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A Time-Dependent Free Boundary Problem Modeling the Visual Image in Electrophotography

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Abstract

The formation of a visual image in electrophotography can be modeled as a time-dependent free boundary problem. The electric potential $-u$ satisfies $\Delta u = 1$ in the toner region and $\Delta u = 0$ outside this region, whereas on the interface (which is a moving boundary)

$$-\frac{\partial u}{\partial N} = \text{velocity of the interface},$$

N being the outward normal to the toner region. It is proved that this problem has a smooth solution for a small time interval; furthermore, for a certain version of the free boundary condition, the solution is unique.

0. Introduction

The formation of visual images in electrophotography is accomplished by means of a toner injected onto the surface of the photoconductor. The toner is carried by biased carriers and, as a consequence, tends to accumulate in those regions of the photoconductor surface which carry surface charge corresponding to dark spots in the document which is being copied (the distribution of this charge represents the *electric image* of the document); for more details see [1, 2, 5, 6, 7].

We consider here a 2-dimensional model wherein a pixel is represented by an interval $-a \leqq x \leqq a$. We denote by $-u$ the potential of the electric field which is responsible for the motion and settling of the toner. A small potential difference M $(M > 0)$ is maintained between boundaries $y = b$ (where $u = M$) and $y = -h$ (where $u = 0$). The surface of the photoconductor is $\{y = 0\}$, and the electric image is assumed, for simplicity, to be a surface-distribution of uniform density σ, $\sigma > 0$, supported on an interval

$$I = \{(x, 0)\,;\ -\gamma < x < \gamma\} \quad \text{where } 0 < \gamma < a\,.$$

Figure 1 shows the formation of the visual image: The domain D where $\Delta u = 1$ is precisely the region occupied by the toner. Off $D \cup I$ the function u is harmonic, and it satisfies the boundary conditions shown in Figure 1; further,

$$[u_y]_I \equiv u_y(x, 0+) - u_y(x, 0-) = -\sigma, \quad -\gamma < x < \gamma.$$

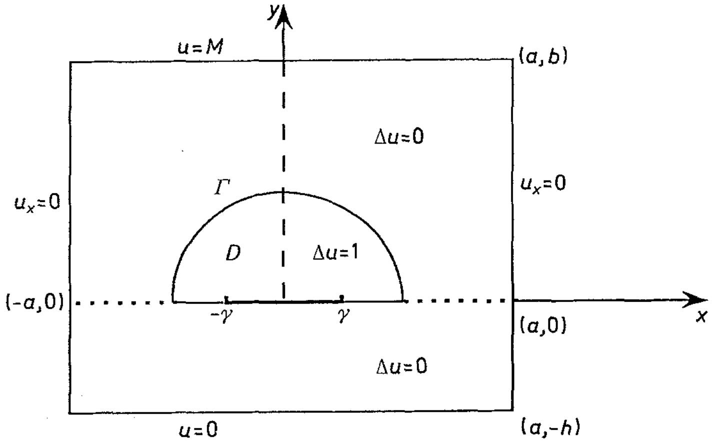

Figure 1

The domain D lies in $\{y > 0\}$, and $\Gamma = \partial D \cap \{y > 0\}$ is called the *free boundary.* After the development of the visual image has been completed, u becomes time-independent, and the equilibrium condition

$$\frac{\partial u}{\partial N} = 0 \quad \text{on } \Gamma$$

must hold.

The above model was studied by FRIEDMAN & HU [3] under the conditions

$$h \leqq b, \quad M < \sigma h. \tag{0.1}$$

Both conditions are satisfied in the physical model. Indeed, in the units of volt-ampere-coulomb and microns, $\Delta u = k$ where $k \sim 3$, $\sigma = \sigma_0/(\kappa\varepsilon_0)$ where $\sigma_0 \sim 4 \times 10^{-10}$ and $\kappa\varepsilon_0$ is the dielectric coefficient, $\kappa\varepsilon_0 \sim 24 \times 10^{-12}$. Thus $\sigma \sim 15$. Also, $h = 20$, $b = 600$ and $M = 50$. It follows that $h < b$ whereas the inequality $M < \sigma h$ becomes $50 < 15 \times 20$.

It was proved in [3] that if γ/a is close to 1, then the problem has a unique solution with Γ initiating at $x = -a$ and terminating at $x = a$, and $u \equiv M$ above Γ. On the other hand, if γ/a is small, then there exist infinitely many solutions for which the toner set D consists of two symmetric components [3]. The case of small γ/a was also studied, more recently, by HU & WANG [4] who proved the existence of a solution with D a connected region. The uniqueness of such a solution has not been proved and it is altogether unclear which of the solutions is dynamically stable, i.e., physical. It should be noted that both cases, γ/a near 1 and γ/a near 0, are physically not very interesting since most pixels are neither nearly all dark nor nearly all light. For the case of intermediate γ/a, no existence results are known.

As the photocopying machines are becoming faster, there may not be enough time for the visual image to develop fully. Thus there is a need to study the time-dependent problem: How does the visual image evolve in time?

In the time-dependent case u is a function of both (x, y) and t, and the toner region $D = D(t)$ and the free boundary $\Gamma = \Gamma(t) = \partial D(t) \cap \{y > 0\}$ also depend on t. Denote by V_n the velocity of points in $\Gamma(t)$ in the direction of the outward normal $N = N(t)$ to $D(t)$. The continuity equation for the charged toner is $\partial\rho/\partial t = -\nabla \cdot \vec{J}$ where $\vec{J} = \mu\rho\vec{E} = -\mu\rho\nabla u$ is the current density; here $\vec{E}$ is the electric field, ρ the charge density and μ the mobility. Since $\rho \approx \text{const.} = \rho_0 > 0$ in the toner and $\rho = 0$ outside the toner, the continuity equation means that the free boundary $\Gamma(t)$ must move according to the law

$$V_n = -\frac{\partial u}{\partial N} \quad \text{on } \Gamma(t), \tag{0.2}$$

provided we take $\mu\rho_0 = 1$.

In this paper we consider this evolutionary toner problem and prove the existence and uniqueness of a solution for a small time interval $0 \leqq t \leqq t_0$. We also establish some geometric features of the free boundary. The main results are stated, more precisely, in § 1, where the structure of the paper is also outlined.

Throughout this paper it is assumed that (0.1) is satisfied.

1. Statement of the main result

Set

$$R = \{(x, y)\,;\ -a < x < a,\ -h < y < b\}.$$

For simplicity we take $\gamma = 1$ in Figure 1; then, of course, $a > 1$.

Consider a family of curves

$$\Gamma(t) : y = f(x, t), \qquad -x_0(t) < x < x_0(t)$$

for $0 \leqq t \leqq t_0$ satisfying the following properties:

$$f(x, t) = f(-x, t),$$

$$f(x, t) > 0 \text{ if } |x| < x_0(t),\ \ f(x_0(t), t) = 0 \text{ where}$$

$$1 < x_0(t) < a \text{ if } 0 < t \leqq t_0,\ x_0(0) = 1,$$

$$|f_x(x, t)| \leqq \frac{C}{|\log t|} \text{ if } |x| < x_0(t),$$

$$f_x(x, t) \leqq \frac{-c}{|\log t|} \text{ if } 1 < x < x_0(t) \quad (C > c > 0), \tag{1.1}$$

$$
\begin{aligned}
&(1.1) \qquad [f_x(\cdot, t)]_{0,\alpha} \leqq \frac{C_0}{|t \log t|^\alpha} \quad (0 < \alpha < \tfrac{1}{2},\ C_0 > 0),\\
&\forall\, \delta > 0,\ \gamma_1 < f_t(x, t) < \gamma_2 \text{ if } |x| < 1 - \delta \quad (\gamma_i = \gamma_i(\delta) > 0),\\
&c_1 t\,|\log t| \leqq x_0(t) - 1 \leqq C_1 t\,|\log t| \quad (C_1 > c_1 > 0),\\
&\{(x, f(x, t))\} \text{ lies between the polygonal lines } l_1(t),\ l_2(t) \text{ where}\\
&\quad l_i(t) \text{ has vertices } (-1 - Q_i t\,|\log t|, 0),\ (-1, A_i t),\ (1, A_i t) \text{ and}\\
&\quad (1 + Q_i t\,|\log t|, 0), \text{ for some constants } A_2 > A_1 > 0,\ Q_2 > Q_1 > 0.
\end{aligned}
$$

Here $[\ \ ]_{0,\alpha}$ denotes the α-Hölder coefficient in the variable x.

Denote by $D(t)$ the domain bounded by $\Gamma(t)$ and the x-axis, and consider the following problem:

$$
\Delta u = \begin{cases} 1 & \text{in } D(t), \\ 0 & \text{in } R\backslash\overline{D(t)}, \end{cases} \tag{1.2}
$$

the limits $u_y(x \pm 0)$ exist for $-1 < x < 1$ and

$$
u_y(x, 0+) - u_y(x, 0-) = -\sigma \quad \text{if } -1 < x < 1, \tag{1.3}
$$

$$
\begin{aligned}
&u \text{ is continuous in } \bar{R} \times [0, t_0]\ \forall\, t \in [0, t_0], \text{ and } u\\
&\text{is continuously differentiable in } R\backslash\{(x, 0);\ -1 \leqq x \leqq 1\},
\end{aligned} \tag{1.4}
$$

$$
u(x, b, t) = M, \qquad -a < x < a, \tag{1.5}
$$

$$
u(x, -h, t) = 0, \qquad -a < x < a, \tag{1.6}
$$

$$
u_x(\pm a, y, t) = 0, \qquad -h < y < b. \tag{1.7}
$$

By uniqueness $u(x, y, t) = u(-x, y, t)$.

We now wish to consider $\Gamma(t)$ as unknown, and impose the free boundary condition (0.2). We recast (0.2), however, in a way which depends more directly on u:

Suppose we write

$$
\Gamma(t) : x = x(t, \lambda),\ y = y(t, \lambda) \qquad (0 \leqq t \leqq t_0) \tag{1.8}
$$

where λ is a parameter such that

$$
\begin{aligned}
&x(0, \lambda) = \lambda, \quad y(0, \lambda) = 0, \quad -\lambda_0(t) < \lambda < \lambda_0(t),\\
&\lambda_0(0) = 1, \quad \lambda_0(t) < 1 \text{ if } t > 0,\\
&y(t, \lambda) > 0 \quad \text{if } t > 0,\ |\lambda| < \lambda_0(t),\\
&y(t, \pm\lambda_0(t)) = 0 \quad \text{if } t > 0.
\end{aligned} \tag{1.9}
$$

Notice that the condition (0.2) means that

$$
\left(\frac{dx}{dt}, \frac{dy}{dt}\right) \cdot N = -\frac{\partial u}{\partial N}. \tag{0.2'}
$$

or

$$\frac{dx}{dt} + u_x = a\frac{\partial x}{\partial \lambda}, \quad \frac{dy}{dt} + u_y = a\frac{\partial y}{\partial \lambda} \tag{0.2''}$$

where a is an *arbitrary* function of (x, y, t), odd in x. We henceforth make the choice $a = 0$, which is mathematically the simplest (see Remark 1.1 below for explanation). This means that we replace (0.2) by the condition

$$\frac{dx}{dt} = -u_x(x, y, t), \quad \frac{dy}{dt} = -u_y(x, y, t). \tag{1.10}$$

Remark 1.1. Theorem 1.1 below asserts the existence and uniqueness of a solution (u, Γ) to (1.1)–(1.10) (for small time). Similarly one can prove the existence and uniqueness of a solution to (1.1)–(1.9) and (0.2″) for any smooth function $a(x, y, t)$ odd in x. This solution however results in the same function u and in a reparametrization of Γ given by the functions $x = \tilde{x}(t, \lambda)$, $y = \tilde{y}(t, \lambda)$ defined as follows:

$$\frac{d\tilde{x}}{dt} + u_x(\tilde{x}, \tilde{y}, t) = a(\tilde{x}, \tilde{y}, t)\frac{\partial \tilde{x}}{\partial \lambda}, \quad \tilde{x}(0, \lambda) = \lambda,$$

$$\tilde{y} = \tilde{y}(t, \lambda) = f(\tilde{x}(t, \lambda), t) \quad (\Gamma : y = f(x, t)).$$

Indeed, after solving the differential equation for $\tilde{x}$, we compute, using (0.2),

$$\begin{aligned}\frac{d\tilde{y}}{dt} + u_y(\tilde{x}, \tilde{y}, t) &= f_x\frac{d\tilde{x}}{dt} + f_t + u_y \\ &= \left(-u_x + a\frac{\partial \tilde{x}}{\partial \lambda}\right)f_x + f_t + u_y \\ &= a\frac{\partial \tilde{x}}{\partial \lambda}f_x = a\frac{\partial \tilde{y}}{\partial \lambda}.\end{aligned}$$

We conclude that the choice $a = 0$ made above is nothing but a certain parametrization of Γ.

Remark 1.1 justifies the following definition:

Definition 1.1. If u, Γ satisfy (1.1)–(1.10), then we say that they form a solution to the *evolutionary toner problem* for $0 \leqq t \leqq t_0$.

The main result of this paper is the following:

Theorem 1.1. *If* (0.1) *holds, then there exists a unique solution to the evolutionary toner problem for some time interval* $0 \leqq t \leqq t_0$ $(t_0 > 0)$.

In § 2 we study the function $\bar{u}(x, y)$, which is the initial state $u(x, y, 0)$ of the solution u. In § 3 we establish interior $C^{2+\alpha}$ estimates for the potential

$$\iint_D \log [(x-\xi)^2 + (y-\eta)^2]^{1/2} \, d\xi \, d\tau$$

along Γ, where D is a domain in $\{y > 0\}$ bounded by Γ and the y-axis; it is assumed that Γ is a $C^{1+\alpha}$ curve. These estimates are crucial for the proof of Theorem 1.1. Section 4 outlines the strategy for proving Theorem 1.1: It is based on establishing a fixed point for a mapping $\mathscr{M}$ of a family $\mathscr{A}$ of curves $\tilde{\Gamma}(t): x = \tilde{x}(t, \lambda)$ $y = \tilde{y}(t, \lambda)$ into itself. The images $\Gamma(t): x = x(t, \lambda), y = y(t, \lambda)$ are defined by the ordinary differential equation (1.10) where u is the solution to (1.2)–(1.7) and where $D(t)$ are the domains bounded by $\tilde{\Gamma}(t)$ and the x-axis. The precise definition of $\mathscr{A}$ is given in § 5. In § 6 we begin with the analysis of the ordinary differential equation, constructing barriers that are used to find simple yet sufficiently good approximations to the solution.

Next, in §§ 7 and 8 we establish, respectively, C^1 and $C^{1+\alpha}$ estimates for the curves $\Gamma(t)$, showing that $\mathscr{M}$ maps $\mathscr{A}$ into itself. Finally, in § 9 we prove that a sequence of iterates $\mathscr{M}^n D(t)$ converges to a unique fixed point of $\mathscr{M}$, thereby completing the proof of Theorem 1.1.

The estimates

$$\frac{c}{|\log t|} \leqq f_x(x, t) \leqq \frac{C}{|\log t|} \quad \text{for } 1 < x < x_0(t) \qquad (C > c > 0) \tag{1.11}$$

for the free boundary provide an interesting bound for the slope of $\Gamma(t)$ as it descends toward the x-axis. The lower bound is critical to the proof of Theorem 1.1. The solution can in fact be continued, in the time, as long as such a bound can be a priori established, provided $f(x, t)$ remains uniformly positive for $-1 \leqq x \leqq 1$. Indeed, this is so because the estimates of § 7 work with arbitrary initial time rather than just with initial time 0.

In § 10 we study the shape of the free boundary for $|x| < 1$. We discover the rather surprising fact that, for any small $\eta > 0$, the free boundary $y = f(x, t)$ for $0 \leqq x \leqq 1 - \eta$ cannot be monotone decreasing in x. In fact, in the "average" sense it is actually monotone increasing! On the other hand, $f(x, t)$ is monotone decreasing for $x > 1$. The form of the free boundary is shown in Figure 2. It indicates that in a fast image-development of a document with one black spot, the photocopy appears lighter at the center of the spot than in the rim. This phenomenon is known as the "edge effect."

