Arch. Rational Mech. Anal. 129 (1995) 1–10. © Springer-Verlag 1995

Rayleigh's Conjecture on the Principal Frequency of the Clamped Plate

NIKOLAI S. NADIRASHVILI

Communicated by J. BALL

0. Introduction

The theory of isoperimetric inequalities in mathematical physics goes back to Lord RAYLEIGH's paper (1877) (cf. [1, 2]) in which he conjectured three inequalities between physical and geometrical quantities of plane domains. Namely, he conjectured that:

(1) Of all fixed membranes with a given area, the circle has the minimal principal frequency.
(2) Of all clamped plates with a given area, the circle has the minimal principal frequency.
(3) Of all conducting plates with a given area, the circle has the minimal electrostatic capacity.

FABER [3] and KRAHN [4] found essentially the same proof for the first of Rayleigh's conjectures. The third of Rayleigh's conjectures was proved by POLYA & SZEGÖ [5]. The second Rayleigh conjecture remained open up to now. Partial results were obtained by POLYA & SZEGÖ [2] who proved that the conjecture is true provided that a corresponding ground-state eigenfunction is of constant sign. Unfortunately, in general the ground state of a clamped plate can change its sign. This was observed, e.g., for a suitable annular plate by DUFFIN & SHAFFER [6], for a semi-infinite strip, by DUFFIN [7] and for a polygonal domain by KONDRATIEV, KOZLOV & MAZ'YA [8]. Recently TALENTI proved in [9] that the conjecture (2) is true up to a factor of 0.98.

The aim of this paper is to prove the second Rayleigh conjecture. A sketch of this proof was given in the announcements [10, 11]. In our proof we use a decomposition introduced by TALENTI [9].

1. Statement of the result

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Let us consider in the domain Ω the Dirichlet spectral problem for the biharmonic operator:

$$\Delta^2 u_i = \lambda_i u_i \quad \text{in } \Omega, \quad u_i = |\nabla u_i| = 0 \quad \text{on } \partial\Omega. \tag{1.1}$$

Here the λ_i are the eigenvalues (arranged in increasing order) and the u_i are the corresponding eigenfunctions of the positive self-adjoint operator H defined in $L_2(\Omega)$ by

$$(H\varphi, \varphi) = \int_\Omega |\Delta\varphi|^2 \, dx, \quad \varphi \in C_0^\infty(\Omega).$$

If $\partial\Omega$ is smooth, then the u_i are classical solutions of (1.1). Let us denote

$$\lambda(\Omega) := \lambda_1.$$

Theorem. *Let $\Omega^* \subset \mathbb{R}^2$ be a disk having the same area as the domain Ω. Then*

$$\lambda(\Omega) \geqq \lambda(\Omega^*),$$

and furthermore, equality holds only when the domain Ω is a disk.

2. Notation

We denote by $B_r \subset \mathbb{R}^2$ the open disk of radius r with center at 0, $S_r := \partial B_r$, $B := B_1$, $S := S_1$.

If $G \subset \mathbb{R}^2$ is a bounded domain and $f \in L^2(G)$, we denote by $f^* \in L^2(G^*)$ the decreasing circular rearrangement of the function f (see, e.g., [9]).

For the problem

$$\Delta u = f \quad \text{in } G, \quad u = 0 \quad \text{on } \partial G,$$

we denote $\mathscr{D} f := u$ (so that $\mathscr{D}$ is the inverse Dirichlet Laplacian).

By $\partial/\partial n$ we denote the inner normal derivative.

3. Proof of the theorem

Consider the

3.1. Variational Problem. For $r, R > 0$, $r^2 + R^2 = 1$, minimize the functional

$$I(f_1, f_2) = \frac{\int_{B_r} f_1^2 \, dx + \int_{B_R} f_2^2 \, dx}{\int_{B_r} (\mathscr{D} f_1)^2 \, dx + \int_{B_R} (\mathscr{D} f_2)^2 \, dx}$$

under the constraints: f_1 and f_2 belong to $L^2(B_r)$, are functions of $|x|$ only and obey

$$\int_{B_r} f_1 \, dx = \int_{B_R} f_2 \, dx.$$

We denote by I_r the infimum of $I(f_1, f_2)$ under the given constraints.

