If $\|\dot{u}\|_2 \leqq \varepsilon(\bar{c})$, we have from (6.8) that

$$h_i(u, \lambda)(t) - h_j(u, \lambda) = (1-\lambda)\,u_i + \lambda\,[u_i] - (1-\lambda)\,u_j - \lambda\,[u_j] \neq 0$$

$$\forall\, i \neq j,\ \forall\, \lambda,\ \forall\, t,$$

since in any case $\|u_i - [u_i]\|_\infty \leqq \sqrt{\frac{T}{12}}\,\|\dot{u}\|_2$. To conclude the proof, we just argue as in the proof of Proposition 5.4. □

As a consequence of this proposition we obtain an estimate from below of the critical level:

Proposition 6.3. *Let* $b > b^*(\bar{c})$ *and let*

$$c_b = \inf\{\gamma : (\{g \geqq b\} \cup \{f \leqq \gamma\}) \cap \{f \leqq \bar{c}\} \in \mathscr{A}.$$

Then $c_b > \frac{1}{2}\,\varepsilon^2(\bar{c})$.

Proof. This result follows immediately from Proposition 6.2, by taking into account of the fact that $\{f \leqq \gamma\} \subset \{\|\dot{u}\|_2 \leqq \sqrt{2\gamma}\,\}$ since $V_{ij} \leqq 0$. □

Remark 6.2. It follows from Remark 6.1 that if $\bar{V}_{ij}^R$ are defined as in (6.6), then the estimate from below on the critical level holds independently of the truncation for $R < 1$.

Proposition 6.4. *Corresponding to each* $\bar{c} > 0$ *there is* $\bar{\rho} > 0$ *such that the nonlinear eigenvalue problem*

$$\nabla f(u) = \lambda \nabla g(u), \quad \lambda > 0$$

has no solution u *with*

$$\tfrac{1}{2}\,\varepsilon(\bar{c}) \leqq f(u) \leqq \bar{c}, \quad \min_{i<j} |[u_i] - [u_j]| > \bar{\rho}.$$

Proof. Let $v = (v_i)$ with $v_i = u_i - [u_i]$. Then we have $\nabla g(u) \cdot v = 0$ and $\nabla f(u) \cdot v$

$$= \int_0^T |\dot{u}|^2 - \tfrac{1}{2} \sum_{i,j=1, i \neq j}^{n} \int_0^T \nabla V_{ij}(u_i - u_j, t) \cdot (v_i - v_j)$$

$$= 2f(u) + \sum_{i,j=1, i \neq j}^{n} \int_0^T V_{ij}(u_i - u_j, t) - \tfrac{1}{2} \sum_{i,j=1, i \neq j}^{n} \int_0^T \nabla V_{ij}(u_i - u_j, t) \cdot (v_i - v_j)$$

$$\geqq \varepsilon^2(\tilde{c}) - n(n-1)\, T \sup_{i,j} \sup_{t, |x| > \bar{\rho} - 2\sqrt{T\tilde{c}/6}} \left(|V_{i,j}(x, t)| + \frac{1}{2}\sqrt{\frac{T\tilde{c}}{6}}\, |\nabla V_{ij}(x, t)| \right).$$

Indeed, we have $\|v_i\|_\infty \leqq \sqrt{T\tilde{c}/6}$ and $\|u_i - u_j\|_\infty \geqq \bar{\rho} - 2\sqrt{T\tilde{c}/6}$. It is clear that whenever $\bar{\rho}$ is sufficiently large, the last expression is strictly positive. Having found $v \in H$ such that $\nabla g(u) \cdot v = 0$ and $\nabla f(u) \cdot v > 0$, we have obviously proved this proposition. □

Proposition 6.5. *If $\bar{\rho}$ is as in Proposition 6.4, then the nonlinear eigenvalue problem*

$$\nabla f(u) = \lambda \nabla g(u), \quad \lambda > 0$$

has no solution u with

$$\tfrac{1}{2}\varepsilon(\tilde{c}) \leqq f(u) \leqq \tilde{c},$$

$$\sum_{i,j=1}^{n} \left(\frac{1}{T} \int_0^T u_i - u_j \right)^2 > 4(n-1)^2 R(n-1, \theta)^2 \left(\frac{\bar{\rho} + \sqrt{\dfrac{T\tilde{c}}{6}}}{\cos\theta} + \frac{\rho}{2} \right)^2 + 2(n-1)\,\bar{\rho}^2. \tag{6.10}$$

