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Axioms and Fundamental Equations of Image Processing

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Abstract

Image-processing transforms must satisfy a list of formal requirements. We discuss these requirements and classify them into three categories: "architectural requirements" like locality, recursivity and causality in the scale space, "stability requirements" like the comparison principle and "morphological requirements", which correspond to shape-preserving properties (rotation invariance, scale invariance, etc.). A complete classification is given of all image multiscale transforms satisfying these requirements. This classification yields a characterization of all classical models and includes new ones, which all are partial differential equations. The new models we introduce have more invariance properties than all the previously known models and in particular have a projection invariance essential for shape recognition. Numerical experiments are presented and compared. The same method is applied to the multiscale analysis of movies. By introducing a property of Galilean invariance, we find a single multiscale morphological model for movie analysis.

Contents

1. Introduction

Define a black and white image as a bounded real function u defined on $\mathbb{R}^N$. $u(x)$ represents the grey level or brightness at the point x. From the physical (or psychophysical) point of view, images are obtained as the result of the impression left by the light sent or reflected by real objects on a surface, which can be a retina, or any photosensitive surface in analogic or digital cameras. One of the main challenges of disciplines like psychophysics, computer vision, robotics, etc., is to understand how, from the local properties of an image, and in a way which is largely independent of the natural or artifical perception devices, stable and reliable information on the shape of the surrounding objects can be obtained.

Most practical devices for image analysis assume a preliminary step in the processing of the pictures which consists in passing from the original picture to smoothed versions, which still contain significant information. At this stage of the processing, the image is therefore transformed into new images, and the main parameter of this preliminary transform is the "scale", which measures the degree of smoothing, or more trivially, the size of the neighbourhoods which are used to give an estimate of the brightness of the picture at a given point. A "multiscale analysis", as it is generally called in image-processing theory, tends to give less local and therefore more reliable information on the grey level than the original fluctuating "pixel". It is also considered by many authors as a necessary preliminary step in any task of artificial intelligence. Indeed, one of the aims of artificial intelligence is to "understand" pictures, that is, to identify the objects lying in them. Now, there can be objects of any size in a picture. So the preprocessing is assumed to be able to yield a reliable "résumé" of the picture at any level of scale reduction and subsequent simplification.

We define, without loss of generality, a "multiscale analysis" to be a family of transforms $(T_t)_{t \geqq 0}$ which, when applied to the original picture $f(x)$, yield a sequence of pictures $u(t,x) = (T_t f)(x)$. Roughly speaking, $T_t f$ is a semi-local version of f where a neighbourhood of size t around x has been scanned for determining the value of $T_t f(x)$. We do not now define the "size" t of the neighbourhood; its meaning will be clear from the mathematical formulations below.

The aim of this paper is to list a series of formal properties, or axioms, which are likely to be satisfied by the "multiscale analysis" T_t and then to give the explicit formulae which can be deduced from these axioms. We shall see that under reasonable assumptions (including the comparison principle) on a multiscale analysis T_t, all sequences of pictures $u(t,x) = (T_t f)(x)$ are solution of a partial differential equation of second order:

$$\frac{\partial u}{\partial t} = F(D^2 u, Du, t) \qquad \text{with } u(0,x) = f(x).$$

Moreover, we shall list a series of invariance properties which must be satisfied by image-analysis operators (in particular "shape-preserving" properties). Then we shall translate each invariance property for the multiscale analysis by describing all the F's which yield the given property. This will allow us to characterize axiomatically the multiscale theories which have arisen in computer vision and to classify them completely. The Marr-Hildreth-Koenderink-Witkin multiscale analysis ("scale space") based on the heat equation $\partial u/\partial t = \Delta u$, the Malik-Perona anisotropic diffusion $\partial u/\partial t = \operatorname{div}(g(|Du|\,Du))$, the "mean-curvature motion" $\partial u/\partial t = \operatorname{div}(Du/|Du|)\,|Du|$ proposed earlier by some of the authors of the present paper, and the basic transforms arising in mathematical morphology (dilation and erosion), given by $\partial u/\partial t = \pm|Du|$ will be therefore axiomatically characterized.

Last but not least, the axiomatic classification of all multiscale analyses yields several new possibilities. Among the new axioms, we give a detailed study of "scale invariance" for multiscale analysis. (A multiscale analysis is scale-invariant if it is, in some sense, independent of the size of the analyzed objects). Within the new multiscale analyses, we find a class of "morphological multiscale analyses", that is, a class of multiscale analyses having the same invariants as the classical morphological operators, including scale invariance. The associated equations are

$$\frac{\partial u}{\partial t} = \beta(t\operatorname{curv}(u))\,|Du|,$$

where β is an arbitrary nondecreasing real function. $\operatorname{curv}(u)$ is the curvature of the level set of u passing through x, so that this equation has a geometric interpretation. This new multiscale analysis can be interpreted as an interpolation between the heat equation of Marr, Hidreth, Koenderink, and Witkin and the classical morphological operators like opening and closing. Its main virtue is to combine the advantages of both theories, but not their drawbacks. Indeed, it keeps the same noise-elimination properties as the heat equation but is shape-preserving like morphological operators.

We show there is a unique multiscale analysis satisfying all classical invariance properties, as well as projection invariance. *This new multiscale analysis therefore has properties that no previously proposed theory has:* It allows the analysis of planar shapes in a way which depends on neither their orientation nor their location in the three-dimensional space. The equation is

$$\frac{\partial u}{\partial t} = |Du| \ (t\,\mathrm{curv}(u))^{1/3}.$$

Of course, we can remove some of the above invariance properties in order to obtain more general operators. We obtain the form of all multiscale analysis T_t satisfying the projection-invariance property, without imposing other shape-preserving properties.

Finally, we study the multiscale analysis of movies. We add two new invariance properties which take into account the particular nature of movies. Firstly, we assume the analysis to be invariant when the movie is uniformly slowed down or accelerated. Secondly, we assume that the analysis is invariant under "travelling", that is, a motion of the whole picture at constant velocity does not alter the analysis. (Thus the analysis is assumed to be able to "follow" such a uniform motion.) For obvious reasons, we call this axiom Galilean invariance.

We show that the equation associated with the multiscale analysis T_t satisfying all these invariance properties is given by

$$\frac{\partial u}{\partial t} = |\nabla u| \ (t\,\mathrm{curv}(u))^{1/3} \left(\left(\frac{\mathrm{curv}(u)\,\mathrm{accel}(u)}{|t\,\mathrm{curv}(u)|^{4/3}} \right)^{+} \right)^{q}$$

where $0 \leqq q < 1$, and $\mathrm{accel}(u)$ represents the acceleration of the movie in the direction of the spatial gradient.

All of these equations have unique solutions in the "viscosity" sense [12]. They correspond to order-preserving semigroups and are obtained by identifying the generator of semigroups having all formal properties discussed above. These new partial differential equation models include several recently introduced "geometric equations", like the mean curvature motion $\partial u/\partial t = \mathrm{div}(Du/|Du|)\,|Du|$.

Let us mention that our axiomatic approach to image processing is inspired by the systematic derivation of models in continuum mechanics from first principles.

2. Axioms of multiscale analysis

We shall distinguish three types of axioms: those which deal with the architecture of the multiscale image analysis (causality, recursivity, regularity, locality), which we call the "architectural axioms". We shall add to the architectural axioms a single axiom for characterizing the fact that the multiscale analysis under study corresponds to a smoothing of the original picture. This axiom is known by physicists and mathematicians as the "comparison principle" or "maximum principle" and is satisfied by the classical multiscale analysis.

We finally consider the axioms which correspond to the shape-preserving properties of image analysis [3, 36, 54], that is, the invariance of the result of the analysis when fluctuations of brightness occur because of the geometrical and technical conditions of the perception. Those axioms are called "morphological axioms", since most of them have been formalized by the school of mathematical morphology.

Let us now make an (informal) list of the axioms of the three groups and their main consequences.

2.1. *Architectural axioms*

We first consider an axiom which is called in vision theory the "causality" property, or the "pyramidal architecture" property. According to this assumption, T_t can be computed from T_s for any $s \leqq t$, and T_0 is of course the identity. This is natural, since a coarser analysis of the original picture is likely to be deduced from a finer one without any dependence upon the original picture. Of course, the finest picture analysis is the identity. A strong version of causality is the semigroup property

[Recursivity] $T_0(f) = f, \quad T_s \circ T_t(f) = T_{s+t}(f)$ *on* $\mathbb{R}^N$, *for all* $s,\ t \geqq 0$ *and* f.

If [Recursivity] is satisfied, the visual process is reduced to a single loop if the scales are discretized. Indeed, T_t can be deduced from the n-th iteration of $T_{t/n}$.

A weaker version of the pyramidal hypothesis is following: We include $T_t = T_{t,0}$ in a family of transition operators $T_{s,t}$ indexed by $0 \leqq s, t < \infty$ and satisfying

[Causality] $\qquad T_{t+s} = T_{t+s,s} \circ T_s$ *for all* $0 \leqq s, t < \infty$.

In order to get back to [Recursivity], one needs to assume that $T_{t+s,s} = T_{t,0}$. From the viewpoint of the theory of perception, [Causality] is a sound hypothesis, if the image perceptual analysis consists in a sequence of filters which are applied sequentially. Since new images are constantly arriving at the retina, the image-analysis process is thought of as a flow of the picture through different filters, each associated with a scale t. If we discretize such a process, it yields a series of operators $S_n = T_{(n+1)h,nh}$ applied successively, [47, 30].

Indeed, we notice that if $t = nh$, then $T_t = T_{nh}$ can be deduced from the n-th iteration of T_h. Now, we need an axiom which states the independence of the multiscale analysis of the choice of h. So we assume the existence of a so-called infinitesimal generator A for the semigroup T_t, defined by

[Generator] $(T_h f - f)/h \to A[f]$ *as* $h \to 0^+$ (*or* $(T_{t+h,t} f - f)/h \to A_t[f]$), *for smooth* f.

A way of justifying [Generator] is to deduce it from axioms more natural from the viewpoint of perception. An example of such an axiom, which, combined with the other axioms of the theory, implies [Generator] is

[Regularity] $\| T_t(f + hg) - (T_t(f) + hg) \|_\infty \leqq Cht$ *for all* h, t *in* $[0, 1]$, *for smooth* f *and* g, *where of course* C *depends on* f *and* g.

This last axiom states a natural assumption of continuity of T_t and is therefore a strong justification for the existence of an infinitesimal generator for the multiscale analysis.

We next require an axiom on the local character of the multiscale analysis T_t for t small (and therefore the local character of the infinitesimal generator A):

[Locality] $\{T_t(f) - T_t(g)\}(x) = o(t)$ *as* $t \to 0^+$, *for all smooth* f *and* g *such that* $D^\alpha f(x) = D^\alpha g(x)$ *for all* $|\alpha| \geqq 0$ *and for all* x.

Roughly speaking, this last axiom means that the value of $T_t(f)$ for t small, at any point x, is determined by the behaviour of f near x.

2.2. Comparison principle

The comparison principle is an obvious order-preserving property (the "maximum principle"). It means that no enhancement is made, but just a smoothing of the original image. Thus if one image g is everywhere brighter than another f, this ordering is preserved.

[Comparison principle] $T_t(f) \leqq T_t(g)$ *on* $\mathbb{R}^N$ *for all* $t \geqq 0$ *and* f, g *such that* $f \leqq g$.

This axiom is equivalent, in the case where T_t is a linear filter, defined by $T_t f = f * F_t$, to the inequality $F_t \geqq 0$. *Thus, this axiom is the most obvious generalization in the nonlinear case of a nonnegative smoothing kernel.*

We want to emphasize here that the comparison principle is quite natural for transforming a grey-level image into a grey-level image. If another kind of structure (such as an edge map, depth map) is sought, the comparison principle is no longer valid, because, for instance, $g \geqq f$ can occur with a constant g without edges while f has many edges.

2.3. *Morphological axioms*

Most of the next axioms, which we call the "morphological axioms", are well-known in mathematical morphology. They state that image analysis must be invariant under fluctuations of light and under changes of position, orientation and scale of the planar shapes.

A first requirement essential to the understanding of images and shapes is to take into account how arbitrary the grey scale of perceptual or digital pictures is. This scale is due to fluctuations of illumination of the *perceptum*: indeed, sun, clouds, artifical lights, reflections and the change of position of *percipiens* and *perceptum* strongly perturb the amount of light which is sent from any point of the world to the perception device. Then, the state of *the percipiens* can also be altered: Aperture, focus, trace of anterior perception, etc., of the perception device also drastically change the effect of the light on the photosensitive surface.

In the case of digital pictures, many electronic devices are applied successively to an image before its arrival at the human eye or at some automatic image-analysis device: Since the grey scale of the resulting image has been changed by each device, the only sound assumption about the information-preserving properties of the whole chain of captors and transmittors is that they might preserve the order of grey levels. In other terms, if some point or some region was brighter than an another in the original picture, this order should be preserved in the final picture. (This property is, however, phenomenologically false: see KANIZSA [29].)

We begin by stating that the image analysis must be independent of the (arbitrary) grey-level scale. In the following, we shall always assume the following weak form of this axiom:

[Grey-level-shift invariance] $T_t(0) = 0$, $T_t(f + C) = T_t(f) + C$ *for any* f *and any constant* C.

This axiom means that no a priori assumption is made on the range of the brightness of a picture to be observed. Of course, this is not absolutely true for natural or artificial photosensitive systems. It is however true that the interpretation of a photograph is widely independent of its exposure time: The photograph can be dark or light and yet be identified as essentially the same picture. This axiom is equivalent, in the case where T_t is a linear filter, defined by $T_t f = f * F_t$, to the requirement that $\int F_t(x)\,dx = 1$.

The strong form of the first morphological axiom is

[Grey-scale invariance] $T_t(h(f)) = h(T_t(f))$ *for all* f *and all* $t \geqq 0$, *where* h *is any nondecreasing real function.*

A stronger form of this axiom assumes the same relation when h is nondecreasing or nonincreasing. In this case, we only assume in the above argument that only the fact that two points have the same brightness is preserved by the whole chain of captors and transmittors.

The function h is simply an order-preserving rearrangement of the grey level. Notice that the second relation of [Grey-level-shift invariance] is a particular case of [Grey-scale invariance].

We now list a series of axioms which preserve the invariance of image analysis under *the respective positions of the percipiens and perceptum.* Indeed, animals, humans and cameras are constantly changing position. Therefore, the *stability* of the resulting representation must be preserved when objects *change scale, position and orientation* before their *projection* on the perceptive surface.

We first require that the multiscale analysis be translation-invariant, that is,

[Translation invariance] $T_t(\tau_h \cdot f) = \tau_h \cdot (T_t f)$ *for all* h *in* $\mathbb{R}^N$, $t \geqq 0$, *where* $(\tau_h \cdot f)(x) = f(x+h)$.

This only means that all points of the space are a priori equivalent. There is no a priori knowledge about location of any feature of the picture: This is clear from our above remarks about the arbitrary respective positions of the *percipiens* and *perceptum.*

We add an optional axiom of isometry-invariance:

[Isometry invariance] $T_t(R \cdot f) = R \cdot T_t(f)$ *for all* f, $t \geqq 0$ *and for all transforms* R *defined by* $(R \cdot f)(x) = f(Rx)$ *where* R *is an orthogonal transform of* $\mathbb{R}^N$.

The next optional morphological axiom is the scale-invariance of image analysis. Set $D_\lambda f(x) = f(\lambda x)$. The scale invariance can be stated as

[Scale invariance] *For any* λ *and* t, *there exists* t' *such that* $D_\lambda T_{t'} = T_t D_\lambda$.

This relation means that the result of the multiscale analysis T_t is independent of the size of the analyzed features: This is very important in the world where we live, since the same object can be seen at very different distance and therefore at very different scales. Thus, it is essential for the stability of shape analysis that the result of an analysis of this object should not yield a different "shape" at different distances. Thus, the sequence of the shapes obtained by multiscale filtering must be independent of the (a priori unknown) size of the object in the picture. Otherwise, objects seen at different distances would have a different multiscale analysis, and therefore recognition of an object would depend on the unknown distance from the object!

Finally, we state an axiom which implies [Isometric invariance], [Scale invariance] and also the invariance of the multiscale analysis under any planar projection of a planar shape. Combining those transformations leads to an arbitrary linear transform A of the plane. For any such transform set $Af(x) = f(Ax)$. With the same formalism as in [Scale invariance], we get

[Projection invariance] *For any* A *and* t, *there exists* $t'(t, A)$ *such that* $AT_{t'} = T_t A$.

3. General form of regular multiscale analysis operators

In this section, we characterize each multiscale filtering as the solution of a generic partial differential equation: In Theorem 2 we obtain a result explaining why the main multiscale image processing models are parabolic partial differential equations of order 2.

More precisely, we shall prove that under the "architectural" conditions [Recursivity], [Regularity], [Locality], together with [Comparison principle], and the most obvious morphological conditions [Translation invariance] and the [Grey-level-shift invariance], there exists a continuous function F, such that for any given picture f, $u(x,t) = T_t f$ satisfies

$$\frac{\partial u}{\partial t} = F(D^2 u, Du).$$

Moreover, for any symmetric matrices A and B, the inequality $A \geqq B$ implies that $F(A,p) \geqq F(B,p)$ for any p. Conversely, any partial differential equation of the preceding kind corresponds to a multiscale analysis satisfying the above-mentioned axioms. If, instead of the semigroup hypothesis [Recursivity], we have the weaker causality axiom [Causality], and the obvious adaptations of the other axioms to $T_{t,s}$, we shall prove the same result with a time-dependent F:

$$\frac{\partial u}{\partial t} = F(D^2 u, Du, t).$$

We do not examine all the combinations of the above axioms. Indeed, if we remove one of the three "architectural" axioms plus the comparison principle, we may lose the partial differential equation form of the multiscale analysis. If we remove [Translation invariance], the equation becomes x-dependent, and F has the form $F(D^2u, Du, x)$. Notice in the same way that a dependence of F on u, so that it has the form $F(D^2u, Du, u, t)$, only contradicts [Grey-level-shift invariance].

We consider a multiscale analysis T_t as a family parametrized by $t \geqq 0$ of possibly nonlinear operators on functions defined on $\mathbb{R}^N$. In practice, $N = 2$ or $N = 3$ but the results below do not depend upon these specific choices. We denote this family by T_t and we assume that it is well defined on C_b^∞, i.e., the space of bounded functions having bounded derivatives at any order, and that $(T_t f)(x)$ is a bounded continuous function on $\mathbb{R}^N \times [0, \infty[$ whenever f is in C_b^∞. This definition is not restrictive, since we shall of course extend the domain of T_t to more general and less smooth functions. In practice, an image is not necessarily defined on the whole Euclidian space, but our axiomatic presentation needs this assumption for simplicity. If not, most invariance properties presented above would be difficult to treat simply and rigorously. We shall see anyway that no loss of generality is made: Indeed, an image, if, e.g., rectangular, can be extended into an image defined in the whole space (for instance by reflection) and the partial differential equation models which we shall find at the end are compatible with such extensions.

We first reformulate the axioms of image processing in a more precise functional framework. We change the order of presentation of the axioms with the aim of inserting useful mathematical comments on their consequences. We always assume

[Translation invariance] $T_t(\tau_h \cdot f) = \tau_h \cdot (T_t f)$ *for all* h *in* $\mathbb{R}^N$, $t \geqq 0$, f *in* C_b^∞, *where* $(\tau_h \cdot f)(x) = f(x+h)$.

Next, we require the order-preserving property ("maximum principle"):

[Comparison principle] $T_t(f) \leqq T_t(g)$ *on* $\mathbb{R}^N$ *for all* $t \geqq 0$ *and* f, g *in* C_b^∞ *such that* $f \leqq g$.

Further, we assume invariance under the addition of constants:

[Grey-scale-shift invariance] $T_t(0) = 0$, $T_t(f + C) = T_t(f) + C$ *on* $\mathbb{R}^N$ *for all* $t \geqq 0$, *and* f, g *in* C_b^∞.

Let $\|f\|_\infty = \sup_{x \in \mathbb{R}^n} |f(x)|$. Since $f \leqq g + \|f - g\|_\infty$, [Comparison principle] and [Grey-scale-shift invariance] immediately imply that

(1) $$\|T_t(f) - T_t(g)\|_\infty \leqq \|f - g\|_\infty \quad \text{for all } t \geqq 0,\ f,\ g \text{ in } C_b^\infty.$$

Of course, (1) yields that T_t can be extended by continuity as a mapping from $BUC(\mathbb{R}^N)$, the space of bounded, uniformly continuous functions on $\mathbb{R}^N$ into $C_b(\mathbb{R}^N)$ (the space of continuous bounded functions on $\mathbb{R}^N$). And then [Translation invariance] and (1) imply that T_t maps $BUC(\mathbb{R}^N)$ into itself. To simplify notation, we now set $X = BUC(\mathbb{R}^N)$ and we remark that by density, axioms [Translation invariance], [Comparison principle], [Grey-scale-shift invariance] and (1) still hold for f, g in X. We also state the main architectural axiom for functions f in X.

