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Some Applications of the Method of Moments for the Homogeneous Boltzmann and Kac Equations

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Abstract

Using the method of moments, we prove that any polynomial moment of the solution of the homogeneous Boltzmann equation with hard potentials or hard spheres is bounded provided that a moment of order strictly higher than 2 exists initially. We also give partial results of convergence towards the Maxwellian equilibrium in the case of soft potentials. Finally, exponential as well as Maxwellian estimates are introduced for the Kac equation.

1. Introduction

The homogeneous Boltzmann equation of gas dynamics has the form

$$\frac{\partial f}{\partial t}(t, v) = Q(f)(t, v), \tag{1.1}$$

where f is a nonnegative function of the time t and the velocity v, and Q is a quadratic collision kernel accounting for any collisions preserving momentum and kinetic energy:

$$Q(f)(t, v) = \int_{v_1 \in \mathbb{R}^3} \int_{\omega \in S^2} \{f(t, v')f(t, v_1') - f(t, v)f(t, v_1)\}$$

$$\times B\left(|v - v_1|, \left|\omega \cdot \frac{v - v_1}{|v - v_1|}\right|\right) d\omega\, dv_1, \tag{1.2}$$

with

$$v' = v - (\omega \cdot (v - v_1))\,\omega, \tag{1.3}$$

$$v_1' = v_1 + (\omega \cdot (v - v_1))\,\omega, \tag{1.4}$$

and the nonnegative cross section B depends on the type of interaction between molecules (cf. [Ce], [Ch, Co], [Tr, Mu]).

In a gas of hard spheres, the cross section is

$$B(x, y) = xy. \tag{1.5}$$

However, for inverse s^{th}-power forces with angular cut-off (cf. [Ce], [Gr]),

$$B(x, y) = x^\alpha \beta(y), \tag{1.6}$$

where $\alpha = \frac{s-5}{s-1}$, and there exists $\beta_1 > 0$ such that for almost every $y \in [0, 1]$,

$$0 < \beta(y) \leqq \beta_1. \tag{1.7}$$

When $s > 5$, the potentials are said to be hard and $0 < \alpha < 1$. But when $3 < s < 5$, the potentials are said to be soft and $-1 < \alpha < 0$. The intermediate case when $s = 5$ is that of "Maxwellian molecules"; it makes exact computations possible (cf. [Tr], [Tr, Mu] and [Bo]).

Since hard and soft potentials are fairly involved (the function β is defined implicitly), engineers often use in numerical computations the simpler variable hard spheres (VHS) model, in which

$$B(x, y) = x^\alpha y, \tag{1.8}$$

and $0 < \alpha \leqq 1$.

Note that, at least formally, for every function $\psi(v)$,

$$\begin{aligned}\int_{v \in \mathbb{R}^3} Q(f)\,(t, v)\,\psi(v)\,dv &= \int_{v \in \mathbb{R}^3} \int_{v_1 \in \mathbb{R}^3} \int_{\omega \in S^2} \{\psi(v') - \psi(v)\}\; f(t, v)\, f(t, v_1) \\ &\quad \times B\left(|v - v_1|, \left|\omega \cdot \frac{v - v_1}{|v - v_1|}\right|\right) d\omega\, dv_1\, dv,\end{aligned} \tag{1.9}$$

and also

$$\begin{aligned}&\int_{v \in \mathbb{R}^3} Q(f)\,(t, v)\,\psi(v)\,dv \\ &= -\frac{1}{4} \int_{v \in \mathbb{R}^3} \int_{v_1 \in \mathbb{R}^3} \int_{\omega \in S^2} \{\psi(v') + \psi(v_1') - \psi(v) - \psi(v_1)\} \\ &\quad \times \{f(t, v') f(t, v_1') - f(t, v) f(t, v_1)\} B\left(|v - v_1|, \left|\omega \cdot \frac{v - v_1}{|v - v_1|}\right|\right) d\omega\, dv_1\, dv.\end{aligned} \tag{1.10}$$

When $\psi(v) = 1,\ v,\ \frac{1}{2}|v|^2$ in (1.10), one obtains the conservation of mass, momentum and evergy for the Boltzmann kernel:

$$\int_{v\in\mathbb{R}^3} Q(f)\,(t,v)\,(1,\ v,\ \tfrac{1}{2}|v|^2)\,dv = 0. \tag{1.11}$$

Moreover, using (1.10) with $\psi = \log f$, one obtains the entropy estimate:

$$\int_{v\in\mathbb{R}^3} Q(f)\,(t,v)\,\log f(t,v)\,dv \leqq 0. \tag{1.12}$$

According to [A 1], [A 2], for any of the cross sections previously presented, there exists a nonnegative solution $f(t, v)$ of (1.1) satisfying $f(0, v) = f_0(v)$ provided that f_0 is nonnegative and

$$\int_{v\in\mathbb{R}^3} f_0(v)\,\left(1 + \tfrac{1}{2}|v|^r + |\log f_0(v)|\right)\,dv < +\infty \tag{1.13}$$

for some $r > 2$. Moreover, estimates (1.11) and (1.12) hold for this solution, and therefore f satisfies

$$\int_{v\in\mathbb{R}^3} f(t,v)\,(1,\ v,\ \tfrac{1}{2}|v|^2)\,dv = \int_{v\in\mathbb{R}^3} f_0(v)\,(1,\ v,\ \tfrac{1}{2}|v|^2)\,dv, \tag{1.14}$$

$$\int_{v\in\mathbb{R}^3} f(t,v)\,\log f(t,v)\,dv \leqq \int_{v\in\mathbb{R}^3} f_0(v)\,\log f_0(v)\,dv \tag{1.15}$$

when $t \geqq 0$.

Note that condition (1.13) can be relaxed by taking $r = 2$ for the proof of existence, but in that case (1.14) may not hold (at least for hard potentials). Note also the results in [DP, L 1] of existence and weak stability for the inhomogeneous equation.

In this work, the solutions of the Boltzmann equation (1.1), are always the nonnegative solutions of [A 1] or [A 2].

It is now well known that in the case of VHS models (including hard spheres) and hard potentials (including Maxwellian molecules), the moments of the solution of the Boltzmann equation

$$l_r(t) = \int_{v\in\mathbb{R}^3} f(t,v)\,|v|^r\,dv \tag{1.16}$$

for $r > 2$ are bounded on $[0, +\infty[$ provided that they exist at time $t = 0$ (cf. [El 1]). The same estimate holds for soft potentials, except that $l_r(t)$ is bounded only on $[0, T]$ for $T > 0$ and may blow up when t goes to infinity (cf. [A 2]). Note finally that the case of Maxwellian molecules is treated extensively in [Tr, Mu] and [Bo].