If we set $\mathscr{D} f_i = v_i$, then Problem 3.1 is equivalent to

3.2. Variational Problem. Minimize the functional

$$\frac{\int_{B_r} (\Delta v_1)^2\,dx + \int_{B_R} (\Delta v_2)^2\,dx}{\int_{B_r} v_1^2\,dx + \int_{B_R} v_2^2\,dx}$$

under the constraints: $v_1 \in W^{2,2}(B_r)$ and $v_2 \in W^{2,2}(B_R)$ are functions of $|x|$ only, $v_1 = 0$ on S_r, $v_2 = 0$ on S_R and

$$\int_{S_r} \frac{\partial v_1}{\partial n}\,ds = \int_{S_R} \frac{\partial v_2}{\partial n}\,ds \geqq 0.$$

Let us denote by $G(p, a)$ the Green function of the Dirichlet problem for the biharmonic operator in the disk B_R, where the singularity is at the point a. Then (cf. [2, p. 141])

$$G(p, a) = |z - a|^2 \ln \left|\frac{R(z-a)}{R^2 - \bar{a}z}\right| - \frac{1}{2R^2}(|R(z-a)|^2 - |R^2 - \bar{a}z|^2) \tag{3.3}$$

where z and a are the complex numbers representing the points p and a. This representation formula implies that $G(p, a) > 0$ for $p, a \in B_R$.

The Krein-Rutman theorem implies

3.4. Lemma. *The principal eigenvalue of problem* (1.1) *in a disk B_R is simple; the corresponding eigenfunction is positive and hence is a function of the radius only.*

Reference [12] proves

3.5. Lemma. *If $\Delta^2 u = 0$ in B, $u = 0$ on S, and $\partial u/\partial n > 0$ on S, then $u > 0$ in B.*

Consider the problem

$$\Delta^2 u = cu \quad \text{in } B, \quad u = 0 \quad \text{on } S, \quad \frac{\partial u}{\partial n} = 1 \quad \text{on } S. \tag{3.6}$$

If $c < \lambda(B)$, this problem has a unique solution $u(c, x)$ which depends smoothly on the parameter c.

From (3.3) it follows that

$$u(0, x) > 0. \tag{3.7}$$

3.8. Lemma. *Let $\dot{u} := \partial u/\partial c$. Then*

$$\dot{u}(c, x) > 0 \quad \textit{for all } 0 \leqq c < \lambda(B), \quad x \in B. \tag{3.8'}$$

Proof. If we assume the contrary, then there exists $c_0 \in [0, \lambda(B))$, $x_0 \in B$ with

$$\dot{u}(c_0, x_0) = 0$$

such that for all $c \in [0, c_0)$, $x \in B$

$$\dot{u}(c, x) > 0.$$

Then from the last inequality and from (3.7) it follows that

$$u(c_0, x) > 0.$$

From (3.6) we derive

$$\Delta^2 \dot{u}(c_0, x) = c_0 \dot{u}(c_0, x) + u(c_0, x) \text{ in } B, \quad \dot{u} = |\nabla \dot{u}| = 0 \text{ on } S.$$

Since by our assumption $\dot{u}(c_0, x) \geqq 0$ and $u(c_0, x) > 0$, from (3.3) it follows that

$$\dot{u}(c_0, x) > 0 \quad \text{in } B.$$

This contradiction proves the assertion.

Let v be a minimizer of the functional $\int_B (\Delta v)^2 \, dx$ under the constraints: $v \in W^{2,2}(B)$ is radial, $v = 0$ on S, $\partial v/\partial n = a$ on S, $\int_B v^2 \, dx \geqq b$, where $a, b > 0$ are given constants. By the multiplier rule there is a number α such that the Euler-Lagrange equation for this problem is

$$\Delta^2 v = \alpha v \text{ in } B. \tag{3.9}$$

Let $u(c, x)$ be a solution of boundary-value problem (3.6). Then u minimizes the integral

$$\int_B ((\Delta u)^2 - cu^2) \, dx \tag{3.10}$$

under the constraints: $u \in W^{2,2}(B)$, $u = 0$ on S, $\partial u/\partial n = a$ on S. Hence if $c < 0$, then

$$\int_B u^2(c, x) \, dx \leqq \int_B u^2(0, x) \, dx.$$

From the last inequality it follows that constant α of (3.9) is non-negative.