[Recursivity] $T_0(f) = f$, $T_s \circ T_t(f) = T_{s+t}(f)$ *on* $\mathbb{R}^N$, *for all* s, $t \geqq 0$ *and* f *in* X.

It is tempting to believe that [Translation invariance], [Comparison principle], [Grey-scale-shift invariance] and [Recursivity] suffice to ensure that T_t admits an infinitesimal generator, but this is an open question in nonlinear semigroup theory. We say that T_t admits an infinitesimal generator A if

[Generator] $(T_t f - f)/t \to A[f]$ *uniformly on* $\mathbb{R}^N$, *as* $t \to 0^+$, *for all* f *in* C_b^∞.

As we shall see, [Generator] can in fact be deduced from the following axiom:

[Regularity] $\|T_t(f + hg) - (T_t(f) + hg)\|_\infty \leqq Cht$ *for all* h, t *in* $[0, 1]$ *and all* f, g *in* Q, *where* $C \geqq 0$ *is a constant depending only on* Q, Q *is the subset of* C_b^∞ *defined by*

(2) $$Q = \{f \text{ in } C_b^\infty, \forall n \geqq 0,\ \|D^\alpha f\|_\infty \leqq C_n \text{ for all } |\alpha| = n\},$$

and C_n *is an arbitrary increasing sequence of nonnegative constants.*

Set $\delta_t(f) = (T_t f - f)/t$. Then we can rewrite the last axiom as $\|\delta_t(f+hg) - \delta_t(f)\|_\infty \leqq Ch$ for all h, t in $[0,1]$, which clearly gives some stability to the difference quotients approximating the infinitesimal generator.

If we want to deduce the existence of an infinitesimal generator for any t in the case of a time-dependent multiscale analysis T_t, then of course we need to impose [Regularity] for any s and any family of operators $T_{t+s,t}$. Thus, in the axiom [Regularity], the time origin $s = 0$ is replaced by an arbitrary $s \geqq 0$, and [Regularity] becomes

[Regularity-bis] $\|T_{t+s,s}(f+hg) - T_{t+s,s}(f) + hg)\|_\infty \leqq Chs$.

We also need an additional axiom giving some temporal stability to the differences $(T_{t+s,t}f - f)/s$:

[Temporal Regularity] $\|T_{t+s,t}(f) - T_{s,0}(f)\|_\infty \leqq Csn(t)$, *where* $n(t)$ *is a positive function going to* 0 *as* t *goes to* 0, *uniformly in* s *in* $]0,1]$ *and where* f *is in* Q.

3.1. *Existence of a generator*

The following theorem (proved in Appendix 1) states that this stability assumption is enough to ensure the existence of a generator for the multiscale analysis.

Theorem 1. (i) *Let* T_t *be a multiscale analysis satisfying* [Translation invariance], [Comparison principle], [Grey-scale-shift invariance], [Recursivity] *and* [Regularity]. *Then it also satisfies* [Generator]. *Moreover, the convergence in* [Generator] *is uniform for* f *in* Q, *and thus, in particular,* $A[f_n] \to A[f]$ *as* $n \to \infty$ *uniformly if* f_n, f *are in* Q *and* $D^\alpha f_n \to D^\alpha f$ *uniformly on* $\mathbb{R}^N$ *for all* $|\alpha| \geqq 0$.

(ii) *Let* T_t *be a multiscale analysis satisfying* [Translation invariance], [Comparison principle], [Grey-scale-shift invariance], [Causality], [Regularity-bis] *and* [Temporal regularity]. *Then, the preceding conclusions are still valid with the obvious adaptation of* [Generator]:

[Generator] $(T_{t+s,t}f - f)/s \to A_t[f]$ *uniformly on* $\mathbb{R}^N$, *as* $s \to 0^+$, *for all* f *in* C_b^∞.

Remark. If T_t is a *linear* operator, [Regularity] obviously reduces to

$$\|T_t(f) - f\|_\infty \leqq Ct \text{ for all } t \text{ in } [0,1],\ f \text{ in } Q, \tag{3}$$

which is a very natural condition from the viewpoint of linear semigroup theory: It only means that for smooth functions, orbits are Lipschitz continuous at $t = 0$.

We require a final axiom which concerns the local character of the operator T_t for t small (and therefore the local character of the infinitesimal generator A).

[Locality] $\{T_t(f) - T_t(g)\}(x) = o(t)$ *as* $t \to 0^+$, *for all* f *and* g *in* C_b^∞ *such that* $D^\alpha f(x) = D^\alpha g(x)$ *for all* x *and* $|\alpha| \geqq 0$.

In the case of a causal multiscale analysis, [Locality] becomes of course

[Locality-bis] $\{T_{t,s}(f) - T_{t,s}(g)\}(x) = o(t-s)$ *as* $t - s \to 0^+$.

If [Generator] holds, this axiom can be replaced without changes in the proofs below by the axiom

[Locality] $A[f](x) = A[g](x)$ *for all* f *and* g *in* C_b^∞ *such that* $D^\alpha f(x) = D^\alpha g(x)$ *for all* x *and* $|\alpha| \geqq 0$.

This last axiom means that the value of $T_t(f)$ for t small, at any point x, is determined by the behaviour of f near x.

Notice that [Regularity] implies that if $f(x) = g(x)$, then $\{T_t(f) - T_t(g)\}(x) = o(t)$ as $t \to 0^+$. Thus [Regularity] is in some weaker sense also a locality assumption; but using [Generator] we can deduce from [Locality] that $A[f](x) = A[g](x)$, while this cannot be deduced from [Regularity]. Thus, roughly speaking, the meaning of [Locality] is that if two functions have the same derivatives at some point, then they have the same infinitesimal generator at this point.

The result which follows shows that the architectural axioms and the comparison principle, plus the translation-invariance properties, imply the existence of a continuous function F defined on $S^N \times \mathbb{R}^N \times \mathbb{R}$ (where S^N denotes the space of $N \times N$ symmetric matrices) which satisfies

$$F(A,p,t) \geqq F(B,p,t) \quad \text{for all } p \text{ in } \mathbb{R}^N,\ A,\ B \text{ in } S^N \text{ with } A \geqq B \tag{4}$$

and such that

$$A_t[f] = F(D^2 f, Df, t) \quad \text{for all } f \text{ in } C_b^\infty. \tag{5}$$

Here and below, $A \geqq B$ means that $A - B \geqq 0$ in the sense of symmetric matrices. Then, we also relate, as we should expect, the semigroup T_t to the solution of the following fully nonlinear, second-order, "parabolic" equation

$$\frac{\partial u}{\partial t} - F(D^2u, Du, t) = 0 \text{ in } \mathbb{R}^N \times [0, \infty[, \quad u(0, \cdot) = u_0(\cdot) \text{ in } \mathbb{R}^N. \tag{6}$$

Existence and uniqueness of solutions of (6) is known provided the equation is understood in the "viscosity" sense, a notion of weak solutions which makes possible a rather complete understanding of such general equations. We refer to the survey article by CRANDALL, ISHII & LIONS [12] for more details.

3.2. Why a second-order partial differential equation?

Theorem 2. (i) *Under conditions* [Translation invariance], [Comparison principle], [Grey-scale-shift invariance], [Recursivity], [Regularity] *and* [Locality], *there exists a continuous function* F *on* $S^N \times \mathbb{R}^N$ *satisfying* (4) *such that* $(T_t f - f)/t \to F(D^2 f, Df)$, *and therefore* (5) *holds. Moreover,* $(T_t f - f)/t \to F(D^2 f, Df)$ *uniformly on* $\mathbb{R}^N$ *as* t *tends to* 0^+, *uniformly on* $\mathbb{R}^N$, *uniformly for* f *in* C_b^∞ *with a uniform modulus of continuity for its second derivatives.*

(ii) *Let* T_t *be a multiscale analysis where the operators* $T_{t,s}$ *satisfy* [Translation invariance], [Comparison principle], [Grey-scale-shift invariance], [Regularity-bis], [Temporal Regularity], [Locality-bis] *and* [Causality]. *Then there exists a continuous function* $F(A, p, t)$ *satisfying* (4) *and* (5) *for all* $t \geqq 0$.

(iii) *In addition, for any* u_0 *in* $BUC(\mathbb{R}^N)$, $u(t,x) = T_t(u_0)(x)$ *is the unique viscosity solution of* $\partial u/\partial t = F(D^2 u, Du, t)$ *in* $BUC_x([0, \infty[\times \mathbb{R}^N)$.

Remark. 1) Here and below, we denote by $BUC_x(\mathbb{R}^N \times [0, \infty[)$ the space of bounded continuous functions on $\mathbb{R}^N \times [0, \infty[$ that are uniformly continuous in x, uniformly in t.
2) It can be shown that any continuous function F satisfying (4) leads to a unique "viscosity semigroup" T_t which satisfies [Translation invariance], [Generator] and [Locality].

Proof of Theorem 2. Let f, g in C_b^∞ satisfy

(7)
$$f(0) = g(0) = 0, \quad Df(0) = Dg(0) = p \in \mathbb{R}^N,$$
$$D^2 f(0) = D^2 g(0) = A \in S^N.$$

We are first going to show that $A[f](0) = A[g](0)$. Once this is proved, we remark that [Translation invariance] and [Grey-scale-shift invariance] then easily imply

(8)
$$A[\tau_h \cdot f] = \tau_h \cdot (A[f]), \quad A[f + C] = A[f] \quad \text{for all } f \text{ in } C_b^\infty,\ h \text{ in } \mathbb{R}^N,\ C \text{ in } \mathbb{R}.$$

And thus we deduce

(9)
$$A[f] = F(D^2 f, Df)$$

for some function F on $S^N \times \mathbb{R}^N$. We then prove the rest of Theorem 2. In order to prove that $A[f](0) = A[g](0)$, we introduce z in $C_b^\infty(\mathbb{R}^N)$ such that $z \geqq 0$, $z(x) = |x|^2$ near 0 and we set $f^\varepsilon = f + \varepsilon z$. Obviously, $f^\varepsilon \geqq g$ for $|x| \leqq c\varepsilon$ with $c > 0$. Next, we set $w_\varepsilon = w(x/\varepsilon)$ where w is in $C_b^\infty(\mathbb{R}^N)$, $0 \leqq w \leqq 1$ on $\mathbb{R}^N$, $w(x) = 1$ if $|x| \leqq c/2$, $w(x) = 0$ if $|x| \geqq c$. We finally introduce $\underline{f}^\varepsilon = w_\varepsilon f^\varepsilon + (1 - w_\varepsilon)\, g$. The function $\underline{f}^\varepsilon$ has been constructed in order to satisfy two properties: First, all its derivatives at 0 are equal to the derivatives of f^ε, and second, $\underline{f}^\varepsilon \geqq g$ on $\mathbb{R}^N$. This last property implies that $T_t(\underline{f}^\varepsilon) \geqq T_t(g)$ on $\mathbb{R}^N$ for all $t \geqq 0$, because of [Comparison principle]. Since $\underline{f}^\varepsilon(0) =$

$f^\varepsilon(0) = f(0) = g(0) = 0$, we deduce that

$$A[f^\varepsilon](0) \geqq A[g](0).$$

But, in view of the first mentioned property of f^ε and of [Locality], we obtain $A[f^\varepsilon](0) = A[\bar{f}^\varepsilon](0)$; we easily deduce from Theorem 1(i) that $A[f^\varepsilon](0) \to A[f](0)$, and by symmetry the assertion is proved.

That F satisfies (4) is shown by a similar argument by considering $A \geqq B$ and setting

$$f(x) = [(p,x) + \tfrac{1}{2}(Ax,x)]\,w(x), \quad g(x) = [(p,x) + \tfrac{1}{2}(Bx,x)]\,w(x).$$

Indeed, $f \geqq g$ on $\mathbb{R}^N$ while $f(0) = g(0)$ and therefore

$$F(A,p) = A[f](0) = \lim_{t\to 0^+} (T_t(f)(0) - f(0))/t$$

$$\geqq \lim_{t\to 0^+} (T_t(g)(0) - g(0))/t = A[g](0) = F(B,p).$$

Using the last conclusion of Theorem 1(i), namely, $A[f_n] \to A[f]$ as $n \to \infty$ uniformly if f_n, f are in Q and $D^\alpha f_n \to D^\alpha f$ uniformly on $\mathbb{R}^N$ for all $|\alpha| \geqq 0$, we also deduce by this argument that F is continuous on $S^N \times \mathbb{R}^N$.

Proof of (ii). The proof is identical. We use part (ii) of Theorem 1 instead of part (i). The continuity of $F(D^2u, Du, t)$ with respect to t comes from [Temporal regularity].

Proof of (iii). Finally, that $T_t(u_0)(x)$ is a viscosity solution of (6) is shown as in [12] (see also LIONS [34]) and thus we only sketch the proof in the time-independent case. The rest of Theorem 2 then follows from general facts on viscosity solutions (see [12], for instance). In order to prove that $u(t,x) = T_t(u_0)(x)$ is a viscosity solution of (6), it suffices to check that u is a viscosity subsolution of (6), the proof that it is a supersolution being shown in the same way. Let (t_0, x_0) in $[0,\infty] \times \mathbb{R}^N$ be a global maximum point of $u - \phi$ where ϕ is in $C_b^\infty(\mathbb{R}^N \times [0,T])$ for any $T < \infty$. We need to show that

$$\frac{\partial\phi}{\partial t}(t_0,x_0) - F(D^2\phi(t_0,x_0), D\phi(t_0,x_0)) \leqq 0. \tag{10}$$

Without loss of generality, we may assume that $u(t_0,x_0) = \phi(t_0,x_0)$ and thus $u \leqq \phi$ on $[0,\infty] \times \mathbb{R}^N$. Again, without loss of generality, by standard approximation arguments in viscosity theory, we may assume that ϕ is of the form

$$\phi(t,x) = f(x) + g(t) \tag{11}$$

where $g(t_0) = 0$, $f(x_0) = u(t_0,x_0)$ and f is in $C_b^\infty(\mathbb{R}^N)$, g is in $C^\infty([0,\infty[)$. Then for $h > 0$, we consider $\phi(t_0,x_0) = u(t_0,x_0) = T_h(u(t_0 - h))(x_0)$ where we set $u(t) = u(t,\cdot)$ for all functions u on $\mathbb{R}^N \times [0,\infty[$. In this equality, we used the semigroup property [Recursivity]. Next, using [Comparison principle] and [Grey-scale-shift invariance] we obtain

$$T_h(u(t_0 - h)) \leqq T_h(\phi(t_0 - h)) \leqq T_h(f) + g(t_0 - h).$$

Therefore, we finally deduce that

$$\frac{1}{h}\{g(t_0) - g(t_0 - h)\} + \frac{1}{h}\{f - T_h(f)\} \leqq 0,$$

and letting h go to 0, using [Generator] or Theorem 1, we recover

$$g'(t_0) - F(D^2 f(x_0), Df(x_0)) \leqq 0,$$

and (10) is proved since $\frac{\partial \phi}{\partial t}(t_0, x_0) = g'(t_0)$ and $D^\alpha f(t_0, x_0) = D^\alpha f(x_0)$ for $|\alpha| = 1, 2$.

Corollary 1. *Let the assumptions of Theorem 2 hold, but with* [Generator] *replacing* [Regularity-bis]. *Further assume that $A_t[f_n](x)$ tends to $A_t[f](x)$ when f_n and all its derivatives respectively tend uniformly to f and all its derivatives. Then the same conclusion is still valid. These assumptions can be weakened slightly: If the continuity property is true except at points where, e.g., $Df(x) = 0$, then the conclusion $A_t[f] = F(D^2 f, Df, t)$ remains true, except at points x where $Df(x) = 0$.*

The last statement of Corollary 1 is very important in Sections 5 and 6, since the morphological invariants force a discontinuity of $F(A, p, t)$ at $p = 0$.

4. Axiomatic analysis of classical multiscale models

4.1. Axiomatization of the Marr-Hildreth-Koenderink-Witkin theory

We now deduce from Theorem 2 a characterization of the heat equation $\partial u / \partial t = \Delta u$ as the unique linear and isometrically invariant multiscale model. Thus, we get a formal justification of a theory based on many formal and heuristic arguments which has always pointed to the heat equation as the only possible multiscale analysis. We give here a proof of this intuition: The heat equation is the only linear isometrically invariant multiscale analysis. Thus, for image models, *linearity and grey-scale invariance are incompatible*, and we also obtain an explanation for the coexistence of (at least) two different schools in image processing: mathematical morphology on one side and classical multiscale analysis on the other.

The classical model comes from Marr & Hildreth [40] and has been formalized by Witkin [61], Koenderink [30]. Canny [8] proposed an efficient variant. The basic step of the multiscale analysis is the convolution of the original image with gaussians of increasing variance. Koenderink [30] noticed that the convolution of the signal with gaussians at each scale is equivalent to the solution of the heat equation with the signal as initial datum. If this datum is denoted by u_0, the "scale-space" analysis associated with u_0 consists in solving the system

$$\frac{\partial u(t, x, y)}{\partial t} = \Delta u(t, x, y), \quad u(0, x, y) = u_0(x, y).$$

The solution of this equation for an initial datum with bounded quadratic norm is $u(t,x,y) = G_t * u_0$ where

$$G_t(x,y) = (4\pi)^{-1} t^{-1} \exp\left(-(x^2+y^2)/4t\right)$$

is the Gauss function. One of the uses of the theory is "edge detection". According to MARR & HILDRETH, (x,y) is an edge point for the "scale" t if $\Delta u(t,x,y)$ changes sign and $|Du(t,x,y)|$ is "large". Of course, this last condition introduces some a priori defined threshold. Unfortunately, it is well known (and it is enough to look at the "edges" found by this method to observe it [39]) that the edges at low scales give an inexact account of the boundaries which, according to our perception, should be considered as correct. This is still true for the low-pass filtering of CANNY [7, 47] which is generally used as the best linear filter for white noise elimination and edge detection. On the other hand, if one makes a sharp low filtering, with small variance, all the edges keep their correct location. Now, the "main" edges are embedded in a crowd of "spurious" edges due to noise, texture, etc. The "scale space" theory of WITKIN [61] proposes therefore to identify the main edges at a low scale, and then to "follow them backward" by making the scale decreasing again. This method could theoretically give the exact location of all main edges. However, its implementation is rather heavy from the computational viewpoint and is unstable, because of the follow-up of edges across scales and the multiple thresholdings involved in the edge detection at each scale.

Theorem 3. *Let T_t be a multiscale analysis satisfying* [Causality], [Regularity-bis], [Locality-bis], [Translation invariance], [Grey-scale-shift invariance], [Temporal regularity], *and* [Comparison principle]. *If the $T_{t,s}$ are linear and satisfy* [Isometric invariance], *then (up to a rescaling $t' = h(t)$) $u(t,x) = (T_t u_0)(x)$ is the solution of the heat equation $\partial u/\partial t - c\Delta u = 0$ in $\mathbb{R}^N \times [0, \infty[$, $u(0,\cdot) = u_0(\cdot)$ in $\mathbb{R}^N$, where c is some positive constant.*

Proof. Let us begin with the scale-independent case. Since $F(D^2u, Du) = \lim_{t\to 0} (T_t u - u)/t$, F is linear in u and therefore satisfies

$$F(rD^2u + sD^2v, rDu + sDv) = rF(D^2u, Du) + sF(D^2v, Dv)$$

for any real numbers r and s and any functions u and v in X and at any point x. Since the values of Du, Dv, D^2u, D^2v are arbitrary and can be independently taken to be 0, we obtain for any vectors p, p' and symmetric matrices A, A' that

$$F(rA + sA', rp + sp') = rF(A,p) + sF(A',p'), \quad F(A,p) = F(A,0) + F(0,p).$$

Thus $F(A,p) = F'(p) + F''(A)$, and F' and F'' are linear.