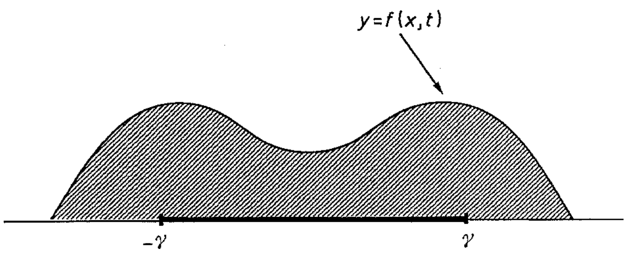

Figure 2

§ 2. The initial state

Initially the toner set is empty, i.e., $D(0) = \phi$. Hence $\bar{u}(x, y) \equiv u(x, y, 0)$ satisfies

$$(2.1) \qquad \begin{aligned} -\Delta \bar{u} &= \sigma \chi_{[-1,1]}(x)\, \delta(y) \quad && \text{in } R, \\ \bar{u}(x, b) &= M, && -a < x < a, \\ \bar{u}(x, -h) &= 0, && -a < x < a, \\ \bar{u}_x(\pm a, y) &= 0, && -h < y < b \end{aligned}$$

where $\delta(y)$ is the Dirac function.

Theorem 2.1. *The following inequalities hold*:

$$(2.2) \qquad \bar{u}_y(x, 0+) < 0 \quad \textit{if } |x| < 1,$$

$$(2.3) \qquad \bar{u}_y(x, 0) > 0 \ \textit{if } 1 < |x| < a.$$

Proof. By the maximum principle, $\bar{u}(x, y) > 0$ in R. Introduce the function

$$v(x, y) = \bar{u}(x, y) - \bar{u}(x, -y) \quad \text{in } -a < x < a,\ 0 < y < h.$$

Notice that

$$v(x, h) = \bar{u}(x, h) > 0.$$

Since $v(x, 0) = 0$ and v is harmonic, it follows, by the maximum principle, that $v_y(x, 0+) > 0$. Since $v_y(x, 0+) = \bar{u}_y(x, 0+) + \bar{u}_y(x, 0-)$ and since, for $|x| > 1$, u is smooth and thus $\bar{u}_y(x, 0-) = \bar{u}_y(x, 0+)$, it follows that

$$2\bar{u}_y(x, 0) = v_y(x, 0+) > 0 \quad \text{if } |x| > 1,$$

i.e., (2.3) holds.

If $|x| < 1$, then $\bar{u}_y(x, 0+) - \bar{u}_y(x, 0-) = -\sigma$, so that

$$(2.4) \qquad 2\bar{u}_y(x, 0+) = -\sigma + v_y(x, 0+).$$

In order to estimate $v_y(x, 0+)$ from above we first obtain an upper bound for $v(x, h)$. Let $w(x, y)$ be the solution to

$$-\Delta w = \sigma \chi_{[-a,a]}(x)\, \delta(y) \quad \text{in } R$$

with the same boundary condition as $\bar{u}$. By the comparison theorem,

$$(2.5) \qquad \bar{u}(x, y) \leqq w(x, y).$$

Observe that w is independent of x and, in fact, as easily verified,

$$w(x, y) = \begin{cases} \dfrac{M + \sigma b}{b + h}(y + h) & \text{for } -h < y < 0, \\ \dfrac{M + \sigma b}{b + h} h + \left(\dfrac{M + \sigma b}{b + h} - \sigma\right) y & \text{for } 0 < y < b. \end{cases}$$

Hence, by (2.5),

$$v(x, h) = \bar{u}(x, h) \leqq w(x, h) = 2\,\frac{M + \sigma b}{b + h}\,h - \sigma h \equiv B.$$

The harmonic function $V(x, y) = By/h$ majorizes v on $y = h$, and $v(x, 0) = V(x, 0) = 0$. Also $v_x = V_x = 0$ on $x = \pm a$. By the comparison theorem it then follows that $v \leqq V$ and

$$v_y(x, 0+) < V_y(x, 0) = \frac{B}{h} = -\sigma + 2\,\frac{M + \sigma b}{b + h}.$$

Recalling (2.4) we get, for $|x| < 1$, that

$$2\bar{u}_y(x, 0+) < -2\sigma + \frac{2(M + \sigma b)}{b + h} = \frac{2(M - \sigma h)}{b + h} \leqq 0 \quad \text{by (0.1)},$$

and (2.2) follows.

Remark 2.1. Theorem 2.1 implies that the velocity of the free boundary is initially positive for all $x \in (-1, 1)$, but not for $|x| > 1$. This means that $\Gamma(t)$ begins growing only from points of the interval

$$I = \{(x, 0);\ -1 < x < 1\}. \tag{2.6}$$

Set

$$\Phi(x, y) = -\frac{\sigma}{2\pi} \int_{-1}^{0} \log\,[(x - \xi)^2 + y^2]^{1/2}\, d\xi. \tag{2.7}$$

Then Φ is a harmonic function off I, and it satisfies the jump relation

$$[\Phi_y] \equiv \Phi_y(x, 0+) - \Phi_y(x, 0-) = -\sigma, \qquad -1 < x < 1. \tag{2.8}$$

We can write

$$\bar{u}(x, y) = \Phi(x, y) + \psi(x, y) \tag{2.9}$$

where ψ is harmonic in R.

3. $C^{1+\alpha}$ estimates on $\nabla G(x, f(x))$

In this section we study the function

$$G(x, y) = \frac{1}{2\pi} \iint_D \log\,[(x - \xi)^2 + (y - \eta)^2]^{1/2}\, d\xi\, d\eta \tag{3.1}$$

where D is a domain given by

$$D = \{0 < y < f(x),\ -a_0 < x < a_0\}, \tag{3.2}$$

for some $a_0 \in (0, a)$. Denote by $B_\mu(x_0, y_0)$ the disc $\{(x - x_0)^2 + (y - y_0)^2 < \mu^2\}$, and set, for simplicity, $B_\mu(x_0) = B_\mu(x_0, f(x_0))$. We assume that f satisfies the following conditions:

$$f(x)=f(-x) \quad \text{and} \quad f(x)>0 \quad \text{if } -a_0<x<a_0,\ f(\pm a_0)=0, \tag{3.3}$$

and there exist (x_0, y_0) and $\mu>0$ such that

$$-a_0+\mu<x_0<a_0-\mu, \quad \mu<y_0<b-\mu, \tag{3.4}$$

$$|f'(x_1)| \leqq 1, \tag{3.5}$$

$$\frac{|f'(x_1)-f'(x_2)|}{|x_1-x_2|^{\alpha}} \leqq \frac{\theta}{\mu^{\alpha}} \quad (\mu^{\alpha}<\theta), \tag{3.6}$$

for any $(x_i, f(x_i))$ $(i=1, 2)$ in $B_{\mu}(x_0)$, where θ, α are positive constants and $\alpha<1$.

Set

$$\begin{aligned} &\nabla G(x, f(x)) \\ &= \frac{1}{2\pi}\left(\iint_D \frac{x-\xi}{(x-\xi)^2+(f(x)-\eta)^2}\,d\xi\,d\eta,\ \iint_D \frac{f(x)-\eta}{(x-\xi)^2+(f(x)-\eta)^2}\,d\xi\,d\eta\right) \\ &\equiv \frac{1}{2\pi}(A(x), E(x)). \end{aligned} \tag{3.7}$$

In this section we prove:

Lemma 3.1. *Let the assumptions* (3.3)–(3.6) *hold. Then*

$$\left|\frac{dA(x)}{dx}\right| + \left|\frac{dE(x)}{dx}\right| \leqq C(|\log\mu|+\theta), \tag{3.8}$$

$$\frac{1}{|x-\bar{x}|^{\alpha}}\left\{\left|\frac{dA(x)}{dx}-\frac{dA(\bar{x})}{dx}\right| + \left|\frac{dE(x)}{dx}-\frac{dE(\bar{x})}{dx}\right|\right\} \leqq \frac{C}{\mu^{\alpha}} + \frac{C\theta|\log\mu|}{\mu^{\alpha}} \tag{3.9}$$

for all $(x, f(x))$, $(\bar{x}, f(\bar{x}))$ *in* $B_{\mu/8}(x_0)$ *where* C *is a positive constant independent of* θ, α *and* μ.

Proof. For any small $\rho>0$ introduce the truncated integrals

$$A_{\rho}(x) = \iint_{D\backslash B_{\rho}(x)} \frac{x-\xi}{(x-\xi)^2+(f(x)-\eta)^2}\,d\xi\,d\eta, \tag{3.10}$$

$$E_{\rho}(x) = \iint_{D\backslash B_{\rho}(x)} \frac{f(x)-\eta}{(x-\xi)^2+(f(x)-\eta)^2}\,d\xi\,d\eta. \tag{3.11}$$

As is easily verified,

(3.12)

$$\frac{dA_\rho(x)}{dx} = \iint\limits_{D\setminus B_\rho(x)} \frac{-(x-\xi)^2+(f(x)-\eta)^2-2(f(x)-\eta)(x-\xi)f'(x)}{(x-\xi)^2+(f(x)-\eta)^2}\,d\xi\,d\eta$$

$$+\iint\limits_{\partial B_\rho(x)\cap D} \frac{x-\xi}{(x-\xi)^2+(f(x)-\eta)^2} N\cdot e_1\,dS_{\xi\eta} \equiv I_{1\rho}(x)+I_{2\rho}(x)$$

where N is the exterior normal to $\partial B_\rho(x)$, and e_1 is the unit vector in the direction of the x-axis. Writing

$$(\xi,\eta) = (x, f(x)) + \rho(\omega_1, \omega_2) \qquad (\omega_1^2+\omega_2^2=1)$$

in $I_{2\rho}$, we get

$$\text{(3.13)} \qquad I_{2\rho} = \int\limits_{\partial B_1(x)\cap\left\{\frac{D-(x,f(x))}{\rho}\right\}} \frac{\omega_1(N\cdot e_1)}{|\omega|^2}\,d\omega$$

$$= -\int\limits_{\partial B_1(x)\cap\{\omega_2-f(x)<f'(x)(\omega_1-x)\}} \frac{\omega_1^2}{|\omega|^2}\,d\omega + \int\limits_S \frac{\omega_1^2}{|\omega|^2}\,d\omega$$

where

$$S=\partial B_1(x)\cap\left\{\frac{D-(x,f(x))}{\rho}\right\}\Delta\{\omega_2-f(x)<f'(x)\;(\omega_1-x)\}$$

and $A\,\Delta\,B=(A\setminus B)\cup(B\setminus A)$. The first integral on the right-hand side of (3.13) is equal to $-\frac{\pi}{2}$. Since the set S is contained in the set of points (ξ,η) such that

$$|\eta-f(x)-f'(x)\;(\xi-x)| \leqq \frac{\theta}{\mu^\alpha}|\xi-x|^{1+\alpha},$$

we deduce from (3.13) that

$$\text{(3.14)} \qquad |I_{2\rho}+\tfrac{\pi}{2}| \leqq C\frac{\theta}{\mu^\alpha}\rho^\alpha.$$

To estimate $I_{1\rho}$ assume first that $f'(x)=0$ and replace $x-\xi$ by $-\xi$ and $f(x)-\eta$ by $-\eta$. Then

$$I_{1\rho} = \iint\limits_{D\setminus B_\rho(0)} \frac{-\xi^2+\eta^2}{(\xi^2+\eta^2)^2}\,d\xi\,d\eta.$$

For $\mu > \rho > \sigma > 0$ we have

$$
\begin{aligned}
&I_{1\sigma} - I_{1\rho} \\
&= \iint\limits_{[B_\rho(0)\setminus B_\sigma(0)]\cap D} \frac{-\xi^2+\eta^2}{(\xi^2+\eta^2)^2}\, d\xi\, d\eta = \iint\limits_{[B_\rho(0)\setminus B_\sigma(0)]\cap\{\eta<0\}} \frac{-\xi^2+\eta^2}{(\xi^2+\eta^2)^2}\, d\xi\, d\eta \\
&\quad + \left\{ \iint\limits_{[B_\rho(0)\setminus B_\sigma(0)]\cap D} \frac{-\xi^2+\eta^2}{(\xi^2+\eta^2)^2}\, d\xi\, d\eta - \iint\limits_{[B_\rho(0)\setminus B_\sigma(0)]\cap\{\eta<0\}} \frac{-\xi^2+\eta^2}{(\xi^2+\eta^2)^2}\, d\xi\, d\eta \right\} \\
&\equiv J_1 + J_2 .
\end{aligned}
$$

Clearly $J_1 = 0$. In J_2 the symmetric difference between the two domains of integration is contained in the set of points (ξ, η) such that

$$|\eta| \leqq \frac{\theta}{\mu^\alpha} |\xi|^{1+\alpha}$$

(recall our assumption that f' vanishes at x). Hence

$$|J_2| \leqq C \int\limits_\sigma^\rho r\, dr \int\limits_0^{C\theta/\mu^\alpha} \frac{d\varphi}{r^2} = C \frac{\theta}{\mu^\alpha} (\rho^\alpha - \sigma^\alpha)$$

where (r, φ) are polar coordinates, and thus

$$|I_{1\rho} - I_{1\sigma}| \leqq C \frac{\theta}{\mu^\alpha} (\rho^\alpha - \sigma^\alpha) . \tag{3.15}$$

It follows that $\{I_{1\rho}\}$ forms a Cauchy sequence and therefore it has a limit, say L; further,

$$|I_{1\rho} - L| \leqq \frac{C\theta}{\mu^\alpha} \rho^\alpha . \tag{3.16}$$

So far we have assumed that $f'(x) = 0$. If $f'(x) \neq 0$, then we obtain another term

$$C|f'(x)|\ (\rho^\alpha - \sigma^\alpha)$$

on the right-hand side of (3.15), and since $|f'(x)| < 1$, inequalities (3.15) and (3.16) remain valid with another constant C.

From what we have proved so far it follows that

(3.17)

$$
\begin{aligned}
\frac{dA(x)}{dx} &= -\tfrac{\pi}{2} \\
&\quad + \lim_{\rho\to 0} \iint\limits_{D\setminus B_\rho(x)} \frac{-(x-\xi)^2 + (f(x)-\eta)^2 - 2(f(x)-\eta)(x-\xi)f'(x)}{(x-\xi)^2 + (f(x)-\eta)^2}\, d\xi\, d\eta \\
&= -\tfrac{\pi}{2} + \iint\limits_{D\setminus B_\mu(x)} \cdots + \lim_{\rho\to 0} \iint\limits_{[B_\mu(x)\setminus B_\rho(x)]\cap D} \cdots
\end{aligned}
$$

where the limit exists, and the last term is actually bounded by

$$\lim_{\rho\to 0}|I_{1\rho} - I_{1\mu}| = C\theta,$$

by (3.15). Since

$$\left|\iint\limits_{D\setminus B_\mu(x)} \cdots\right| \leqq \iint\limits_{D\setminus B_\mu(x)} \frac{C}{r^2}\, r\, dr\, d\varphi \leqq C|\log\mu|,$$

it follows that $|dA(x)/dx| \leqq C(|\log\mu| + \theta)$.

We next consider $E_\rho(x)$. We can write

$$\frac{dE_\rho(x)}{dx} = \iint\limits_{D\setminus B_\rho(x)} \frac{f'(x)(x-\xi)^2 - 2(x-\xi)(f(x)-\eta) - (f(x)-\eta)^2 f'(x)}{(x-\xi)^2 + (f(x)-\eta)^2}\, d\xi\, d\eta$$

$$+ \iint\limits_{\partial B_\rho(x)\cap D} \frac{(f(x)-\eta)\, N\cdot e_1}{(x-\xi)^2 + (f(x)-\eta)^2}\, dS_{\xi\eta} \equiv J_{1\rho} + J_{2\rho}.$$

We can proceed as in the case of $A_\rho(x)$; the only difference is in evaluating $\lim J_{2\rho}$:

$$\lim_{\rho\to 0} J_{2\rho} = \iint\limits_{\partial B_1(x)\cap\{\omega_2 - f(x) < f'(x)(\omega_1 - x)\}} \frac{\omega_2\omega_1}{|\omega|^2}\, d\omega_1\, d\omega_2 = 0.$$

We obtain, analogously to (3.17),

(3.18)

$$\frac{dE(x)}{dx} = \lim_{\rho\to 0} \iint\limits_{D\setminus B_\rho(x)} \frac{f'(x)(x-\xi)^2 - 2(x-\xi)(f(x)-\eta) - (f(x)-\eta)^2 f'(x)}{(x-\xi)^2 + (f(x)-\eta)^2}\, d\xi\, d\eta$$

where the last limit in fact exists; furthermore, $|dE/dx|$ is bounded by $C(|\log\mu| + \theta)$.