Let (v_1, v_2) be a minimizer of Problem 3.2. Then each of the functions v_1, v_2 is a minimizer of the problem leading to (3.9) in the disks B_r, B_R, respectively. Hence

$$\Delta^2 v_1 = \alpha_1 v_1 \text{ in } B_r, \quad \Delta^2 v_2 = \alpha_2 v_2 \text{ in } B_R, \tag{3.11}$$

with $\alpha_1, \alpha_2 > 0$.

Let (v_1, v_2) be a minimizer of Problem 3.2. We prove that $v_1, v_2 \geqq 0$: Let us define a function v in B_r which is radial and satisfies $\partial v/\partial r \leqq 0$, $-\partial v/\partial r = |\partial v_1/\partial r|$ in B_r. Then

$$\begin{aligned} \int_{B_r} (\Delta v)^2 \, dx &= \int_{B_r} (\Delta v_1)^2 \, dx, \\ \int_{B_r} v^2 \, dx &\geqq \int_{B_r} v_1^2 \, dx, \end{aligned} \tag{3.12}$$

and the second inequality is strict if v is not identically equal to v_1. Hence, since (v_1, v_2) is a minimizer, $v \equiv v_1$ and so $v_1 \geqq 0$. A similar proof shows that $v_2 \geqq 0$.

3.13. Lemma. *Let $r, R \neq 0$, $\partial v_1/\partial r < 0$ on S_r. Then $\alpha_1 < \lambda(B_r)$, $\alpha_2 < \lambda(B_R)$.*

Proof. Let $\lambda_1 := \lambda(B_r)$, and let $u_1 > 0$ be the principal eigenfunction of problem (1.1) in B_r. Then

$$\int_{B_r} u_1 v_1 \, dx = \frac{1}{\lambda_1} \int_{B_r} v_1 \Delta^2 u_1 \, dx \tag{3.14}$$

$$= \frac{\alpha_1}{\lambda_1} \int_{B_r} v_1 u_1 \, dx + \frac{1}{\lambda_1} \int_{S_r} \frac{\partial v_1}{\partial n} \Delta u_1 \, ds.$$

Note that

$$0 = \int_{B_r} u_1 \Delta^2 (r^2 - |x|^2) \, dx = \lambda_1 \int_{B_r} u_1 (r^2 - |x|^2) \, dx - 2r \int_{S_r} \Delta u_1 \, ds$$

and therefore $\Delta u_1 > 0$ on S_r. Hence

$$\int_{B_r} u_1 v_1 \, dx > \frac{\alpha_1}{\lambda_1} \int_{B_r} u_1 v_1 \, dx;$$

consequently from (3.12) it follows that

$$\alpha_1 < \lambda_1 .$$

A similar proof shows that $\alpha_1 < \lambda(B_R)$.

3.15. Lemma. *Let $a, b, c, d > 0$ and either $\gamma > 1$, $a/b > c/d$ or $0 < \gamma < 1$, $a/b < c/d$. Then*

$$p(\gamma) := \frac{\gamma a + c}{\gamma b + d} > \frac{a + c}{b + d}. \tag{3.16}$$

Proof.

$$\frac{\partial p}{\partial \gamma} = \frac{ad - bc}{(\gamma b + d)^2}$$

and so the inequality evidently follows.