From the isometrical invariance, we also obtain $F({}^tRAR, {}^tRp) = F(A,p)$ for any isometry R of $\mathbb{R}^N$. Taking $A = 0$, we obtain from the preceding relations that $F'(Rp) = F'(p)$ and therefore $F'(Rp) = F'(p)$ for any isometry R. Since F' is linear, this is only possible if $F'(p) = 0$ for any p. Thus

$F(A,p) = F''(A)$. By the isometry invariance again, we have $F''({}^tRAR) = F''(A)$ for any isometry R and any symmetric matrix A. Since every symmetric matrix can be diagonalized in an orthonormal basis and every pair of orthonormal bases can be exchanged by some isometry, we see that F'' only depends on the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A. Thus $F''(A) = F''(\lambda_1, \ldots, \lambda_n)$. Since the eigenspaces can also be exchanged by isometries, $F''(\lambda_1, \ldots, \lambda_n)$ must be invariant under any permutation of the eigenvalues. Thus F only depends on the symmetric functions of the eigenvalues. Now, the only linear symmetric function is the sum. Thus $F''(A)$ only depends (linearly) on the trace of A and therefore $F''(A) = c\,\mathrm{trace}(A)$ for some constant c. We conclude that $F(D^2u, Du) = c\Delta u$. Since F must be increasing in A, the constant c is nonnegative. This completes the proof in the case of a recursive architecture. In case of a causal architecture, we can apply exactly the same reasoning and we obtain $F(D^2u, Du, t) = c(t)\,\Delta u$ for some continuous nonnegative function $c(t)$. Thus, with the rescaling $\partial t'(t)/\partial t = c(t)$, we obtain again the heat equation $\partial u/\partial t - \Delta u = 0$.

Remark on other invariants of the heat equation. It is easy to check that the equation is scale invariant with $t' = t\lambda^2$. Let us prove that the grey-scale invariance does not hold. Let $\partial u/\partial t = \Delta u$ and $u = h(v)$. Then $h'(v)\,\partial v/\partial t = h'(v)\,\Delta v + h''(v)\,|Dv|^2$. If the equation were grey-scale invariant, then v would satisfy $\partial v/\partial t = \Delta v$. Combining these equations yields $h''(v)\,|Dv|^2 = 0$, and since v is arbitrary, $h'' \equiv 0$. Thus, the equation is only invariant under affine transforms of the grey scale.

4.2. Basic operators of classical mathematical morphology

The basic operations of mathematical morphology, upon which all others are built, are dilations and erosions. In this subsection, we prove, using the mathematical tools obtained in Section 3, that the erosions are a multiscale analysis in the axiomatic sense, as well as the dilations.

Let B be a bounded subset of $\mathbb{R}^N$, which is called in mathematical morphology a "structuring element". Let $t \geqq 0$ be a scale parameter. The dilation at scale t of function f with structuring element B is defined by

$$D_t f(x) = \sup\{f(y), y - x \text{ in } tB\}.$$

Similarly, the erosion at scale t of function f with structuring element B is defined by

$$E_t f(x) = \inf\{f(y), y - x \text{ in } -tB\}.$$

One also defines the opening of f at scale t, by composing $D_t E_t$ and closing of f by $E_t D_t$.

We first prove that the scale-dependent erosions and dilations satisfy the semigroup axiom if and only if the structuring element B is convex. Indeed, let us apply D_t to the characteristic function f of a single point, 0 for instance. Then $D_t f$ is the characteristic function of $-tB$. Similarly, $D_{t+s} f$ is the characteristic function of $-(t+s)B$. Now, D_s applied to $D_t f$ yields $-tB - sB$.

If the semigroup property is true, then we must have $(t+s)B = tB + sB$. Conversely, it is obvious from the definition that if this property is satisfied, then dilation and erosion satisfy the semigroup property.

Lemma. *$(t+s)B = tB + sB$ for any s and $t \geqq 0$ if and only if B is convex.*

Proof. If B is convex, then by the definition of convexity, for any s and t, $sB + tB$ is included in $(s+t)B$, and the reverse inclusion is obvious. Conversely, assume that $sB + tB$ is contained in $(s+t)B$. Thus, for any x and y in B one can find z in B such that $(s+t)z = sx + ty$. This means that the barycenter of x and y with weights $s/(s+t)$ and $t/(s+t)$ is also in B. Since this is true for any positive s and t, this means that B is convex.

Thus, we can characterize the structuring elements B yielding families of erosions and dilations which satisfy the architectural axioms as the convex sets. Let us now prove that these families have an infinitesimal generator and can be obtained by a partial differential equation. We begin by defining the operator associated with a convex set B (which we assume, without loss of generality, to contain 0) by

$$\|y\|_B = \sup_{z \in B} y \cdot z,$$

where $y \cdot z$ denotes the Euclidean scalar product of y and z. When B is a ball, $\|\cdot\|_B$ is the usual Euclidean norm.

Proposition. *The multiscale analyses $D_t f(x)$ and $E_t f(x)$ satisfy all above-mentioned axioms (except* [Projection invariance]*). $u(t,x) = D_t f(x)$ is the viscosity solution of $\partial u/\partial t = \|Du\|_B$ and $u(t,x) = E_t f(x)$ is the viscosity solution of $\partial u/\partial t = -\|Du\|_B$.*

Proof. Of course, if we prove that these multiscale analyses satisfy all axioms, we can use Theorem 1 and deduce that $\partial u/\partial t = F(D^2 u, Du)$. Let us assume that this has been proved and find what F is. Let u_0 be a C^2 function on $\mathbb{R}^N$ and set $u(t,x) = D_t u_0(x)$. Of course, $u(0,x) = u_0(x)$ and

$$\begin{aligned} u(h,x) - u(0,x) &= \sup\{u_0(y), y - x \text{ in } hB\} - u_0(x) \\ &= \sup\{u_0(y) - u_0(x), y - x \text{ in } hB\}. \end{aligned}$$

Since $u_0(x)$ is differentiable in x, we get

$$\begin{aligned} u(h,x) - u(0,x) &= \sup\{Du_0(x) \cdot z, z \in hB\} + o(h) \\ &= h \sup\{Du_0(x) \cdot z, z \in B\} + o(h). \end{aligned}$$

By dividing by h and passing to the limit as h tends to zero, we get $\partial u/\partial t(0) = \|Du_0\|_B$. Thus the infinitesimal generator is $\|Du\|_B$. The proof for the erosions is the same.

We now check that dilations satisfy all the above-mentioned axioms (except [Projection invariance]). [Recursivity] has been proved in the preceding lemma. Let us prove [Regularity]. (The proof for erosions is identical because $D_t f = -E_t(-f)$.)

$$
\begin{aligned}
&D_t(f+hg)(x) - D_t f(x) - hg(x) \\
&\qquad = \sup\{f(y) + hg(y) \text{ for } y-x \text{ in } tB\} - \sup\{f(y) \text{ for } y-x \text{ in } tB\} - hg(x) \\
&\qquad \leqq \sup\{f(y) + hg(y) - f(y) \text{ for } y-x \text{ in } tB\} - hg(x) \\
&\qquad = h\sup\{g(y) - g(x) \text{ for } y-x \text{ in } tB\} \leqq Cht
\end{aligned}
$$

if g is in a family Q having uniformly bounded derivatives. By the same argument with $-f$ and $-g$ instead of f and g, we obtain

$$D_t(f+hg)(x) - D_t f(x) - hg(x) \geqq h\sup\{g(x) - g(y) \text{ for } y-x \text{ in } tB\} \geqq -Cht.$$

Let us now check [Locality]:

$$
\begin{aligned}
D_t f(x) - D_t g(x) &= \sup\{f(y) \text{ for } y-x \text{ in } tB\} - \sup\{g(y) \text{ for } y-x \text{ in } tB\} \\
&\leqq \sup\{f(y) - g(y) \text{ for } y-x \text{ in } tB\} = o(t).
\end{aligned}
$$

Using this relation with $-f$ and $-g$ yields

$$D_t f(x) - D_t g(x) \geqq \sup\{g(y) - f(y) \text{ for } y-x \text{ in } tB\} = o(t).$$

[Comparison principle] is obvious, as well as [Grey-scale invariance], [Translation invariance]. Thus [Generator] is also true by Theorem 1. Of course, [Isometric invariance] is true if and only if B is a ball. [Scale invariance] is trivially true with $t'(t,\lambda) = t$.

Remark on openings and closings. Set $O_t f(x) = v$. Then v can obviously be computed as $v(2t)$, where $v(t)$ is solution of the partial differential equation

$$\frac{\partial u}{\partial s} = -\operatorname{sign}^+(t-s)\|Du\|_B + \operatorname{sign}^+(s-t)\|Du\|_B,$$

where the first term acts as an erosion until time t and then vanishes, while the second term becomes active when s is greater than t and yields a dilation of scale t at time $2t$. Of course, closure is obtained by exchanging the signs of both terms. These last equations have the drawback of being discontinuous at $t = 1$. This can be easily remedied by making a change of time scale which makes the equation "slow down" when t approaches 1. If we set

$$\frac{\partial u}{\partial s} = -(T-s) + \|Du\|_B + (s-T)^+\|Du\|_B,$$

then it is easily seen that $O_t f$ is $v(2T)$, where T is chosen such that $t = Ts - s^2/2$ for $s = T$. In other terms, $O_t f = v(2(2T)^{1/2})$ where $v(s)$ is the solution of the preceding partial differential equation. Thus openings, closings (and every concatenation of closings and openings) can be obtained as the result of an evolution equation $\partial u/\partial t = F(t, \|Du\|)$.

4.3. The anisotropic diffusion of Perona & Malik

An important improvement of the classical linear multiscale analysis, with a more accurate edge detection, was proposed by PERONA & MALIK [47] who also proposed a partial differential equation model and were the first authors, to our knowledge, to state explicitly a maximum principle as a basic requirement in image processing. (Notice, by the way, that the comparison principle is also stated in mathematical morphology [54] as an important and useful property of erosion, dilation, opening and closing.)

The main idea of PERONA & MALIK is to introduce a part of the edge detection step in the filtering itself, allowing an interaction between scales from the beginning in the algorithm. They propose to replace the heat equation by a nonlinear equation:

$$\frac{\partial u}{\partial t} = \operatorname{div}(g(|Du|)\,Du), \quad u(0) = u_0. \tag{6}$$

In this equation, g is a smooth nonincreasing function with $g(0) = 1$, $g(s) \geqq 0$ and $g(s)$ tending to zero at infinity. The motivation is that the smoothing process obtained by the equation is "conditional": If $Du(x,y)$ is large, then the diffusion is small and therefore the exact localization of the "edges" is kept. If $Du(x,y)$ is small, then the diffusion tends to regularize still more around (x,y). Thus the choice of g corresponds to a sort of thresholding which has to be compared to the thresholding of $|Du|$ used in the final step of the classical theory explained above. Since this thresholding introduced anyway a nonlinear device, it was natural to use it earlier in the method, in the smoothing process itself. The experimental results obtained by PERONA & MALIK are perceptually impressive and show that an "edge detector" based on this theory gives edges which remain much more stable across the scales, making therefore the backward following of edges across scales unnecessary. However, the "anisotropic diffusion" of PERONA & MALIK is not properly a diffusion in the direction orthogonal to the gradient, and creates diffusion in the direction of the gradient. Indeed,

$$\operatorname{div}(g(|Du|)\,Du) = g(|Du|)\,\Delta u + g'(|Du|)\,|Du|^{-1} D^2u(Du, Du).$$

Set

$$u_{\xi\xi} = (u_{xx}u_y^2 - 2u_{xy}u_xu_y + u_{yy}u_x^2)/|Du|^2 = D^2u(Du^\perp/|Du^\perp|, Du^\perp/|Du^\perp|),$$

which is the linear diffusion term in the direction orthogonal to Du, and

$$u_{\eta\eta} = (u_{xx}u_x^2 - 2u_{xy}u_xu_y + u_{yy}u_y^2)/|Du|^2 = D^2u(Du/|Du|, Du/|Du|),$$

which is the diffusion term in the direction of $|Du|$. Then the anisotropic diffusion term can be rewritten as

$$\operatorname{div}(g(|Du|)\,Du) = g(|Du|)\,u_{\xi\xi} + G'(|Du|)\,u_{\eta\eta}, \quad \text{where } G(s) = sg(s).$$

This new form permits an easy interpretation. Since g is a positive function, the first term always is a diffusion "along the edges". The coefficient of the diffusion in the direction perpendicular to the edge can be:

positive if $G'(|Du|) > 0$. In this case, the equation behaves locally as a diffusion in both directions.

zero if $G'(|Du|) = 0$. Then we have a diffusion exactly in the direction orthogonal to the gradient.

negative if $G'(|Du|) < 0$. This corresponds to a local inversion of the heat equation. In this case, there is a combination of smoothing and "shock-capturing" in the equation. Related models have been developed by OSHER & RUDIN [45].

Let us discuss the well-posedness of the equation. If $G'(t) < 0$ for some t, then some images ask for a reverse heat equation. Then no uniqueness of the solution and no stability of the process can be expected. If $G'(t) \geqq 0$, then the equation has two "good" properties: uniqueness of the solution and the comparison principle.

Let us now check which other axioms of our list are true for the model. Of course, all architectural and all elementary invariance properties (shift in space and in grey scale, isometry) are true. One easily verifies that the scale invariance is true if and only if g is a power function. (The proof is similar to that given in Section 5 below for morphological operators.) The grey-scale invariance can never hold. Indeed, the equation can be rewritten as

$$\frac{\partial u}{\partial t} = F(D^2u, Du) \quad \text{with } F(A,p) = g'(|p|)|p|^{-1}A(p,p) + g(|p|)\,\text{trace}(A).$$

The grey-scale invariance of the multiscale analysis implies that $F(tA, tp) = tF(A,p)$ (see Section 5). Choosing for A a symmetric matrix such that $Ap = 0$ and $\text{tr}(A) \neq 0$, we get $g(t|p|) = g(|p|)$ for all $t > 0$. Thus, g is a constant function and the model is the heat equation, which is not grey-scale invariant.

4.4. The constrained smoothing of Osher & Rudin

OSHER & RUDIN [45, 51, 52] proposed a variational method for "denoising" images. Indeed, to denoise an image u_0, they propose to minimize $\int |Du(x)|\, dx$ under constraints corresponding to the a priori knowledge of the statistics of the noise (for instance, the variance). The minimization algorithm leads to the evolution equation $\partial u/\partial t = \text{div}(Du/|Du|) + \lambda(u)\,(u - u_0)$ where λ is a Lagrange multiplier depending on u.

4.5. The Alvarez-Lions-Morel "pure" anisotropic diffusion: mean curvature motion

In [2], ALVAREZ, LIONS & MOREL propose and study a class of nonlinear parabolic differential equations for image-processing which originates from the original idea of PERONA & MALIK:

$$\text{(1)} \qquad \frac{\partial u}{\partial t} = |Du|\,\text{div}(Du/|Du|), \quad u(0,x,y) = u_0(x,y)$$

where $u_0(x,y)$ is the grey level of the image to be processed, $u(t,x,y)$ is its smoothed version depending on the "scale parameter" t. Roughly speaking, the interpretation of the terms of the equation are as follows:
a) The term $|Du|\operatorname{div}(Du/|Du|) = \Delta u - D^2u(Du, Du)/|Du|^2$ represents a degenerate diffusion term, which diffuses u in the direction orthogonal to its gradient Du and does not diffuse at all in the direction of Du. The aim of the degenerate diffusion term is to make u smooth on both sides of an "edge" with a minimal smoothing of the edge itself. This means that for such a theory edges are nothing but the boundaries of the level lines of the image. Thus the equation answers exactly the requirements of PERONA & MALIK for an anisotropic diffusion, but there is no more an adaptive speed of the diffusion, while in the model of PERONA & MALIK the diffusion is multiplied by $1/|Du|$. Thus, the main difference between both models is that in the model of PERONA & MALIK, the diffusion is lowered when $|Du|$ is big. Thus "important" edges are better conserved: It is a contrast-dependent smoothing. But the pure anisotropic diffusion satisfies, as we shall see, [Grey-scale invariance]. Obviously, this property contradicts the use of contrast as a significant piece of information. Let us observe that this implies that "edge detection" and mathematical morphology are incompatible.

In the expression $\Delta u - (1/|Du|^2)\, D^2u(Du,Du)$, notice that the first term, the Laplacian, is the same as in scale-space theory and the second is an "inhibition" of the diffusion in the direction of the gradient.

Let us denote by ξ the coordinate associated with the direction orthogonal to Du. Therefore, a formulation of the preceding equation with respect to this new coordinate is

$$\frac{\partial u}{\partial t} = u_{\xi\xi},$$

where, of course, ξ depends on Du. In a more literal formulation,

$$\frac{\partial u}{\partial t} = \frac{1}{u_x^2 + u_y^2}\,(u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy})\,. \tag{5}$$

This equation has recently received a lot of attention because of its geometrical interpretation: Indeed, at least formally (see OSHER & SETHIAN [46], EVANS & SPRUCK [16], e.g.) the level sets of the solution move in the normal direction with a speed proportional to their mean curvature (curvature in two dimensions). (This "mean-curvature motion" effect will be shown in the experimental results presented below.) As we shall see, this property of mean-curvature motion implies that the proposed multiscale analysis satisfies the main "morphological" axiom, that is, the invariance of the analysis under any rearrangement of the grey-level scale. Thus, the mean-curvature motion is a morphological operator, which has been ignored by mathematical morphology. It is however quite close to the basic operations of mathematical morphology. Indeed, dilation and erosion can be interpreted as a (forward and backward) motion of level curves at constant spead in the direction of the normal.

A theory of weak solutions based upon the so-called viscosity solutions theory has been proposed by CHEN, GIGA & GOTO [10], EVANS & SPRUCK [16], GIGA, GOTO, ISHII & SATO [20], and SONER [58], among others.

5. Axioms and new operators of multiscale mathematical morphology

General assumptions on the multiscale analysis. In this section, we consider a multiscale analysis defined by an equation $\partial u/\partial t = F(D^2u, Du, t)$ and we assume that $F(A,p,t) \geqq F(B,p,t)$ if $A \geqq B$ (comparison principle). The F's under consideration are always assumed to be continuous for $p \neq 0$. We also assume that the preceding equation uniquely defines $T_t f$ as its solution with initial condition $u(0) = f$. The uniqueness can be obtained for instance by using the concept of viscosity solutions. (Thus, we extend our study to multiscale analyses which are more general than those deduced from the axioms of Theorems 1 and 2, but always correspond to the assumptions of Corollary 1.) Indeed, we shall obtain a very interesting multiscale analysis when the continuity assumption on F is weakened. (This weakening implies that the proposed analysis somehow satisfies weaker versions of [Regularity] and [Locality].)

If the multiscale analysis T_t satisfies the morphological axiom [Grey-scale invariance], and if the dimension N is 2, we shall prove, as in [10], that F can be rewritten so that the preceding equation becomes

$$\frac{\partial u}{\partial t} = |Du|\, G\left(\operatorname{curv}(u), \frac{Du}{|Du|}, t\right),$$

where G is nondecreasing with respect to its first term and

$$\operatorname{curv}(u)(x) = \operatorname{div}\left(\frac{Du}{|Du|}\right) = |Du|^{-1}\left(\Delta u - D^2u\left(\frac{Du}{|Du|}, \frac{Du}{|Du|}\right)\right).$$

This term can be interpreted as the curvature of the level line $\{y, u(y) = u(x)\}$ passing through x. *Thus we obtain a general description of morphological multiscale operators.*

The only operators already known in this class are the morphological operators (dilation, contraction and their combinations) which correspond to the case where G does not depend on $\operatorname{curv}(u)$ and has the special form: $\partial u/\partial t = \pm\|Du\|_B$. If we add [Isometric invariance], then the resulting equation in dimension 2 is, as we shall see,

$$\frac{\partial u}{\partial t} = |Du|\, G(\operatorname{curv}(u), t).$$

If, in addition, we assume [Scale invariance], then (modulo a possible rescaling of time $t' = h(t)$) the equation is

$$\frac{\partial u}{\partial t} = |Du|\, \beta(t \operatorname{curv}(u)),$$

where β is a continuous nondecreasing function. Imposing the projection invariance leads to a single equation

$$\frac{\partial u}{\partial t} = |Du| \, (t \operatorname{curv}(u))^{1/3}.$$

Let us now examine the case of higher dimensions. If we assume the isometry invariance, then we arrive at the equation

$$\frac{\partial u}{\partial t} = G(\lambda_1, \dots, \lambda_{N-1}, |Du|, t),$$

where $\lambda_1, \dots, \lambda_{N-1}$ are the principal curvatures of the level hypersurface, for some continuous function G defined on $\mathbb{R}^{N-1} \times [0, +\infty[\times [0, +\infty[$ which is symmetric with respect to $(\lambda_1, \dots, \lambda_{N-1})$, positively homogeneous with respect to $(\lambda_1, \dots, \lambda_{N-1}, |Du|)$ and nondecreasing with respect to each λ_i $(1 \leqq i \leqq N-1)$ for all t in $[0, +\infty[$.