We now proceed to estimate the α-Hölder coefficient of dA/dx. Introducing the function

(3.19)

$$\Phi(x,\xi,\eta) = \frac{-(x-\xi)^2 + (f(x)-\eta)^2 - 2(f(x)-\eta)(x-\xi)f'(x)}{(x-\xi)^2 + (f(x)-\eta)^2}$$

$$= \frac{-(x-\xi)^2 + (f(x)-\eta)^2}{(x-\xi)^2 + (f(x)-\eta)^2} - \frac{2(f(x)-\eta)(x-\xi)f'(x)}{(x-\xi)^2 + (f(x)-\eta)^2}$$

we can write

$$\frac{dA_\rho(x)}{dx} \iint\limits_{[B_\mu(x_0)\setminus B_\rho(x)]\cap D} \Phi(x, \xi, \eta)\, d\xi\, d\eta + \iint\limits_{D\setminus B_\mu(x_0)} \Phi(x, \xi, \eta)\, d\xi\, d\eta \tag{3.20}$$

$$+ \int\limits_{\partial B_\rho(x)\cap D} \frac{x-\xi}{(x-\xi)^2 + (f(x) - \eta)^2} N \cdot e_1\, dS_{\xi\eta}$$

$$\equiv E_1(x, \rho, \mu) + E_2(x, \rho, \mu) + E_3(x, \rho),$$

where we shall take $0 < \rho < \mu/8$.

We begin by estimating

$$L = \frac{1}{|x - \bar{x}|^\alpha} |E_1(x, \rho, \mu) - E_1(\bar{x}, \rho, \mu)|, \tag{3.21}$$

distinguishing two cases:

Case (i): $[(x-\bar{x})^2 + (f(x) - f(\bar{x}))^2]^{1/2} \leqq \rho/4$,

Case (ii): $[(x-\bar{x})^2 + (f(x) - f(\bar{x}))^2]^{1/2} \geqq \rho/4$.

By (3.19) and the mean value theorem,

$$|\Phi(x, \xi, \eta) - \Phi(\bar{x}, \xi, \eta)|$$

$$\leqq \sup \left| \frac{d}{dx} \left(\frac{-(x-\xi)^2 + (f(x) - \eta)^2}{(x-\xi)^2 + (f(x) - \eta)^2} \right) \right| |x - \bar{x}|$$

$$+ \sup \left| \frac{2(f(x) - \eta)(x - \xi)}{(x-\xi)^2 + (f(x) - \eta)^2} \right| |f'(x) - f'(\bar{x})|$$

$$+ \sup |f'(x)| \cdot \sup \left| \frac{d}{dx} \left(\frac{2(f(x) - \eta)(x - \xi)}{(x-\xi)^2 + (f(x) - \ )^2} \right) \right| |x - \bar{x}|$$

where $\sup |h(x)|$ here means the sup of $|h(\tilde{x})|$ when $\tilde{x}$ varies over the interval with endpoints $x, \bar{x}$. Setting

$$R = [(x-\xi)^2 + (f(x) - \eta)^2]^{1/2}$$

and using the assumption of case (i), we easily find that

$$|\Phi(x, \xi, \eta) - \Phi(\bar{x}, \xi, \eta)| \leqq \frac{C}{R^3} |x - \bar{x}| + \frac{C\theta}{\mu^\alpha R^2} |x - \bar{x}|^\alpha \tag{3.22}$$

$$\leqq \left(\frac{C\rho^{1-\alpha}}{R^3} + \frac{C\theta}{\mu^\alpha R^2} \right) |x - \bar{x}|^\alpha,$$

since $|x - \bar{x}| < \rho < R$.

For $x \in B_{\mu/4}(x_0)$ we have

$$\iint\limits_{[B_{\mu/4}(x)\setminus B_\rho(x)]\cap\{\eta - f(x) < f'(x)(x-\xi)\}} \Phi(x, \xi, \eta)\, d\xi\, d\eta = 0.$$

Therefore

$$E_1(x, \rho, \mu) = \iint\limits_{[B_\mu(x_0)\setminus B_{\mu/4}(x)]\cap D} \Phi(x, \xi, \eta)\, d\xi\, d\eta + \iint\limits_{S_\rho(x)} \Phi(x, \xi, \eta)\, d\xi\, d\eta \equiv J_1(x) + J_2(x) \tag{3.23}$$

where

$S_\rho(x) = \{[B_{\mu/4}(x)\setminus B_\rho(x)] \cap D\} \triangle \{[B_{\mu/4}(x)\setminus B_\rho(x)] \cap \{\eta - f(x) < f'(x)(x-\xi)\}\}.$

Set

$$\Omega_1 = [B_\mu(x_0)\setminus B_{\mu/4}(x)] \cap D, \quad \Omega_2 = [B_\mu(x_0)\setminus B_{\mu/4}(\bar{x})] \cap D.$$

To estimate $J_1(x) - J_1(\bar{x})$ we use the estimate (3.22) if $(\xi, \eta) \in \Omega_1 \cap \Omega_2$; if $(\xi, \eta) \notin \Omega_1 \cap \Omega_2$, then we simply use the estimate

$$|\Phi(x, \xi, \eta)| \leqq \frac{C}{R^2}$$

and a similar estimate for $\Phi(\bar{x}, \xi, \eta)$ (with $\bar{R}$). We get

$$\frac{1}{|x-\bar{x}|^\alpha} |J_1(x) - J_1(\bar{x})| \leqq C \iint\limits_{B_\mu(x_0)\setminus B_{\mu/8}(x_0)} \left(\frac{\rho^{1-\alpha}}{R^3} + \frac{\theta}{\mu^\alpha R^2}\right) d\xi\, d\eta + \frac{C}{|x-\bar{x}|^\alpha} \iint\limits_{\Omega_1 \triangle \Omega_2} \left(\frac{1}{R^2} + \frac{1}{\bar{R}^2}\right) d\xi\, d\eta.$$

The first integral is bounded by

$$\frac{C\rho^{1-\alpha}}{\mu^\alpha} + \frac{C\theta}{\mu^\alpha}.$$

In the second integral r and $\bar{R}$ are $\approx \mu$ and the domain of integration has area $O(\mu|x-\bar{x}|)$. Hence the integral is bounded by $C|x-\bar{x}|/\mu$. We conclude that

$$\frac{1}{|x-\bar{x}|^\alpha} |J_1(x) - J_1(\bar{x})| \leqq \frac{C}{\mu^\alpha} + \frac{C\theta}{\mu^\alpha}. \tag{3.24}$$

To estimate $|J_2(x) - J_2(\bar{x})|$ we first examine the difference of the sets $S_\rho(x)$ and $S_\rho(\bar{x})$. These sets have the form $S_\rho(x) = \Omega_1 \cap \tilde{\Omega}_1$ and $S_\rho(\bar{x}) = \Omega_2 \cap \tilde{\Omega}_2$ where

$$\tilde{\Omega}_1 = [B_{\mu/4}(x)\setminus B_\rho(x)] \cap \{\eta - f(x) \leqq f'(x)\,(x-\xi)\},$$

$$\tilde{\Omega}_2 = [B_{\mu/4}(\bar{x})\setminus B_\rho(\bar{x})] \cap \{\eta - f(\bar{x}) < f'(\bar{x})\,(\bar{x}) - \xi)\}.$$

We can write

$$
(3.25)\quad |J_2(x) - J_2(\bar{x})| \leqq \iint\limits_{S_\rho(x)\cap S_\rho(\bar{x})} |\Phi(x, \xi, \eta) - \Phi(\bar{x}, \xi, \eta)| + \iint\limits_{S_\rho(x)\cap S_\rho(\tilde{x})} (|\Phi(x, \xi, \eta)| + |\Phi(\bar{x}, \xi, \eta)|).
$$

Using (3.22) we find that the first integral is bounded by

$$
C|x-\bar{x}|^\alpha \int_\rho^{\mu/4} R\, dR \int_0^{\theta R^\alpha/\mu^\alpha} \left(\frac{\rho^{1-\alpha}}{R^3} + \frac{\theta}{\mu^\alpha R^2}\right) d\varphi
$$

$$
\leqq C|x-\bar{x}|^\alpha \left(\frac{1}{\mu^\alpha} + \frac{\theta}{\mu^\alpha}\right).
$$

In the second integral the integrand is bounded by C/R^2. Furthermore, the domain of integration, $(\Omega_1 \cap \tilde{\Omega}_1) \mathbin{\triangle} (\Omega_2 \cap \tilde{\Omega}_2)$, has an interior piece with $R \approx \rho$ and area $\approx |x-\bar{x}|\, \rho^{1+\alpha}$ and an exterior piece with $R \approx \mu$ and area $\approx |x - \bar{x}|\, \mu^{1+\alpha}$. Hence this integral is bounded by

$$
C|x-\bar{x}| \left(\frac{\rho^{1+\alpha}}{\rho^2} + \frac{\mu^{1+\alpha}}{\mu^2}\right) \leqq C|x-\bar{x}|^\alpha.
$$

We conclude that

$$
\frac{1}{|x-\bar{x}|^\alpha} |J_2(x) - J_2(\bar{x})| \leqq \frac{C}{\mu^\alpha} + \frac{C\theta}{\mu^\alpha}.
$$

Combining this with (3.24) and recalling (3.23), we obtain

$$
(3.26)\quad \frac{1}{|x-\bar{x}|^\alpha} |E_1(x, \rho, \mu) - E_1(\bar{x}, \rho, \mu)| \leqq \frac{C(1+\theta)}{\mu^\alpha},
$$

if case (i) holds.

Consider next case (ii) and set

$$
\lambda = 4[(x-\bar{x})^2 + (f(x) - f(\bar{x}))^2]^{1/2}.
$$

The same argument that was used to prove (3.24) shows that

$$
\frac{1}{|x-\bar{x}|^\alpha} \left| \int\limits_{[B_\mu(x_0)\setminus B_\lambda(x)]\cap D} \Phi(x, \xi, \eta)\, d\xi\, d\eta - \int\limits_{[B_\mu(x_0)\setminus B_\lambda(\bar{x})]\cap D} \Phi(\bar{x}, \xi, \eta)\, d\xi\, d\eta \right|
$$

$$
\leqq \frac{C(1+\theta)}{\mu^\alpha}.
$$

On the other hand, by the proof of (3.15),

$$\frac{1}{|x-\bar{x}|^\alpha}\left|\iint_{[B_\lambda(x)\setminus B_\rho(x)]\cap D}\Phi(x,\xi,\eta)\,d\xi\,d\eta-\iint_{[B_\lambda(\bar{x})\setminus B_\rho(\bar{x})]\cap D}\Phi(\bar{x},\xi,\eta)\,d\xi\,d\eta\right|$$

$$\leqq\frac{1}{|x-\bar{x}|^\alpha}\left|\iint_{[B_\lambda(x)\setminus B_\rho(x)]\cap D}|\Phi(x,\xi,\eta)|+\iint_{[B_\lambda(\bar{x})\setminus B_\rho(\bar{x})]\cap D}|\Phi(\bar{x},\xi,\eta)|\right|$$

$$\leqq C\frac{1}{|x-\bar{x}|^\alpha}\frac{\theta}{\mu^\alpha}\lambda^\alpha\leqq C\frac{\theta}{\mu^\alpha}.$$

We conclude that (3.26) also holds in case (ii).

From the definition of E_2 in (3.20) we see that we can apply (3.22) provided that $(x,f(x))$ and $(\bar{x},f(\bar{x}))$ belong to $B_{\mu/2}(x_0)$. We then get

$$\text{(3.27)}\quad \frac{1}{|x-\bar{x}|^\alpha}|E_2(x,\rho,\mu)-E_2(\bar{x},\rho,\mu)\leqq\int_{D\setminus B_\mu(x_0)}C\left(\frac{\mu^{1-\alpha}}{R^3}+\frac{\theta}{\mu^\alpha R^2}\right)d\xi\,d\eta$$

$$\leqq C\left(\frac{1}{\mu^\alpha}+\frac{\theta}{\mu^\alpha}|\log\mu|\right).$$

Consider finally

$$E_3(x,\rho)-E_3(\bar{x},\rho)=\int_{\partial B_\rho(x)\cap D}G(x,\xi,\eta)\,dS_{\xi\eta}-\int_{\partial B_\rho(\bar{x})\cap D}G(\bar{x},\xi,\eta)\,dS_{\xi\eta}$$

where

$$G(x,\xi,\eta)=\frac{x-\xi}{(x-\xi)^2+(f(x)-\eta)^2}N\cdot e_1=-\frac{(x-\xi)^2}{[(x-\xi)^2+(f(x)-\eta)^2]^{3/2}}$$

in the first integral, and $G(\bar{x},\xi,\eta)$ is similarly defined in the second integral. Notice that

$$\text{(3.28)}\qquad \int_{\partial B_\rho(x)\cap\{\eta-f(x)<f'(x)(\xi-x)\}}G(x,\xi,\eta)\,dS_{\xi\eta}=-\tfrac{\pi}{4}.$$

Also, with $v=(v_1,v_2)=(\bar{x}-x,f(\bar{x})-f(x))$,

$$E_3(x,\rho)-E_3(\bar{x},\rho)=\int_{\partial B_\rho(x)\cap D(t)}G(x,\xi,\eta)\,dS_{\xi\eta}$$

$$-\int_{\partial B_\rho(x)\cap\{D(t)-v\}}G(\bar{x},\xi+v_1,\eta+v_2)\,dS_{\xi\eta}.$$

Since, as is immediately seen,

$$G(\bar{x},\xi+v_1,\eta+v_2)=G(x,\xi,\eta),$$

we deduce, after using (3.28), that

$$|E_3(x,\rho)-E_3(\bar{x},\rho)|\leqq\int_{\partial B_\rho(x)\cap\Omega\cap\{\eta-f(x)<f'(x)(\xi-x)\}}|G(x,\xi,\eta)|\,dS_{\xi\eta}$$

where Ω is the symmetric difference of the sets $D(t)$ and $D(t) - v$. In the last integral, the integrand is bounded by C/ρ and the domain of integration has measure

$$\leqq \frac{C\theta}{\mu^{\alpha}} |x - \bar{x}|^{\alpha} \rho .$$

Hence

$$\frac{1}{|x - \bar{x}|^{\alpha}} |E_3(x, \rho) - E_3(\bar{x}, \rho)| \leqq \frac{C\theta}{\mu^{\alpha}} . \tag{3.29}$$

Combining (3.26), (3.27), (3.29) and using (3.20), we deduce the Hölder estimate (3.9) for dA/dx. The same analysis can be used to derive the Hölder estimate (3.9) for dE/dx.

4. Outline of the proof of the main result

To prove Theorem 1.1 we shall proceed as follows:
Choose a family of curves

$$\begin{gathered} \tilde{\Gamma}(t) : x = \tilde{x}(t, \lambda), y = \tilde{y}(t, \lambda), \quad 0 \leqq t \leqq t_0, \\ \tilde{x}(0, \lambda) = \lambda, \quad \tilde{y}(0, \lambda) = 0, \quad -\lambda_0(t) < \lambda < \lambda_0(t), \\ \tilde{y}(t, \lambda) = \tilde{y}(t, -\lambda), \quad \tilde{x}(t, \lambda) = -\tilde{x}(t, -\lambda) \end{gathered} \tag{4.1}$$

with $\lambda_0(0) = \lambda$; $\lambda_0(t) < 1$ if $t > 0$; $\tilde{y}(t, \lambda) > 0$ if $|\lambda| < \lambda_0(t)$, $t > 0$; and $\tilde{y}(t, \lambda_0(t)) = 0$. Denote by $\tilde{D}(t)$ the region bounded by $\tilde{\Gamma}(t)$ and the x-axis.