We prove in Section 2 that in fact, for VHS models (including hard spheres) as well as in the case of hard potentials (but not including Maxwellian molecules) and under assumption (1.13), the moments $l_q(t)$ (for $q > 2$) are bounded when $t \geqq \bar{t}$ (for any $\bar{t} > 0$). In other words, all polynomial moments of f exist for $t > 0$ provided that one of them (of order strictly higher than 2) exists initially.

In Section 3, we give some estimates for the solution f of equation (1.1) with soft potentials. We write the cross section B in the form

$$B(x, y) = x^{-\gamma}\beta(y) \tag{1.17}$$

with $\gamma > 0$ ($\gamma = -\alpha$ in equation (1.6)).

We prove that if $l_r(0)$ exists (with $r > 2$), then we can find $K_0 > 0$ such that

$$l_r(t) \leqq K_0 t + K_0 . \tag{1.18}$$

This estimate is a little more explicit than that of [A 2]. Moreover, we also get

$$\int_0^t l_{r-\gamma}(s)\, ds \leqq K_0 t + K_0 , \tag{1.19}$$

which means that $l_{r-\gamma}$ is bounded in the Cesaro sense. Note that the same kind of estimates can be found in [Pe 1] and [Pe 2], in a linear context. Note also that the estimates can be derived from the works of ELMROTH (cf. [El 1], [El 2]). However, for the sake of completness we give here a self-contained proof. These estimates are then used to prove partial results of convergence towards the equilibrium when t goes to infinity (the reader can find a survey on this subject in [De 2]).

Finally, in Section 4, we introduce KAC's model (cf. [K], [MK]) and, using monotonicity results, we prove exponential and Maxwellian estimates for its solution.

2. Hard potentials

The bounds that we present in this work are based on (1.9). The exploitation of this estimate is called the "method of moments". We begin by putting (1.9) into a new form. We write

$$\omega = \cos\theta \frac{v_1 - v}{|v_1 - v|} + \sin\theta(\cos\phi\, i_{v,v_1} + \sin\phi\, j_{v,v_1}), \tag{2.1}$$

where

$$\left(\frac{v_1 - v}{|v_1 - v|}, i_{v,v_1}, j_{v,v_1}\right) \tag{2.2}$$

is an orthonormal basis of $\mathbb{R}^3$. Then estimate (1.9) becomes

$$\begin{aligned}\int_{v\in\mathbb{R}^3} Q(f)\,(t, v)\,\psi(v)\,dv = \int_{v\in\mathbb{R}^3}\int_{v_1\in\mathbb{R}^3}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2} &\Big\{\psi\Big(v + \cos\theta\,|v - v_1| \\ &\times\Big[\cos\theta\frac{v_1 - v}{|v_1 - v|} + \sin\theta(\cos\phi\, i_{v,v_1} + \sin\phi\, j_{v,v_1})\Big]\Big) - \psi(v)\Big\} \\ &\times f(t, v)\, f(t, v_1)\, 2\sin\theta B(|v - v_1|, \cos\theta)\, d\theta\, d\phi\, dv_1\, dv .\end{aligned} \tag{2.3}$$

Introducing in (2.3) the change of variables $\theta = \frac{\delta}{2}$, and defining

$$R_{\delta,\phi}\left(\frac{v_1 - v}{|v_1 - v|}\right) = \cos\delta\,\frac{v_1 - v}{|v_1 - v|} + \sin\delta(\cos\phi\, i_{v,v_1} + \sin\phi\, j_{v,v_1}), \tag{2.4}$$

one obtains

$$\int_{v\in\mathbb{R}^3} Q(f)\ (t, v)\ \psi(v)\ dv$$

$$= \int_{v\in\mathbb{R}^3}\int_{v_1\in\mathbb{R}^3}\int_{\phi=0}^{2\pi}\int_{\delta=0}^{\pi} \left\{\psi\left(\frac{v+v_1}{2} + \frac{|v-v_1|}{2} R_{\delta,\phi}\left(\frac{v_1 - v}{|v_1 - v|}\right)\right) - \psi(v)\right\}$$

$$\times f(t, v)\, f(t, v_1)\, \sin\tfrac{\delta}{2}\, B\left(|v - v_1|, \cos\tfrac{\delta}{2}\right)\, d\delta\, d\phi\, dv_1\, dv, \tag{2.5}$$

which is in fact a classical form for the Boltzmann collision term (cf. [Bo] or [De 3], for example).

We state now three useful lemmas.

Lemma 1. *Assume that* $\varepsilon > 0$ *and that* Λ *is a strictly positive function in* $L^\infty([0, \pi])$. *Then there exists* $K_1 > 0$ *and two functions* $T_1(v, v_1)$, $T_2(v, v_1)$ *such that*

$$W(v, v_1)$$

$$= \int_{\phi=0}^{2\pi}\int_{\delta=0}^{\pi} \left|1 + \frac{|v - v_1|\,|v + v_1|}{v^2 + v_1^2}\left[R_{\delta,\phi}\left(\frac{v_1 - v}{|v_1 - v|}\right)\cdot\frac{v + v_1}{|v + v_1|}\right]\right|^{1+\varepsilon} \Lambda(\delta)\, d\delta\, d\phi$$

$$= T_1(v, v_1) + T_2(v, v_1), \tag{2.6}$$

with

$$T_1(v, v_1) = -T_1(v_1, v), \tag{2.7}$$

$$0 \leqq T_2(v, v_1) \leqq K_1 < 2^{1+\varepsilon}\pi \int_{\delta=0}^{\pi} \Lambda(\delta)\, d\delta. \tag{2.8}$$

Proof. We adopt the following notations for $i = 1, 2$:

$$T_i(v, v_1)$$

$$= \int_{\phi=0}^{2\pi}\int_{\delta=0}^{\pi} \chi_i\left(\frac{|v - v_1|\,|v + v_1|}{v^2 + v_1^2}\left\{R_{\delta,\phi}\left(\frac{v_1 - v}{|v_1 - v|}\right)\cdot\frac{v + v_1}{|v + v_1|}\right\}\right) \Lambda(\delta)\, d\delta\, d\phi, \tag{2.9}$$

with

$$\chi_i(x) = \frac{(1+x)^{1+\varepsilon} + (-1)^i (1-x)^{1+\varepsilon}}{2}. \tag{2.10}$$

We can see that

$$W(v, v_1) = T_1(v, v_1) + T_2(v, v_1), \tag{2.11}$$

$$T_1(v, v_1) = -T_1(v_1, v). \tag{2.12}$$

But χ_2 is even, strictly increasing from $x = 0$ to $x = 1$, and

$$\chi_2(0) = 1, \quad \chi_2(1) = 2^{\varepsilon}. \tag{2.13}$$

Therefore, using the inequality

$$|v - v_1|\,|v + v_1| \leqq v^2 + v_1^2, \tag{2.14}$$

we obtain the estimate

$$0 \leqq T_2(v, v_1) \leqq 2^{1+\varepsilon}\pi \int_{\delta=0}^{\pi} A(\delta)\, d\delta. \tag{2.15}$$

Then, a simple argument of compactness ensures that Lemma 1 holds.