5.1. General form of morphological multiscale analysis

The main axiom which we introduce in this section is

[Grey-scale invariance] $T_t(h(f)) = h(T_t(f))$ *for all* f *in* X *and all* $t \geqq 0$, *where* h *is any* C^∞ *nondecreasing real function.*

In the calculations below, we shall ignore the t-dependence of F because it does not change the calculations and results; we write $F(D^2u, Du)$ instead of $F(D^2u, Du, t)$.

In view of the results of the preceding section, it is necessary and sufficient to consider a locally bounded function F on $S^N \times \mathbb{R}^N$ satisfying (4) and to ask when do we have

$$F(h'(f)\, D^2 f + h''(f)\, Df \otimes Df, h'(f)\, Df) = h'(f)\, F(D^2 f, Df) \quad \text{on } \mathbb{R}^N$$

for every f in $C_b^\infty(\mathbb{R}^N), h$ in $C^\infty(\mathbb{R}^N), h$ nondecreasing. This last relation is obtained by using the fact that if $u(t) = T_t(f)$ is solution of (6), that is, if $\partial u/\partial t - F(D^2u, Du) = 0$, then $h(u(t))$ is also solution of (6).

Next, it is easy to see that [Grey-scale invariance] is in fact equivalent to

[Grey-scale invariance]$'$ $F(\mu A + \lambda p \otimes p, \mu p) = \mu F(A, p)$ *for all* $\mu \geqq 0$, A *in* S^N, p *in* $\mathbb{R}^N$, $p \neq 0$, λ *in* $\mathbb{R}$.

For $\lambda = 0$, this yields in particular that F is positively homogeneous:

(13) $\quad F(tA, tp) = tF(A, p) \quad$ for all $t \geqq 0$, A in S^N, p in $\mathbb{R}^N$, $p \neq 0$.

(In particular, $F(0,0) = 0$ by continuity.) Now, if (13) holds, it is clear that [Grey-scale invariance]′ reduces to

(14) $F(A + \lambda p \otimes p, p) = F(A,p)$ for all λ in $\mathbb{R}$, A in S^N, p in $\mathbb{R}^N$, $p \neq 0$.

Of course, if $N = 1$, this implies that F depends only on p, and in view of (13), F is necessarily given by

(15)
$$F(A,p) = ap^+ + bp^- \quad \text{for some } a,\, b \text{ in } \mathbb{R},$$
$$\text{where } p^+ = \max(p,0),\ p^- = \min(p,0).$$

Nontrivial situations occur if $N \geqq 2$, an assumption we make in all that follows. Our main result is

Theorem 4. *Let T_t be a multiscale analysis satisfying* [Grey-scale invariance]. *Then the associated F satisfies*

(16) $F(A,p) = F(Q_p A Q_p, p)$ *for all A in S^N, p in $\mathbb{R}^N$, $p \neq 0$,*

where Q_p is the projection matrix given by $Q_p = I_N - p \otimes p/|p|^2$.

Remark. Q_p is the matrix that corresponds to the projection onto the orthogonal complement of $\mathbb{R}p$, that is, onto the hyperplane orthogonal to p. The condition (4) is essential: Indeed, if $N = 2$, $F(A,p) = a_{11}p_2/|p| - a_{12}p_1/|p|$ satisfies (14) (and (13)!) but (16) does not hold (choose $p_1 = 1$, $p_2 = 0$, $A^t(1,0) = {}^t(0,1)$ and $A^t(0,1) = {}^t(1,0)$ so that $Q_pAQ_p = 0$ while $F(A,p) = -1$).

Proof of Theorem 4. In order to show (16), it is enough to fix p in $\mathbb{R}^N$, $p \neq 0$. We can select a coordinate system such that $p = |p|\,(0,\dots,0,1)$, in which case Q_pAQ_p becomes $(A'_{ij})_{1 \leqq i,j \leqq N}$ where $A'_{ij} = A_{ij}$ if $1 \leqq i,\ j \leqq N-1$, $A'_{ij} = 0$ otherwise, while $p \otimes p = |p|^2\,(\delta_{Ni}\delta_{Nj})_{1 \leqq i,j \leqq N}$. Then clearly (14) implies that $F(A)$ does not depend on A_{NN}. Set $M = a_{N1}^2 + \dots + a_{N,N-1}^2$ and $I_\varepsilon = \varepsilon I_N + (M/\varepsilon - \varepsilon)\ (\delta_{Ni}\delta_{Nj})_{1 \leqq i,j \leqq N}$. Then one easily checks that $Q_pAQ_p \leqq A + I_\varepsilon$ and $A \leqq Q_pAQ_p + I_\varepsilon$. Using $F(A,p) \geqq F(B,p)$ if $A \geqq B$, and letting ε tend to zero, we obtain $F(A,p) = F(Q_pAQ_p, p)$ since F does not depend on A_{NN} and is continuous for $p \neq 0$.

Corollary 2. *Let $N = 2$ and let T_t be a multiscale analysis satisfying* [Grey-scale invariance]. *Then the associated F satisfies*

(17)
$F(A,p) = G(\operatorname{tr}(A) - |p|^{-2}\operatorname{tr}(A \cdot p \otimes p), p)$ *for all A in S^2, p in $\mathbb{R}^2$, $p \neq 0$*

for some function G on $\mathbb{R} \times \mathbb{R}^2$, continuous on $\mathbb{R} \times (\mathbb{R}^2 - \{0\})$, satisfying

(18) *$G(t,p)$ is nondecreasing with respect to t in $\mathbb{R}$ for all p in $\mathbb{R}^2$, $p \neq 0$.*

Proof. One just needs to observe that if we use an orthogonal coordinate system on $\mathbb{R}^2$ whose second basis vector is given by $p/|p|$, then

$$Q_pAQ_p = \begin{pmatrix} \mathrm{tr}(Q_pAQ_p) & 0 \\ 0 & 0 \end{pmatrix},$$

and moreover,

$$\begin{aligned} \mathrm{tr}(Q_pAQ_p) &= \mathrm{tr}(A) - \mathrm{tr}(AP) - \mathrm{tr}(PA) + \mathrm{tr}(PAP) \\ &= \mathrm{tr}(A) - 2\mathrm{tr}(AP) + \mathrm{tr}(AP^2) \\ &= \mathrm{tr}(A) - \mathrm{tr}(AP) \end{aligned}$$

where $P = p \otimes p/|p|^2$ (so that $P^2 = P$).

Another important application of Theorem 4 is to the situation where we add an isometry invariance axiom:

[Isometric invariance] $T_t(R \cdot f) = R \cdot T_t(f)$ *for all* f *in* X, $t \geqq 0$ *and for all transforms* R *defined on* X *by*

(19) $$(R \cdot f)(x) = f(Rx)$$

where R *is an orthogonal transform of* $\mathbb{R}^N$.

Proceeding as we did for obtaining [Grey-scale invariance]$'$, we immediately translate this condition in terms of F into

(20) $$F({}^tRAR, {}^tRp) = F(A,p) \quad \text{for all } A \text{ in } S^N,\ p \text{ in } \mathbb{R}^N,\ p \neq 0,\ R \text{ in } O^N,$$

where O^N denotes the group of orthogonal transforms of $\mathbb{R}^N$.

We need to introduce some notation. Since Q_pAQ_p leaves $(\mathbb{R}p)^\perp$ invariant (for $p \neq 0$ fixed) and since Q_pAQ_p vanishes on $\mathbb{R}p$, Q_pAQ_p admits N real eigenvalues which include 0 and $(\lambda_1, \dots, \lambda_{N-1})$, which are the real eigenvalues of the restriction of Q_pAQ_p to $(\mathbb{R}p)^\perp$. Of course, if $N = 2$, then $\lambda_1 = \mathrm{tr}(Q_pAQ_p) = \mathrm{tr}(A) - \mathrm{tr}(Ap \otimes p/|p|^2)$.

With this notation we can prove the

Corollary 3. *Let* $N \geqq 2$. *Let* T_t *be a multiscale analysis satisfying* [Grey-scale invariance]. *If* [Isometric invariance] *holds, then*

(21) $F(A,p) = G(\lambda_1, \dots, \lambda_{N-1}, |p|)$, *for all* A *in* S^N, p *in* $\mathbb{R}^N$, $p \neq 0$, *for some continuous function* G *defined on* $\mathbb{R}^{N-1} \times [0, +\infty[$ *which satisfies*

(22) $G(\lambda_1, \dots, \lambda_{N-1}, q)$ *is symmetric with respect to* $(\lambda_1, \dots, \lambda_{N-1})$ *and nondecreasing with respect to each* λ_i $(1 \leqq i \leqq N-1)$ *for all* q *in* $[0, \times \infty[$.

Remark. We shall present below some examples of such nonlinear functions. At this stage, let us mention only the special one where $G = g(\Sigma_{i=1,\dots,N-1} \lambda_i, q)$, i.e., $F(A,p) = g(\mathrm{tr}(A) - \mathrm{tr}(Ap \otimes p/|p|^2, |p|))$ where $g(r,q)$ is nondecreasing with respect to r for all q in $[0, \infty[$.

Proof of Corollary 3. The first step consists in showing that for $p \neq 0$ fixed, (20) implies that F is a function only (of p and) of $(\lambda_1, \ldots, \lambda_{N-1})$ which satisfies (22). In order to do so, we consider the subgroup of O^N defined by those transforms R that leave p invariant, i.e., $Rp = p$ so that ${}^tRp = p$. Then, clearly Q_p and R commute and thus (20) implies that $F({}^tRQ_pAQ_pR) = F(Q_pAQ_p)$, where we ignore the dependence upon p, which is fixed. Therefore, $F(A,p) = G_1(\lambda_1, \ldots, \lambda_{N-1}, p)$, where $(\lambda_1, \ldots, \lambda_{N-1})$ are the eigenvalues of Q_pAQ_p. Let R be in O^N and $q = {}^tRp$. We notice that $Q_q{}^tRARQ_q = {}^tRQ_pAQ_pR$, and then $Q_q{}^tRARQ_q$ and Q_pAQ_p have the same eigenvalues. Thus by using [Isometric invariance], we obtain $F(A,p) = F({}^tRAR, {}^tRp) = G_1(\lambda_1, \ldots, \lambda_{N-1}, {}^tRp)$. Thus F only depends on the modulus of p. Therefore we can write $F(A,p) = G_1(\lambda_1, \ldots, \lambda_{N-1}, |p|)$. Moreover, F is a symmetric function of $(\lambda_1, \ldots, \lambda_{N-1})$, and, since F satisfies (4), it is clearly nondecreasing with respect to each λ_i for $1 \leqq i \leqq N-1$.

5.2. Geometric interpretation of the derived equations

Let $u \in C^2(\mathbb{R}^N)$ and let $x_0 \in \mathbb{R}^N$. If $Du(x_0) \neq 0$, there is, near x_0, a C^2 hypersurface containing x_0 on which u is constant: It is of course the level surface of u going through x_0 (or level curve if $N = 2$). Then, $(\mathbb{R}Du(x_0))^{\perp}$ is the tangent hyperplane and $|Du|^{-1}Q_{Du}D^2uQ_{Du}$ restricted to $(\mathbb{R}Du)^{\perp}$ is the curvature tensor. In particular, the $\lambda_i/|Du|$ are the principal curvatures $(1 \leqq i \leqq N-1)$ while $\operatorname{Curv}(u)$ is the mean curvature. We explain now how the above classification can be understood and recovered (at least formally) in a geometrical way.

The "mathematical morphology" axiom [Grey-scale invariance] means that the evolution equation reflects in fact only an evolution of the level curves or surfaces $\Gamma = \{u = \lambda\}$ for some λ in $\mathbb{R}$. Of course, all this is a bit formal since the evolution of the curve or the surface has to be carefully defined. It is possible to view mathematical morphology as the motion of level sets of the level of grey of the image and then the axioms say that this motion should be translation-invariant, local (one might also add isometry invariance but this is not needed in the main line of arguments). At a mathematical level, we may parametrize the level surface with curvilinear $X(y,t)$ coordinates in $\mathbb{R}^N$ with y in $\mathbb{R}^{N-1}$, $t \geqq 0$, and the motion can then formally be described by

$$X' = -g(X)\,n \tag{23}$$

where $n = n(X)$ is the unit normal, g depends upon $X(\cdot, t)$ and corresponds to the "velocity prescription" while $X' = \partial X/\partial t$. With arguments similar to those developed in Section 3, one can show that (if the evolution is well defined on smooth surfaces for a short time) g can depend only on the normal and the curvature tensor at the point X (i.e., the first and second derivatives of X with respect to y). In addition, g must be nondecreasing with respect to the curvature tensor in order to avoid the crossing of two different level surfaces or curves. (Otherwise the reconstruction of u would be meaningless!) Recall next that $n = Du/|Du|$ and that the curvature tensor is essentially given by $|Du|^{-1} Q_{Du} D^2 u Q_{Du}$. The formulation in terms of u can be understood as follows (See OSHER & SETHIAN [46], CHEN, GIGA & GOTO [19], BARLES [5], EVANS & SPRUCK [16], SONER [58], among others): By definition, we have

$$(24) \qquad u(X(t),t) = \lambda \quad \text{for } t \geqq 0.$$

Hence, if we differentiate (24) with respect to t, we obtain $\frac{\partial u}{\partial t} - (Du \cdot n)\, g = 0$, or

$$(25) \qquad \frac{\partial u}{\partial t} - |Du|\, g(|Du|^{-1} Q_{Du} D^2 u Q_{Du}, Du/|Du|) = 0$$

and we recover precisely the classification we obtained above!

6. Scale and projection-invariant multiscale analysis ($N = 2$)

General assumptions on the multiscale analysis. In this section, as in the preceding one, we consider a multiscale analysis defined by an equation $\partial u/\partial t = F(D^2 u, Du, t)$ and we assume the comparison principle that $F(A, p, t) \geqq F(B, p, t)$ if $A \geqq B$. The F's under consideration are always continuous for $p \neq 0$. We also assume that the preceding equation uniquely defines $T_t f$ as its solution with initial condition $u(0) = f$. The uniqueness can, for instance, be obtained by using the concept of viscosity solution. We also assume, to avoid spurious cases, that the analysis is not cyclic, that is, $T_t = T_s$ implies $t = s$. In this section, by "multiscale analysis" we always mean a family of operators T_t satisfying the preceding assumptions. This section is devoted to a study of the effect of the scale-invariance axiom, and more generally of the projection invariance on the multiscale analysis. For simplicity, we restrict the study to the dimension $N = 2$ and we also assume the isometry invariance of the multiscale analysis.

We state the scale invariance more precisely as

[Scale invariance] *For any positive λ and t, there exists $t' > 0$ such that $D_\lambda T_{t'} = T_t D_\lambda$. Moreover, $t'(t, \lambda)$ is differentiable with respect to λ at $\lambda = 1$, and the function $g(t) = \partial t'/\partial \lambda (t, 1)$ is continuous and positive for $t > 0$.*

Remark. The assumption on t', $g(t) = \partial t'/\partial \lambda (t, 1) > 0$, can be interpreted by looking at the relation $D_\lambda T_{t'} = T_t D_\lambda$ when the scale λ increases, i.e., when the size of the image is reduced before analysis by T_t. Then, the corresponding analysis time before reduction is increased. In less formal terms, we can say that the analysis time somehow increases with the size of the picture. This is a natural assumption, satisfied by all classical models. The continuity and differentiability assumptions on t' are also satisfied by all classical models and seem natural.

Notice that no condition has been imposed on the relation between t' and (t, λ). In order to fix ideas, let us examine the classical multiscale analysis with respect to this axiom. In the case of the basic morphological operators, dilation and erosion, it is easily seen that $t'(t,\lambda) = \lambda t$. In the case of the heat equation and of anisotropic diffusion, one has $t'(t,\lambda) = \lambda^2 t$. The meaning of these relations is clear: When, for instance, λ is bigger than 1, D_λ contracts the image. Thus, the analysis scale must be smaller. Now, it is clear that the dependence of t' upon the scale t and the contraction ratio λ cannot be a priori fixed since it is different in those classical examples. So we shall only make regularity assumptions on the function t' and try to deduce its form.

For any linear transform A, set $Af(x) = f(Ax)$. With the same formalism as in [Scale invariance], we shall finally consider

[Projection invariance] *For any A and t, there exists $t'(t,A)$ such that $AT_{t'} = T_t A$.*

Notice that scale invariance is a particular case of projection invariance. Therefore, we keep the notation t' in [Projection invariance] and we prove in the next lemma that the functions t' in both axioms can be identified.

Lemma 1 ($N = 2$). *Let $(T_t)_{t>0}$ be a multiscale analysis satisfying* [Projection invariance] *and* [Scale invariance] *for continuous scale space functions $t'(t,B)$ and $t'(t,\lambda)$. Then*

(i) *For any linear transforms B and C and any t one has the semigroup property $t'(t,BC) = t'(t'(t,B),C)$ (or $t'(t,\lambda\mu) = t'(t'(t,\lambda),\mu)$).*

(ii) *$t'(t,B)$ is increasing with respect to t. Moreover, if R is an isometry, then $t'(t,B) = t$. (Thus* [Projection invariance] *implies* [Isometric invariance].)

(iii) *The function $t'(t,B)$ only depends on t and $|\det B|$: $t'(t,B) = t'(t,|\det B|^{1/2})$.*

(iv) *There exists a unique differentiable increasing function h on $\mathbb{R}^+$ such that $h(0) = 0$, $h(1) = 1$, $h(t) \to +\infty$ when $t \to +\infty$, and $t'(t,\lambda) = h^{-1}(\lambda h(t))$. Therefore, if $S_t = T_{h^{-1}(t)}$, then the projection invariance holds with $t'(t,B) = |\det B|^{1/2} t$.*

Proof. (i) We have $BCT_{t'(t,BC)} = T_t BC = BT_{t'(t,B)}C = BCT_{t'(t'(t,B),C)}$. The map which associates T_t with t being one-to-one, this implies the stated relation.

(ii) Let us prove that $t'(t,A)$ is one-to-one with respect to t for any A. Indeed, if not, there would exist some A and some (s,t) such that $t'(t,A) = t'(s,A)$. Thus, by projection invariance, $AT_{t'(t,A)} = T_t A$ and $AT_{t'(s,A)} = T_s A$. Hence, $T_s = T_t$, which contradicts the fact that T_t is one-to-one. Notice that $t'(0,\lambda) = 0$ (which follows immediately from the injectivity of the family T_t). Since t' is continuous, one-to-one and nonnegative, we deduce that it is increasing with respect to t.

Let R be an isometry. Then, iterating the formula of (i) we have $t'(t'(t'\ldots, t'(t,R),R,R,\ldots,R))) = t'(t,R^n)$. Since there exists a subsequence of R^n tending to Id and since t' is continuous, we have for this subsequence $\lim t'(t'(t'(\ldots, t'(t,R),R,R,\ldots,R))) = t'(t,\mathrm{Id}) = t$. (Notice that $t'(t,\mathrm{Id}) = t$ because of [Projection invariance] and the fact that the map $t \to T_t$ is one-to-one.)

Assume for contradiction that $t'(t,R) = t'' < t$. Then $t'(t'(t,R),R) = t'(t'',R) \leqq t'(t,R) = t''$ by the monotonicity of t' with respect to t. Iterating, we obtain $t'(t'(t',\dots,t'(t,R),R,R,\dots,R))) \leqq t'' < t$. This is a contradiction, since the left member tends to t. Thus $t'(t,R) \geqq t$. We prove the converse inequality in the same way; therefore $t'(t,R) = t$.

(iii) We begin by noting that any linear transform B of $\mathbb{R}^2$ can be obtained as a product of isometries and of affine transforms of the kind $A(\lambda)$: $(x,y) \to (\lambda x, y)$ where λ is nonnegative. Thus $B = R_1 A(\lambda_1) R_2 A(\lambda_2) \dots A(\lambda_n) R_n$. Iteratively using (i) and (ii), we obtain

$$t'(t,B) = t'(t'(t'(\dots,t'(t,A(\lambda_1)),A(\lambda_2)),\dots,A(\lambda_n)))\dots).$$

Again using (i) and the relation $A(\lambda_1)A(\lambda_2)\dots A(\lambda_n) = A(\lambda_1\lambda_2\dots\lambda_n)$, we obtain

$$t'(t,B) = t'(t,\lambda_1\lambda_2\dots\lambda_n) = t'(t,|\det(B)|).$$

(iv) Using (i) and (iii), we have $t'(t,\lambda\mu) = t'(t'(t,\mu),\lambda)$ for any positive λ and μ. Differentiating this relation with respect to μ and setting $\mu = 1$ yields

$$\lambda \frac{\partial t'}{\partial t}(t,\lambda) = \frac{\partial t'}{\partial t}(t,1)\frac{\partial t'}{\partial t}(t,\lambda). \tag{26}$$

Remark. The last relation is an identity, for each positive t, between Radon measures on $\mathbb{R}$. Indeed, since $t'(t,\lambda)$ is a nondecreasing function with respect to t, $\partial t'/\partial t(t,\cdot)$ is a Radon measure on $\mathbb{R}$. Thus by (26) the derivative with respect to λ makes sense and we see that t' is nondecreasing with respect to λ. In (26) and in the following, we use the following easy fact: If $u(\lambda) = k(h(\lambda))$ is a nondecreasing function on $\mathbb{R}$, and if h is a C^1 function with $h' > 0$, then $u'(\lambda) = k'(h(\lambda))h'(\lambda)$ and $k'(h)$ is a Radon measure.