Recall that the function $\bar{u}(x, y)$ $(= u(x, y, 0))$ studied in § 1 has the form (2.9) where $\Phi(x, y)$ is defined in (2.7), and

$$\begin{gathered} \Delta \psi = 0 \quad \text{in } R, \\ \psi(x, b) = M - \Phi(x, b), \quad \psi(x, -h) = -\Phi(x, -h), \\ \psi_x(\pm a, y) = -\Phi_x(\pm a, y) . \end{gathered} \tag{4.2}$$

Let w be the solution of

$$\begin{gathered} \Delta w = \chi_{\tilde{D}(t)}(x, y) \quad \text{in } R, \\ w(x, b) = 0, \quad w(x, -h) = 0, \quad w_x(\pm a, y) = 0 . \end{gathered} \tag{4.3}$$

We can write

$$w(x, y, t) = G(x, y, t) + \tilde{\zeta}(x, y, t) \tag{4.4}$$

where

$$G(x, y, t) = \frac{1}{2\pi} \iint_{\tilde{D}(t)} \log [(x-\xi)^2 + (y-\eta)^2]^{1/2} \, d\xi \, d\eta, \tag{4.5}$$

$$\Delta\tilde{\zeta} = 0 \quad \text{in } R,$$

$$\tilde{\zeta}(x, b, t) = -G(x, b, t), \quad \tilde{\zeta}(x, -h, t) = -G(x, -h, t), \tag{4.6}$$

$$\tilde{\zeta}_x(\pm a, y, t) = -G_x(\pm a, y, t).$$

Set

$$u = \bar{u} + w = \Phi + \psi + G + \tilde{\zeta} \tag{4.7}$$

and consider the differential equation (1.10).

In the following sections it will be shown that this system has a solution $x = x(t, \lambda)$, $y = y(t, \lambda)$ for $0 \leqq t \leqq t_0$; it determines a family of curves $\Gamma(t)$ as in (1.8) and a family of sets $D(t)$:

$$D(t) \text{ is bounded by } \Gamma(t) \text{ and } \{y = 0\}. \tag{4.8}$$

Our plan is to show that the mapping from $\{\tilde{D}(t)\}$ to $\{D(t)\}$ has a unique fixed point. The solution $u = u(x, y, t)$ corresponding to this fixed point, together with the corresponding family $\Gamma(t)$, forms a solution to the evolutionary toner problem, i.e., to (1.1)–(1.10).

The existence proof also establishes an asymptotic behavior of the curves $\Gamma(t)$ near the critial points $(\pm 1, 0)$, for $t \to 0$.

5. The ordinary differential equation

We assume that the functions $\tilde{x}(t, \lambda)$, $\tilde{y}(t, \lambda)$ are defined in

$$Q_{t_0} = \{(t, \lambda);\ -\lambda_0(t) < \lambda < \lambda_0(t),\ 0 < t < \min(t_0, \tilde{t}(\lambda))\}$$

for some $t_0 > 0$, and satisfy the following conditions:

$$\begin{gathered} \tilde{x}(0, \lambda) = \lambda, \quad \tilde{y}(0, \lambda) = 0, \quad \tilde{y}(t, \lambda) > 0 \text{ if } 0 < t < \tilde{t}(\lambda), \\ \tilde{y}(\tilde{t}(\lambda), \lambda) = \tilde{y}(t, \lambda_0(t)) = 0, \\ \tilde{x}(t, -\lambda) = -\tilde{x}(t, \lambda), \quad \tilde{y}(t, -\lambda) = \tilde{y}(t, -\lambda), \end{gathered} \tag{5.1}$$

$$\begin{gathered} \lambda \to \tilde{y}(t, \lambda) \text{ and } \lambda \to \tilde{y}(t, \lambda) \text{ belong to } C^{1+\alpha} \text{ for each } t \in [0, t_0], \\ \frac{1}{2} < \frac{\partial}{\partial \lambda} \tilde{x}(t, \lambda) < 2. \end{gathered} \tag{5.2}$$

Consequently, the inverse function $\lambda = \tilde{\lambda}(x, t)$ of $x = \tilde{x}(t, \lambda)$ is well defined. If we set

$$y = \tilde{y}(t, \lambda) = \tilde{y}(t, \tilde{\lambda}(x, t)) \equiv f(x, t),$$

the function $f(x, t)$ is then defined for $-x_0(t) < x < x_0(t)$, for some $x_0(t) > 1$, and

$$f(x, t) = f(-x, t), \tag{5.3}$$

$$f(x, t) \begin{cases} >0 & \text{if } -x_0(t) < x < x_0(t), \\ =0 & \text{if } |x| = x_0(t), \end{cases} \tag{5.4}$$

$$1 + Q_1 t |\log t| < x_0(t) < 1 + Q_2 t |\log t| \qquad (Q_2 > Q_1 > 0), \tag{5.5}$$

(5.6) the curve $\{(x, f(x, t))\}$ lies between the polygonal lines $l_1(t)$, $l_2(t)$ where $l_i(t)$ has vertices $(-1 - Q_i t |\log t|, 0)$, $(-1, \gamma_i t)$, $(1, \gamma_i t)$, $(1 + Q_i t |\log t|, 0)$

with $\gamma_2 > \gamma_1 > 0$,

$$|f_x(x, t)| < \frac{L_1}{|\log t|} \quad \text{if } |x| \leqq x_0(t), \tag{5.7}$$

$$f_x(x, t) > \frac{L_0}{|\log t|} \quad \text{if } 1 < x < x_0(t) \tag{5.8}$$

where $0 < L_0 < L_1$, and

$$\frac{|f_x(x, t) - f_x(\bar{x}, t)|}{|x - \bar{x}|^\alpha} \leqq \frac{L_2}{\min[f(x, t), f(\bar{x}, t)]^\alpha} \quad \text{if } -x_0(t) < x, \bar{x} < x_0(t) \tag{5.9}$$

where $L_2 > 0$, $0 < \alpha < \frac{1}{2}$.

Set

$$W = (Q_1, Q_2, L_0, L_1, L_2, \alpha)$$

and denote by $\mathscr{A}_W$ the class of functions $(\tilde{x}, \tilde{y})$ satisfying (5.1)–(5.9).

We shall need an explicit form for the first derivatives of the function Φ defined in (2.7):

$$\frac{\partial \Phi(x, y)}{\partial x} = -\frac{\sigma}{4\pi} \int_{-1}^{1} \frac{\partial}{\partial x} \log[(x - \xi)^2 + y^2]\, d\xi \tag{5.10}$$

$$= \frac{\sigma}{4\pi} \log[(x - \xi)^2 + y^2]\,\Big|_{-1}^{1} = \frac{\sigma}{4\pi} \log \frac{(x - 1)^2 + y^2}{(x + 1)^2 + y^2},$$

$$\frac{\partial \Phi(x, y)}{\partial y} = -\frac{\sigma}{2\pi} \int_{-1}^{1} \frac{y}{(x - \xi)^2 + y^2}\, d\xi$$

$$= -\frac{\sigma}{2\pi} \left[\arctan \frac{1 - x}{y} + \arctan \frac{1 + x}{y}\right]$$

where arctan is taken to have values in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ $(\arctan(-x) = -\arctan x)$.

In view of (4.7), the ordinary differential equations (1.10) can then be written in the form

$$\text{(5.11)}\qquad \begin{aligned} \dot{x} &= -\frac{\sigma}{4\pi}\log\frac{(1-x)^2+y^2}{(1+x)^2+y^2} - \psi_x(x,y) - w_x(x,y,t)\,, \\ \dot{y} &= \frac{\sigma}{2\pi}\left\{\arctan\frac{1-x}{y} + \arctan\frac{1+x}{y}\right\} - \psi_y(x,y) - w_y(x,y,t)\,. \end{aligned}$$

From (4.5) we have

$$|\nabla G(x,y,t)| \leqq C \iint\limits_{\tilde{D}(t)} \frac{d\xi\, d\eta}{R} \leqq \left(\iint\limits_{\tilde{D}(t)} d\xi\, d\eta\right)^{1/p'} \left(\iint\limits_{\tilde{D}(t)} \frac{d\xi\, d\eta}{R^p}\right)^{1/p}$$

where $1/p + 1/p' = 1$. Since $|\tilde{D}(t)| \leqq Ct$ and since the last integral is bounded if $1 < p < 2$, we get

$$|\nabla G(x,y,t)| \leqq C_\delta t^{\frac{1}{2}-\delta} \qquad \forall\, \delta > 0\,.$$

The same bound holds also for $G(x, y, t)$ and thus, by the definition of $\tilde{\zeta}$ in (4.6), also for $\nabla\tilde{\zeta}$. We conclude that

$$\text{(5.12)}\qquad |\nabla w(x,y,t)| \leqq C_\delta t^{\frac{1}{2}-\delta}\,.$$

Set

$$\text{(5.13)}\qquad B = \psi_y(1,0)\,.$$

We wish to consider the behavior of the solution of the ordinary differential equations (5.11) near the point (1, 0). For simplicity we set $\xi = x - 1$. Then, by (5.12), (5.13),

$$\text{(5.14)}\qquad \begin{aligned} \dot{\xi} &= -\frac{\sigma}{4\pi}\log(\xi^2+y^2) + o(1)\,, \\ \dot{y} &= \frac{\sigma}{2\pi}\left\{-\arctan\frac{\xi}{y} + \arctan\frac{2+\xi}{y}\right\} - B + o(1) \\ &= \frac{\sigma}{2\pi}\left\{-\arctan\frac{\xi}{y} + \frac{\pi}{2}\right\} - B + o(1) \end{aligned}$$

as $t \to 0$, $(\xi, y) \to (0, 0)$.

By Theorem 2.1,

$$\text{(5.15)}\qquad B > 0, \quad \frac{\sigma}{2} - B > 0\,.$$

We want to approximate (5.14) by the system

$$\dot{\xi} = -\frac{\sigma}{4\pi}\log \xi^2, \tag{5.16}$$

$$\dot{y} = \begin{cases} \dfrac{\sigma}{2} - B & \text{if } \xi < 0, \\ -B & \text{if } \xi > 0. \end{cases} \tag{5.17}$$

If we set

$$F(z) = \int_0^z \frac{d\xi}{|\log \xi^2|}, \tag{5.18}$$

the solution to (5.16) with $\xi(0) = \xi_0$ is given by

$$F(\xi(t)) - F(\xi_0) = \frac{\sigma}{4\pi} t. \tag{5.19}$$

In the next section we shall use this approximate solution to construct barriers (or invariant domains) for solutions of the complete system (5.11), and then derive a simple but sufficiently effective approximation to the solution of the system.

6. Construction of barriers

We shall prove that the solution of (5.11) with $\xi = x - 1$, $\xi(0) = \xi_0 < 0$, $|\xi_0|$ small, must remain in the region R_{ξ_0} indicated in Figure 3.

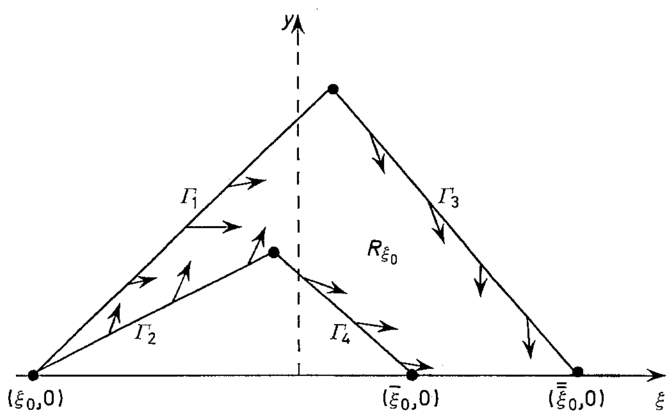

Figure 3

We need to construct upper barriers Γ_1 and Γ_3 and lower barriers Γ_2 and Γ_4. From (5.17) we see that

$$y = \left(\frac{\sigma}{2} - B\right) t + \gamma \qquad (\gamma \text{ constant})$$

for $\xi(t) < 0$. Combining this with (5.19) we get the solution

$$\lambda_0 y = F(\xi) + \gamma_0, \quad \lambda_0 = \frac{\sigma}{2\pi} \cdot \frac{1}{\sigma - 2B} \tag{6.1}$$

where γ_0 is a constant. We now construct upper and lower barriers Γ_1, Γ_2 having a form similar to (6.1).

We begin with an upper barrier Γ_1 defined as

$$\begin{gathered} \Gamma_1 : y = \frac{1}{\lambda} [F(\xi - \varepsilon\xi_0) - F(\xi_0 - \varepsilon\xi_0)], \qquad \xi_0 \leqq \xi \leqq \varepsilon|\xi_0| \\ \text{where } \lambda = \frac{\sigma}{2\pi} \cdot \frac{1}{\sigma - 2B} - \varepsilon' > 0, \end{gathered} \tag{6.2}$$

for any small $\varepsilon' > 0$ and for sufficiently small $\varepsilon > 0$.

We need to show that the vector field along Γ_1 as determined by (5.14) has smaller slope than Γ_1. The slope of this vector field is

$$\frac{dy}{d\xi} = \frac{\dfrac{\sigma}{2\pi}\left\{-\arctan\dfrac{\xi}{y} + \arctan\dfrac{2+\xi}{y}\right\} - B + o(1)}{-\dfrac{\sigma}{4\pi}\log\left\{\xi^2 + \dfrac{1}{\lambda^2}[F(\xi - \varepsilon\xi_0) - F(\xi_0 - \varepsilon\xi_0)]^2\right\}}$$

$$< \frac{4\pi}{\sigma}\left(\frac{\sigma}{2} - B + o(1)\right) \frac{1}{\left|\log\left\{\xi^2 + \dfrac{1}{\lambda^2}[F(\xi - \varepsilon\xi_0) - F(\xi_0 - \varepsilon\xi_0)]^2\right\}\right|}$$

$$< \frac{1}{\lambda} \frac{1}{|\log(\xi - \varepsilon\xi_0)^2|} = \text{slope of } \Gamma_1 \qquad (\text{by } (6.2)).$$

The last inequality is a consequence of

$$\xi^2 + \frac{1}{\lambda^2}[F(\xi - \varepsilon\xi_0) - F(\xi_0 - \varepsilon\xi_0)]^2 > (\xi - \varepsilon\xi_0)^2$$

which is clearly valid if $\xi_0 \leqq \xi \leqq \varepsilon\xi_0$; if $\varepsilon\xi_0 < \xi < \varepsilon|\xi_0|$, then the second term on the left-hand side majorizes the right-hand side.

We next construct the lower barrier Γ_2:

$$\begin{gathered} \Gamma_2 : y = \frac{1}{\lambda} [F(\xi + \varepsilon|\xi_0|) - F(\xi_0 + \varepsilon|\xi_0|)], \qquad \xi_0 \leqq \xi \leqq -4\varepsilon|\xi_0|, \\ \text{where } \lambda = \frac{\sigma}{2\pi} \cdot \frac{1}{\sigma - 2B} + \varepsilon', \qquad \varepsilon' > 0. \end{gathered} \tag{6.3}$$

We compute the slope $dy/d\xi$ from (5.14), along Γ_2:

$$\frac{dy}{d\xi} = \frac{\dfrac{\sigma}{2\pi}\left\{-\arctan\left(\dfrac{\lambda\xi}{F(\xi+\varepsilon|\xi_0|)-F(\xi_0+\varepsilon|\xi_0|)}\right)+\dfrac{\pi}{2}\right\}-B+o(1)}{-\dfrac{\sigma}{4\pi}\log\left\{\xi^2+\dfrac{1}{\lambda^2}[F(\xi+\varepsilon|\xi_0|)-F(\xi_0+\varepsilon|\xi_0|)]^2\right\}}.$$

If $\xi_0 \leqq \xi \leqq -\varepsilon|\xi_0|$, then the expression in the arctan is negative and large in absolute value, if $|\xi_0| \to 0$ (ε is small but fixed). Hence the arctan $(\cdots)$ is approximately equal to $-\frac{\pi}{2}$. Also the denominator is positive and smaller than

$$(1+\eta_1)\frac{\sigma}{4\pi}|\log \xi^2| \leqq \frac{\sigma}{4\pi}(1+\eta_1)(1+\eta_2)|\log(\xi+\varepsilon|\xi_0|)^2|$$

if $\xi_0 \leqq \xi \leqq -4\varepsilon|\xi_0|$, where η_1, η_2 are positive numbers which converge to zero if $\xi_0 \to 0$. Since

$$\frac{\dfrac{\sigma}{2\pi}\pi - B}{\dfrac{\sigma}{4\pi}} > \frac{1}{\lambda},$$

we deduce that along Γ_2,

$$\frac{dy}{d\xi} > \frac{1}{\lambda}\frac{1}{|\log(\xi+\varepsilon|\xi_0|)^2|} = \text{slope of } \Gamma_2$$

if $\xi_0 \leqq \xi \leqq -4\varepsilon|\xi_0|$.