Lemma 2. *Assume that* $\varepsilon > 0$ *and that the cross section* B *in* (1.2) *satisfies*

$$B(x, y) = B_0(x)\, B_1(y), \tag{2.16}$$

where $B_1 \in L^\infty([0, \pi])$ *is strictly positive. Then, there exists* $K_2 > 0$ *and* $K_3 \in]0, 1[$ *such that*

$$\int_{v \in \mathbb{R}^3} Q(f)\,(t, v)\, |v|^{2+2\varepsilon}\, dv$$

$$\leqq K_2 \int_{v \in \mathbb{R}^3} \int_{v_1 \in \mathbb{R}^3} \{\tfrac{1}{2} K_3 (v^2 + v_1^2)^{1+\varepsilon} - |v|^{2+2\varepsilon}\} f(t, v)\, f(t, v_1)\, B_0(|v - v_1|)\, dv_1\, dv. \tag{2.17}$$

Proof. According to equation (2.5), for $\varepsilon > 0$,

$$\int_{v \in \mathbb{R}^3} Q(f)\,(t, v)\, |v|^{2+2\varepsilon}\, dv = \int_{v \in \mathbb{R}^3} \int_{v_1 \in \mathbb{R}^3} \left\{ \left(\frac{v^2 + v_1^2}{2} \right)^{1+\varepsilon} \right.$$

$$\times \int_{\phi=0}^{2\pi} \int_{\delta=0}^{\pi} \left| 1 + \frac{|v - v_1|\,|v + v_1|}{v^2 + v_1^2} \left[R_{\delta, \phi} \left(\frac{v_1 - v}{|v_1 - v|} \right) \cdot \frac{v + v_1}{|v + v_1|} \right] \right|^{1+\varepsilon}$$

$$\times \sin \tfrac{\delta}{2} B_1(\cos \tfrac{\delta}{2})\, d\delta\, d\phi - |v|^{2+2\varepsilon} \int\limits_{\phi=0}^{2\pi} \int\limits_{\delta=0}^{\pi} \sin \tfrac{\delta}{2} B_1(\cos \tfrac{\delta}{2})\, d\delta\, d\phi \Bigg\}$$

$$\times f(t, v)\, f(t, v_1)\, B_0(|v - v_1|)\, dv_1\, dv. \tag{2.18}$$

Moreover, using Lemma 1 with

$$\Lambda(\delta) = \sin \tfrac{\delta}{2} B_1(\cos \tfrac{\delta}{2}), \tag{2.19}$$

we have

$$\int\limits_{v \in \mathbb{R}^3} Q(f)\ (t, v)\, |v|^{2+2\varepsilon}\, dv$$

$$\leqq \int\limits_{v \in \mathbb{R}^3} \int\limits_{v_1 \in \mathbb{R}^3} \Bigg\{ \left(\frac{v^2 + v_1^2}{2} \right)^{1+\varepsilon} \{K_1 + T_1(v, v_1)\}$$

$$- |v|^{2+2\varepsilon} \int\limits_{\phi=0}^{2\pi} \int\limits_{\delta=0}^{\pi} \sin \tfrac{\delta}{2} B_1(\cos \tfrac{\delta}{2})\, d\delta\, d\phi \Bigg\}$$

$$\times f(t, v) f(t, v_1)\, B_0(|v - v_1|)\, dv_1\, dv, \tag{2.20}$$

with K_1 and $T_1(v, v_1)$ as in Lemma 1. Therefore, taking

$$K_2 = 2\pi \int\limits_{\delta=0}^{\pi} \sin \tfrac{\delta}{2} B_1(\cos \tfrac{\delta}{2})\, d\delta, \tag{2.21}$$

$$K_3 = \frac{K_1}{2^{1+\varepsilon}\pi \int_{\delta=0}^{\pi} \sin \tfrac{\delta}{2} B_1(\cos \tfrac{\delta}{2})\, d\delta} < 1, \tag{2.22}$$

and using the change of variables $(v, v_1) \to (v_1, v)$, we obtain Lemma 2.

Lemma 3. *Let B and ε be as in Lemma 2. Then there exist $K_4, K_5 > 0$ such that*

$$\int\limits_{v \in \mathbb{R}^3} Q(f)(t, v)\, |v|^{2+2\varepsilon}\, dv$$

$$\leqq -K_4 \int\limits_{v \in \mathbb{R}^3} \int\limits_{v_1 \in \mathbb{R}^3} |v|^{2+2\varepsilon} f(t, v)\, f(t, v_1)\, B_0(|v - v_1|)\, dv_1\, dv$$

$$+ K_5 \int\limits_{v \in \mathbb{R}^3} \int\limits_{v_1 \in \mathbb{R}^3} |v|^2\, |v_1|^{2\varepsilon} f(t, v)\, f(t, v_1)\, B_0(|v - v_1|)\, dv_1\, dv. \tag{2.23}$$

Proof. Note that there exists $K_6 > 0$ such that

$$(v^2 + v_1^2)^{1+\varepsilon} \leqq |v|^{2+2\varepsilon} + |v_1|^{2+2\varepsilon} + K_6(|v|^2\,|v_1|^{2\varepsilon} + |v|^{2\varepsilon}\,|v_1|^2). \quad (2.24)$$

Using Lemma 2 and the change of variables $(v, v_1) \to (v_1, v)$, one easily obtains Lemma 3 with $K_4 = K_2(1 - K_3)$ and $K_5 = K_2 K_3 K_6$.

We now come to the main theorem of this section.