Proof. Thanks to Proposition 6.4 we only have to examine the case when

$$\min_{i<j} |[u_i] - [u_j]| \leqq \bar{\rho}.$$

For simplicity of notation we can assume that

$$|[u_n] - [u_{n-1}]| \leqq \bar{\rho}; \tag{6.11}$$

thus we have

$$\|u_n - [u_{n-1}]\|_\infty \leqq \bar{\rho} + \sqrt{\frac{T\tilde{c}}{6}}. \tag{6.12}$$

Now we apply the covering Lemma 4.1 with $x_i = [u_i]$, $i = 1, \ldots, n-1$ and $r = \bar{\rho} + \sqrt{T\tilde{c}/6}$. We then find the cover $\mathscr{B}' = \{\bar{B}(x_i', r_i')\}_{i=1,\ldots,n'-1}$ and the surjective map $\sigma: \{1, \ldots, n-1\} \to \{1, \ldots, n'-1\}$ are as in Lemma 4.1. We put $\sigma(n) = \sigma(n-1)$.

Since

$$\sum_{i,j=1}^{n} ([u_i] - [u_j])^2 = \sum_{i,j=1}^{n-1} ([u_i] - [u_j])^2 + 2 \sum_{i=1}^{n-1} ([u_i] - [u_n])^2$$

from (6.11), we then deduce that

$$\sum_{i,j=1}^{n} ([u_i] - [u_j])^2 \leqq 2 \sum_{i,j=1}^{n-1} ([u_i] - [u_j])^2 + 2(n-1)\bar{\rho}^2.$$

Thus (6.10) yields

$$\sum_{i,j=1}^{n-1} ([u_i] - [u_j])^2 > 2(n-1)^2 R(n-1, \theta)^2 \left(\frac{\bar{\rho} + \sqrt{\frac{T\tilde{c}}{6}}}{\cos\theta} + \frac{\rho}{2} \right)^2$$

and therefore $n' - 1 \geqq 2$, by virtue of (ii) of Lemma 4.1. The assertion then follows by arguing as in the proof of Proposition 4.1. □

Remark 6.3. Let $\bar{V}_{ij}^R(x, t)$ be the family of truncated potentials defined as in (6.6). We remark that, for $R \geqq 1$, there is $C > 0$ independent of R such that $|\bar{V}_{ij}^R| + |\nabla \bar{V}_{ij}^R| \leqq C(|V_{ij}| + |\nabla V_{ij}|)$. Therefore the constant $\bar{\rho}$ appearing in Propositions 6.4 and 6.5 can be chosen to be independent of $R \geqq 1$; indeed, as we have already pointed out in Remark 6.2, the estimate from below of the critical level $\varepsilon(\tilde{c})$ is also independent of $R \geqq 1$.

Proof of Theorem 6.2. The proof of Theorem 6.2 works exactly as the proof of Theorem 1. Indeed, when applying the deformation Lemma 2.1, thanks to the estimates of Propositions 6.3 and 6.4, we can use Proposition 6.5 instead of Proposition 4.1; by virtue of Proposition 6.5, estimate (1.2) is then changed to estimate (6.7). □

Proof of Theorem 3. Let $\bar{V}_{ij}^R(x, t)$ be the family of truncated potentials defined as in (6.6). We can apply Theorem 6.1 to the $\bar{V}_{ij}^R(x, t)$'s obtaining solutions u^R of the approximate problem. Moreover, from (6.7), the u^R satisfy an estimate of the type of

$$\|u_i^R - u_j^R\|_\infty \leqq K_1(n, T)(c^*, \rho, \bar{\rho}, R)(\cos\theta)^{-(n-2)}.$$