We proceed to construct the barrier Γ_3, Γ_4. From (5.17) we have that $y = -Bt + \gamma$ (with γ a constant) for $\xi(t) > 0$. Recalling (5.19) we see that

$$-\frac{\sigma}{4\pi B}y = F(\xi) + \gamma_0, \qquad \gamma_0 \text{ constant}, \tag{6.4}$$

is an approximate solution. This suggests the upper barrier

$$\begin{gathered}\Gamma_3: y = -\frac{1}{\lambda}[F(\xi-\varepsilon\xi_0) - F(\bar{\bar{\xi}}_0-\varepsilon\xi_0)] \quad \text{for } \varepsilon|\xi_0| \leqq \xi \leqq \bar{\bar{\xi}}_0 \\ \text{where } \lambda = \frac{\sigma}{4\pi B} - \varepsilon', \quad \bar{\bar{\xi}}_0 \text{ is small;}\end{gathered} \tag{6.5}$$

ε' is any small number and ε is positive and sufficiently small. To verify this we compute along Γ_3 the slope of the vector field (5.14):

$$\frac{dy}{d\xi} = \frac{\dfrac{\sigma}{2\pi}\left\{-\arctan\left(\dfrac{\xi}{-\dfrac{1}{\lambda}[F(\xi-\varepsilon\xi_0)-F(\bar{\bar{\xi}}_0-\varepsilon\xi_0)]}\right)+\dfrac{\pi}{2}\right\}-B+o(1)}{-\dfrac{\sigma}{4\pi}\log\left\{\xi^2+\dfrac{1}{\lambda^2}(F(\xi-\varepsilon\xi_0)-F(\bar{\bar{\xi}}_0-\varepsilon\xi_0-\varepsilon\xi_0)]^2\right\}}.$$

For $\xi > \varepsilon|\xi_0|$ the expression in the arctan is positive and large so that arctan $(\cdots)$ is approximately $\frac{\pi}{2}$. The denominator is negative and

$$> -(1+\eta_1)\frac{4\pi B}{\sigma}\frac{1}{|\log(\xi-\varepsilon\xi_0)|}$$

(cf. the argument in the case of Γ_2); $\eta_1 \to 0$ if $\bar{\bar{\xi}}_0 \to 0$. It follows that

$$\frac{dy}{d\xi} < -\frac{1}{\lambda}\frac{1}{|\log(\xi-\varepsilon\xi_0)|} = \text{slope of } \Gamma_3 .$$

Consider next

$$\Gamma_4 : y = -\frac{1}{\lambda}[F(\xi+\varepsilon_0) - F(\bar{\xi}_0+\varepsilon\xi_0] \quad \text{for } -\varepsilon|\xi_0| \leqq \xi \leqq \bar{\xi}_0$$
$$\text{where } \lambda = \frac{\sigma}{4\pi B} + \varepsilon' \tag{6.6}$$

with any small $\varepsilon' > 0$ and sufficiently small $\varepsilon > 0$. We can argue as in the case of Γ_1 to deduce that along Γ_4

$$\frac{dy}{d\xi} > \frac{-B + o(1)}{-\frac{\sigma}{4\pi}\log\left\{\xi^2 + \frac{1}{\lambda^2}[F(\xi+\varepsilon\xi_0) - F(\bar{\xi}_0+\varepsilon\xi_0)]^2\right\}}$$
$$> -\frac{1}{\lambda}\frac{1}{|\log(\xi+\varepsilon\xi_0)^2|} = \text{slope of } \Gamma_4;$$

hence Γ_4 is a lower barrier.

We finally have to fit endpoints of Γ_1 with Γ_3 and Γ_2 with Γ_4. This gives the approximate equations for $\bar{\bar{\xi}}_0$, $\bar{\xi}_0$:

$$(\sigma - 2B)\frac{2\pi}{\sigma}F(|\xi_0|) \approx \frac{4\pi B}{\sigma}F(\bar{\bar{\xi}}_0),$$
$$(\sigma - 2B)\frac{2\pi}{\sigma}F(|\xi_0|) \approx \frac{4\pi B}{\sigma}F(\bar{\xi}_0), \tag{6.7}$$

the ratio of the left-hand side to the right-hand side (in both expressions) goes to 1 as $\xi_0 \to 0$.

From the relation

$$\int_0^x \frac{d\xi}{\log\xi} = \frac{x}{\log x} + \int_0^x \frac{d\xi}{(\log\xi)^2}$$

we deduce that

$$F(x) \equiv \int_0^x \frac{d\xi}{|\log\xi^2|} = \frac{x}{|\log x^2|}\left(1 + O\left(\frac{1}{|\log x|}\right)\right) \quad \text{for } x \to 0. \tag{6.8}$$

Hence (6.7) implies that

$$\bar{\bar{\xi}}_0 \approx \frac{\sigma - 2B}{2B} |\xi_0|, \quad \bar{\xi}_0 \approx \frac{\sigma - 2B}{2B} |\xi_0|. \tag{6.9}$$

From the form of the barriers we conclude that the solution of (5.11), or (5.14), with $\xi(0) = \xi_0$, $y(0) = 0$ behaves approximately like a solution to (5.16), (5.17). More precisely:

Lemma 6.1. *The solution to* (5.11) *with* $\xi(0) = \xi_0 < 0$, $y(0) = 0$ *satisfies*

$$\begin{gathered} F(\xi + o(|\xi_0|)) - F(\xi_0 + o(|\xi_0|)) = \frac{\sigma}{4\pi} t, \\ y = \begin{cases} \left(\frac{\sigma}{2} - B\right) t(1 + o(1)) & \text{if } \xi < 0, \\ -Bt(1 + o(1)) + \gamma & \text{if } \xi > 0 \end{cases} \end{gathered} \tag{6.10}$$

where $o(1) \to 0$, $\frac{1}{\xi_0} o(|\xi_0|) \to 0$ *if* $\xi_0 \to 0$, *uniformly in* (ξ, y).

7. C^1 estimate of $\Gamma(t)$

By Lemma 3.1 $\nabla w(x, y, t)$ is in $C^{1+\alpha}$ on $y = f(x, t)$; it is not, however, that smooth elsewhere. Since we shall need to work with $C^{1+\alpha}$ functions of (x, y) on the right-hand sides in (5.11), we are forced to modify the function $\nabla w(x, y, t)$. Figure 4 is a visual aid in describing the modification. The internal longitudinal curve is the curve $y = f(x, t)$. Each transversal line segment l is of lenght $\varepsilon_0 t$, where ε_0 is small enough. Along each l we take a cutoff function ζ_l, $\zeta_l = 1$ on $l \cap \{y = f(x, t)\}$, and define

$$\tilde{A}(x, y, t) = w_x(x, f(x, t), t)\, \zeta_l. \tag{7.1}$$

We construct the l and ζ_l to be symmetric with respect to the y-axis, $\zeta_l = 0$ at the endpoints of l, and along the segment l the directional derivative $D_l \zeta_l$

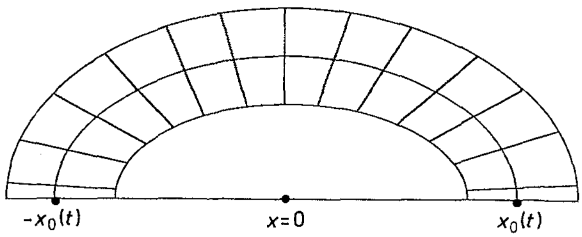

Figure 4

of ζ_l is $O(1/t)$. Then, by (5.12),

$$|D_l\tilde{A}| = O\left(\frac{1}{t^{\frac{1}{2}+\delta}}\right) \quad \text{along } l,$$

for any small $\delta > 0$. On the other hand, the derivative of $\tilde{A}$ along the curve $\Gamma(t)$ is bounded by $C(|\log y| + \theta)$, as a consequence of Lemma 3.1 with $\mu = \varepsilon_1|\log y|$, $\varepsilon_1 > 0$ and small (the assumptions (5.6)–(5.8) enable us to choose ε_0 and ε_1). The angle between l and Γ is at least $\geqq L_0/|\log t|$ by (5.8), and therefore

$$|\nabla\tilde{A}(x, y, t)| \leqq \frac{C_\delta}{t^{\frac{1}{2}+\delta}} + C|\log y|\,|\log t| \tag{7.2}$$

for any small $\delta > 0$. Similarly we define

$$\tilde{B}(x, y, t) = w_y(x, f(x, t), t)\,\zeta_l \tag{7.3}$$

and, then,

$$|\nabla\tilde{B}(x, y, t)| \leqq \frac{C_\delta}{t^{\frac{1}{2}+\delta}} + C|\log y|\,|\log t|\,. \tag{7.4}$$

The reason we choose the line segments l to be horizontal near $y = 0$ is that otherwise, the singularity in the $C^{1+\alpha}$ estimates at $(x_0(t), t)$ (which occurs as we take $\mu \to 0$ in Lemma 3.1) propagates into the entire subinterval of l lying in $\{y > 0\}$ (with one endpoint at $(x_0(t), 0)$). The factor $|\log t|$ on the right-hand sides of (7.2), (7.4), which results from this choice of l, does not cause any difficulties.

We henceforth replace (5.11) by

$$\begin{aligned} \dot{x} &= -\frac{\sigma}{4\pi}\log\frac{(1-x)^2+y^2}{(1+x)^2+y^2} - \psi_x(x, y) - \tilde{A}(x, y, t)\,, \\ \dot{y} &= \frac{\sigma}{2\pi}\left\{\arctan\frac{1-x}{y} + \arctan\frac{1+x}{y}\right\} - \psi_y(x, y) - \tilde{B}(x, y, t)\,. \end{aligned} \tag{7.5}$$

Suppose we follow the procedure outlined in § 4 and establish a fixed point, using (7.5) instead of (5.11). At the fixed point, $\tilde{x}(t, \lambda) = x(t, \lambda)$ and $\tilde{y}(t, \lambda) = y(t, \lambda)$, and therefore $\tilde{A}$ and $\tilde{B}$ coincide with w_x and w_y respectively, i.e., (7.5) reduces to (5.11). This fixed point would then be a fixed point also for the mapping based on the ordinary differential equation (5.11), and thus the existence proof would be completed.

We proceed to prove that the curves $\Gamma(t)$ belong to the class $\mathscr{A}_W$. We begin by studying the C^1 nature of these curves near $x = 1$, $y = 0$, and again resort to the change of variable $\xi = x - 1$. If we differentiate (7.5) with respect to the parameter λ and set $X = \partial\xi/\partial\lambda$, $Y = \partial y/\partial\lambda$, we obtain

$$\dot{X} = -\frac{\sigma}{2\pi}\frac{\xi}{\xi^2+y^2}X - \frac{\sigma}{2\pi}\frac{y}{\xi^2+y^2}Y$$

$$+ g_1(\xi, y)\,X + g_2(\xi, y)\,Y - \frac{\partial\tilde{A}}{\partial\xi}X - \frac{\partial\tilde{A}}{\partial y}Y,$$

(7.6)

$$\dot{Y} = -\frac{\sigma}{2\pi}\frac{y}{\xi^2+y^2}X + \frac{\sigma}{2\pi}\frac{\xi}{\xi^2+y^2}Y$$

$$+ g_3(\xi, y)\,X + g_4(\xi, y)\,Y - \frac{\partial\tilde{B}}{\partial\xi}X - \frac{\partial\tilde{B}}{\partial y}Y,$$

(7.7) $$X(0) = 1, \quad Y(0) = 0;$$

the functions g_i have uniformly bounded derivatives.

The slope of $\Gamma(t)$ is given by $W = Y/X$, as λ varies. From (7.6) we easily obtain an equation for $W(t) \equiv W(\xi(t), y(t)) \equiv W(\xi(t), y(t), \lambda)$:

(7.8)

$$\dot{W} = \left(-\frac{\sigma}{2\pi}\frac{y}{\xi^2+y^2} + g_3 - \frac{\partial\tilde{B}}{\partial\xi}\right) + \left(\frac{\sigma}{2\pi}\frac{\xi}{\xi^2+y^2} + (g_4 - g_1) + \left(\frac{\partial\tilde{A}}{\partial\xi} - \frac{\partial\tilde{B}}{\partial y}\right)\right)W$$

$$+ \left(\frac{\sigma}{2\pi}\frac{y}{\xi^2+y^2} + \frac{\partial\tilde{A}}{\partial y} - g_2\right)W^2, \quad W(0) = 0.$$

Hence

(7.9)

$$W(t) = \int_0^t \left[-\frac{\sigma}{2\pi}\frac{y(s)}{\xi^2(s)+y^2(s)} + g_3(\xi(s), y(s)) - \frac{\partial\tilde{B}}{\partial\xi}(\xi(s), y(s), s)\right]$$

$$\times\exp\left\{\int_s^t \left[\frac{\sigma}{2\pi}\frac{\xi(\tau)}{\xi^2(\tau)+y^2(\tau)} + (g_4 - g_1)\,(\xi(\tau), y(\tau))\right.\right.$$

$$\left. + \left(\frac{\partial\tilde{A}}{\partial\xi} - \frac{\partial\tilde{B}}{\partial y}\right)(\xi(\tau), y(\tau), \tau)\right] d\tau$$

$$+ \int_s^t \left[\frac{\sigma}{2\pi}\frac{y(\tau)}{\xi^2(\tau)+y^2(\tau)} + \frac{\partial\tilde{A}}{\partial y}(\xi(\tau), y(\tau), \tau) - g_2(\xi(\tau), y(\tau))\right]$$

$$\left.\times\, W(\xi(\tau), y(\tau))\right\} ds.$$

By Lemma 6.1, the trajectories of (7.5) with $|\xi_0|$ small approximately satisfy the equations (see (6.10))

$$
\begin{aligned}
&F(\xi) - F(\xi_0) = \frac{\sigma}{4\pi} t, \\
(7.10)\qquad &y = \left(\frac{\sigma}{2} - B\right) t \quad \text{if } \xi < 0, \\
&y = -Bt + \gamma \quad \text{if } \xi > 0,\ \gamma \text{ constant},
\end{aligned}
$$

where F is given by (6.8).

Choose any M large. By (7.10) (or, more precisely, by (6.10)), if $|\xi_0|$ is sufficiently small, we have:

$$
(7.11)\qquad
\begin{aligned}
&\text{If } |\xi(t)| \leqq My(t), \text{ then } |F(\xi(t))| \ll |F(\xi_0)|, \\
&\text{so that } |F(\xi_0)| \approx \frac{\sigma}{4\pi} t,\ y(t) \approx \lambda |F(\xi_0)|
\end{aligned}
$$

where λ is a positive constant λ_1 if $\xi(t) < 0$ and another positive constant λ_2 if $\xi(t) > 0$.

Lemma 7.1. *The following inequality holds:*

$$
(7.12)\qquad \int_0^t \frac{|\xi(s)|}{\xi^2(s) + y^2(s)}\, ds \leqq C \frac{\log|\log|\xi_0||}{|\log|\xi_0||}.
$$

Proof. We split the integral into integrals over $\{|\xi(s)| > y(s)\}$ and $\{|\xi(s)| \leqq y(s)\}$, and use (7.11). We obtain

$$
(7.13)\quad \int_0^t \frac{|\xi(s)|}{\xi^2(s) + y^2(s)}\, ds \leqq C \int_{\{|\xi(s)| > y(s)\}} \frac{ds}{|\xi(s)|} + \frac{C}{|F(\xi_0)|} |\{s; |\xi(s)| < y(s)\}|.
$$

Here and in the sequel, when we write integrals such as

$$
\int_{\{|\xi(s)| \geqq y(s)\}} \quad \text{or} \quad \int_{\{|\xi(s)| \leqq y(s)\}},
$$

it is always to be understood that s varies in the interval $0 \leqq s \leqq t$.