Theorem 1. *Let f_0 satisfying* (1.13) *be a nonnegative initial datum for the Boltzmann equation* (1.1) *with hard potentials* (*but not with Maxwellian molecules*) *or with the VHS model* (*including hard spheres*). *Denote by $f(t, v)$ a solution of the equation with this initial datum. Then, for all $r' > 0$, $\bar{t} > 0$, there exists $C(r', \bar{t}) > 0$ such that*

$$\int_{v \in \mathbb{R}^3} f(t, v)\,|v|^{r'}\,dv \leqq C(r', \bar{t}) \quad (2.25)$$

when $t \geqq \bar{t}$.

Proof. According to (1.6) and (1.8), the cross section for hard potentials (but not Maxwellian molecules) or for the VHS model (including hard spheres) is of the form (2.23) with $B_0(x) = |x|^\alpha$, and $\alpha \in]0, 1]$. Therefore, we can apply Lemma 3.

For $\varepsilon > 0$, we write

$$\int_{v \in \mathbb{R}^3} Q(f)\;(t, v)\,|v|^{2+2\varepsilon}\,dv$$

$$\leqq -K_4 \int_{v \in \mathbb{R}^3} \int_{v_1 \in \mathbb{R}^3} |v|^{2+2\varepsilon}\,|v - v_1|^\alpha f(t, v)\, f(t, v_1)\,dv_1\,dv$$

$$+ K_5 \int_{v \in \mathbb{R}^3} \int_{v_1 \in \mathbb{R}^3} |v|^2\,|v_1|^{2\varepsilon}\,|v - v_1|^\alpha f(t, v)\, f(t, v_1)\,dv_1\,dv \quad (2.26)$$

$$\leqq -K_4 2^{-\alpha} l_{2+2\varepsilon+\alpha}(t)\; l_0(t) + K_4 2^{2-\alpha} l_2(t)\; l_{2\varepsilon+\alpha}(t)$$

$$+ K_5 l_{2+\alpha}(t)\; l_{2\varepsilon}(t) + K_5 l_{2\varepsilon+\alpha}(t)\; l_2(t)\,, \quad (2.27)$$

with the notation (1.16).

Since f is the solution of (1.1), the conservations of mass and energy (1.14) ensure that for $\theta \in [0, 2]$, $l_\theta(t)$ is bounded (for $t \geqq 0$). Therefore, there exist $K_7, K_8, K_9 > 0$ such that

$$\int_{v \in \mathbb{R}^3} Q(f)\;(t, v)\,|v|^{2+2\varepsilon}\,dv \leqq -K_7 l_{2+2\varepsilon+\alpha}(t) + K_8 l_{2+\alpha}(t)\; l_{2\varepsilon}(t) + K_9 l_{2\varepsilon+\alpha}(t)\,. \quad (2.28)$$

Remember that $\bar{t} > 0$ and $r > 2$ are given in the hypothesis of Theorem 1 (r is defined in (1.13)). We can always suppose that $r \leqq 4$. We prove in a first step that there exists $t_0 \in]0, \bar{t}[$ such that $l_{r+\alpha}(t)$ is bounded on $[t_0, +\infty[$.

According to Hölder's inequality, when $0 < \mu < \nu$,

$$l_\mu(t) \leqq l_0^{1-\mu/\nu}(t)\; l_\nu^{\mu/\nu}(t)\,. \quad (2.29)$$

Therefore, using estimate (2.28) with $\varepsilon = \frac{r}{2} - 1$, one obtains

$$\int_{v\in\mathbb{R}^3} Q(f)\ (t, v)\ |v|^r\, dv$$
$$\leqq -K_7 l_{r+\alpha}(t) + K_8 l_{r+\alpha}^{\frac{2+\alpha}{r+\alpha}}(t)\ l_0^{1-\frac{2+\alpha}{r+\alpha}}(t)\ l_{r-2}(t) + K_9 l_{r+\alpha}^{\frac{r+\alpha-2}{r+\alpha}}(t)\ l_0^{1-\frac{r+\alpha-2}{r+\alpha}}(t). \tag{2.30}$$

Remember that since $r-2 \in]0, 2]$, the moments $l_0(t)$ and $l_{r-2}(t)$ are bounded on $[0, +\infty[$. Moreover, we can find $K_{10}, K_{11} > 0$ such that when $x \geqq 0$, $t \geqq 0$,

$$-K_7 x + K_8 x^{\frac{2+\alpha}{r+\alpha}} l_0^{1-\frac{2+\alpha}{r+\alpha}}(t)\ l_{r-2}(t) + K_9 x^{\frac{r+\alpha-2}{r+\alpha}}\ l_0^{1-\frac{r+\alpha-2}{r+\alpha}}(t) \leqq -K_{10}x + K_{11}. \tag{2.31}$$

Therefore,

$$\int_{v\in\mathbb{R}^3} Q(f)\ (t, v)\ |v|^r\, dv \leqq -K_{10} l_{r+\alpha}(t) + K_{11}. \tag{2.32}$$

Integrating the Boltzmann equation (1.1) on $[0, \bar{t}] \times \mathbb{R}^3$ against $|v|^r$ and using estimate (2.32), one gets

$$l_r(\bar{t}) + K_{10} \int_0^{\bar{t}} l_{r+\alpha}(s)\, ds \leqq K_{11}\bar{t} + l_r(0). \tag{2.33}$$

According to (1.13) and (2.33), we can see that there exists $t_0 \in]0, \bar{t}[$ such that $l_{r+\alpha}(t_0) < +\infty$. But it is well known that if a moment exists at a given time t_0, then it is bounded for $t \geqq t_0$ (cf. [El 1] or the remark at the end of Section 2); therefore $l_{r+\alpha}(t)$ is bounded for $t \geqq t_0$.

We come back now to estimate (2.28). Using equation (2.29), an estimate similar to (2.31), and the result of boundedness for $l_{r+\alpha}(t)$, one obtains $K_{12}, K_{13} > 0$ such that

$$\int_{v\in\mathbb{R}^3} Q(f)\ (t, v)\ |v|^{2+2\varepsilon}\, dv \leqq -K_{12} l_{2+2\varepsilon+\alpha}(t) + K_{13} \tag{2.34}$$

for $t \geqq t_0$. Now we integrate (when $t_0 \leqq t_- < \bar{t}$) the Boltzmann equation (1.1) on $[t_-, \bar{t}] \times \mathbb{R}^3$ against $|v|^{2+2\varepsilon}$ and we use estimate (2.34) to obtain

$$l_{2+2\varepsilon}(\bar{t}) + K_{12} \int_{t_-}^{\bar{t}} l_{2+2\varepsilon+\alpha}(s)\, ds \leqq K_{13}(\bar{t} - t_-) + l_{2+2\varepsilon}(t_-). \tag{2.35}$$

Therefore, if $t_0 \leqq t_-$ and if $l_{2+2\varepsilon}(t_-) < +\infty$, then there exists $\tau \in [t_-, \bar{t}[$ such that $l_{2+2\varepsilon+\alpha}(\tau) < +\infty$.