Set $g(t) = \partial t'/\partial\lambda(t,1)$. In order to show that there is a function h such that $t'(t,\lambda) = h^{-1}(\lambda h(t))$ for every $t > 0$ and $\lambda > 0$, we set $G(x,y) = t'(x,y/h(x))$, so that $t'(t,\lambda) = G(t,h(t)\lambda)$. This change of function is permissible if $h(t) > 0$ for $t > 0$. Then (26) becomes

$$h(t)\lambda\frac{\partial G}{\partial y}(t,h(t)\lambda) = g(t)\left(\lambda h'(t)\frac{\partial G}{\partial y}(t,\lambda h(t)) + \frac{\partial G}{\partial x}(t,\lambda h(t))\right).$$

We now choose h so that $\partial G/\partial x = 0$. It is enough to set $h(t) = \exp\left(\int_1^t ds/g(s)\right)$. Since $gh' = h$, we obtain $\partial G(t,\lambda h(t))/\partial x = 0$ and therefore $\partial G(x,y)/\partial x = 0$. Thus $G(x,y) = \beta(y)$ for some differentiable nondecreasing function β. We obtain that $t'(t,\lambda) = \beta(h(t)\lambda)$. Returning to the definition of $g(t)$, we have $g(t) = \partial t'(t,1)/\partial\lambda = h(t)\beta'(h(t))$. Since $gh' = h$, we deduce that $1 = h'(t)\beta'(h(t))$. Integrating this last relation between 0 and t yields $\beta(h(t)) = t + \beta(h(0))$. Using the fact that $t'(0,\lambda) = 0$ (which derives immediately from the injectivity of the family T_t), we obtain $b(h(0)) = 0$ and therefore $t'(t,\lambda) = h^{-1}(\lambda h(t))$. Notice that $h(0) = 0$ because if not, $t'(t,\lambda)$ could not be defined for small λ, and $h(+\infty) = +\infty$ because if not, $t'(t,\lambda)$ could not be defined for large λ.

To finish the proof, we set $S_t = T_{h^{-1}(t)}$ and we prove that the projection invariance holds with $t'(t,\lambda) = \lambda t$:

$$S_t B = T_{h^{-1}(t)} B = B T_{t'(h^{-1}(t),\lambda)} = B T_{h^{-1}(\lambda h(h^{-1}(t)))} = B T_{h^{-1}(\lambda t)} = B S_{\lambda t}.$$

Theorem 5. $(N = 2)$. (i) *Let T_t be a multiscale analysis. If the multiscale analysis satisfies* [Projection invariance] *and* [Scale invariance], *then, with a possible rescaling $\tau = h(t)$,*

$$\frac{\partial u}{\partial t} = \frac{1}{t} G\left(t^4 |Du|^3 \operatorname{curv}(u), t^4 |Du|^2 \left(u_{\eta\eta} - \frac{(u_{\eta\xi})^2}{u_{\xi\xi}}\right)\right)$$

where G is a continuous function, η represents the direction of the gradient and ξ represents the orthogonal direction of the gradient.

(ii) *Let T_t be a multiscale analysis defined by an equation $\partial u/\partial t = |Du|\, G(\operatorname{curv}(u), t)$ (which derives from the fact that the multiscale analysis satisfies* [Grey-scale invariance], [Isometric invariance]). *If the multiscale analysis satisfies* [Scale invariance] *and the assumptions of Lemma 1, then, with a possible rescaling $\tau = h(t)$, the equation of the multiscale analysis is*

$$\frac{\partial u}{\partial t} = |Du|\, \beta(t \operatorname{curv}(u)),$$

where β is a continuous nondecreasing real function.

(iii) *If both hypothesis* (i) *and* (ii) *hold, then the only possible equation is, up to a rescaling,*

$$\frac{\partial u}{\partial t} = |Du|\ (t \operatorname{curv}(u))^{1/3},$$

Remark. If we want [Recursivity] (after some rescaling) to be satisfied by the model obtained in (i) and (ii), then β is necessarily a power function, $\beta(s) = |s|^{p-1} s$.

Proof. (i) By the preceding lemma, we can assume that $t'(t,B) = \lambda^{1/2} t$, where $\lambda = |\det(B)|$. Set $u(t) = T_t u_0$ and $v(t,x) = u(\lambda^{1/2} t, Bx) = B T_{t'} u_0$. Since by assumption, $B T_{\lambda t} u_0 = T_t B u_0$, we also have $v(t) = T_t B u_0$ and $v(t,x)$ is solution of $\partial v/\partial t = F(D^2 v, Dv, t)$. Thus

$$\lambda \frac{\partial u}{\partial t}(\lambda t, Bx) = F(B D^2 u^t B, BDu, t).$$

Now $u(t,x)$ is solution of $\partial u(\lambda^{1/2} t, Bx)/\partial t = F(D^2 u, Du, \lambda^{1/2} t)$ and by matching both equations we get

$$F(B D^2 u\, {}^t B, BDu, t) = \lambda^{1/2} F(D^2 u, Du, \lambda^{1/2} t).$$

Thus

$$F(BA^tB, Bp, t) = |\det B|^{1/2} F(A, p, |\det B|^{1/2} t) \tag{27}$$

for any vector p and symmetric matrix A. Let us choose B such that $\det B = 1$ and $Bp = (1,0)$. This is equivalent to taking $p/\|p\|^2 - ap^\perp$ as the first row for B for an arbitrary real number a and taking $p^\perp$ as second row. Then, by using (27), we obtain

$$F(A,p,t) = F\big(A(p^\perp,p) - aA(p^\perp,p^\perp), A(p^\perp,p^\perp), a^2A(p^\perp,p^\perp) - 2aA(p,p^\perp) + A(p,p), (1,0), t\big)$$

Since this relation is true for any a we can choose $a = A(p^\perp,p)/A(p^\perp,p^\perp)$, and we obtain

$$F(A,p,t) = G\left(A(p^\perp,p^\perp), A(p,p) - \frac{A(p^\perp,p)^2}{A(p^\perp,p^\perp)}, t\right).$$

Again using (27) for $B = \lambda\,\mathrm{Id}$, we obtain

$$F(BA^tB, Bp, t) = G\left(\lambda^4 A(p^\perp,p^\perp), \lambda^4\left(A(p,p) - \frac{A(p^\perp,p)^2}{A(p^\perp,p^\perp)}\right), t\right)$$

$$= \lambda F(A,p,t\lambda) = \lambda G\left(A(p^\perp,p^\perp), A(p,p) - \frac{A(p^\perp,p)^2}{A(p^\perp,p^\perp)}, t\lambda\right).$$

Therefore if we replace t by t/λ and we choose $\lambda = t$, we get

$$F(A,p,t) = \frac{1}{t} G\left(t^4 A(p^\perp,p^\perp), t^4\left(A(p,p) - \frac{A(p^\perp,p)^2}{A(p^\perp,p^\perp)}\right)\right).$$

(ii) We consider the equation $\partial u/\partial t = |Du|\, G'(\mathrm{curv}(u), t)$. We can make the same initial calculations as above and obtain $G'(s,t\lambda) = G'(\lambda s, t)$. Changing t to t/λ, and taking $\lambda = 1/t$ we get $G'(s,t) = G'(ts,1) = \beta(ts)$ for some function β.

(iii) By (i), (ii) and Lemma 1(iv), there exists a rescaling h for which F assumes the two representations given by (i) and (ii). Then,

$$F(A,p,t) = \frac{1}{t} G\left(t^4 A(p^\perp,p^\perp), t^4\left(A(p,p) - \frac{A(p^\perp,p)^2}{A(p^\perp,p^\perp)}\right)\right)$$

$$= \beta\left(\frac{tA(p^\perp,p^\perp)}{|Du|^3}\right)|Du|.$$

Thus, G does not depend on its second argument, and we have

$$G(s) = tk\beta\left(\frac{s}{(kt)^3}\right) \qquad \text{for any } k,\ t > 0 \text{ and } s.$$

Letting $tk = |s|^{1/3}$, we obtain $\beta(s) = |s|^{1/3}\beta'(\mathrm{sign}(s))$. Then (iii) follows by using that $F(-A,-p,t) = -F(A,p,t)$.

7. Multiscale analysis of movies

We formalize a movie as a bounded function $u_0(x, y, \theta)$ defined on $\mathbb{R}^3$, where x and y are the spatial variables and θ the time variable. This section is concerned with the axiomatic characterization of the multiscale analysis T_t of movies. As in the preceding section, we assume that the map $t \to T_t$ is one-to-one.

Of course, all axioms (including comparison principle) considered in the second section make sense in this context, but we need to specify them and to take into account the special role of time (θ). In all that follows, the multiscale analyses to be considered are of the form $\partial u/\partial t = F(D^2u, Du, t)$, where F now has ten scalar arguments. The assumptions on F are the same as those in Sections 3 and 5, that is, $F(A, p, t)$ is nondecreasing with respect to its first argument, and $F(A, p, t)$ is continuous at all points where $p \neq 0$.

Finally, we assume that the solution $u(x, y, \theta, t)$ of the equation is uniquely defined as a viscosity solution. (This will of course be checked a posteriori for the models we derive).

Let us now consider the morphological invariants: We keep [Grey-scale invariance], which yields the relations obtained in Section 3:

(13) $$F(\lambda A, \lambda p, t) = \lambda F(A, p, t) \quad \text{for all } \lambda \geqq 0,\ A \text{ in } S^3,\ p \text{ in } \mathbb{R}^3,\ p \neq 0.$$

(16) $$F(A, p, t) = F(Q_p A Q_p, p, t) \quad \text{for all } A \text{ in } S^3,\ p \text{ in } \mathbb{R}^3,\ p \neq 0,$$

where Q_p is the projection matrix given by $Q_p = I_N - p \otimes p/|p|^2$.

We assume the analysis to be invariant under all linear transforms of the spatial plane $\mathbb{R}^2 \times \{0\}$. Therefore, by Lemma 1, we can find a rescaling of the multiscale analysis such that the corresponding F satisfies

[Projection invariance] $F(BA\,{}^tB, Bp, t) = |\det(B)|^{1/2} F(A, p, t|\det(B)|^{1/2})$.

7.1. *Axioms for multiscale analysis of movies*

We now state two new axioms which take into account the particular nature of movies. First, we assume the analysis to be invariant when the movie is uniformly slowed down or accelerated. Set $S_\lambda(x, y, \theta) = (x, y, \lambda\theta)$ and, with the usual notation, $S_\lambda u = u(x, y, \lambda\theta)$.

[Time-scale invariance] *For any real number λ and $t \geqq 0$, there exists $t'(t, \lambda)$ such that $S_\lambda T_{t'} = T_t S_\lambda$. In addition, $t'(t, \lambda)$ is differentiable with respect to λ.*

By the same obvious argument as in the beginning of the proof of Theorem 5, this relation is equivalent to

[Time-scale invariance] $F(S_\lambda A\,{}^tS_\lambda, S_\lambda p, t) = \dfrac{\partial t'}{\partial t}(t, \lambda)\, F(A, p, t')$.

We finally assume that the analysis is invariant under "travelling", that is, a motion of the whole picture at constant velocity does not alter the analysis. (Thus the analysis is assumed to be able to "follow" such a uniform motion.) For obvious reasons, we call this axiom Galilean invariance. In order to formalize it, we denote by $B(v) = B(v_1, v_2)$ the uniform translation operator defined by

$$B(x, y, \theta) = (x - v_1\theta, y - v_2\theta, \theta).$$

$B(v)$ represents a uniform motion with constant velocity $v = (v_1, v_2)$.

[Galilean invariance] *For any* v *and* t, *there exists* $t''\left(t, B(v)\right)$ *such that* $B(v)\, T_{t''} = T_t B(v)$. *Moreover,* $t''\left(t, B(-v)\right) = t''\left(t, B(v)\right)$ *and* t'' *is nondecreasing with respect to* t.

The second statement means that reversing time should not alter the analysis. Let us simplify the axiom. By using Lemma 1(i), we have

$$t''\left(t''\left(t, B(v)\right), B(v)\right) = t''\left(t''\left(t, B(v)\right), B(-v)\right) = t''\left(t, B(v)\, B(-v)\right)$$

$$= t''\,(t, \mathrm{Id}) = t.$$

Repeating the argument of Lemma 1(ii), we deduce from this relation that $t''\left(t, B(v)\right) = t$. Thus [Galilean invariance] reduces to the simpler relation (to which we give the same name)

[Galilean invariance] $F\left({}^tB(v)\, A B(v), {}^tB(v) p, t\right) = F(A, p, t)$ *for any* v *in* $\mathbb{R}^2$.

We are going to introduce some notation in order to state our results. We write $\nabla u = (u_x, u_y)$. Thus ∇u is associated with the spatial gradient of $u(x, y, \theta)$. We associate with Du two other vectors $Du^{\perp}$ and $Du^{\pm}$ defined by

$$Du^{\perp} = MDu = (-u_y, u_x, 0), \quad Du^{\pm} = NDu = (-u_x u_\theta, -u_y u_\theta, u_x^2 + u_y^2).$$

We define $\gamma_1(D^2u, Du) = D^2u(Du^{\perp}, Du^{\perp})$, $\gamma_2(D^2u, Du) = D^2u(Du^{\perp}, Du^{\pm})$ and $\gamma_3(D^2u, Du) = D^2u(Du^{\pm}, Du^{\pm})$. Since $Du^{\perp}$ and $Du^{\pm}$ are orthogonal to Du and to each other, we see that $Q_{Du} A Q_{Du}$ is entirely characterized by the values of γ_1, γ_2 and γ_3. Moreover, the spatial curvature $\mathrm{curv}(u)$ and the acceleration $\mathrm{accel}(u)$ (see Appendix 3) are given by

$$\mathrm{curv}(u) = \frac{\gamma_1}{|\nabla u|^3},$$

$$\mathrm{accel}(u) = \frac{(\gamma_3 - \gamma_2^2/\gamma_1)}{|\nabla u|^5}.$$

7.2. Characterization of multiscale analysis of movies

Theorem 6. (i) *If the multiscale analysis satisfies* [Grey-scale invariance], [Isometric invariance] *with respect to the spatial variables x and y, and* [Galilean invariance], *then, with a possible rescaling* $\tau = h(t)$, *we have*

$$\frac{\partial u}{\partial t} = |\nabla u| \, F(\operatorname{curv}(u), \operatorname{accel}(u), t).$$

(ii) *If the hypothesis of* (i) *and* [Projection invariance], [Time-scale invariance], *all hold, then, with a possible rescaling* $\tau = h(t)$,

$$\frac{\partial u}{\partial t} = (t \operatorname{curv}(u))^{1/3} |\nabla u| \left((\operatorname{curv}(u) \operatorname{accel}(u) \, |t \operatorname{curv}(u)|^{-4/3})^{+} \right)^{q}$$

where $0 \leqq q < 1$.

Proof of Theorem 6. (i) The proof essentially consists in reducing the number of scalar arguments of F. As we stated above, we start with ten parameters and we wish to arrive at at most two final and invariant parameters. There are certainly other ways to arrive at the final equations, but the one that we propose is rather quick. We begin the parameter reduction by using the consequence (16) of [Grey-scale invariance]. Let us fix some notation.

We write $p = (p_1, p_2, p_3) = (p^*, p_3)$. Thus, p^* is associated with the spatial gradient of $u(x, y, \theta)$. We associate with p two other vectors $p^{\perp}$ and $p^{\pm}$ defined by

$$p^{\perp} = Mp = (-p_2, p_1, 0), \quad p^{\pm} = Np = (-p_1 p_3, -p_2 p_3, p_1^2 + p_2^2).$$

Since $p^{\perp}$ and $p^{\pm}$ are orthogonal to p and to each other, we see that $Q_p A Q_p$ is entirely characterized by the values of $\gamma_1(A, p) = A(p^{\perp}, p^{\perp})$, $\gamma_2(A, p) = A(p^{\perp}, p^{\pm})$ and $\gamma_3(A, p) = A(p^{\pm}, p^{\pm})$. Thus, (16) implies that $F(A, p, t)$ can be rewritten as

$$F(A, p, t) = F(\gamma_1, \gamma_2, \gamma_3, p, t). \tag{27}$$

Of course the F on the right side is not the same as that on the left side, but we keep the same notation because it does not lead to ambiguous situations. The next step in the reduction of the number of arguments is to use the spatial isometry invariance to reduce (p_1, p_2, p_3) to $(|p^*|, p_3)$. Indeed, we remark that the operators M and N defining $p^{\perp}$ and $p^{\pm}$ commute with spatial isometries. Thus, if R is a spatial isometry (that is, an isometry leaving $(0, 0, 1)$ invariant), then

$$\gamma_1(RA^tR, Rp) = RA^tR(MRp, MRp) = RA^tR(RMp, RMp) = A(Mp, Mp)$$

because $R^tR = \mathrm{Id}$. In the same way, γ_2 and γ_3 are invariant under spatial isometries. Thus choosing R such that $Rp = (0, |p^*|, p_3)$, using [Isometric invariance], $F(RA^tR, Rp, t) = F(A, p, t)$ and (27), and setting $\tilde{A} = RA^tR$ we obtain

$$F(A, p, t) = F(\gamma_1(\tilde{A}, Rp), \gamma_2(\tilde{A}, Rp), \gamma_3(\tilde{A}, Rp), Rp, t). \tag{28}$$

Since Rp belongs to $\mathbb{R}^2$, six arguments are left to F from the initial ten. In order to get rid of two more, let us now use the Galilean invariance, $F({}^tB(v)AB(v), {}^tB(v)p, t) = F(A,p,t)$. We choose $B(v) = B(bp_3|p^*|, p_3/|p^*|)$ where b is some free parameter. $B(v)$ has been chosen in order to yield ${}^tB(v)\,Rp = (0, |p^*|, 0)$. By (28), we have to compute $\gamma_1({}^tB(v)\tilde{A}B(v), {}^tB(v)\,Rp)$ and the corresponding γ_2 and γ_3. An easy calculation yields

$$\gamma_1(A,p) = \gamma_1(\tilde{A}, Rp) = \tilde{a}_{11}|p^*|^2,$$

$$\gamma_2(A,p) = \gamma_2(\tilde{A}, Rp) = \tilde{a}_{12}|p^*|^2 p_3 - \tilde{a}_{13}|p^*|^3,$$

$$\gamma_3(A,p) = \gamma_3(\tilde{A}, Rp) = \tilde{a}_{22}|p^*|^2 p_3^2 + \tilde{a}_{33}|p^*|^4 - 2\tilde{a}_{23}|p^*|^3 p_3;$$

$$\begin{aligned}
\gamma_1({}^tB(v)AB(v), {}^tB(v)\,p) &= \gamma_1({}^tB(v)\,\tilde{A}B(v), {}^tB(v)\,Rp) = \tilde{a}_{11}|p^*|^2 \\
&= \gamma_1(A,p);\\
\gamma_2({}^tB(v)AB(v), {}^tB(v)\,p) &= \kappa_2({}^tB(v)\,\tilde{A}B(v), {}^tB(v)\,Rp) \\
&= b\tilde{a}_{11}|p^*|^4 p_3 + \tilde{a}_{21}|p^*|^2 p_3 - \tilde{a}_{31}|p^*|^3 \\
&= b|p^*|^2 p_3 \gamma_1(A,p) + \gamma_2(A,p);\\
\gamma_3({}^tB(v)AB(v), {}^tB(v)\,p) &= \gamma_3({}^tB(v)\,\tilde{A}B(v), {}^tB(v)\,Rp) \\
&= b^2\tilde{a}_{11}|p^*|^6 p_3^2 + 2b\tilde{a}_{12}|p^*|^4 p_3^2 - 2b\tilde{a}_{13}|p^*|^5 p_3 \\
&\quad + \tilde{a}_{22}|p^*|^2 p_3^2 + \tilde{a}_{33}|p^*|^4 - 2\tilde{a}_{23}|p^*|^3 p_3 \\
&= b^2|p^*|^4 p_3^2 \gamma_1(A,p) + 2b|p^*|^2 p_3 \gamma_2(A,p) + \gamma_3(A,p).
\end{aligned}$$