By (5.14),

$$
\dot{\xi} \approx -\frac{\sigma}{4\pi}(\xi^2 + y^2),
$$

and $\log(\xi^2 + y^2) \approx \log \xi^2$ if $y(s) \leqq |\xi(s)|$. Hence, in this range,

$$
\frac{\sigma}{4\pi} ds \approx \frac{d\xi}{|\log \xi^2|}.
$$

Also, from (6.10) we see (using (6.8)) that the range of $|\xi(s)|$ when $|\xi(s)| > y(s)$ lies in an interval $\approx [|F(\xi_0)|, c_0|\xi_0|]$, $c_0 > 0$. Hence

$$(7.14)\qquad \int\limits_{\{|\xi(s)|>y(s)\}} \frac{ds}{|\xi(s)|} \leqq C \int\limits_{c_1|\xi_0|/|\log|\xi_0||}^{c_0|\xi_0|} \frac{d\xi}{|\xi|\,|\log\xi|}$$

$$= \log \frac{\log c_0|\xi_0|}{\log c_1|\xi_0| - \log|\log|\xi_0||} \leqq C\frac{\log|\log|\xi_0||}{|\log|\xi_0||}.$$

On the complementary set $\{|\xi(s)| \leqq y(s)\}$ we have

$$\frac{4\pi}{\sigma}\dot{\xi} \approx -\log(\xi^2 + y^2) \sim -\log y^2 \sim -\log|\xi_0|$$

by (7.11), whereas $|\xi(s)| \leqq y(s) \leqq |F(\xi_0)|$. It follows that

$$(7.15)\qquad |\{s; |\xi(s)| < y(s)\}| \leqq C\frac{|F(\xi_0)|}{|\log|\xi_0||}.$$

Assertion (7.12) follows from using (7.15) and (7.14) in (7.13).

Lemma 7.2. *The following estimates hold:*

$$(7.16)\qquad \int\limits_0^t \frac{y(s)}{\xi^2(s) + y^2(s)}\,ds \leqq \frac{C}{|\log t|},$$

(7.17)

$$\int\limits_0^t \frac{y(s)}{\xi^2(s) + y^2(s)}\,ds \geqq \frac{c}{|\log t|} \quad \textit{at points } (\xi(t,\lambda), y(t,\lambda)) \textit{ with } \xi(t,\lambda) > 0,$$

where C, c are positive constants.

Proof. We first establish both (7.16), (7.17) in case $\xi(t,\lambda) > 0$. For any large $M > 0$, we write

$$(7.18)\qquad \int\limits_0^t \frac{y(s)}{\xi^2(s) + y^2(s)}\,ds = \int\limits_{\{|\xi(s)|>My(s)\}} \cdots + \int\limits_{\{|\xi(s)|\leqq My(s)\}} \cdots.$$

Then (cf. (7.14))

$$\int_{\{|\xi(s)|>My(s)\}} \frac{y(s)}{\xi^2(s)+y^2(s)}\,ds \leqq C|F(\xi_0)| \int_{\{|\xi(s)|>My(s)\}} \frac{ds}{|\xi(s)|^2}$$

$$\leqq C|F(\xi_0)| \int_{C_1M|\xi_0|/|\log|\xi_0||}^{c_0|\xi_0|} \frac{d\xi}{\xi^2|\log \xi^2|}.$$

Since

$$\int_{\varepsilon}^{1} \frac{d\xi}{\xi^2|\log \xi|} \approx \frac{1}{\varepsilon|\log \varepsilon|} \quad \text{if } \varepsilon \to 0$$

(cf. the proof of (6.8)), it follows that

$$\int_{\{|\xi(s)|>My(s)\}} \frac{y(s)}{\xi^2(s)+y^2(s)}\,ds \leqq \frac{C}{M}\frac{1}{|\log|\xi_0||}. \tag{7.19}$$

Next, by (7.11),

$$\int_{\{|\xi(s)|\leqq My(s)\}} \frac{y(s)}{\xi^2(s)+y^2(s)}\,ds \approx \tilde{C}|F(\xi_0)| \int_{\{|\xi(s)|\leqq My(s)\}} \frac{ds}{\xi^2(s)+\tilde{C}^2|F(\xi_0)|^2} \tag{7.20}$$

where $\tilde{C}$ is a constant independent of M; here "$\approx$" means that "$\leqq$" holds with constant $\tilde{C}$ and "$\geqq$" holds with another positive constant $\tilde{C}$, of course not the same. Also, by (7.11),

$$\frac{4\pi}{\sigma}\dot{\xi} = \log(\xi^2+y^2) \approx -\log(F(\xi_0))^2 \approx -\log t^2.$$

Hence

$$\int_{\{|\xi(s)|\leqq My(s)\}} \frac{y(s)}{\xi^2(s)+y^2(s)}\,ds \approx \frac{C|F(\xi_0)|}{|\log t|} \int_{\substack{0<|\xi|<\tilde{C}|F(\xi_0)| \\ \xi<0}} \frac{d\xi}{\xi^2+\tilde{C}^2(F(\xi_0))^2} \tag{7.21}$$

$$\approx \frac{C}{|\log t|}.$$

(Since $\xi(t,\lambda)>0$, the integral on the right-hand side should actually extend also to a range of ξ's with $\xi>0$; however this portion of the integral is bounded by the same integral taken only over $\xi<0$.) Using (7.21) and (7.19) in (7.18), and choosing M large enough, we obtain both (7.16) and (7.17).

Notice that the condition $\xi(t,\lambda)>0$ was implicitly used in the proof of (7.17). This condition implies that the trajectory $(\xi(s), y(s)$ for $0 \leqq s \leqq t$ contains at least the part of the full trajectory in $\{y>0\}$ which lies in $\{\xi<0\}$, and this allows us to assert that the domain of integration for the ξ-variable in (7.21) contains an interval $-\tilde{C}|F(\xi)| < \xi < 0$ and is contained in another interval $-\tilde{C}|F(\xi_0)| < \xi < \tilde{C}|F(\xi_0)|$.

If we drop the restriction that $\xi(t, \lambda) > 0$, we can only establish (7.4) with "$\leqq$" instead of "$\approx$", and this together with (7.19), (7.21), establishes the assertion (7.16).

We now proceed to estimate W from (7.9). We prove that

$$|W(t)| \leqq \frac{C}{|\log t|}, \tag{7.22}$$

$$|W(t)| > \frac{c}{|\log t|} \quad \text{at } (\xi(t, \lambda), y(t, \lambda)), \text{ if } \xi(t, \lambda) > 0. \tag{7.23}$$

Recall that $W = W(t, \lambda)$ is a function of both t and λ, and $W(0, \lambda) = 0$. Hence for any λ there exists a constant $C(\lambda)$ such that

$$|W(t, \lambda)| \leqq \frac{C(\lambda)}{|\log t|}. \tag{7.24}$$

We are considering here values of λ near $x = 1$, which correspond to values of $\xi_0 < 0$ with $|\xi_0|$ small. Although $C(\lambda)$ may possibly become unbounded as $\lambda \to 1$, the inequality (7.24) is still useful.

By (7.2), (7.4),

$$\int_0^t (|\nabla \tilde{A}| + |\nabla \tilde{B}|)\, ds \leqq Ct^{\frac{1}{2}-\delta} + C \int^{*}_{\{|\log y(s)| > 1/(|\log s|\, s^{\frac{1}{2}+\delta})\}} |\log s|\, |\log y(s)|\, ds.$$

The asterisk over the last integral indicates that we only integrate over the range of s for which $\nabla\tilde{A}$ or $\nabla\tilde{B}$ do not vanish. From the definition of $\tilde{A}$, $\tilde{B}$ it follows that, in $\int^*$, $s \approx t$. The last integral can now be estimated by substituting $y = y(s)$ and using the relation $dy/ds \approx 1$. We get the bound

$$C|\log t| \int_0^{\exp[-1/t^{\frac{1}{2}+\delta}|\log t|]} |\log y|\, dy \leqq Ct^{\frac{1}{2}-\delta}.$$

Hence

$$\int_0^t (|\nabla \tilde{A}| + |\nabla \tilde{B}|) \leqq Ct^{\frac{1}{2}-\delta}. \tag{7.25}$$

Using (7.25), the bounds $|g_i| \leqq C$ and Lemmas 7.1, 7.2, we see that the exponent in (7.9) is bounded by

$$\frac{C}{|\log t|} + \frac{C(\lambda)}{|\log t|} \frac{C}{|\log t|}.$$

If $0 \leqq t \leqq t_\lambda$, where t_λ is small enough so that $C(\lambda) \leqq |\log t|$ in (7.24), then we deduce, using (7.9) and Lemma 7.2, that

$$|W(t)| \leqq \left\{1 + \frac{C}{|\log t|}\left(1 + \frac{C(\lambda)}{|\log t|}\right)\right\}\frac{C}{\log t}. \tag{7.26}$$

We may decrease $C(\lambda)$ if necessary so that it actually satisfies

$$C(\lambda) = \sup_{0 \leqq t \leqq t_\lambda} |W(t, \lambda) \log t|.$$

Then (7.25) implies that $C(\lambda) \leqq C$ where C is independent of λ. This allows us to repeat the above argument with $t_\lambda \leqq t \leqq 2t_\lambda$, etc. We conclude that (7.22) holds for small t, say $0 \leqq t \leqq t_*$ where $t_*$is positive and independent of λ.

The proof of (7.23) follows from (7.9) upon using (7.22), (7.25) and (7.17).

So far we considered only what happens if $|\xi_0|$ is small, i.e., if λ is near 1. The same considerations apply also to the case where λ is near -1. We therefore known how the trajectories of the ordinary differential equations behave in $\{y \geqq 0\}$ when $1 - \delta_1 < |\lambda| < 1$ and $0 \leqq t \leqq t_1$, where δ_1 and t are sufficiently small positive numbers.

All the other trajectories, i.e., those with $-1 + \delta_1 \leqq \lambda \leqq 1 - \delta_1$, are very smooth if $0 \leqq t \leqq t_0$ where t_0 is a sufficiently small positive number such that all these trajectories stay in a region $-1 + \delta_0 < x < 1 - \delta_0$ for some $0 < \delta_0 < \delta_1$; here we take $0 < t_0 < t_1$. For all these trajectories

$$\frac{\partial x}{\partial \lambda} \sim 1, \quad \frac{\partial y}{\partial \lambda} \sim 0 \quad \text{if } t \to 0, \tag{7.27}$$

and the corresponding portions of the $\Gamma(t)$ are uniformly $C^{1+\alpha}$. Furthermore, by continuity,

$$|f_x(x, t)| \leqq Ct, \quad [f_x(\cdot, t)]_{0,\alpha} \leqq Ct \quad \text{if } |x| < 1 - \delta_0. \tag{7.28}$$

One can easily check that with the exception of (5.2) and the Hölder condition (5.9), the family $(x(t, \lambda), y(t, \lambda))$ satisfies all the conditions imposed on $(\tilde{x}(t, \lambda), \tilde{y}(t, \lambda))$ in §5, provided $0 \leqq t \leqq t_0$. Indeed, (5.7) and (5.8) for $(\tilde{x}, \tilde{y})$ follow from (7.22), (7.23) and (7.28), and (5.4)–(5.6) follow from Lemma 6.1. Furthermore, the new constants L_i, Q_i, t_0, are actually independent of the constants L_i, Q_i, t_0 in §5. By appropriate choice of these initial constants, we conclude that, with the exception of (5.2) and (5.9), the mapping

$$\mathscr{M}\{\tilde{x}(t, \lambda), \tilde{y}(t, \lambda)\} \to \{x(t, \lambda), y(t, \lambda)\} \tag{7.29}$$

maps $\mathscr{A}_W$ into itself.

The proof of (5.9) is given in §8. Here again it suffices to consider only the portion of $\Gamma(t)$ for which the trajectories initiate at $(\xi_0, 0)$ with $\xi_0 < 0$ and $|\xi_0|$ sufficiently small. The proof of (5.2) for $(\tilde{x}, \tilde{y})$ will follow from (7.27) and the proof of (5.9), and therefore need not be further discussed.

8. $C^{1+\alpha}$ estimate of $\Gamma(t)$

Writing $Y = XW$ in the first differential equation of (7.6), we can immediately solve for X:

$$X(t) = \exp\left\{\int_0^t \left\{-\frac{\sigma}{2\pi}\frac{\xi(s)}{\xi^2(s)+y^2(s)}\right.\right.$$

$$-\frac{\sigma}{2\pi}\frac{y(s)}{\xi^2(s)+y^2(s)} W(s) + (g_1+g_2)\,(\xi(s), y(s))$$

$$+ g_2(\xi(s), y(s))\, W(s) - \frac{\partial \tilde{A}}{\partial \xi}(\xi(s), y(s), s)$$

$$\left.\left. - \frac{\partial \tilde{A}}{\partial y}(\xi(s), y(s), s)\, W(s)\right\}\right\} ds.$$

Using (7.22) and Lemmas 7.1, 7.2 we find that $X(t) \approx 1$ if $\lambda \to 1$ (i.e., if $\xi_0 < 0$, $|\xi_0| \to 0$), that is,

$$\frac{\partial \xi(t, \lambda)}{\partial \lambda} \approx 1, \quad \text{or} \quad \xi(t, \lambda) - \xi(t, \tilde{\lambda}) \approx \lambda - \tilde{\lambda}. \tag{8.1}$$

This implies that

$$\text{if } \lambda_1 < \lambda_2, \text{ then } \xi(t, \lambda_1) < \xi(t, \lambda_2) \text{ along } \Gamma(t). \tag{8.2}$$

The right end-point $(x_0(t), t)$ of $\Gamma(t)$ corresponds to some value $\lambda = \lambda_0(t)$. From (8.2) it follows that $\Gamma(t)$ is formed precisely by the points $(x(t, \lambda), y(t, \lambda))$ with $-\lambda_0(t) < \lambda < \lambda_0(t)$.