3.17. Lemma. *Let (v_1, v_2) be a minimizer of Problem 3.2. Let $0 < r < R$ and $\partial v_1/\partial r < 0$ on S_r. Then*

$$\alpha_1 \leqq \left(\frac{R}{r}\right)^4 \alpha_2 .$$

Proof. Let us denote

$$\Delta v_i = f_i,\ i = 1, 2, \quad (R/r)^2 = k > 1, \quad \int f_i^2 \, dx = a_i, \quad \int v_i^2 \, dx = b_i, \quad v_2(\sqrt{k} x) = u(x),$$

$$\Delta u = g, \quad v_1(x/\sqrt{k}) = w(x), \quad \Delta w = h .$$

Then

$$\int u^2 \, dx = b_2/k, \quad \int g^2 \, dx = k a_2, \quad \int w^2 \, dx = k b_1, \quad \int h^2 \, dx = a_1/k.$$

For contradiction, let us assume that $\alpha_1 > k^2\alpha_2$. Since

$$\int_{S_r} \frac{\partial u}{\partial n} ds = \int_{S_r} \frac{\partial v_1}{\partial n} ds,$$

the pair (u, v_2) satisfies the constraints of Problem 3.2. We have

$$\Delta^2 u = k^2 \alpha_2 u.$$

From Lemma 3.8, identity (3.14) and our assumption that $\alpha_1 > k^2\alpha_2$, we conclude that

$$b_1 > \frac{b_2}{k}.$$

Since the pair (v_1, v_2) is a minimizer, we have

$$\frac{a_1 + a_2}{b_1 + b_2} \leqq \frac{ka_2 + a_2}{b_2/k + b_2}.$$

Now (3.16) implies that

$$\frac{ka_2 + a_2}{b_2/k + b_2} < \frac{\gamma k a_2 + a_2}{b_1 + b_2}$$

with $\gamma = kb_1/b_2 > 1$. So $a_1 < \gamma k a_2$, or

$$\frac{a_1}{b_1} < k^2 \frac{a_2}{b_2}.$$

Since

$$\int_{S_R} \frac{\partial w}{\partial n} ds = \int_{S_R} \frac{\partial v_2}{\partial n} ds,$$

the pair (v_1, w) satisfies the constraints of Problem 3.2. By inequality (3.16), we have

$$\frac{a_1 + a_1/k}{b_1 + kb_1} < \frac{a_1 + \kappa a_1/k}{b_1 + b_2}$$

with $\kappa = b_2/kb_1 < 1$. Since we proved that

$$\frac{\kappa a_1}{k} = \frac{b_2 a_1}{k^2 b_1} < a_2,$$

we conclude that

$$\frac{a_1 + a_1/k}{b_1 + kb_1} < \frac{a_1 + a_2}{b_1 + b_2}.$$

But this inequality contradicts the assumption that (v_1, v_2) is a minimizer.

Inequalities (3.17) and (3.8′) imply that

$$\frac{\int_{B_r} v_1\,dx}{r^2} \leqq \frac{\int_{B_R} v_2\,dx}{R^2}. \tag{3.18}$$

3.19. Main Lemma.

$$\frac{\partial}{\partial r} I_r \geqq 0 \quad \textit{for } 0 < R.$$

Proof. Let (f_1, f_2) be a minimizer of Problem 3.1. Recall that $r^2 + R^2 = 1$, and let $0 < \rho < r$. Let $g_1(\rho, x)$ be the restriction of the function f_1 to the disk B_ρ. We assume that $\rho^2 + \rho'^2 = 1$. Let $g_2(\rho, x)$ be an extension of the function f_2 onto the disk $B_{\rho'}$ such that $g_2(\rho, x)$ is a function of ρ and $|x|$ only, monotonically nondecreasing for $|x| \in (R, \rho')$ and such that for all $a \in \mathbb{R}$,

$$\operatorname{meas}\{x \in B_{\rho'} \setminus B_R, g_2(\rho, x) > a\} = \operatorname{meas}\{x \in B_r \setminus B_\rho, -f_1(x) > a\}.$$