Finally, we note that any moment is bounded on $[\tau, +\infty[$ provided that it is defined at time τ (cf. [El 1]), and we use a proof by induction to get Theorem 1.

Remark. Note that, using equation (2.35), we can produce explicitly the maximum principle for $l_{2+2\varepsilon}$. Namely, when $\varepsilon > 0$, estimate (2.29) ensures that there exists $K_{14} > 0$ such that

$$\frac{d}{dt} l_{2+2\varepsilon}(t) \leqq -K_{14} l_{2+2\varepsilon}^{\frac{2+2\varepsilon+\alpha}{2+\varepsilon}}(t) + K_{13}, \tag{2.36}$$

which gives

$$l_{2+2\varepsilon}(t) \leqq \sup \left(l_{2+2\varepsilon}(t_-), \left(\frac{K_{13}}{K_{14}}\right)^{\frac{2+2\varepsilon}{2+2\varepsilon+\alpha}} \right), \tag{2.37}$$

for $t \geqq t_-$ (this is another proof of the result of [El 1]).

3. Soft potentials

In this section we consider the Boltzmann equation (1.1) with a cross section B of the form

$$B(x, y) = x^{-\gamma}\beta(y), \tag{3.1}$$

with $\gamma > 0$ ($\gamma = -\alpha$ in formula (1.6)), and β satisfying (1.7). This is exactly the hypothesis of soft potentials.

We begin by proving the

Theorem 2. *Consider the operator Q defined in* (1.2) *with B satisfying* (3.1). *Then for $\varepsilon > 0$, there exist $K_{20}, K_{21} > 0$ such that*

$$\int_{v\in\mathbb{R}^3} Q(f)\ (t, v)\, |v|^{2+2\varepsilon}\, dv \leqq K_{20} - K_{21} \int_{v\in\mathbb{R}^3} f(t, v)\, |v|^{2+2\varepsilon-\gamma}\, dv \tag{3.2}$$

when $f(t, v)$ satisfies the conservations of mass and energy (1.14). (*The constants K_{20} and K_{21} depend in fact on this mass and this energy.*)

Proof. According to equation (2.5), for $\varepsilon > 0$,

$$\int_{v\in\mathbb{R}^3} Q(f)\ (t, v)\, |v|^{2+2\varepsilon}\, dv$$

$$= \int_{v\in\mathbb{R}^3} \int_{v_1\in\mathbb{R}^3} \int_{\phi=0}^{2\pi} \int_{\delta=0}^{\pi} \left\{ \left| \frac{v+v_1}{2} + \frac{|v-v_1|}{2} R_{\delta,\phi}\left(\frac{v_1 - v}{|v_1 - v|}\right) \right|^{2+2\varepsilon} - |v|^{2+2\varepsilon} \right\}$$

$$\times f(t, v)\, f(t, v_1)\, |v - v_1|^{-\gamma} \sin\tfrac{\delta}{2}\, \beta\left(\cos\tfrac{\delta}{2}\right) d\delta\, d\phi\, dv_1\, dv. \tag{3.3}$$

We make the change of variables $u = v_1 - v$, and consider the integral in (3.3) when $|u| \geqq \frac{1}{2}$ and when $|u| \leqq \frac{1}{2}$. Then, we use Lemma 3 for the first term and get

$$\int_{v\in\mathbb{R}^3} Q(f)\,(t, v)\, |v|^{2+2\varepsilon}\, dv$$

$$\leqq -K_4 \int_{v\in\mathbb{R}^3} \int_{v_1\in\mathbb{R}^3} |v|^{2+2\varepsilon} f(t, v)\, f(t, v_1)\, B_0(|v - v_1|)\, dv_1\, dv$$

$$+ K_5 \int_{v\in\mathbb{R}^3} \int_{v_1\in\mathbb{R}^3} |v|^2\, |v_1|^{2\varepsilon} f(t, v)\, f(t, v_1)\, B_0(|v - v_1|\, dv_1\, dv$$

$$+ \left(\tfrac{1}{2}\right)^{1-\gamma} (2+2\varepsilon) \int_{v\in\mathbb{R}^3} \int_{|u|\leqq\frac{1}{2}} \int_{\phi=0}^{2\pi} \int_{\delta=0}^{\pi} \left(|v| + \tfrac{1}{2}\right)^{1+2\varepsilon} f(t, v)$$

$$\times f(t, v+u) \sin\tfrac{\delta}{2} \beta\left(\cos\tfrac{\delta}{2}\right) d\delta\, d\phi\, du\, dv, \tag{3.4}$$

where

$$B_0(x) = 1_{x\geqq\frac{1}{2}} x^{-\gamma}. \tag{3.5}$$

With the notation (1.16), one obtains after computations that

$$\int_{v\in\mathbb{R}^3} Q(f)\ (t, v)\ |v|^{2+2\varepsilon}\, dv$$

$$\leqq -\frac{K_4}{4} l_{2+2\varepsilon-\gamma}(t) \frac{(l_0(t))^2}{l_0(t) + l_2(t)} + \frac{K_4}{2} (l_0(t))^2 + 2K_4 l_{2\varepsilon-\gamma}(t) \left(l_2(t) + \tfrac{1}{4} l_0(t)\right)$$

$$+ K_5 2^\gamma l_2(t)\ l_{2\varepsilon}(t) + 2^{\gamma+2\varepsilon}(2+2\varepsilon)\,\pi^2\beta_1 l_{1+2\varepsilon}(t) l_0(t)$$

$$+ 2^{\gamma-1}(2+2\varepsilon)\ \pi^2\beta_1 (l_0(t))^2. \tag{3.6}$$

Since we supposed that $l_0(t) = l_0(0)$ and $l_2(t) = l_2(0)$, there exist $K_{15}, K_{16}, K_{17}, K_{18}, K_{19} > 0$ such that

$$\int_{v\in\mathbb{R}^3} Q(f)\ (t, v)\ |v|^{2+2\varepsilon}\, dv \leqq -K_{15} l_{2+2\varepsilon-\gamma}(t) + K_{16} l_{1+2\varepsilon}(t)$$

$$+ K_{17} l_{2\varepsilon}(t) + K_{18} l_{2\varepsilon-\gamma}(t) + K_{19}. \tag{3.7}$$

Using estimate (2.29) and working as in (2.31), we obtain Theorem 2.