Taking into account Remark 6.3 we conclude that $K_1(n, T)(c^*, \rho, \bar{\rho}, R)(\cos\theta)$ does not actually depend on R when $R \geqq 1$. In order to obtain an a priori estimate on the u^R's we argue as in the proof of Proposition 6.1, just replacing n with $n-1$ in all the exponents of $\cos\theta$ and replacing $K(n, T)$ with $K_1(n, T)$. □

7. Proof of Lemma 2.1

Here we prove Lemma 2.1 in the particular case when ∇f and ∇g are bounded and locally Lipschitz continuous, and ∇g is bounded away from zero near to the set $\{f = c, g = b\}$. We refer to [14] for the general case functionals on manifolds with boundary. The boundedness of the gradients is assumed here in order to use a gradient-flow type of deformation. So, let us assume that (2.1), (2.3), (2.4) and (2.5) hold in addition to

$$\begin{gathered}\exists\, \varepsilon > 0 \text{ such that if } |f(x) - c| \leqq \varepsilon, |g(x) - b| \leqq \varepsilon, \\ \text{then } \|\nabla f(x)\| \leqq \varepsilon^{-1},\ \varepsilon \leqq \|\nabla g(x)\| \leqq \varepsilon^{-1}.\end{gathered} \tag{7.1}$$

We point out that, in the setting of Theorem 1, our particular functionals f and g in fact satisfy (7.1) (see, e.g., the remark after Proposition 3.2).

Proof of Lemma 2.1 (Sketch). Assume for contradiction that

$$K_{c,b} = \{x \in \Lambda : f(x) = c,\ g(x) \leqq b,\ \nabla f(x) = 0\} = \emptyset.$$

Then we show that for some $\varepsilon \in\,]0, \tilde{c} - c[$, the set $\{f \leqq c + \varepsilon\} \cup (\{f \leqq \tilde{c}\} \cap \{g \geqq b\})$ is deformable into the set $\{f \leqq c - \varepsilon\} \cup (\{f \leqq \tilde{c}\} \cap \{g \geqq b\})$ in Λ (cf. Definition 2.1). If $K_{c,b} = \emptyset$, it follows from (2.3) that

$$\exists\, \varepsilon > 0 \text{ such that if } |f(x) - c| \leqq \varepsilon,\ g(x) \leqq b + \varepsilon, \text{ then } \|\nabla f(x)\| \geqq \varepsilon. \tag{7.2}$$

Now, from (2.4), (2.5) and the additional assumption (7.1), we deduce that

$$\begin{gathered}\exists\, \varepsilon > 0 \text{ such that if } |f(x) - c| \leqq \varepsilon,\ g(x) - b| \leqq \varepsilon, \\ \text{then } \operatorname{ang}(\nabla f(x), \nabla g(x)) \geqq \varepsilon.\end{gathered} \tag{7.3}$$

Indeed, if not, there would exist a sequence $\{x_\nu\}_\nu$ such that

$$f(x_\nu) \to c, \quad g(x_\nu) \to b, \quad \operatorname{ang}(\nabla f(x_\nu), \nabla g(x_\nu)) \to 0,$$

and therefore

$$\left(\frac{-\nabla f(x_\nu)}{\|\nabla f(x_\nu)\|} + \frac{\nabla g(x_\nu)}{\|\nabla g(x_\nu)\|}\right)^2 \to 0.$$

From the compactness assumption (2.4) and from the boundedness of the gradients (7.1) we derive the existence of a subsequence converging to some limit x. Of course, we have $f(x) = c$, $g(x) = b$ and, from the last formula, also $\nabla f(x) = \lambda \nabla g(x)$, for the positive $\lambda = \|f(x)\|\, \|\nabla g(x)\|^{-1}$, in contradiction to assumption (2.5).