To prove the Hölder continuity of the slope of $\Gamma(t)$, we take two values λ and $\tilde{\lambda}$ near $\lambda_0(t)$, but smaller than $\lambda_0(t)$ and their corresponding trajectories. For simplicity we set

$$\xi(t) = \xi(t, \lambda), \quad y(t) = y(t, \lambda), \quad W(t) = W(t, \lambda),$$

$$\tilde{\xi}(t) = \xi(t, \tilde{\lambda}), \quad \tilde{y}(t) = y(t, \tilde{\lambda}), \quad \tilde{W}(t) = W(t, \tilde{\lambda}).$$

Set

$$\Phi(\xi, y) = \frac{y}{\xi^2 + y^2}, \qquad G(\xi, y) = \frac{\xi}{\xi^2 + y^2}.$$

We easily compute that

$$\Phi_\xi = G_y = -\frac{2\xi y}{(\xi^2 + y^2)^2}, \qquad \Phi_y = -G_\xi = \frac{\xi^2 - y^2}{(\xi^2 + y^2)^2}. \tag{8.3}$$

From (7.8) we deduce that

$$\frac{d}{dt}\frac{W-\tilde{W}}{|\lambda-\tilde{\lambda}|^{\alpha}}=\left(\frac{\sigma}{\pi}\frac{\xi}{\xi^2+y^2}+\mu(\xi,y)\right)\frac{W-\tilde{W}}{|\lambda-\tilde{\lambda}|^{\alpha}}$$
$$+\left(\frac{\sigma}{2\pi}\frac{y}{\xi^2+y^2}+\frac{\partial\tilde{A}}{\partial y}(\xi,y,t)\right)(W+\tilde{W})\frac{W-\tilde{W}}{|\lambda-\tilde{\lambda}|^{\alpha}}+J(t)$$

where

(8.4)
$$J(t)=-\frac{\sigma}{2\pi}\frac{1}{|\lambda-\tilde{\lambda}|^{\alpha}}[\Phi(\xi,y)-\Phi(\tilde{\xi},\tilde{y})]+\frac{1}{|\lambda-\tilde{\lambda}|^{\alpha}}[g_3(\xi,y)-g_3(\tilde{\xi},\tilde{y})]$$
$$-\frac{1}{|\lambda-\tilde{\lambda}|^{\alpha}}\left[\frac{\partial\tilde{B}}{\partial\xi}(\xi,y,t)-\frac{\partial\tilde{B}}{\partial\xi}(\tilde{\xi},\tilde{y},t)\right]$$
$$+\frac{\tilde{W}}{|\lambda-\tilde{\lambda}|^{\alpha}}\left[\left(\frac{\partial\tilde{A}}{\partial\xi}-\frac{\partial\tilde{B}}{\partial y}\right)(\xi,y,t)-\left(\frac{\partial\tilde{A}}{\partial\xi}-\frac{\partial\tilde{B}}{\partial y}\right)(\tilde{\xi},\tilde{y},t)\right]$$
$$+\frac{\tilde{W}}{|\lambda-\tilde{\lambda}|^{\alpha}}\frac{\sigma}{\pi}[G(\xi,y)-G(\tilde{\xi},\tilde{y})]+\frac{\tilde{W}}{|\lambda-\tilde{\lambda}|^{\alpha}}[\mu(\xi,y)-\mu(\tilde{\xi},\tilde{y})]$$
$$+\frac{\sigma}{2\pi}\frac{\tilde{W}^2}{|\lambda-\tilde{\lambda}|^{\alpha}}[\Phi(\xi,y)-\Phi(\tilde{\xi},\tilde{y})]+\frac{\tilde{W}^2}{|\lambda-\tilde{\lambda}|^{\alpha}}\left[\frac{\partial\tilde{A}}{\partial y}(\xi,\eta)-\frac{\partial\tilde{A}}{\partial y}(\tilde{\xi},\tilde{y})\right]$$
$$-\frac{\tilde{W}^2}{|\lambda-\tilde{\lambda}|^{\alpha}}[g_2(\xi,y)-g_2(\tilde{\xi},\tilde{y})]$$

where $\mu=g_4-g_1$. Hence

(8.5)
$$\frac{W(t)-\tilde{W}(t)}{|\lambda-\tilde{\lambda}|^{\alpha}}$$
$$=\int_0^t ds\,J(s)\exp\left[\int_0^t d\tau\left\{\frac{\sigma}{2\pi}\frac{\xi}{\xi^2+y^2}+\mu(\xi,y)+\left(\frac{\sigma}{2\pi}\frac{y}{\xi^2+y^2}+\frac{\partial\tilde{A}}{\partial y}\right)(W+\tilde{W})\right\}\right].$$

From (8.1) and (7.22) we have

$$\left|\frac{\partial y(t,\lambda)}{\partial\lambda}\right|\leqq\frac{C}{|\log t|}. \tag{8.6}$$

Hence

$$|\xi-\tilde{\xi}|^{\alpha}+|y-\tilde{y}|^{\alpha}\leqq C|\lambda-\tilde{\lambda}|^{\alpha}. \tag{8.7}$$

Since g_i is in C^1, it follows that

$$\frac{1}{|\lambda-\tilde{\lambda}|^{\alpha}}|g_i(\xi,y)-g_i(\tilde{\xi},\tilde{y})|\leqq C. \tag{8.8}$$

We next show that

$$(8.9)\quad \frac{1}{|\lambda-\tilde{\lambda}|^{\alpha}}\{[|\nabla\tilde{A}(\xi,y,t)-\nabla\tilde{A}(\tilde{\xi},\tilde{y},t)|]+[|\nabla\tilde{B}(\xi,y,t)-\nabla\tilde{B}(\tilde{\xi},\tilde{y},t)|]\}$$
$$\leqq \frac{C}{t^{\frac{1}{2}+\delta+\alpha}}+\frac{C}{|\hat{y}|^{\alpha}}|\log\hat{y}||\log t|$$

where $\hat{y}=\min(y,\tilde{y})$. Indeed, from the definition of $\tilde{A}$ in (7.1) we have deduced (7.2), and similarly we can show, using Lemma 3.1, that

$$[\nabla\tilde{A}]_{0,\alpha}\leqq\frac{C}{t^{\frac{1}{2}+\delta+\alpha}}+\frac{C|\log\hat{y}||\log t|}{|\hat{y}|^{\alpha}},$$

where $[\ ]_{0,\alpha}$ stands here for the α-Hölder coefficient in (ξ,η). In view of (8.7), the assertion (8.9) for $\tilde{A}$ follows. The proof for $\tilde{B}$ is the same.

From Lemmas 7.1, 7.2 and the estimate (7.24) we deduce that the expression in the exponent in (8.9) is uniformly bounded if $|\xi_0|$ is small enough.

Notice that if $\lambda<\tilde{\lambda}$, then $\hat{y}\approx y$, and, as in the proof of (7.25),

$$\int_0^t\frac{1}{\hat{y}(s)^{\alpha}}|\log\hat{y}(s)||\log s|\,ds\quad\text{if } t\to 0.$$

Using this in (8.9), and then also using (8.8), we conclude from (8.5), (8.4) that

$$(8.10)\quad \frac{|W(t)-\tilde{W}(t)|}{|\lambda-\tilde{\lambda}|^{\alpha}}$$
$$\leqq C+\int_0^t ds\left\{C\int_0^t\left|\frac{\partial\Phi(x_r)}{\partial\xi}\right|\frac{|\xi(s)-\tilde{\xi}(s)|+|y(s)-\tilde{y}(s)|/|\log s|}{|\lambda-\tilde{\lambda}|^{\alpha}}\,dr\right.$$
$$\left.+C\int_0^t\left|\frac{\partial\Phi(x_r)}{\partial y}\right|\frac{|\xi(s)-\tilde{\xi}(s)|/|\log s|+|y(s)-\tilde{y}(s)|}{|\lambda-\tilde{\lambda}|^{\alpha}}\,dr\right\}$$

where $x_r=r\xi(s)+(1-r)\tilde{\xi}(s),\ ry(s)+(1-r)\tilde{y}(s)$; here we used the mean value theorem and (8.3) in evaluating the difference of the Φ's and G's in (8.4), and we also used the fact that $\frac{1}{2}+\delta+\alpha<1$ in (8.9) (which is valid if $0<\alpha<\frac{1}{2}$ and δ is sufficiently small).

By (8.1), (8.6),

$$\frac{|\xi(s)-\tilde{\xi}(s)|}{|\lambda-\tilde{\lambda}|^{\alpha}}\leqq C|\xi(s)-\tilde{\xi}(s)|^{1-\alpha},$$

$$\frac{|y(s)-\tilde{y}(s)|}{|\lambda-\tilde{\lambda}|^{\alpha}}\leqq\frac{C}{|\log s|^{\alpha}}|y(s)-\tilde{y}(s)|^{1-\alpha}\leqq\frac{C}{|\log s|}|\xi(s)-\tilde{\xi}(s)|^{1-\alpha}.$$

Consequently, using (8.3), we obtain

$$\frac{|W(t)-\tilde{W}(t)|}{|\lambda-\tilde{\lambda}|^{\alpha}} \tag{8.11}$$

$$\leqq C+C\int_0^t ds\left\{\int_0^1 \frac{|\xi_r y_r|}{(\xi_r^2+y_r^2)^2}\,dr+\frac{C}{|\log t|}\int_0^1 \frac{|\xi_r^2-y_r^2|}{(\xi_r^2+y_r^2)^2}\,dr\right\}|\xi(t)-\tilde{\xi}(t)|^{1-\alpha};$$

here we used the fact (which follows from (8.1)) that

$$|\xi(t)-\tilde{\xi}(t)|\approx|\xi(s)-\tilde{\xi}(s)|. \tag{8.12}$$

To prove the Hölder continuity we only need to show, in view of (8.12), that

$$\frac{|W(t)-\tilde{W}(t)|}{|\lambda-\tilde{\lambda}|^{\alpha}}\leqq\frac{C}{\min[y(t),\tilde{y}(t)]^{\alpha}}. \tag{8.13}$$

The initial point of $\Gamma(t)$ is given by $(\bar{\xi}_0(t),0)$ where $\bar{\xi}_0(t)=\lambda_0(t)-1$. Recall also that

$$\bar{\xi}_0(t)\approx t\log t. \tag{8.14}$$

To prove (8.13) we consider two cases:

Case (i): $|\xi_0(0)-\tilde{\xi}_0(0)|>\tilde{\varepsilon}|\bar{\xi}_0(t)|$,

Case (ii): $|\tilde{\xi}_0(0)-\xi_0(0)|\leqq\tilde{\varepsilon}|\bar{\xi}_0(t)|$

where $\tilde{\varepsilon}$ is a small positive number.

In case (i),

$$\frac{|W(t)-\tilde{W}(t)|}{|\lambda-\tilde{\lambda}|^{\alpha}}\leqq\frac{|W(t)|+|\tilde{W}(t)|}{(\tilde{\varepsilon}|\bar{\xi}_0(t)|)^{\alpha}}\leqq\frac{C}{|\log t|\,|t\log t|^{\alpha}} \tag{8.15}$$

by (8.14), (7.22).

In case (ii) we have, by (8.12) and the approximate equations (6.10), that

$$\xi_r(t)\approx\xi(t),\qquad y_r(t)\approx y(t)$$

in (8.11) and, therefore,

$$\frac{|W(t)-\tilde{W}(t)|}{|\lambda-\tilde{\lambda}|^{\alpha}} \tag{8.16}$$

$$\leqq C+C|\bar{\xi}_0(t)|^{1-\alpha}\left\{\int_0^t\frac{|\xi(s)\,y(s)|}{(\xi^2(s)+y^2(s))^2}\,ds+\frac{1}{|\log t|}\int_0^t\frac{|\xi^2(s)-y^2(s)|}{(\xi^2(s)+y^2(s))^2}\,ds\right\}.$$

We evaluate the integrals by the same method used to prove Lemmas 7.1, 7.2. First

$$(8.17)\quad \int_0^t \frac{|\xi(s)\,y(s)|}{(\xi^2(s)+y^2(s))^2}\,ds = \int\limits_{\{|\xi(s)|\geqq y(s)\}} \cdots + \int\limits_{\{|\xi(s)|\leqq y(s)\}} \cdots \equiv J_1 + J_2 .$$

By (7.11), (7.15),

$$(8.18)\quad J_2 \leqq C \int\limits_{\{|\xi(s)|\leqq y(s)\}} \frac{ds}{y^2(s)} \leqq \frac{C}{|F(\bar{\xi}_0(t))|^2}\,\frac{|F(\bar{\xi}_0(t))|}{|\log|\bar{\xi}_0(t)||} = \frac{C}{|\xi_0(t)|}.$$

Next, since $|y(s)| \leqq C|F(\bar{\xi}_0(t))|$,

$$J_1 \leqq C|F(\bar{\xi}_0(t))| \int\limits_{\{|\xi(s)|\geqq y(s)\}} \frac{ds}{|\xi(s)|^3} \leqq C|F(\bar{\xi}_0(t))| \int\limits_{C|\bar{\xi}_0(t)|/|\log|\bar{\xi}(t)||}^{c|\bar{\xi}_0(t)|} \frac{d\xi}{|\xi|^3\,|\log \xi|}.$$

The last integral is

$$\approx 1 \Big/ \left\{ \left(\frac{C\bar{\xi}_0(t)}{\log|\bar{\xi}_0(t)|} \right)^2 \left| \log \frac{|C\xi_0(t)|}{|\log|\bar{\xi}_0(t)||} \right| \right\}$$

(by integration by parts; cf. the proof of (6.8)) and, therefore,

$$J_1 \leqq \frac{C}{|\bar{\xi}_0(t)|}$$

Combining this with (8.18), we conclude from (8.17) that

$$(8.19)\quad \int_0^t \frac{|\xi(s)\,y(s)|}{(\xi^2(s)+y^2(s))^2}\,ds \leqq \frac{C}{|\bar{\xi}_0(t)|}.$$

Next we estimate

$$\int_0^t \frac{\xi^2}{(\xi^2(s)+y^2(s))^2}\,ds = \int\limits_{\{|\xi(s)|\geqq y(s)\}} \cdots + \int\limits_{\{|\xi(s)|\leqq y(s)\}} \equiv L_1 + L_2 .$$

$$L_1 \leqq C \int\limits_{\{|\xi(s)|\geqq y(s)\}} \frac{ds}{\xi^2(s)} \leqq C \int\limits_{C|\bar{\xi}_0(t)|/|\log|\bar{\xi}(t)||}^{c|\bar{\xi}_0(t)|} \frac{d\xi}{|\xi|^3\,|\log \xi|}$$

$$\leqq C \Big/ \left\{ \frac{|\bar{\xi}_0(t)|}{|\log|\bar{\xi}_0(t)||} \left| \log \frac{|\bar{\xi}_0(t)|}{|\log|\bar{\xi}_0(t)||} \right| \right\} \leqq \frac{C}{|\bar{\xi}_0(t)|}.$$

Also,

$$L_2 \leqq C \int_{\{|\xi(s)| \leqq y(s)\}} \frac{ds}{y^2(s)} \leqq \frac{C}{|\bar{\xi}_0(t)|}$$

by (8.18). It follows that

$$\int_0^t \frac{\xi^2(s)}{(\xi^2(s) + y^2(s))^2} \, ds \leqq \frac{C}{|\bar{\xi}_0(t)|}. \tag{8.20}$$

Finally,

$$\int_0^t \frac{y^2(s)}{(\xi^2(s) + y^2(s))^2} \, ds \leqq \frac{C}{|\bar{\xi}_0(t)|}. \tag{8.21}$$

Indeed, the left-hand side is bounded by

$$C \int_{\{|\xi(s)| \geqq y(s)\}} \frac{ds}{\xi^2(s)} + C \int_{\{|\xi(s)| \leqq y(s)\}} \frac{ds}{y^2(s)}.$$

The first integral bounded by

$$C \int_{C|\bar{\xi}_0(t)|/|\log|\bar{\xi}(t)||}^{c|\bar{\xi}_0(t)|} \frac{d\xi}{|\xi|^3 \, |\log \xi|} \leqq \frac{C}{|\bar{\xi}_0(t)|},$$

and the second integral has already been estimated in (8.18) by $C/|\bar{\xi}_0(t)|$.

Using (8.19)–(8.21) in (8.16) we find that

$$\frac{|W(t) - \tilde{W}(t)|}{|\lambda - \tilde{\lambda}|^\alpha} \leqq \frac{C}{|\bar{\xi}_0(t)|} \leqq \frac{C}{|t \log t|^\alpha}, \tag{8.22}$$

where we have used also (8.14). Recalling (8.15), we conclude that (8.22) holds in both cases. This implies (8.13), since the right-hand side of (8.13) is larger than C/t^α.

So far we have (tacitly) assumed in the above analysis that $\xi(0)$ and $\tilde{\xi}(0)$ are near 0, i.e., $x(0)$ and $\tilde{x}(0)$ are near $+1$. The same estimate holds if $x(0)$ and $\tilde{x}(0)$ are near -1. Finally if $x(0)$, $\tilde{x}(0)$ are in some interval $[-1 + \delta_1, 1 - \delta_1]$ with $\delta_1 > 0$, then the $C^{1+\alpha}$ estimate is rather immediate (see the paragraph containing (7.27)).

A review of the proof shows that the constant C in (8.13) is independent of the constant L_2 in (5.9). Hence by choosing L_2 larger than this constant C, we conclude that the mapping (7.27) maps $\mathscr{A}_W$ into itself.

Remark 8.1. Since we have proved (8.22), which is stronger than (8.13), it follows that if we replace (5.9) by

$$[f_x(\cdot, t)]_{0,\alpha} \leqq \frac{L_2}{|t \log t|^\alpha}, \tag{8.23}$$

then $\mathscr{M}$ still maps the new class $\mathscr{A}_W$ into itself; we denote this new class by $\mathscr{A}_W^0$.