These relations, (28) and [Galilean invariance] yield

$$\begin{aligned}
F(A,p,t) &= F(\gamma_1, \gamma_2, \gamma_3, |p^*|, p_3) \\
&= F(\gamma_1, b|p^*|^2 p_3\gamma_1 + \gamma_2, b^2|p^*|^4 p_3^2\gamma_1 + 2b|p^*|^2 p_3\gamma_2 + \gamma_3, |p^*|, 0).
\end{aligned}$$

Since this relation is true for any b, we can fix $b = -\gamma_2/(|p^*|^2 p_3\gamma_1)$ and we obtain

$$F(A,p,t) = F\left(\gamma_1, \gamma_3 - \frac{\gamma_2^2}{\gamma_1}, |p^*|, t\right)$$

and four little scalar arguments are left. Using the second relation deduced from [Grey-scale invariance],

$$\text{(13)} \qquad F(\lambda A, \lambda p, t) = \lambda F(A,p,t) \quad \text{for all } \lambda \geqq 0,$$

and the relations

$$\gamma_1(\lambda A, \lambda p) = \lambda^3 \gamma_1(A,p), \quad \gamma_2(\lambda A, \lambda p) = \lambda^4 \gamma_2(A,p),$$
$$\gamma_3(\lambda A, \lambda p) = \lambda^5 \gamma_3(A,p),$$

we finally get

$$\lambda F(A,p,t) = F(\lambda^3 g_1, \lambda^5(\gamma_3 - \gamma_2^2/\gamma_1), \lambda|p^*|, t).$$

With $\lambda|p^*| = 1$, this yields

$$F(A,p,t) = |p^*|\, F\left(\frac{\gamma_1}{|p^*|^3}, \frac{(\gamma_3 - \gamma_2^2/\gamma_1)}{|p^*|^5}, t\right). \tag{29}$$

(ii) Let us now satisfy the curiosity of the reader by giving the explicit formulas of γ_1, γ_2, γ_3 and $\gamma_1\gamma_3 - \gamma_2^2$:

$$\gamma_1(A,p) = a_{11}p_2^2 - 2a_{12}p_1p_2 + a_{22}p_1^2,$$

$$\gamma_2(A,p) = a_{22}p_1p_2p_3 - a_{11}p_1p_2p_3 + a_{12}(p_1^2 - p_2^2)p_3 + a_{13}(p_1^2 + p_2^2)p_2 - a_{23}(p_1^2 + p_2^2)p_1,$$

$$\gamma_3(A,p) = a_{11}p_1^2p_3^2 + a_{22}p_2^2p_3^2 + 2a_{12}p_1p_2p_3^2 + a_{33}(p_1^2 + p_2^2)^2 - 2a_{13}(p_1^2 + p_2^2)p_1p_3 - 2a_{23}(p_1^2 + \ p_3,$$

$$\gamma_1\gamma_3 - \gamma_2^2 = (p_1^2 + p_2^2)^2\,(-a_{13}^2p_2^2 - a_{23}^2p_1^2 - a_{12}^2p_3^2 + a_{11}a_{22}p_3^2 + a_{11}a_{33}p_2^2 - 2a_{11}a_{23}p_2p_3 + a_{22}a_{33}p_1^2 - 2a_{22}a_{13}p_1p_3 - 2a_{33}a_{12}p_1p_2 + 2a_{12}a_{13}p_2p_3 + 2a_{12}a_{23}p_1p_3 + 2a_{13}a_{23}p_1p_2).$$

Notice that in obtaining this form for F, we have used [Grey-scale invariance], [Isometry invariance] and [Galilean invariance]. It is easy to check that conversely the equation (29) defines a multiscale analysis satisfying all of these invariants (provided its solutions are uniquely defined as, e.g., viscosity solutions). Indeed, this can be calculated directly by using the above formula for γ_2 and γ_3. (By the way, $\gamma_1/|p^*|^3$ is nothing but the spatial curvature of u.) Now, this is not necessary since we have used only equivalent formulations except the step concerning [Galilean invariance]. In this case, the computation is straightforward by using the above formulas.

We now state the invariance of the analysis for a spatial dilation, which is a particular case of

[Projection invariance] $F(BA^tB, Bp, t) = |\det(B)|^{1/2} F(A, p, t|\det(B)|^{1/2})$,

where we take $B(x,y,z) = (\lambda x, \lambda y, z)$. Therefore $\det(B) = \lambda^2$. The explicit formula given above for γ_1, γ_2, γ_3 easily yield

$$\gamma_1(BA^tB, Bp) = \gamma^4\gamma_1(A,p), \quad \gamma_2(BA^tB, Bp) = \lambda^4\gamma_2(A,p),$$

$$\gamma_3(BA^tB, Bp) = \lambda^4\gamma_3(A,p).$$

Thus we deduce from [Projection invariance] that

$$\lambda F(A,p,t\lambda) = \lambda|p^*|F\left(\frac{\gamma_1}{|p^*|^3}, \frac{(\gamma_3-\gamma_2^2/\gamma_1)}{|p^*|^5}, t\lambda\right)$$

$$= F(BA^tB, Bp, t) = \lambda|p^*|F\left(\frac{\lambda\gamma_1}{|p^*|^3}, \frac{(\gamma_3-\gamma_2^2/\gamma_1)}{\lambda|p^*|^5}, t\right).$$

This yields

$$F(A,p,t) = |p^*|F\left(\frac{\gamma_1}{|p^*|^3}, \frac{(\gamma_3-\gamma_2^2/\gamma_1)}{\lambda|p^*|^5}, \frac{t}{\lambda}\right).$$

Taking $\lambda = t$, we obtain

$$F(A,p,t) = |p^*|F\left(\frac{t\gamma_1}{|p^*|^3}, \frac{(\gamma_3-\gamma_2^2/\gamma_1)}{t|p^*|^5}\right)$$

and only two arguments are left.

We conclude the reduction process by again using [Projection invariance] with orthogonal affinities. Because of the isometry invariance, we only need to look at a special orthogonal affinity, say $B(x,y,\theta) = (\lambda x, y, \theta)$. Take, for simplicity, $p_2 = 0$. Then

$$\gamma_1(BA^tB, Bp) = \lambda^2\gamma_1(A,p), \quad \gamma_2(BA^tB, Bp) = \lambda^3\gamma_2(A,p),$$
$$\gamma_3(BA^tB, Bp) = \lambda^4\gamma_3(A,p).$$

Thus, by [Projection invariance], $F(BA^tB, Bp, t) = \lambda^{1/2}F(A,p,t\lambda^{1/2})$, and therefore

$$\lambda|p^*|F\left(\frac{t\gamma_1}{\lambda|p^*|^3}, \frac{(\gamma_3-\gamma_2^2/\gamma_1)}{\lambda t|p^*|^5}\right) = \lambda^{1/2}F(A,p,t\lambda^{1/2})$$

$$= \lambda^{1/2}|p^*|F\left(\frac{\lambda^{1/2}t\gamma_1}{|p^*|^3}, \frac{(\gamma_3-\gamma_2^2/\gamma_1)}{\lambda^{1/2}t|p^*|^5}\right).$$

Therefore, F satisfies a relation of the kind

$$\lambda^{1/2}|p^*|F\left(\lambda^{1/2}s, \frac{s'}{\lambda^{1/2}}\right) = \lambda|p^*|F\left(\frac{s}{\lambda}, \frac{s'}{\lambda}\right),$$

which can be rewritten as

$$\lambda^{-1/2}|p^*|F(\lambda^{3/2}s, s'\lambda^{1/2}) = |p^*|F(s,s').$$

Setting $\lambda = |s|^{-2/3}$, we obtain

$$F(s,s') = |s|^{1/3}F(\operatorname{sign}(s), s'|s|^{-1/3}).$$

This last formula depends essentially on one argument. We obtain

$$F(A,p,t) = |\gamma_1 t|^{1/3} F(\operatorname{sign}(\gamma_1), (\gamma_3 - \gamma_2^2/\gamma_1)|p^*|^{-4} t^{-4/3}|\gamma_1|^{-1/3})$$

and it is easy to check that F is invariant under the affinity being considered even if $p_2 \neq 0$.

The reduction of arguments has concluded and we can now only hope to identify the shape of F by some invariance property. In order to do this we use [Time-scale invariance] and we obtain the relation

$$|\gamma_1 t|^{1/3} F\left(\operatorname{sign}(\gamma_1), \lambda^2\left(\gamma_3 - \frac{\gamma_2^2}{\gamma_1}\right)|p^*|^{-4} t^{-4/3}|\gamma_1|^{-1/3}\right)$$

$$= \frac{\partial t'}{\partial t}(t,\lambda)|\gamma_1 t'|^{1/3} F\left(\operatorname{sign}(\gamma_1), \lambda^2\left(\gamma_3 - \frac{\gamma_2^2}{\gamma_1}\right)|p^*|^{-4} t'^{-4/3}|\gamma_1|^{-1/3}\right).$$

Therefore, $F(\operatorname{sign}(\gamma_1), s)$ satisfies

(30) $$F(\operatorname{sign}(\gamma_1), s) = \frac{\partial t'}{\partial t}(t,\lambda)\left(\frac{t'}{t}\right)^{1/3} F\left(\operatorname{sign}(\gamma_1), \frac{s(t'/t)^{-4/3}}{\lambda^2}\right).$$

Assuming that γ_1 has, e.g., positive sign, let us identify F. Forgetting the sign argument in F we have

(31) $$F(s) = kF\left(\frac{s(t'/t)^{-4/3}}{\lambda^2}\right)$$

where k depends on t and λ. We consider two cases:
(a) If the function $(t'/t)^{-4/3}/\lambda^2$ does not assume two different values, then we obtain $t' = Ct\lambda^{-3/2}$. Substituting this in (30) and letting λ vary yields $F = 0$.
(b) Otherwise, the function $(t'/t)^{-4/3}/\lambda^2$ assumes two different values, and since it is continuous, its range contains an interval I. Thus we can rewrite (30) as $F(as) = k(a)F(s)$, where $a > 0$ can assume an arbitrary value in some interval I. Then, we can conclude that $F(s) = cs^q$.

Indeed, assume that $s > 0$ and set $G(x) = \operatorname{Log} F(\exp(x))$ and $g(y) = \operatorname{Log} k(\exp(y))$. Recall that since F is nondecreasing and $F(0) = 0$, we have $k > 0$ and $F > 0$ for $s > 0$ and $a > 0$. Then G and g satisfy

$$G(x+y) - G(y) = g(x) \quad \text{for any } x \text{ in some interval } I \text{ and any } y > 0.$$

If y and z are two rational numbers in I, one easily deduces that $g(y) = (y/z)g(z)$, and since g is continuous like G, this relation is still true for any y and z in I. We conclude that $g(y) = qy$ for some nonnegative constant q and for any y in I. Thus we can write $G(x+y) - G(y) = qx$ if x is in I for any $y > 0$. Differentiating with respect to x yields $G'(x+y) = q$. Thus, $G(x) = qx + d$. Returning to F, we conclude that $F(s) = e^d s^q$.

Replacing $F(s)$ by cs^q in (30) and using $t'(0,\lambda) = 0$ yields

(32) $$t'(t,\lambda) = t\lambda^{3q/2(1-q)}.$$

We can carry out exactly the same proof for $s < 0$ and we get $F(s) = -e^{d'} s^{q'}$ for some constants d' and q'. Now, the relation (32) implies that q and q' must be equal. Finally, we obtain

$$F(A,p,t)$$
$$= |\gamma_1 t|^{1/3} G\left(\text{sign}(\gamma_1), \text{sign}\left(\gamma_3 - \frac{\gamma_2^2}{\gamma_1}\right)\right) \left| \left(\gamma_3 - \frac{\gamma_2^2}{\gamma_1}\right) |p^*|^{-4} t^{-4/3} |\gamma_1|^{-1/3} \right|^q$$

where $q \geqq 0$.

If we change u to $-u$, we must have that $F(A,p,t) = -F(-A, -p, t)$; then we obtain that

$$F(A,p,t)$$
$$= \text{sign}(\gamma_1) |\gamma_1 t|^{1/3} G(\text{sign}(\gamma_3\gamma_1 - \gamma_2^2)) \, |(\gamma_3 - \gamma_2^2/\gamma_1) |p^*|^{-4} t^{-4/3} |\gamma_1|^{-1/3}|^q.$$

Let $\Pi = \gamma_3\gamma_1 - \gamma_2^2 = \gamma_1 \,\text{accel}(u)$ and

$$H(\gamma_1, \gamma_2, \gamma_3) = \text{sign}(\gamma_1) |\gamma_1|^{1/3} G(\text{sign}(\gamma_3\gamma_1 - \gamma_2^2)) \; ((\gamma_3 - \gamma_2^2/\gamma_1) |\gamma_1|^{-1/3})^q;$$

then we have

$$F(A,p,t) = H(\gamma_1, \gamma_2, \gamma_3) \, t^{(1-4q)/3} |p^*|^{-4q}.$$

We notice that $F(A,p,t)$ must be continuous with respect to t in $t = 0$. This would require $q \leqq \frac{1}{4}$; however, in order to get a well-defined solution, it is enough that for some rescaling $t' = h(t)$ of the equation, F becomes continuous. Such a rescaling is possible if and only if $0 \leqq q \leqq 1$.

Now, to use the [Comparison principle], F must satisfy the growth condition: For all p, t, if $A \geqq B$, then $F(A,p,t) \geqq F(B,p,t)$. Then, we use the results of the Appendix 2. In order to verify the growth of the function H, we must have the following conditions on H:

$$\text{(33)} \qquad \frac{\partial H}{\partial \gamma_1} \geqq 0, \quad \frac{\partial H}{\partial \gamma_3} \geqq 0, \quad 4\frac{\partial H}{\partial \gamma_1}\frac{\partial H}{\partial \gamma_3} \geqq \left(\frac{\partial H}{\partial \gamma_2}\right)^2.$$

First, we are going to simplify the form of F: If H is nondecreasing with respect to γ_3, we can choose $\gamma_1 > 0$ and $\gamma_2 = 0$. Then $\Pi = \gamma_3\gamma_1$ implies that $\text{sign}(\Pi) = \text{sign}(\gamma_3)$. We have $H(\gamma_1, 0, 1) \geqq H(\gamma_1, 0, 0) \geqq H(\gamma_1, 0, -1)$. Hence, $G(1) \geqq 0 \geqq G(-1)$. Secondly, if H is nondecreasing with respect to γ_1, we can choose $\gamma_3 = 0$ and $\gamma_2 \neq 0$ so that if $\Pi = -\gamma_2^2 \leqq 0$, then $\text{sign}(\Pi) = -1$, and $H(1, \gamma_2, 0) \geqq H(-1, \gamma_2, 0)$. Thus if $G(-1) \geqq -G(-1)$, then $G(-1) \geqq 0$. Therefore we obtain $G(-1) = 0$ and $G(1) > 0$, and $H(\gamma_1, \gamma_2, \gamma_3) = \text{sign}(\gamma_1) |\gamma_1|^{1/3} (\Pi^+ |\gamma_1|^{-4/3})^q$, where $\Pi^+ = \sup(0, \Pi)$.

Next, we prove the continuity of H, when $\gamma_1 \to 0$. In the first case: $\gamma_1\gamma_3 \leqq 0$; then $\Pi \leqq 0$, so $H(\gamma_1, \gamma_2, \gamma_3) = 0$. In the second case: $\gamma_1\gamma_3 \geqq 0$; then $\Pi \leqq \gamma_1\gamma_3$, $|H(\gamma_1, \gamma_2, \gamma_3)| \leqq |\gamma_1|^{1/3} (\gamma_3^+ |\gamma_1|^{-1/3})^q$, which tends to 0 when $\gamma_1 \to 0$, because $q < 1$. Thus H is a continuous function, and for all γ_2, γ_3, $\lim_{\gamma_1 \to 0} H(\gamma_1, \gamma_2, \gamma_3) = 0$.

Finally, we check (33): We set $H(\gamma_1, \gamma_2, \gamma_3) = \text{sign}(\gamma_1)\,|\gamma_1|^{1/3}x^q$, where $x = \Pi^+\,|\gamma_1|^{-4/3}$. Then

$$\frac{\partial H}{\partial \gamma_1} = \frac{1}{3}\,|\gamma_1|^{-2/3}x^q + \text{sign}(\gamma_1)\,|\gamma_1|^{1/3}\,qx^{q-1}\,\frac{\delta x}{\delta \gamma_1},$$

$$\frac{\partial x}{\partial \gamma_1} = -\frac{1}{3}\,\gamma_3\,|\gamma_1|^{-4/3} + \frac{4}{3}\,\text{sign}(\gamma_1)\,\frac{\gamma_2^2}{|\gamma_1|^{-7/3}} = \frac{\text{sign}(\gamma_1)}{3|\gamma_1|}\left(x - \frac{3\gamma_2^2}{|\gamma_1|^{-4/3}}\right).$$

Thus, $\partial H/\partial \gamma_1 = (1-q)/3|\gamma_1|^{-2/3}x^q + qx^{q-1}\gamma_2^2/|\gamma_1|^{-4/3}$. And if $0 \leqq q \leqq 1$, we have

$$\frac{\partial H}{\partial \gamma_1} \geqq 0,$$

$$\frac{\partial H}{\partial \gamma_3} = q\,\text{sign}(\gamma_1)\,|\gamma_1|^{1/3}\,x^{q-1}\,\frac{\gamma_1}{|\gamma_1|^{-4/3}} = qx^{q-1} \geqq 0,$$

$$\frac{\partial H}{\partial \gamma_2} = -2qx^{q-1}\,\frac{\gamma_2}{\gamma_1},$$

$$4\,\frac{\partial H}{\partial \gamma_1}\,\frac{\partial H}{\partial \gamma_3} - \left(\frac{\partial H}{\partial \gamma_2}\right)^2 = 4q(x^{q-1})^2\left(\frac{1-q}{3}\,|\gamma_1|^{-2/3}x + q\,\frac{\gamma_2^2}{\gamma_1^2} - \frac{q\gamma_2^2}{\gamma_1^2}\right)$$

$$= \tfrac{4}{3}\,q(x^{q-1})^2\,(1-q)\,|\gamma_1|^{-2/3}x \geqq 0 \quad \text{since } 0 \leqq q \leqq 1 \text{ and } x \geqq 0.$$

We conclude that the function

$$F = \text{sign}(\gamma_1)\,|\gamma_1 t|^{1/3}\,\left((\Pi^+/\gamma_1)\,|p^*|^{-4}\,t^{-4/3}|\gamma_1|^{-1/3}\right)^q$$

satisfies the [Comparison principle].

8. Numerical experiments

In order to show the performances of the different multiscale analysis models introduced above, we present some numerical experimental results. We compare three different multiscale analysis models.

1) Classical Mathematical Morphology. Erosion and Dilatation:

$$\frac{\partial u}{\partial t} = \pm|Du|,$$

2) Mean Curvature Motion:

$$\frac{\partial u}{\partial t} = |Du|\,t\,\text{curv}(u),$$

3) Our New Model:

$$\frac{\partial u}{\partial t} = |Du| \; (t \operatorname{curv}(u))^{1/3}.$$

The three models satisfy [Grey-scale invariance] and thus the evolution of the shapes in the picture under the action of the multiscale analysis depends only on the curvature of the level line (see Section 5.1). To fix ideas we first show analytically the evolution of a circle for the three models.

In model (1) the velocity of propagation of the level lines is constant. If an initial level line is a circle of radius $r(0)$, then, at the scale t, this level line corresponds to a circle of radius $r(t) = r(0) \pm t$. In particular it can be removed in time $t = r(0)$.

In model (2) the velocity of propagation of the level lines is proportional to the curvature of the level line. The evolution of the radius $r(t)$ of a circle is given by the equation: $r(t) = \left((r(0)^2 - t^2)^+\right)^{1/2}$. In this case it is known that asymptotically all the level lines tend to circles.