Now we fix a $\varepsilon > 0$ satisfying (7.1), (7.2), (7.3) and $c + \varepsilon/2 \leqq \tilde{c}$. From (7.2) and (7.3) we then obtain:

$$\text{if } |f(x) - c| \leqq \varepsilon,\ |g(x) - b| \leqq \varepsilon,\ 0 \leqq \lambda \leqq 1, \text{ then}$$

$$\left(\frac{-\nabla f(x)}{\|\nabla f(x)\|} + \frac{\nabla g(x)}{\|\nabla g(x)\|}\right) \cdot \nabla f(x) \leqq (-1 + \lambda \nabla f(x) \cdot \nabla g(x))\, \|\nabla f(x)\|$$

$$\leqq (-1 + \lambda \cos \varepsilon)\, \|\nabla f(x)\| \leqq (-1 + \cos \varepsilon)\, \varepsilon =: -\delta \leqq -\varepsilon. \tag{7.4}$$

We finally observe that

$$\left(\frac{-\nabla f(x)}{\|\nabla f(x)\|} + \frac{\nabla g(x)}{\|\nabla g(x)\|}\right) \cdot \nabla g(x) \geqq 0 \tag{7.5}$$

whenever (7.5) makes sense. Because of our assumption that ∇f and ∇g are locally Lipschitz continuous, it makes sense to consider the flow η defined by the Cauchy Problem:

$$\frac{d}{ds}\eta = \varphi(|f(\eta) - c|)\,\varphi(g(\eta) - b)\left(\frac{-\nabla f(\eta)}{\|\nabla f(\eta)\|} + \varphi(|g(\eta) - b|)\frac{\nabla g(\eta)}{\|\nabla g(\eta)\|}\right), \tag{CP}$$

$$\eta(x, 0) = x,$$

where $\varphi: \mathbf{R} \to [0, 1]$ is a Lipschitz continuous function such that $\varphi(t) = 1$ if $t \leqq \varepsilon/2$ and $\varphi(t) = 0$ if $t \geqq \varepsilon$. By virtue of assumptions (2.1) and (7.2), the flow η is defined for all positive s and sends A into A, since, as we are going to see, f is decreasing along the lines of η.

From (7.2), (7.4) and (7.5) it follows that η enjoys the properties:

$$\frac{d}{ds} f(\eta(x, s)) \leqq 0 \quad \forall x,\ \forall s; \tag{7.6}$$

$$\text{if } |f(\eta(x, s)) - c| \leqq \frac{\varepsilon}{2},\ g(\eta(x, s)) \leqq b + \frac{\varepsilon}{2},\ \text{then } \frac{d}{ds} f(\eta(s)) \leqq -\delta; \tag{7.7}$$

$$\text{if } |g(\eta(x, s)) - b| \leqq \frac{\varepsilon}{2},\ \text{then } \frac{d}{ds} g(\eta(x, s)) \geqq 0. \tag{7.8}$$

Hence, all sublevels of f and the set $\{g \geqq b\}$ are positively invariant under the flow η, so that the sets $\{f \leqq \tilde{c}\} \cap \{g \geqq b\}$ and $\{f \leqq c \pm \varepsilon/2\} \cup (\{f \leqq \tilde{c}\} \cap \{g \geqq b\})$ are also positively invariant. Let us respectively denote by $A_{c+\varepsilon/2}$ and $A_{c-\varepsilon/2}$ the sets $\{f \leqq c + \varepsilon/2\} \cup (\{f \leqq \tilde{c}\} \cap \{g \geqq b\})$ and $\{f \leqq c - \varepsilon/2\} \cup (\{f \leqq \tilde{c}\} \cap \{g \geqq b\})$. Now we assert that for $s_0 = \varepsilon/\delta$, we have $\eta(A_{c+\varepsilon/2}, s_0) \subset A_{c-\varepsilon/2}$. Of course, we only have to prove that if $g(\eta(x_0, s)) \leqq b$ for some $x_0 \in A_{c+\varepsilon/2}$ and for every $s \in [0, s_0]$, then $f(\eta(x_0, s_0)) \leqq c - \varepsilon/2$. Assuming the contrary that $f(\eta(x_0, s_0)) > c - \varepsilon/2$, we have from (7.6) that $c + \varepsilon/2 > f(x_0) \geqq f(\eta(x_0, s)) > c - \varepsilon/2$ for all $s \in [0, s_0]$ and therefore, from (7.7) that

$$-\varepsilon < f(\eta(x_0, s_0)) - f(x_0) = \int_0^{s_0} \frac{d}{ds} f(\eta(x_0, s)) \leqq -\delta s_0 = -\varepsilon,$$

a contradiction. □

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