We give now the main corollaries of this theorem.

Corollary 2.1. *Suppose that $f(t, v)$ is a solution of the Boltzmann equation* (1.1) *with a cross section B satisfying* (3.1) (*i.e., in the case of soft potentials*), *such that $f(0, v) = f_0(v) \geqq 0$ and f_0 satisfies* (1.13). *Then there exists $K_0 > 0$ such that*

$$\int_{v\in\mathbb{R}^3} f(t, v)\ |v|^r\, dv \leqq K_0 t + K_0 \tag{3.8}$$

(*with r defined in* (1.13)).

Proof. Integrating the Boltzmann equation (1.1) on $[0, t]\times\mathbb{R}^3$ against $|v|^r$ and using Theorem 2 with $\varepsilon = \frac{r}{2} - 1$, one obtains

$$\int_{v\in\mathbb{R}^3} f(t, v)\ |v|^r\, dv - \int_{v\in\mathbb{R}^3} f_0(v)\ |v|^r\, dv \leqq K_{20} t - K_{21} \int_0^t \int_{v\in\mathbb{R}^3} f(s, v)\ |v|^{r-\gamma}\, dv\, ds, \tag{3.9}$$

which gives estimate (3.8) for $K_0 = \sup(K_{20}, l_r(0))$.

Corollary 2.2. *Suppose that $f(t, v)$ is solution of the Boltzmann equation* (1.1) *with a cross section B satisfying* (3.1) (*i.e., in the case of soft potentials*), *such that $f(0, v) = f_0(v) \geqq 0$ and*

$$\int_{v\in\mathbb{R}^3} f_0(v)\ \left(1 + |v|^r + |\log f_0(v)|\right)\, dv < +\infty \tag{3.10}$$

for some $r > 2 + \gamma$. *Then there exist* $K_0, K_{22} > 0$ *such that*

$$\int_0^t \int_{v\in\mathbb{R}^3} f(s, v)\, |v|^{r-\gamma}\, dv\, ds \leqq K_0 t + K_0, \tag{3.11}$$

$$\frac{d}{dt} \int_{v\in\mathbb{R}^3} f(t, v)\, |v|^{r-\gamma}\, dv \leqq K_{22}. \tag{3.12}$$

Proof. Estimate (3.11) comes out of equation (3.9). Moreover, making $\varepsilon = \frac{r}{2} - \frac{\gamma}{2} - 1$ in Theorem 2 immediately gives (3.12).

We now give a corollary of formulas (3.11) and (3.12), describing the convergence towards equilibrium for equation (1.1) with soft potentials.

Corollary 2.3. *Suppose that* $f(t, v)$ *is a solution of the Boltzmann equation* (1.1) *with a cross section* B *satisfying* (3.1) (*i.e., in the case of soft potentials*), *such that* $f(0, v) = f_0(v)$ *and*

$$\int_{v\in\mathbb{R}^3} f_0(v)\, \left(1 + |v|^r + |\log f_0(v)|\right)\, dv < +\infty \tag{3.13}$$

for some $r > 2 + \gamma$. *Then there exists a sequence* $(t_n)_{n\in\mathbb{N}}$ *going to infinity such that for all* $T > 0$, $f_n(t, v) = f(t + t_n, v)$ *converges in* $L^\infty([0, T]; L^1(\mathbb{R}^3))$ *weak* to the time-independent Maxwellian*

$$m(v) = \frac{\tilde{\rho}}{(2\pi\tilde{T})^{3/2}}\, e^{-\frac{|v-\tilde{u}|^2}{2\tilde{T}}}, \tag{3.14}$$

with

$$\tilde{\rho} = \int_{v\in\mathbb{R}^3} f_0(v)\, dv, \tag{3.15}$$

$$\tilde{\rho}\tilde{u} = \int_{v\in\mathbb{R}^3} v f_0(v)\, dv, \tag{3.16}$$

$$\frac{1}{2}\tilde{\rho}|\tilde{u}|^2 + \frac{3}{2}\tilde{\rho}\tilde{T} = \int_{v\in\mathbb{R}^3} \frac{1}{2}|v|^2 f_0(v)\, dv. \tag{3.17}$$

Proof. We first note that the solution f of the Boltzmann equation (1.1) with soft potentials satisfies the following entropy estimate:

$$\begin{aligned}
&\sup_{t\in[0,+\infty[} \int_{v\in\mathbb{R}^3} f(t, v)\, |\log f(t, v)|\, dv \\
&+ \int_{s=0}^{+\infty} \int_{v\in\mathbb{R}^3} \int_{v_1\in\mathbb{R}^3} \int_{\omega\in S^2} \{f(s, v')\, f(s, v_1') - f(s, v)\, f(s, v_1)\} \\
&\times \log\left\{\frac{f(s, v')\, f(s, v_1')}{f(s, v)\, f(s, v_1)}\right\} |v - v_1|^{-\gamma}\, \beta\left(\left|\omega \cdot \frac{v - v_1}{|v - v_1|}\right|\right) d\omega\, dv_1\, dv\, ds < +\infty.
\end{aligned} \tag{3.18}$$

This inequality is obtained from (1.12), (1.14), (1.15) and (3.13) as in the space-dependent case (cf. [DP, L 1] and [DP, L 2]).

Now according to Corollary 2.2, there exists a sequence $(t_n)_{n\in\mathbb{N}}$ going to infinity and an $\tilde{r} = r - \gamma > 2$ such that

$$\int_{v\in\mathbb{R}^3} f(t_n, v)\,|v|^{\tilde{r}}\,dv \leqq K_0 + 1. \tag{3.19}$$

Moreover, because of estimate (3.12), we have for $t\in[0, T]$ that

$$\int_{v\in\mathbb{R}^3} f(t_n + t, v)\,|v|^{\tilde{r}}\,dv \leqq K_0 + 1 + K_{22}T. \tag{3.20}$$

Denoting

$$\Gamma(x, y) = (x - y)\log\left(\frac{x}{y}\right), \tag{3.21}$$

and using estimates (3.18), (3.20) and the conservation of mass (1.14), we can find $K_{23} > 0$ such that $f_n(t, v) = f(t + t_n, v)$ satisfies

$$\sup_{t\in[0,T]} \int_{v\in\mathbb{R}^3} f_n(t, v)\,\{1 + |v|^{\tilde{r}} + |\log f_n(t, v)|\}\,dv \leqq K_{23}, \tag{3.22}$$

and

$$\int_{s=0}^{T}\int_{v\in\mathbb{R}^3}\int_{v_1\in\mathbb{R}^3}\int_{\omega\in S^2} \Gamma(f_n(s, v')\,f_n(s, v_1'), f_n(s, v)\,f_n(s, v_1))$$

$$\times|v - v_1|^{-\gamma}\beta\left(\left|\omega\cdot\frac{v - v_1}{|v - v_1|}\right|\right)d\omega\,dv_1\,dv\,ds \tag{3.23}$$

tends to 0 when n goes to infinity.