9. A fixed point

We have shown that $\mathscr{M}$ maps a family $\{\tilde{x}(t, \lambda), \tilde{y}(t, \lambda)\}$ in $\mathscr{A}_W^0$ into another family $\{x(t, \lambda), y(t, \lambda)\}$ in $\mathscr{A}_W^0$, provided $0 \leqq t \leqq t_0$ where t_0 is a sufficiently small positive constant. Denote by $\tilde{\Gamma}(t)$ and $\tilde{D}(t)$ the boundaries and domains corresponding to $\{\tilde{x}(t, \lambda), \tilde{y}(t, \lambda)\}$, and define $\Gamma(t), D(t)$ similarly with respect to $\{x(t, \lambda), y(t, \lambda)\}$. Then we write

$$D(t) = \mathscr{M}\tilde{D}(t).$$

Choose any element $\{x(t, \lambda), y(t, \lambda)\}$ in $\mathscr{A}_W^0$ with the corresponding domains $D(t)$ and boundaries $\Gamma(t)$, and define the iterates

$$D^{n+1}(t) = \mathscr{M}D^n(t) \qquad (n = 1, 2, \ldots)$$

where $D^1(t) = D(t)$.

The modification of ∇w as described in connection with Figure 4 is somewhat arbitrary. We could for instance replace the transversal segments l_y (of direction γ) with concentric segments of larger length; the length being $\approx \varepsilon_0 t$ if the midpoint is in $|x| < 1$, and it grows to $\approx \varepsilon_0 t |\log t|$ as the directions of l_y become horizontal. Let us choose the cutoff functions ζ_y such that they are equal to 1 on each symmetrically situated subinterval of l_y of length $\frac{1}{2}|l_y|$. From the approximate behavior of the trajectories as described in § 6 (see Lemma 6.1) we can deduce that if t is small enough and the internal longitudinals are the curves $\Gamma(t)$ corresponding, say, to the domains $D^2(t)$, then we can carry out the modifications of w_x, w_y by (7.1), (7.3) with the *same* l_y, ζ_{l_y} for all the $D^n(t)$, $n \geqq 2$. Furthermore, for each $n \geqq 2$, each line segment l_y (or its extension) forms angle $\leqq \theta_0 < \frac{\pi}{2}$ (θ_0 constant) with the direction $(x_t(t, \lambda), y_t(t, \lambda))$ of the trajectory at the point $(x(t, \lambda), y(t, \lambda))$.

Since

$$|\nabla \zeta_l| \leqq \frac{C}{t|\log t|} \leqq \frac{\tilde{C}}{t},$$

the analysis in §§ 7, 8 remains unchanged.

Now take two domains

$$D_1(t) = D^{n_1}(t), \quad D_2(t) = D^{n_2}(t)$$

and introduce in addition to the usual Hausdorff distance $\delta_1(t) \equiv \delta_1(D_1(t), D_2(t))$ another distance function based on measuring distances along the half lines $\hat{l}_\gamma$ in $\{x \geqq 0\}$ or in $\{x \leqq 0\}$ containing the line segments l_γ:

$$\delta(t) \equiv \delta(D_1(t), D_2(t)) = \sup_\gamma \operatorname{diam}\{\hat{l}_\gamma \cap [\partial D_1(t) \cup \partial D_2(t)]\}.$$

Clearly (since $\theta_0 < \frac{\pi}{2}$),

$$\delta_1(t) \leqq \delta(t)\,. \tag{9.1}$$

Denote by $r_i = (x_i, y_i) = (x_i(t, \lambda), y_i(t))$ the trajectories defining the boundaries $\Gamma_i(t)$ of $\mathscr{M}D_i(t)$ and by $\vec{\lambda}_i = (\tilde{A}_i, \tilde{B}_i)$ the modifications of ∇w_i.

The ordinary differential equation for r_i is then

$$\dot{r}_i(t) = -\nabla\bar{u}(r_i(t)) - \vec{\lambda}_i(r_i(t), t)\,. \tag{9.2}$$

Recall also that

$$\nabla w_i(x, y, t) = -\frac{1}{2\pi} \iint\limits_{D_i(t)} \left(\frac{x-\xi}{\rho^2}, \frac{y-\eta}{\rho^2}\right) d\xi\, d\tau + \nabla\psi_i \tag{9.3}$$

where

$$\rho^2 = (x-\xi)^2 + (y-\eta)^2$$

and $\nabla\psi_i$ is a smooth function.

We can write

$$\begin{aligned} &|\tilde{A}_1(r, t) - \tilde{A}_2(r, t)| \\ &\leqq \iint\limits_{[D_1(t) \cup D_2(t)] \cap B_{c\delta}(r)} |\cdots| + \iint\limits_{[D_1(t) \cap D_2(t)] \setminus B_{c\delta}(r)} |\cdots| + \int\limits_{[D_1(t) \triangle D_2(t)] \cap B_{c\delta}(r)} |\cdots| \\ &\equiv J_1 + J_2 + J_3\,. \end{aligned} \tag{9.4}$$

Here c is any fixed large positive constant. The integrand in J_1 is bounded by C/ρ (using (9.3)). Hence

$$J_1 \leqq \iint\limits_{B_{c\bar{r}}} \frac{C}{\rho} \leqq C\delta\,. \tag{9.5}$$

In J_3 we estimate the integrand also by C/ρ and conclude that

$$J_3 \leqq C\delta_1(t)\,|\log \delta_1(t)|\,. \tag{9.6}$$

Finally, using the definition of $\vec{\lambda}_i$ above and setting $\Sigma(t) = [D_1(t) \cap D_2(t)] \setminus B_{c\delta}(r)$ we see that

$$J_2 \leqq C \iint\limits_{\Sigma(t)} \left| \frac{x_1(t) - \xi}{\rho_1^2} - \frac{x_2(t)}{\rho_2^2} \right| d\xi\, dy$$

where $(x_i(t), y_i(t)) = r_i(t)$ and $\rho_i^2 = (x_i(t) - \xi)^2 + (y_i(t) - \eta)^2$. Applying the mean value theorem we get

$$J_2 \leqq C\delta_1(t) \iint\limits_{\Sigma(t)} \frac{C}{\rho^2} \leqq C\delta_1(t)\,|\log \delta_1(t)|\,.$$

Combining this estimate with (9.6), (9.5) we get, after using (9.1),

$$|\tilde{A}_1(r,t) - \tilde{A}_2(r,t)| \leqq C\delta(t)\,|\log\delta(t)|\,. \tag{9.7}$$

We now take the difference of (9.2) with $i = 1$ and $i = 2$. After using (9.7) and the corresponding estimate for the $\tilde{B}_i$, we obtain

$$\begin{aligned} |r_1 - r_2|_t \leqq{}& |\nabla\bar{u}(r_1(t)) - \nabla\bar{u}(r_2(t))| \\ &+ |\vec{\lambda}_1(r_1(t),t) - \vec{\lambda}_1(r_2(t),t)| + C\delta(t)\,|\log\delta(t)|\,. \end{aligned} \tag{9.8}$$

By the mean value theorem and (7.2), (7.4),

$$|\vec{\lambda}_1(r_1(t),t) - \vec{\lambda}_1(r_2(t),t)| \leqq \gamma(t)\,|r_1(t) - r_2(t)|$$

where

$$\gamma(t) = \frac{1}{t^{\frac{1}{2}+\delta}} + C|\log y(t)|\,|\log t|\,, \tag{9.9}$$

and where $y(t)$ belongs to the interval $(y_1(t), y_2(t))$.

By the mean value theorem, also

$$|\nabla\bar{u}(r_1(t)) - \nabla\bar{u}(r_2(t),t)| \leqq C|\nabla^2\bar{u}|\,|r_1(t) - r_2(t)|$$

and, as easily verified,

$$|\nabla^2\bar{u}(x,y)| \leqq \frac{C}{|x| + y}\,. \tag{9.10}$$

We can then write

$$|r_1(t) - r_2(t)| \leqq C\int_0^t \delta(s)\,|\log\delta(s)|\; e^{\int_0^s[\cdots]}\,ds \tag{9.11}$$

where the expression in $[\cdots]$ is the sum of $\gamma(t)$ plus the right-hand side of (9.10) evaluated at a point in the interval $(r_1(t), r_2(t))$. By the results of § 8 ((7.22) and Lemmas 7.1, 7.2) it follows that

$$\sup_{0<s<t}\left\{\exp\int_0^s[\cdots]\right\} \to 1 \quad \text{if } t \to 0\,.$$

Hence (9.11) implies that

$$\delta(\mathscr{M}D_1(t), \mathscr{M}D_2(t)) \leqq C\int_0^t \delta(s)\,|\log\delta(s)|\,ds\,. \tag{9.12}$$

We apply (9.12) to the domain $D_1(t) = D^n(t)$, $D_2(t) = D^{n+1}(t)$. Setting

$$g_n(t) = \delta(D^n(t), D^{n+1}(t))\,,$$

we get

$$g_{n+1}(t) \leqq C\int_0^t g_n(s)\,|\log g_n(s)|\,. \tag{9.13}$$

We assert that

$$g_n(t) \leqq A^n t^n |\log t|^n. \tag{9.14}$$

Indeed, this is true for $n = 1$ if A is sufficiently large. Proceeding by induction we assume that (9.14) holds for some n and prove it for $n + 1$.

Applying (9.13) we get

$$g_{n+1}(t) \leqq CA^n \int_0^t s^n |\log s|^n \left|\log |A + \log s + \log|\log s|||\right| ds.$$

Since $\log s$ is negative whereas $A + \log|\log s|$ is positive if s is small,

$$0 < \log|A + \log s + \log|\log s|| \leqq |\log s|.$$

It follows that

$$g_{n+1}(t) \leqq CA^n \int_0^t s^n |\log s|^{n+1} ds. \tag{9.15}$$

But

$$\int_0^t s^n |\log s|^{n+1} = \frac{t^{n+1}|\log t|^{n+1}}{n+1} + \int_0^t s^n |\log s|^n ds$$

and therefore

$$\int_0^t s^n |\log s|^{n+1} \leqq \frac{2t^{n+1}|\log t|^{n+1}}{n+1}$$

if t is small. We substitute this into (9.15) and choose $A > 2C$ to complete the proof of (9.14) for $n + 1$.

By compactness, any subsequence of $\{\Gamma^n(t)\} = \{\partial D^n(t) \cap \{y \geqq 0\}\}$ has a further subsequence which converges to a $C^{1+\alpha}$ curve $\Gamma(t)$. We assert that the complete family $\{\Gamma^n(t)\}$ has a unique limit. Indeed, this follows from the fact that for any two sequences $\{\Gamma^{n_1}(t)\}$, $\{\Gamma^{n_2}(t)\}$,

$$\delta(D^{n_1}(t), D^{n_2}(t)) \to 0 \quad \text{as } n_1, n_2 \to \infty,$$

by (9.14). If we denote by $D(t)$ the limit in the δ-metric of the $D^n(t)$, then

$$\mathscr{M}D(t) = D(t).$$

It can be easily checked that $u(x, y, t)$ is continuous in (x, y, t), and this completes the existence part of Theorem 1.1.

To prove uniqueness suppose we have another solution with domains $\hat{D}(t)$ and set

$$\delta(t) = \delta(D(t), \hat{D}(t)).$$

We can choose the same l_y, ζ_{l_y} for both domains, and therefore (9.12) can be applied. We thus get

$$\delta(t) \leqq C \int_0^t \delta(s) |\log \delta(s)| ds.$$

We deduce as before that

$$\delta(t) \leqq A^n t^n |\log t| \quad \forall\, n \geqq 1,$$

so that $\delta(t) \equiv 0$, i.e., the two solutions coincide.

10. The shape of the free boundary

Theorem 10.1. *The function* $\bar{u}(x, y)$ *satisfies*

$$\bar{u}_{xy}(x, 0) < 0 \quad \text{if } 0 < x < a,\ x \neq 1. \tag{10.1}$$

Proof. Consider $\bar{u}_x$ in

$$R_\delta = \{0 < x < a,\ -h < y < b\} \backslash B_\delta(1, 0), \quad \delta > 0.$$

Notice that $\bar{u}_x$ and $\bar{u}_{xy}$ are continuous across $\{(x, 0), 0 \leqq x \leqq 1 - \delta\}$ $\big(\bar{u}_{xy}(x, 0+) - \bar{u}_{xy}(x, 0-)\big) = -\sigma_x = 0\big)$ so that $\bar{u}_x$ is actually harmonic in R_δ. By (2.9) and (5.10) we see that

$$\bar{u}_x < 0 \quad \text{in } B_\delta(1, 0)$$

if δ is small enough. Since, further, $\bar{u}_x = 0$ on $\partial R_\delta \backslash \partial B_\delta(1, 0)$, it follows by the maximum principle that

$$\bar{u}_x < 0 \quad \text{in } 0 < x < a,\ -h < y < b. \tag{10.2}$$

Consider the harmonic function

$$w(x, y) = \bar{u}_x(x, y) - \bar{u}_x(x, -y)$$

in $R_0^h = \{0 < x < a, 0 < y < h\}$. This function is bounded in a neighborhood of (1, 0), as a consequence of (2.9) and the fact that $\Phi_x(x, y) = \Phi_x(x, -y)$ (cf. (5.10)). Also $w = 0$ on $x = 0$, $x = a$ and $y = 0$, and $w(x, h) < 0$, by (10.2). Therefore, by the maximum principle, $w < 0$ in R_0^h and $w_y(x, 0+) < 0$. This yields (10.1).

Let us define curves $\Gamma_0(t)$ in $\{y > 0\}$ by

$$x = x_0(t, \lambda), \quad y = y_0(t, \lambda)$$

where

$$\begin{gathered} \dot{x}_0 = -\bar{u}_x(x_0, y_0), \quad \dot{y}_0 = -\bar{u}_y(x_0, y_0), \\ x_0(0, \lambda) = \lambda, \quad y_0(0, \lambda) = 0 \quad (-1 < \lambda < 1). \end{gathered} \tag{10.3}$$

Since

$$\frac{\partial x_0}{\partial \lambda} = 1, \quad \frac{\partial y_0}{\partial y} = 0 \quad \text{at } t = 0, \tag{10.4}$$

$\Gamma_0(t)$ can be written in the form

$$\Gamma_0(t) = \{y = f_0(x, t)\} \tag{10.5}$$

provided t is small. From (5.11), (5.12) it follows that

$$|f(x,t) - f_0(x,t)| \leqq Ct^{\frac{3}{2}-\delta}, \quad \delta > 0. \tag{10.6}$$

Set

$$g_0(x) = -\frac{\partial^2 \bar{u}(x,0)}{\partial x\, \partial y}, \quad g(x) = \int_0^x g_0(y)\, dy. \tag{10.7}$$

By Theorem 10.1, $g'(x) > 0$ if $0 < x < 1$.

By (10.4),

$$\frac{\partial}{\partial x} f_0(x,t) = -\frac{\partial y_0/\partial\lambda}{\partial x_0/\partial\lambda} = \frac{\partial y_0}{\partial\lambda}(1 + O(t)).$$

On the other hand, for any $\eta > 0$,

$$\frac{d}{dt}\frac{\partial y_0}{\partial y} = -\frac{\partial^2 \bar{u}}{\partial x\, \partial y}\frac{\partial x_0}{\partial\lambda} - \frac{\partial^2 \bar{u}}{\partial y^2}\frac{\partial y_0}{\partial\lambda} = g_0(x)\,(1 + O(t))$$

if $0 \leqq x \leqq 1 - \eta$, by (10.4). It follows that

$$\frac{\partial}{\partial x} f_0(x,t) = g_0(x)\,(1 + O(t))$$

and, by (10.6), that

$$f(x,t) = g(x)\, t + O\left(t^{\frac{3}{2}-\delta}\right), \quad 0 \leqq x \leqq 1 - \eta. \tag{10.8}$$

We summarize these results:

Theorem 10.2. *For any $\eta > 0$ the free boundary $y = f(x,t)$ satisfies* (10.8), *where $g'(x) > 0$ if $0 < x < 1$.*

Thus $f(x,t)$ is increasing in x in an "average" sense.

From (10.8) we see the precise linear growth in t, of the free boundary, when $0 < x \leqq 1 - \eta$.

We recall that

$$-\frac{C}{|\log t|} \leqq f_x(x,t) < -\frac{c}{|\log t|} \quad \text{if } 1 \leqq x < x_0(t) \tag{10.9}$$

where $C > c > 0$. Thus $f(x,t)$ decreases in x for $1 < x < x_0(t)$, at rate $\approx 1/|\log t|$. The shape of the free boundary, for a small time t, is described in Figure 1 of Section 1.

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