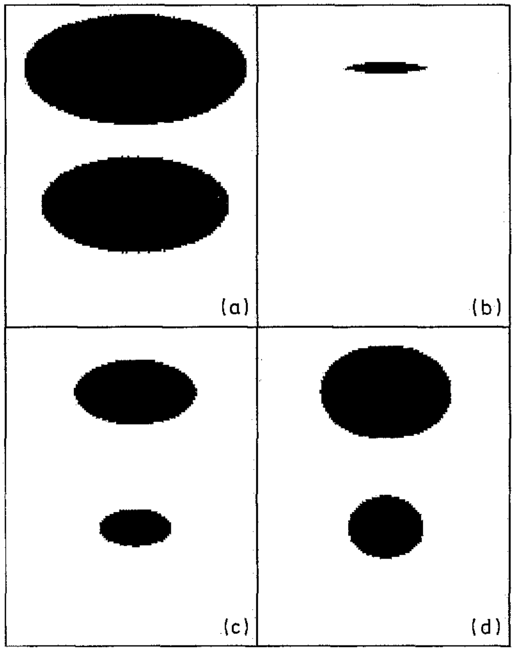


Fig. 1. We present the evolution of a simple synthetic picture (140 × 140 pixels) composed of two ellipses of different size. (a) Original picture. (b) Erosion of original picture (scale = 15). Singularities are developed at the extreme points of the ellipses. (c) Analysis of the original picture by using model (3) (scale = 28). The shape of the ellipses remains stable. (d) Analysis of the original picture by using model (2) (scale = 32). The original ellipses tend to circles.

In model (3) the velocity of propagation of the level lines is proportional to the curvature of the level line to the power $\frac{1}{3}$. The evolution of the radius $r(t)$ of a circle is given by the equation: $r(t) = \left((r(0)^{4/3} - t^{4/3})^+\right)^{3/4}$. Moreover, since this multiscale analysis satisfies [Projection invariance], the more complex forms remain stable. For instance, if an initial level line is an ellipse, then due to [Projection invariance] this level line remains an ellipse for any scale t.

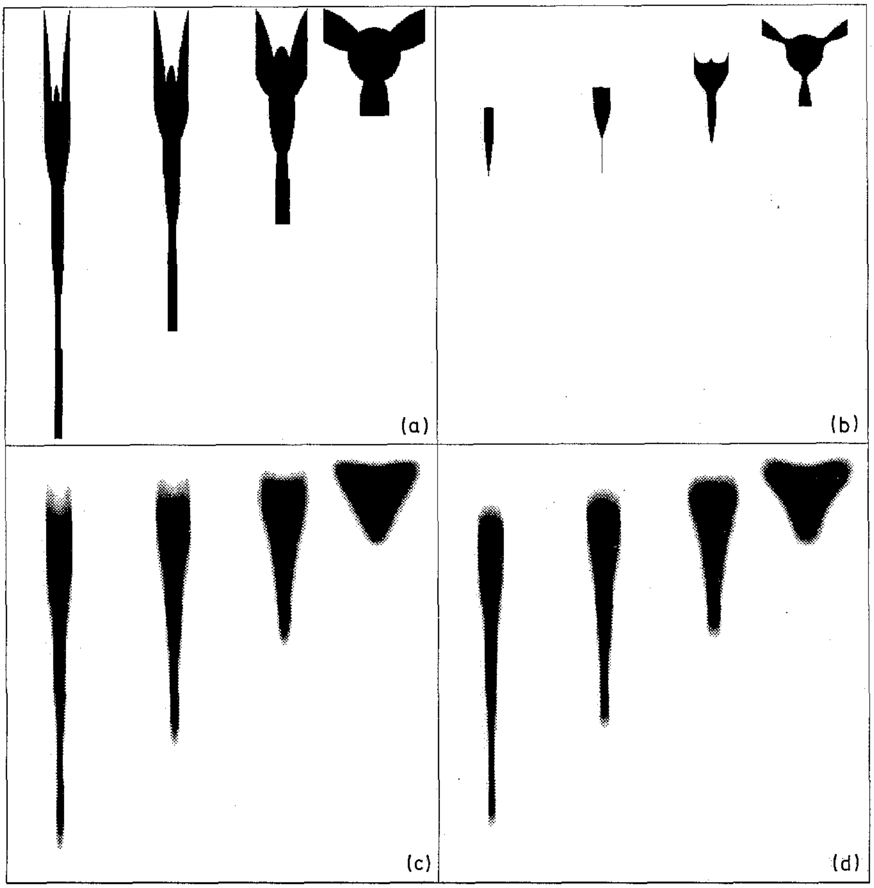


Fig. 2. We present an experiment in order to evaluate the performance of the different multiscale models under the action of projections. We transform a synthetic shape (60×250 pixels) by using the projection $B(x,y) = (\lambda x, y/\lambda)$. We remark that since $\det B = 1$ the scale of the analysis t is the same for the original shape and its projection. (a) The original picture, which consists of the synthetic shape and its projections (from right to left) with $\lambda = \frac{1}{2}, 1, \frac{3}{2}, 2$. (b) Erosion of the original picture (scale = 15). (c) Analysis of the original picture by using model (3) (scale = 28). (d) Analysis of the original picture by using model (2) (scale = 32).

Therefore, in these three models, at the scale t, any detail enclosed in a circle of radius t is removed. However, the evolution of the level lines is quite different for the different models. For instance, at the scale $t_0 = r(0)/2$, the radius of the circle obtained is $r(t_0) = r(0)/2$ for model (1), $r(t_0) = r(0)\,(1-2^{-2})^{1/2}$ for model (2) and $r(t_0) = r(0)\,(1-2^{-4/3})^{3/4}$ for model (3).

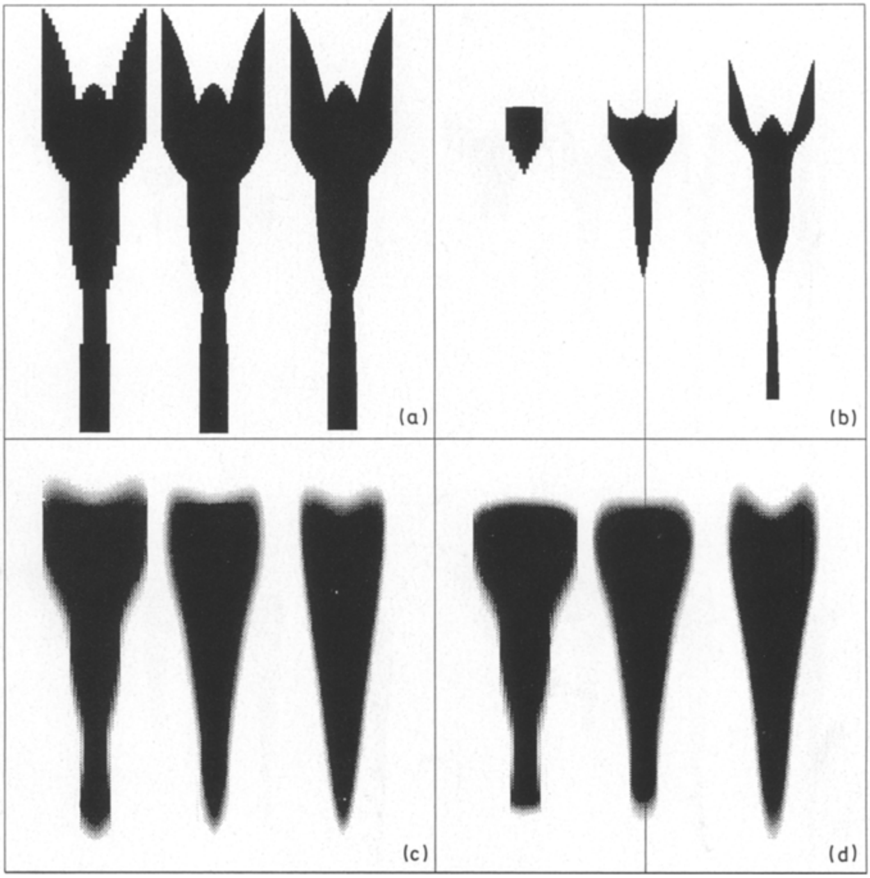


Fig. 3. We reconstruct the original synthetic shape shown in Figure 2 by using the inverse projection $B(x,y) = (x/\lambda, \lambda y)$ with $\lambda = \frac{1}{2}, 1, 2$. (a) Reconstruction based on Figure 2(a). (b) Reconstruction based on Figure 2(b). (c) Reconstruction based on Figure 2(c); in this case we notice that the three shapes are in principle identical, but, of course, the discretization of the equation and the projections on a grid introduce some approximation errors. (d) Reconstruction based on Figure 2(d).

Fig. 4. We present the multiscale analysis of a real picture (200 × 220 pixels) by using model (3). From left to right and from top to bottom we have the original picture and the analyzed versions in the scale $t = 1, 2, 3, 4, 5$.

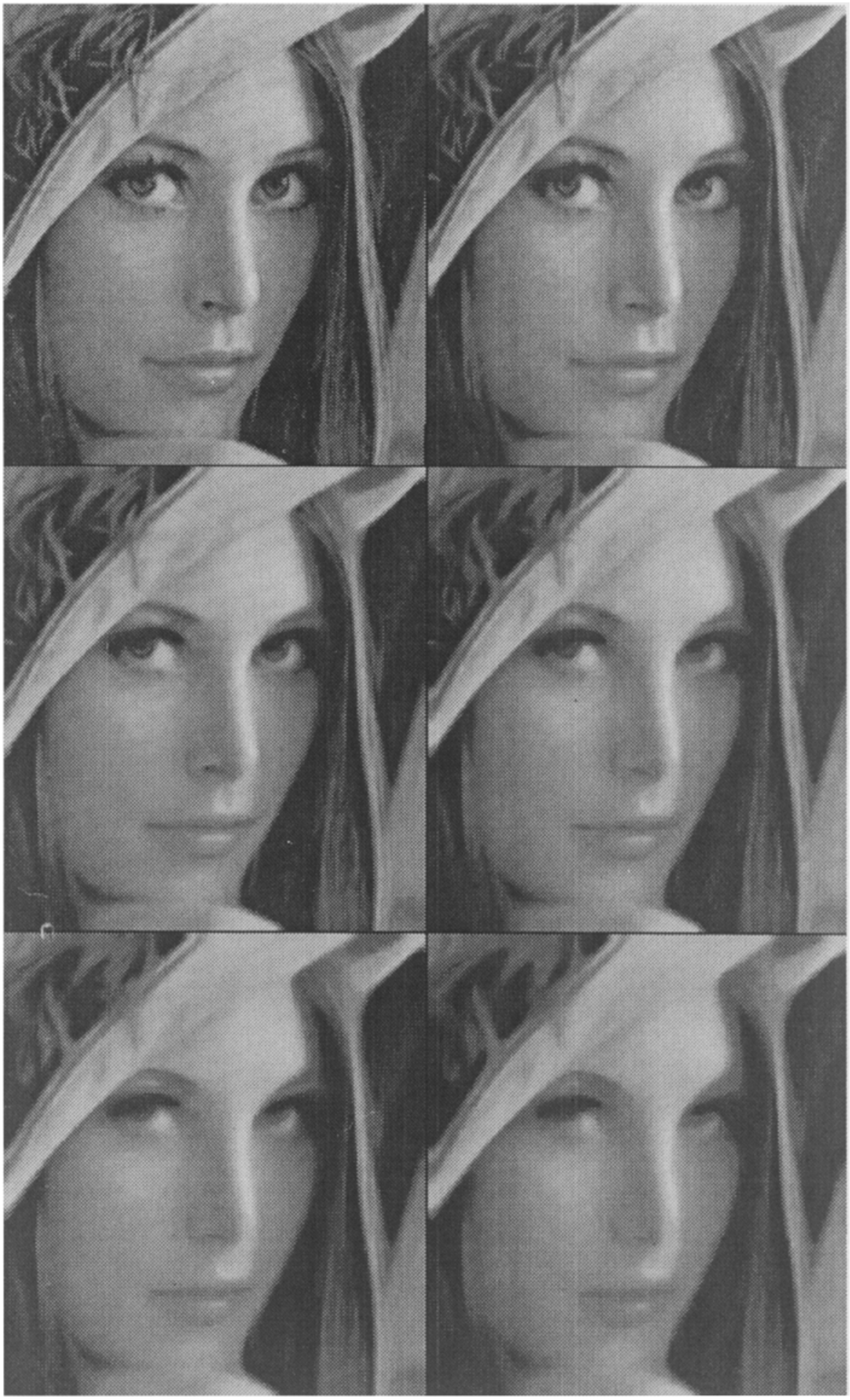


Fig. 4

9. Conclusions

We summarize in Table 1 all multiscale models already known and all the new models we have introduced or discussed. The table is organized as a tree, where axioms are progressively added from left to right. Thus the final models (with all invariance properties) are the leaves of the tree.

Let us now give a brief résumé of our conclusions about multiscale image analysis models.

1) First of all, we have explained why the classical multiscale models could not be improved: In the case of the "American school" (the heat equation), the model cannot be improved because of the incompatibility between the linearity of the model and the morphological invariants like grey-scale invariance or projection invariance.

In the case of the "Fontainebleau school", no partial differential equation model was developed, and the known operators, dilation and erosions, seem to be the only ones to have a simple direct formulation without partial differential equations.

2) By making a *union of the axiomatics of both schools*, we have been able to construct a new class of multiscale analyses which have *all* the properties of the models known before, *and more*, since we also achieve projection invariance. And projection invariance is an essential tool in the analysis of natural scenes and in shape recognition [18].

3) As a consequence of the equations obtained, we have been able to define a concept of "multiscale curvature", which derives immediately from our "best invariant" equation. This leads us to a correct definition of the multiscale analysis of curves and shapes as it has been proposed in [3, 36]. Now, to define a reliable multiscale curvature has been acknowledged by the computer-vision community as a key problem to achieve efficient shape recognition algorithms. Furthermore, our proofs show that there is a *single way* to define a morphologically invariant multiscale curvature.

4) Finally, we have applied the same methodology to the analysis of movies. We have proved that the addition of a single new and obvious axiom, the *Galilean invariance*, leads to a single possibility for multiscale analysis of movies. In the same way as we got a definition of multiscale curvature, we get a natural concept of "multiscale acceleration".

5) Beyond the problems of computer vision, we want to emphasize that somehow the preceding results give the first mathematical hints of *how biological vision is possible* (a problem which has been quite well formulated in MARR's classical book [39]). Indeed, as we stated in Section 1, biological vision presupposes the capability of passing from information from the local retina to global intelligence of shapes. But this process must be invariant under all transforms we listed in Section 1 (change of brightness, perspective, etc.). We have given mathematical evidence that such a process is possible.

Table 1

$\dfrac{\partial u}{\partial t} = F(D^2u, Du, t)$	[Isometric invariance] + linearity $F = c(t)\,\Delta u$ (Heat Equation)		$F = \operatorname{div}\,(g(\lvert Du\rvert)\,Du)$		(MALIK & PERONA)
	[Grey-scale invariance]	+	[Isometric invariance]	+	Classical Mathematical Morphology
	$F = \lvert Du\rvert\, G\left(\dfrac{1}{\lvert Du\rvert} Q_{Du} A Q_{Du}, \dfrac{Du}{\lvert Du\rvert}, t\right)$	→	$F = \lvert Du\rvert\, G\left(\dfrac{\lambda_1}{\lvert Du\rvert}, \dots, \dfrac{\lambda_{N-1}}{\lvert Du\rvert}, t\right)$	→	$G = cte \Rightarrow F = C\lvert Du\rvert$
	$Q_{Du} = \mathrm{Id} - \dfrac{Du \otimes Du}{\lvert Du\rvert^2}$		$\lambda_1, \dots, \lambda_{N-1}$ eigenvalues of $Q_{Du} A Q_{Du}$		
	[Grey-scale invariance] + $(N = 2)$	+	[Scale invariance], [Isometric invariance]	+	[Projection invariance]
	$F = \lvert Du\rvert\, G\left(\operatorname{curv}(u), \dfrac{Du}{\lvert Du\rvert}, t\right)$	→	$F = \lvert Du\rvert\, \beta(t \operatorname{curv}(u))$	→	$F = \lvert Du\rvert \operatorname{curv}(u)^{1/3}$
	[Scale invariance] + $(N = 2)$	+	[Projection invariance]	+	[Grey-scale invariance]
	$F = \dfrac{1}{t} G(tD^2u, tDu)$	→	$F = \dfrac{1}{t} G(t^4 \lvert Du\rvert^3 \operatorname{curv}(u), t^4 A(u))$	→	$F = \lvert Du\rvert \operatorname{curv}(u)^{1/3}$
			$A(u) = \lvert Du\rvert^2 \left(u_{\eta\eta} - \dfrac{u_{\eta\xi}^2}{u_{\xi\xi}}\right);\ \xi = \dfrac{Du}{\lvert Du\rvert}, \eta = \dfrac{Du^{\perp}}{\lvert Du\rvert}$		
	Movies $(N = 3)$ [Grey-scale invariance], [Isometric invariance], [Galilean invariance]	+	[Projection invariance], [Time-scale invariance]		
	$F = \lvert \nabla u\rvert\, F(\operatorname{curv}(u), \operatorname{accel}(u), t)$	→	$\lvert \nabla u\rvert\ (t \operatorname{curv}(u))^{1/3} \left(\left(\dfrac{\operatorname{curv}(u) \operatorname{accel}(u)}{\lvert t \operatorname{curv}(u)\rvert^{4/3}}\right)^{+}\right)^{q}$ $0 \leqq q \leqq 1$		

Appendix 1: Proof of Theorem 1

The proof involves several steps. To preserve some compactness in notation, we set $\delta_{t,s}(f) = (T_t(f) - T_s(f))/(t-s)$ and $\delta_t(f) = \delta_{t,0}(f) = (T_t(f) - f)/t$. Notice that we can rewrite [Regularity] $(\|T_t(f+hg) - (T_t(f) + hg)\|_\infty \leqq Cht$ for all h, t in $[0,1]$ and all f, g in $Q)$, as

(A1) $\|\delta_t(f+hg) - \delta_t(f)\|_\infty \leqq Ch$ for all h, t in $[0,1]$ and all f, g in Q.

Step 1. We begin by proving that the functions $\delta_t(f) = (T_t(f) - f)/t$ are uniformly bounded. Indeed, we have $T_t(0) = 0$ by [Grey-scale-shift invariance]. Using [Regularity] with $h = 1$ and $f = 0$ and $g = f$ we deduce that

(A2) $$\|T_t(f) - f\|_\infty \leqq C_0 t \quad \text{for } t \text{ in } [0,1]$$

for some constant C_0 which depends only on bounds of derivatives of f in C_b^∞.

Step 2. We now prove that the functions $\delta_t(f)$ are uniformly Lipschitz continuous. More precisely,

(A3) $\delta_t(f)$ is Lipschitz continuous on $\mathbb{R}^N$ uniformly for t in $]0,1]$ (and uniformly for f belonging to a set Q).

To this end, let z be in $\mathbb{R}^N$, $|z| = 1$; we wish to estimate $\|\tau_{hz}\{\delta_t(f)\} - \delta_t(f)\|_\infty$ for h in $[0,1]$. We first observe that because of [Translation invariance],

$$\tau_{hz}\{\delta_t(f)\} = \delta_t(\tau_{hz} \cdot f).$$

Then, we recall that $\tau_{hz} \cdot f = f + hg_h$ for some g_h in C_b^∞ (and that bounds uniform in h on derivatives of g_h depend only on bounds on derivatives of f). Therefore, by (A1) ([Regularity]), we have $\|\delta_t(\tau_{hz} \cdot f) - \delta_t(f)\|_\infty \leqq Ch$, and then (A3) is proved.

Step 3. From the preceding steps, we can already deduce a compactness property for $\delta_t(g)$ as t tends to zero. Now, we need more, since we want the whole sequence to converge to the same limit. So we need a Cauchy estimate. Our next assertion is that for all t, s in $[0, \frac{1}{2}]$ we have

(A4) $$\|\delta_{t+s,t}(f) - \delta_s(f)\|_\infty \leqq m(t)$$

for some continuous, nonnegative, nondecreasing function m on $[0, \frac{1}{2}]$ such that $m(0) = 0$, and m depends only on bounds of derivatives of f. Again this follows from [Regularity]. Indeed, we can write

$$\delta_s(f) = \delta_s(f) * \rho_\varepsilon \{\delta_s(f) * \rho_\varepsilon - \delta_s(f)\}$$

where $\rho_\varepsilon = \varepsilon^{-N}\rho(\cdot/\varepsilon)$, $\rho \geqq 0$, $\int_{\mathbb{R}^N} \rho(y)\,dy = 1$, $\rho \in C_0^\infty(\mathbb{R}^N)$. We deduce from (A3) that

(A5) $$\|\delta_s(f) * \rho_\varepsilon - \delta_s(f)\|_\infty \leqq C_1 \varepsilon \quad \text{for all } \varepsilon > 0,$$

for some $C_1 \geqq 0$ depending only on bounds on derivatives of f. Then, because of (1) and [Recursivity], we have

$$\text{(A6)} \qquad \|T_{t+s}(f) - T_t(f + s\delta_s(f) * \rho_\varepsilon)\|_\infty \leqq C_1 \varepsilon s \quad \text{for all } \varepsilon > 0 .$$

On the other hand, since $\delta_s(f) * \rho_\varepsilon$ belongs to C_b^∞, [Regularity] implies that

$$\text{(A7)} \qquad \|T_t(f + s\delta_s(f) * \rho_\varepsilon) - T_t(f) - s\delta_s(f) * \rho_\varepsilon\|_\infty \leqq C_\varepsilon st$$

for some $C_\varepsilon \geqq 0$ depending only upon $\varepsilon > 0$ and bounds on derivatives of f. Collecting the estimates (A5)–(A7), we finally deduce that

$$\|\delta_{t+s,t}(f) - \delta_s(f)\|_\infty \leqq 2C_1\varepsilon + C_\varepsilon t$$

and we conclude that (A4) holds by setting $m(t) = \inf_{\varepsilon \in]0,1]} (2C_1\varepsilon + C_\varepsilon t)$.