According to estimate (3.22), there exists a subsequence of f_n (still denoted by f_n) which converges to a limit $m(t, v)$ in $L^\infty([0, T]; L^1(\mathbb{R}^3))$ weak*.

To prove that m is a Maxwellian function of v which does not depend on t, one can proceed essentially as in [De 1].

Now we must identify $\tilde{\rho}$, $\tilde{u}$, and $\tilde{T}$. Using the conservations of mass, impulse and energy (1.14), one gets for all $t\in[0, T]$, that

$$\int_{v\in\mathbb{R}^3} f_n(t, v)\,(1, v, \tfrac{1}{2}|v|^2)\,dv = \int_{v\in\mathbb{R}^3} f_0(v)\,(1, v, \tfrac{1}{2}|v|^2)\,dv. \tag{3.24}$$

But because of estimate (3.22),

$$\int_{t=0}^{T}\int_{v\in\mathbb{R}^3} (1, v, |v|^2)\,f_n(t, v)\,dv\,dt \xrightarrow[n\to+\infty]{} T\int_{v\in\mathbb{R}^3} (1, v, |v|^2)\,m(v)\,dv, \tag{3.25}$$

and therefore the parameters $\tilde{\rho}$, $\tilde{u}$, $\tilde{T}$ are given by formulas (3.15)–(3.17).

Remark. This is only a partial result. One would expect in fact that the whole function tends when $t\to+\infty$ to the Maxwellian given in (3.14)–(3.17). Note that this is the case for hard potentials, the convergence even being strong and exponential under suitable assumptions (cf. [A 3]). Note also that the existence of a converging subsequence for any sequence t_n going to infinity can be derived from the papers of ARKERYD (cf. [A 2]), but the limits in that case may have less energy than the initial datum.

4. The Kac equation

We now introduce the one-dimensional homogeneous Kac model (cf. [K], [MK]), where all collisions have the same probability. The density $f(t, v) > 0$ of particles which at time t move with velocity v satisfies

$$\frac{\partial f}{\partial t}(t, v) = Q'(f)\ (t, v), \tag{4.1}$$

where Q' is a quadratic collision kernel:

$$Q'(f)\ (t, v) = \int_{v_1 \in \mathbb{R}^3} \int_{\theta=-\pi}^{\pi} \{f(t, v^*) f(t, v_1^*) - f(t, v) f(t, v_1)\} \frac{d\theta}{2\pi} dv_1, \tag{4.2}$$

with

$$v^* = \sqrt{v^2 + v_1^2} \cos\theta, \tag{4.3}$$

$$v_1^* = \sqrt{v^2 + v_1^2} \sin\theta. \tag{4.4}$$

It is easy to prove (at least at the formal level) the conservation of mass and energy

$$\int_{v \in \mathbb{R}} f(t, v)\ (1, \tfrac{1}{2}|v|^2)\ dv = \int_{v \in \mathbb{R}} f(0, v)\ (1, \tfrac{1}{2}|v|^2)\ dv, \tag{4.5}$$

and the entropy estimate

$$\int_{v \in \mathbb{R}} f(t, v) \log f(t, v)\ dv \leqq \int_{v \in \mathbb{R}} f(0, v) \log f(0, v)\ dv. \tag{4.6}$$

Adapting, for example, the proof of ARKERYD (cf. [A 1]) for the Boltzmann equation, one can prove that if f_0 satisfies

$$\int_{v \in \mathbb{R}} f_0(v)\ \big(1 + |v|^r + |\log f_0(v)|\ dv < +\infty \tag{4.7}$$

for some $r > 2$, then there exists a solution of the Kac equation (4.1) such that $f(0, v) = f_0(v)$. Moreover, this solution satisfies estimates (4.5) and (4.6).

It is also easy to adapt the theorems of TRUESDELL (cf. [Tr] and [Tr, Mu]) for this equation. Namely, one can give an explicit induction formula to compute the moments

$$L_n(t) = \int_{v \in \mathbb{R}} f(t, v)\ v^n\ dv \tag{4.8}$$

when $n \in \mathbb{N}$, provided that these moments exist initially. Therefore, we do not deal in this work with the polynomial moments of f, but rather with the Maxwellian moments

$$\mathcal{M}_f(t, \lambda) = \int_{v \in \mathbb{R}} f(t, v)\, e^{\lambda v^2}\, dv, \tag{4.9}$$

for $\lambda > 0$.

We begin by proving the following theorem:

Theorem 3. *Let f_0 satisfy* (4.7), *and consider a solution $f(t, v)$ of the Kac equation* (4.1) *such that $f(0, v) = f_0(v)$. Suppose, moreover, that there exists $\lambda_0 > 0$ such that $\mathcal{M}_f(0, \lambda_0) < +\infty$. Then, there exists $\bar{\lambda} > 0$ and $K_{24} > 0$ such that $\mathcal{M}_f(t, \bar{\lambda}) \leqq K_{24}$ when $t \geqq 0$.*

Proof. We look for an equation satisfied by $\mathcal{M}_f(t, \lambda)$:

$$\begin{aligned}
\frac{\partial}{\partial t} \mathcal{M}_f(t, \lambda) &= \int_{v \in \mathbb{R}} Q'(f)\ (t, v)\ e^{\lambda v^2}\, dv \\
&= \int_{v \in \mathbb{R}} \int_{v_1 \in \mathbb{R}} \int_{\theta = -\pi}^{\pi} f(t, v) f(t, v_1) \{e^{\lambda v^{*2}} - e^{\lambda v^2}\} \frac{d\theta}{2\pi}\, dv_1\, dv \\
&= \int_{v \in \mathbb{R}} \int_{v_1 \in \mathbb{R}} \int_{\theta = -\pi}^{\pi} f(t, v) f(t, v_1) \{e^{\lambda (v^2 + v_1^2) \cos^2\theta} - e^{\lambda v^2}\} \frac{d\theta}{2\pi}\, dv_1\, dv \\
&= \int_{\theta = -\pi}^{\pi} \mathcal{M}_f^2(t, \lambda \cos^2\theta) - \mathcal{M}_f(t, \lambda)\, \mathcal{M}_f(0, 0)\, \frac{d\theta}{2\pi},
\end{aligned} \tag{4.10}$$

since the conservation of mass (4.5) holds.

For any $\bar{\rho}$, $\bar{T} > 0$, we denote by $m_{\bar{\rho}, \bar{T}}$ the steady Maxwellian of density $\bar{\rho}$ and temperature $\bar{T}$:

$$m_{\bar{\rho}, \bar{T}}(t, v) = \frac{\bar{\rho}}{(2\pi\bar{T})^{1/2}}\, e^{-\frac{|v|^2}{2\bar{T}}}. \tag{4.11}$$

It is easy to see that $m_{\bar{\rho}, \bar{T}}$ is a steady solution of the Kac equation (4.1). Therefore

$$\mathcal{M}_{m_{\bar{\rho}, \bar{T}}}(t, \lambda) = \frac{\bar{\rho}}{\sqrt{1 - 2\lambda\bar{T}}} \tag{4.12}$$

is a steady solution of equation (4.10) on $[0, +\infty[\times [0, \frac{1}{2\bar{T}}[$ (this can be seen directly from equation (4.10)).

We now prove that under the hypothesis of Theorem 3, there exist $\tilde{\lambda} > 0$, $\bar{T} > 0$, such that

$$\forall \lambda \in [0, \tilde{\lambda}], \quad \mathcal{M}_f(0, \lambda) \leqq \mathcal{M}_{m_{\bar{\rho}, \bar{T}}}(0, \lambda), \tag{4.13}$$

with

$$\bar{\rho} = \int_{v \in \mathbb{R}} f(0, v)\, dv. \tag{4.14}$$

In order to prove (4.13), we use a development of $\mathscr{M}_f(0, \lambda)$ around 0:

$$\begin{aligned}\mathscr{M}_f(0, \lambda) &= \int_{v\in\mathbb{R}} f(0, v)\, e^{\lambda v^2}\, dv \\ &= \int_{v\in\mathbb{R}} f(0, v) \left(1 + \lambda v^2 + \lambda^2 v^4 \int_{u=0}^{1} (1-u)\, e^{\lambda u v^2}\, du\right) dv \\ &= \int_{v\in\mathbb{R}} f(0, v)\, dv + \lambda \int_{v\in\mathbb{R}} f(0, v)\, |v|^2\, dv + O(\lambda^2), \end{aligned} \tag{4.15}$$

since

$$\int_{v\in\mathbb{R}} f(0, v)\, v^4 \left(\int_{u=0}^{1} (1-u)\, e^{\lambda u v^2}\, du\right) dv \leqq \int_{v\in\mathbb{R}} f(0, v)\, v^4 e^{\lambda v^2}\, dv < +\infty \tag{4.16}$$

when $\lambda < \lambda_0$. But

$$\mathscr{M}_{m_{\bar\rho, \bar T}}(0, \lambda) = \bar\rho + \lambda \bar\rho \bar T + O(\lambda^2), \tag{4.17}$$

and therefore (4.13) holds provided that we take $\tilde\lambda$ small enough and

$$\bar T > \frac{1}{\bar\rho} \int_{v\in\mathbb{R}} f(0, v)\, |v|^2\, dv. \tag{4.18}$$

But equation (4.10) clearly satisfies the following monotonicity property: If $a(0, \lambda)$ and $b(0, \lambda)$ are two initial data for (4.10) and $\bar\lambda$ is a strictly positive number such that

$$\forall \lambda \in [0, \bar\lambda[, \quad a(0, \lambda) \leqq b(0, \lambda), \tag{4.19}$$

$$a(0, 0) = b(0, 0), \tag{4.20}$$

then for all $t \geqq 0$, the solutions $a(t, \lambda)$ and $b(t, \lambda)$ of (4.10) satisfy

$$\forall \lambda \in [0, \bar\lambda[, \quad a(t, \lambda) \leqq b(t, \lambda). \tag{4.21}$$

Using (4.12), (4.13), (4.21), and taking

$$0 < \bar\lambda < \inf\left(\tilde\lambda, \frac{1}{2\bar T}\right), \tag{4.22}$$

we obtain Theorem 3.

We give now estimates for the exponential moments

$$\mathscr{N}_f(t, \lambda) = \int_{v\in\mathbb{R}} f(t, v)\, e^{\lambda v}\, dv, \tag{4.23}$$

for $\lambda \in \mathbb{R}$. We can prove the following theorem:

Theorem 4. *Let f_0 satisfy* (4.7), *and let $f(t, v)$ be a solution of the Kac equation* (4.1) *such that $f(0, v) = f_0(v)$. Suppose, moreover, that there exists $\lambda_0 > 0$ such that $\mathscr{N}_f(0, \lambda_0) < +\infty$, $\mathscr{N}_f(0, -\lambda_0) < +\infty$, and*

$$\int_{v\in\mathbb{R}} f_0(v)\, v\, dv = 0. \tag{4.24}$$

Then, there exists $\bar\lambda > 0$ and $K_{25} > 0$ such that $\mathscr{N}_f(t, \bar\lambda) + \mathscr{N}_f(t, -\bar\lambda) \leqq K_{25}$ for $t \geqq 0$.

Proof. It is easy to see that

$$\frac{\partial}{\partial t}\mathcal{N}_f(t,\lambda) = \int_{\theta=-\pi}^{\pi} \mathcal{N}_f(t,\lambda\cos\theta)\,\mathcal{N}_f(t,\lambda\sin\theta) - \mathcal{N}_f(t,\lambda)\,\mathcal{N}_f(0,0)\,\frac{d\theta}{2\pi}. \tag{4.25}$$

Moreover, since $m_{\bar{\rho},\bar{T}}$ is a steady solution of the Kac equation (4.1),

$$\mathcal{N}_{m_{\bar{\rho},\bar{T}}}(t,\lambda) = \bar{\rho}e^{\frac{\lambda^2}{2}\bar{T}} \tag{4.26}$$

is a steady solution of equation (4.25) on $[0,+\infty[\times\mathbb{R}$ (this can be seen directly in equation (4.25)). Thus the proof is quite similar to the proof of Theorem 3.

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