Step 4. We now are in a position to give a Cauchy estimate in $0 < h \leqq t \leqq \frac{1}{2}$ for $\delta_s(f)$:

$$\text{(A8)} \qquad \|\delta_t(f) - \delta_h(f)\|_\infty \leqq 2\,\frac{C_0 r}{t} + m(t) \quad \text{where } r = t - \left[\frac{t}{h}\right] h .$$

(We denote by $[a]$ the largest integer N such that $N \leqq a$.) Indeed, denoting $N = [t/h]$, we observe that $\delta_t(f) = (Nh/t)\ \delta_{Nh}(f) + (r/t)\ \delta_{Nh+r,Nh}(f)$, and using (A4) with $t = Nh$, $s = r$, we deduce that

$$\text{(A9)} \qquad \left\|\delta_t(f) - \frac{Nh}{t}\delta_{Nh}(f) - \frac{r}{t}\,\delta_r(f)\right\|_\infty \leqq \frac{r}{t}\, m(Nh) .$$

Writing $\delta_{Nh}(f) = \big((N-1)\,h/Nh\big)\ \delta_{(N-1)h}(f) + (h/Nh)\ \delta_{Nh,(N-1)h}(f)$ and using (A4) with $t = (N-1)\,h$, $s = h$, we deduce that

$$\text{(A10)} \qquad \left\|\delta_{Nh}(f) - \frac{N-1}{N}\,\delta_{(N-1)h}(f) - \frac{1}{N}\,\delta_h(f)\right\|_\infty \leqq \frac{1}{N} m\big((N-1)\,h\big) .$$

Combining this inequality with (A9), we obtain

$$\text{(A11)} \qquad \left\|\delta_t(f) - (N-1)\,\frac{h}{t}\,\delta_{(N-1)h}(f) - \frac{h}{t}\,\delta_h(f) - \frac{r}{t}\,\delta_r(f)\right\|_\infty$$

$$\leqq \frac{r}{t}\, m(Nh) + \frac{h}{t}\, m\big((N-1)\,h\big) .$$

Reiterating the argument which leads to (A10), we obtain after $(N-1)$ more steps that

$$\text{(A12)} \qquad \left\|\delta_t(f) - \frac{Nh}{t}\,\delta_h(f) - \frac{r}{t}\,\delta_r(f)\right\|_\infty \leqq \frac{r}{t}\, m(Nh) + \frac{h}{t} \sum_{j=1,\dots,N-1} m(jh) .$$

In particular, in view of (A2) and since m is nondecreasing, we deduce

$$\|\delta_t(f) - \delta_h(f)\|_\infty \leqq \left(\frac{r}{t} + (N-1)\frac{h}{t}\right) m(t) + 2\frac{r}{t} C_0,$$

and (A8) is proved since $t = Nh + r$.

It is now easy to conclude. Using (A2) and (A3), we pick h_n going to 0 so that $\delta_{h_n}(f)$ converges uniformly on compact sets to a bounded Lipschitz function on $\mathbb{R}^N$, which we denote by $A[f]$. Then we deduce from (A8) that

$$\text{(A13)} \qquad \|\delta_t(f) - A[f]\|_\infty \leqq m(t),$$

and Theorem 1(i) is proved.

We now indicate briefly which adjustments are necessary in order to prove the second item of Theorem 1. We consider operators $T_{t,s}$, but, without loss of generality, we assume that $s = 0$. To keep the same notation as in the preceding proof, we set

$$\delta_{t,s}(f) = \frac{T_{0,t}(f) - T_{0,s}(f)}{t-s}, \qquad \delta_t(f) = \delta_{t,0}(f) = \frac{T_{0,t}(f) - f}{t}.$$

The only step to be modified in the preceding proof is Step 3. Indeed, the operators $T_{0,t}$ satisfy the same assumptions as the operators T_t, except [Recursivity] and this axiom is only used in Step 3.

Step 3-bis. We prove again that for all t, s in $[0, \frac{1}{2}]$ we have

$$\text{(A4)} \qquad \|\delta_{t+s,t}(f) - \delta_s(f)\|_\infty \leqq m(t)$$

for some continuous, nonnegative, nondecreasing function m on $[0, \frac{1}{2}]$ such that $m(0) = 0$, and m depends only on bounds of derivatives of f. Again we write

$$\delta_s(f) = \delta_s(f) * \rho_\varepsilon - \{\delta_s(f) * \rho_\varepsilon - \delta_s(f)\}$$

where ρ_ε is defined as above. We deduce from (A3) that

$$\text{(A5)} \qquad \|\delta_s(f) * \rho_\varepsilon - \delta_s(f)\|_\infty \leqq C_1 \varepsilon \quad \text{for all } \varepsilon > 0$$

for some $C_1 \geqq 0$ depending only on bounds on derivatives of f. Then, because of (1) and [Causality],

$$\|T_{0,t+s}(f) - T_{0,t}(f + s\delta_s(f) * \rho_\varepsilon)\|_\infty \leqq \|T_{t,t+s}(f) - (f + s\delta_s(f) * \rho_\varepsilon)\|_\infty$$

since by [Temporal regularity], we have $\|T_{t,t+s}(f) - T_{0,s}(f)\|_\infty \leqq Csn(t)$. Using (A5) and the definition of δ_s, we obtain

$$\text{(A6)} \quad \|T_{0,t+s}(f) - T_{0,t}(f + s\delta_s(f) * \rho_\varepsilon)\|_\infty \leqq C_1 \varepsilon s + Csn(t) \quad \text{for all } \varepsilon > 0.$$

Now, [Regularity] implies that

$$\text{(A7)} \qquad \|T_{0,t}(f + s\delta_s(f) * \rho_\varepsilon) - T_{0,t}(f) - s\delta_s(f) * \rho_\varepsilon\|_\infty \leqq C_\varepsilon st,$$

for some $C_\varepsilon \geqq 0$ depending only upon $\varepsilon > 0$ and bounds on derivatives of f. Collecting the estimates (A5)–(A7), we finally deduce

$$\|\delta_{t+s,t}(f) - \delta_s(f)\|_\infty \leqq 2C_1\varepsilon + Cn(t) + C_\varepsilon t,$$

and we conclude (A4) by setting $m(t) = \inf_{\varepsilon\in]0,1]} 2C_1\varepsilon + C_\varepsilon t) + Cn(t)$.

Appendix 2: Ellipticity conditions on F

In order to have the maximum principle for the equation $\partial u/\partial t = F(D^2u, Du, t)$, F must satisfy the inequality. $F(A,p,t) \geqq F(B,p,t)$ where A, B are symmetric and $A \geqq B$. [Grey-scale invariance] implies that F depends only on Q_pAQ_p, where $Q_p = I_N - p \otimes p/|p|^2$. Moreover, if $A \geqq B$, then $Q_pAQ_p \geqq Q_pBQ_p$ since Q_p is symmetric. Thus we can limit the study to the restriction of A in a plane:

Let $H(A) = H(\gamma_1, \gamma_2, \gamma_3)$ be a differentiable function, where $A = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_3 \end{pmatrix}$.

We begin by finding conditions on H which imply that $H(A) \geqq H(B)$ if $A \geqq B$.

Set $B = \begin{pmatrix} \varepsilon & 0 \\ 0 & 0 \end{pmatrix}$, $\varepsilon > 0$; then $H(A+B) \geqq H(A) \Rightarrow \dfrac{\partial H}{\partial \gamma_1} \geqq 0$.

Now set $B = \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon \end{pmatrix}$, $\varepsilon > 0$; then $H(A+B) \geqq H(A) \Rightarrow \dfrac{\partial H}{\partial \gamma_3} \geqq 0$.

Finally set $B = \begin{pmatrix} |\varepsilon| y & \varepsilon \\ \varepsilon & |\varepsilon| y^{-1} \end{pmatrix}$, $y \geqq 0$ and $\varepsilon \in \mathbb{R}$; then $H(A+B) \geqq H(A) \Rightarrow$

$y \dfrac{\partial H}{\partial \gamma_1} + \operatorname{sign}(\varepsilon) \dfrac{\partial H}{\partial \gamma_2} + y^{-1} \dfrac{\partial H}{\partial \gamma_3} \geqq 0$ for all $y \geqq 0$, or equivalently,

$4 \dfrac{\partial H}{\partial \gamma_3} \dfrac{\partial H}{\partial \gamma_1} \geqq \left(\dfrac{\partial H}{\partial \gamma_2}\right)^2$.

Since H is nondecreasing, we have the three following conditions:

$$(1)\ \frac{\partial H}{\partial \gamma_1} \geqq 0, \quad (2)\ \frac{\partial H}{\partial \gamma_3} \geqq 0, \quad (3)\ 4\frac{\partial H}{\partial \gamma_3}\frac{\partial H}{\partial \gamma_1} \geqq \left(\frac{\partial H}{\partial \gamma_2}\right)^2.$$

On the other hand, these conditions are sufficient. Indeed, for every positive B, we can find a, b, c, y in $\mathbb{R}^+$ such that

$$B = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c\begin{pmatrix} y & \pm 1 \\ \pm 1 & y^{-1} \end{pmatrix}.$$

Then, the conditions (1), (2), (3) imply that $H(A+B) \geqq H(A)$.

Appendix 3: Interpretation of accel(u)

In order to understand the meaning of the term accel(u) in the equation, let us give the following lemma:

Lemma. *Consider a picture in translation motion. Let v be the velocity vector of the translation: $v = (vx(\theta), vy(\theta))$. Define $v_1 = (v, \nabla u)/\|\nabla u\|$, the flow on the direction of ∇u. Then,* accel$(u) = -\partial v_1(\theta)/\partial\theta$. *Thus* accel$(u)$ *is the acceleration in the direction of ∇u, which can be called the apparent acceleration. Moreover,* accel(u) *is the curvature of the level curve in the plane $(\nabla u, V)$ multiplied by the square of the norm of V, where V is the velocity vector $(vx(\theta), vy(\theta), 1)$.*

Proof. Let $u(x,y)$ be the static picture, N a point of the plane and $w(x,y,\theta)$ this picture in movement. We have

$$w(x,y,\theta) = u\left(x - \int_{\theta_0}^{\theta} v_x(\theta), y - \int_{\theta_0}^{\theta} v_y(\theta)\right).$$

Because of the isometry invariance, we can take the axes (i,j,k) so that at the point N: $(i, Du) = 0$ and $(j, Du) \geqq 0$. Then $v_1(\theta) = v_x(\theta)$ and $Du = (|\nabla u|, 0, 0)$. We set $v_2(\theta) = v_y(\theta)$. We have

$$Dw = (|\nabla u|, 0, -v_1|\nabla u|), \quad Dw^{\perp} = (0, |\nabla u|, 0),$$

$$Dw^{\pm} = |\nabla u| \, (-v_1|\nabla u|, 0, -|\nabla u|),$$

$$Aw = \begin{pmatrix} u_{xx} & u_{xy} & -v_1 u_{xx} - v_2 u_{xy} \\ u_{xy} & u_{yy} & -v_1 u_{xy} - v_2 u_{yy} \\ -v_1 u_{xx} - v_2 u_{xy} & -v_1 u_{xy} - v_2 u_{yy} & v_1^2 u_{xx} + v_2^2 u_{yy} + 2 v_1 v_2 u_{xy} - |\nabla u| \dfrac{\partial v_1(\theta)}{\partial\theta} \end{pmatrix}.$$

Thus,

$$\gamma_{1w} = \gamma_1, \quad \gamma_{2w} = v_2 \gamma_1 |\nabla u|, \quad \gamma_{3w} = |\nabla u|^2 v_2^2 \gamma_1 - |\nabla u|^5 \frac{\partial v_1(\theta)}{\partial\theta}.$$

Therefore

$$\text{accel}(w) = \frac{\gamma_{1w}\gamma_{3w} - \gamma_{2w}^2}{|\nabla u|^5 \gamma_{1w}} = -\frac{\partial v_1(\theta)}{\partial\theta}.$$

Let us now consider the velocity vector $V = (v_1, v_2, 1)$. We have $v_1 = -\dfrac{w_\theta}{|\nabla u|}$ and $\gamma_{2w} = v_2\gamma_1|\nabla u|$. Therefore, $v_2 = \dfrac{\gamma_{2w}}{\gamma_{1w}|\nabla u|}$. Then $V = \left(-\dfrac{w_\theta}{|\nabla u|}, \dfrac{\gamma_{2w}}{\gamma_{1w}|\nabla u|}, 1\right)$; we remark that $V = \dfrac{1}{|\nabla u|^2}\left(Dw^{\perp}\dfrac{\gamma_{2w}^2}{\gamma_1} - Dw^{\pm}\right)$ and that

$$(AV, V) = \frac{1}{|\nabla u|^4}\left(\frac{\gamma_{2w}^2}{\gamma_1^2}(ADw^{\perp}, Dw^{\perp}) - 2\,\frac{\gamma_{2w}}{\gamma_1}\,(ADw^{\perp}, Dw^{\pm}) + (ADw^{\pm}, Dw^{\pm})\right).$$

Hence, $(AV, V) = \frac{1}{|\nabla u|^4}\left(\gamma_{3w} - \frac{\gamma_{2w}^2}{\gamma_1}\right)$. Then $\mathrm{accel}(w) = (AV, V)/|\nabla u| = \|V\|^2 \mathrm{curv}2$,

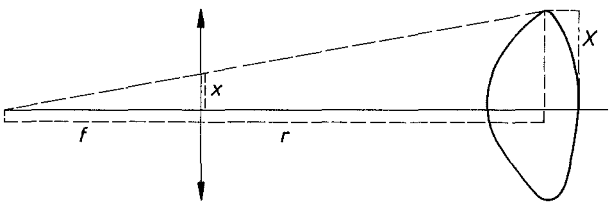
where $\mathrm{curv}2 = \left(\frac{AV}{\|V\|}, \frac{V}{\|V\|}\right)\Big/|\nabla u|$ is the curvature of the level curve included in the plane $(\nabla u, V)$.

A picture is the projection of the three-dimentional space; we now want to prove that $\mathrm{accel}(u)$ is equal to 0 for every uniform translation of an object in the space. We prove the following lemma:

Lemma. *Consider an object at a distance $r(\theta)$ from the retina. Then*

$$\mathrm{accel}(u) = -r''(\theta)\frac{f^2}{(f + r(\theta))^3}.$$

Proof. We must first define the vision system, in other terms, the projection of the three-dimensional space on the plane which defines the picture:



Thus, we obtain the formulae

$$X = \frac{fx}{(f + r)}, \qquad Y = \frac{fy}{(f + r)}.$$

Thus, when the distance decreases, the picture grows, and we have a growth factor $a = f/(f + r)$.

Let $u(x, y)$ be the picture of an static object, and $w(x, y, \theta)$ the movie of that object such that $w(x, y, \theta) = u(a(\theta)x, b(\theta)y)$. We choose the axes, in order to have $(\nabla u, j) = 0$, and $(\nabla u, i) \geqq 0$ at the point $N(x_0, y_0, \theta)$. Then

$$Dw = (a(\theta)|\nabla u|, 0, -xa'(\theta)|\nabla u|), \quad Dw^{\perp} = (0, a|\nabla u|, 0),$$

$$Dw^{\pm} = a|\nabla u|\,(xa'(\theta)|\nabla u|, 0, -a(\theta)|\nabla u|),$$

$$Aw = \begin{pmatrix} a(\theta)^2 u_{xx} & a(\theta)b(\theta)u_{xy} & xa(\theta)\,a'(\theta)u_{xx} + ya(\theta)b'(\theta)u_{xy} + a'(\theta)|\nabla u| \\ w_{xy} & b(\theta)^2 u_{yy} & yb(\theta)b'(\theta)u_{yy} + xb(\theta)a'(\theta)u_{xy} \\ w_{x\theta} & w_{y\theta} & (xa'(\theta)u_{xx} + yb'(\theta)u_{xy})xa'(\theta) + xa''(\theta)|\nabla u| \\ & & + (xa'(\theta)u_{xy} + yb'(\theta)u_{yy})yb'(\theta) \end{pmatrix}.$$

Thus,

$$\gamma_{1w} = a(\theta)^2 b(\theta)^2 |\nabla u|^2 u_{yy},$$

$$\gamma_{2w} = y a(\theta)^3 b(\theta) b'(\theta) |\nabla u|^3 u_{yy},$$

$$\gamma_{3w} = y^2 a(\theta)^4 b'^2(\theta) |\nabla u|^4 u_{yy} + x |\nabla u|^3 a(\theta) (a(\theta) a''(\theta) - 2a'^2(\theta)).$$

We obtain

$$\gamma_{1w}\gamma_{3w} - \gamma_{2w}^2 = \frac{b(\theta)^2}{a(\theta)^2} |\nabla u|^5 \gamma_1 (a(\theta)a''(\theta) - 2a'(\theta)^2).$$

Then

$$\operatorname{accel}(w) = [b(\theta)^2 / a(\theta)^2] \, (a(\theta)a''(\theta) - 2a'(\theta)^2),$$

but $a(\theta) = f/(f + r(\theta))$. So

$$a(\theta)a''(\theta) - 2a'(\theta)^2 = \frac{f}{f + r(\theta)} \left(\frac{2fr'(\theta)^2}{(f + r(\theta))^3} - \frac{fr''(\theta)}{(f + r(\theta))^2} \right) - \frac{2f^2 r'(\theta)}{(f + r(\theta))^4}$$

$$= -\frac{f^2 r'' \theta}{(f + r(\theta))^3}.$$

Thus, with $a(\theta) = b(\theta)$, we obtain $\operatorname{accel}(w) = -r''(\theta) f^2/(f + r(\theta))^3$; thus $\operatorname{accel}(w)$ is proportional to the acceleration. Moreover, $\operatorname{accel}(w) = 0$ if and only if $r''(\theta) = 0$, i.e., if and only if $r'(\theta)$ is constant. Then we obtain that $\operatorname{accel}(u) = 0$, for all objects in uniform translation in the three-dimensional space.

Addendum

After this paper was accepted for publication, we noticed several works on image processing which confirm our conclusions.

1) First of all, the newly introduced model

$$\frac{\partial u}{\partial t} = |Du| \, (\operatorname{curv}(u))^{\frac{1}{3}} \tag{34}$$

has been independently discovered by SAPIRO & TANNENBAUM [SaTa 1,2] and ourselves [AGLM 1,2,3], in different contexts.

SAPIRO & TANNENBAUM consider smooth Jordan curves $C(s)$ where s denotes the affine parameter, defined by $\left|\det\left(\frac{\partial C}{\partial s}, \frac{\partial^2 C}{\partial s^2}\right)\right| = 1$ [SaTa]. Their approach, modelling curve evolution, yields the intrinsic equation

$$\frac{\partial C}{\partial t}(s,t) = \frac{\partial^2 C}{\partial s^2}(s,t). \tag{35}$$

This second equation describes the motion of the level curves of a solution u of (34); therefore both models are equivalent.

Equation (35) has been solved, with existence, regularity and uniqueness results when the initial Jordan curve is smooth in [Gray, GaHa, SaTa 1,2].

2) In image processing the combination of both [Causality] and [Comparison] principles is generally called *causality* [30, 61, Kim, SaTa 1,2, KTZ]. Indeed, the causality assumption in image processing states that when the scale increases, no new structure appears in the image. HUMMEL [Hum] noticed that the causality assumption can be reduced to a maximum principle plus a pyramidal structure. We preferred to divide the *causality* into two principles.

3) The Alvarez-Lions-Morel model of Section 4.5, when restricted to a single curve evolution, was mathematically introduced in [GaHa, Gray, Ang]. They proved existence, uniqueness, regularity of solutions in this context.

This equation (the classical mean-curvature motion) was introduced in image processing in [Kim, KTZ, MaMo, AGLM 1,2,3, 62] in different contexts. However, an early version of an algorithm leading this models is proposed in [KoDo]; see also [BaGe, MBO].

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