Then for all $a < \rho < r$,

$$\int\limits_{B_\rho} g_1\,dx = \int\limits_{B_{\rho'}} g_2\,dx, \quad \int\limits_{B_\rho} g_1^2\,dx + \int\limits_{B_{\rho'}} g_2^2\,dx \equiv C,$$

where C is independent of ρ. Let us denote

$$\tilde{v}_i(\rho, x) := \mathscr{D} g_i(\rho, x).$$

Then

$$\tilde{v}_1(\rho, x) = \tilde{v}_1(0, x)|_{B_\rho} - \tilde{v}_1(0, x)|_{S_\rho}$$

and hence

$$\left.\frac{\partial}{\partial \rho} \int\limits_{B_\rho} \tilde{v}_1^2(\rho, x)\,dx\right|_{\rho = 0} = \frac{1}{\pi r} \frac{\partial \tilde{v}_1(0, x)}{\partial n} \int\limits_{B_r} \tilde{v}_1(0, x)\,dx.$$

By similar reasoning we obtain

$$\left.\frac{\partial}{\partial \rho} \int\limits_{B_{\rho'}} \tilde{v}_2^2(\rho, x)\,dx\right|_{\rho = 0} = \frac{-1}{\pi R} \frac{\partial \tilde{v}_2(0, x)}{\partial n} \int\limits_{B_R} \tilde{v}_2(0, x)\,dx.$$

Since

$$\int\limits_{S_r} \frac{\partial \tilde{v}_1(0, x)}{\partial n}\,dx = \int\limits_{S_R} \frac{\partial \tilde{v}_2(0, x)}{\partial n}\,ds,$$

we have by (3.18) that

$$\left.\frac{\frac{\partial}{\partial \rho} \int_{B_\rho} \tilde{v}_1^2(\rho, x)\,dx}{\frac{\partial}{\partial \rho} \int_{B_{\rho'}} \tilde{v}_2^2(\rho, x)\,dx}\right|_{\rho = 0} = \frac{R^2}{r^2} \frac{\int_{B_r} \tilde{v}_1(0, x)\,dx}{\int_{B_R} \tilde{v}_2(0, x)\,dx} \leqq 1.$$

Thus we conclude that

$$\frac{\partial}{\partial\rho}\left(\int_{B_\rho} \tilde{v}_1^2(\rho, x)\,dx + \int_{B_{\rho'}} \tilde{v}_2^2(\rho, x)\,dx\right)\Big|_{\rho=0} \leqq 0,$$

as required.

3.20. Lemma.

$$\lim_{r\to 0} I_r \geqq \lambda(B).$$

Proof. By the variational principle,

$$\lambda(\Omega) = \inf \frac{\int_\Omega (\Delta u)^2\,dx}{\int_\Omega u^2\,dx} \tag{3.21}$$

under the constraints $u \in W^{2,2}(\Omega)$, $u = |\nabla u| = 0$ on $\partial\Omega$. Let

$$\inf \frac{\int_{B_r} (\Delta u)^2\,dx}{\int_{B_r} u^2\,dx} = \frac{\beta}{r^4}, \tag{3.22}$$

where the infimum is subject to the constraints $u \in W^{2,2}K(B_r)$, $u = 0$ on S_r. Since the operator $\mathscr{D}$ is bounded in $L^2(B_r)$, it follows that $\beta > 0$.

Let $0 < r < R$ and let $(f_1(r, x), f_2(r, x))$ be a minimizer of Problem 3.1. Without loss of generality we may assume that

$$\int_{B_r} f_1(r, x)\,dx = 1$$

for $0 < r < R$. From (3.22) it follows that

$$\frac{\int_{B_r} f_1^2(r, x)\,dx}{\int_{B_r} (\mathscr{D}f_1(r, x))^2\,dx} \geqq \frac{\beta}{r^4}.$$

Since

$$\int_{B_r} f_1^2(r, x)\,dx \geqq \frac{1}{\pi r^2}$$

and $I_r <$ const for all $0 < r$, it follows that

$$\int_{B_R} f_2^2(r, x)\,dx \to \infty$$

as $r \to 0$, and hence

$$\int_{B_R} (\mathscr{D}f_2(r, x))^2\,dx \to \infty$$

as $r \to 0$. Therefore

$$\frac{\int_{B_R} |\mathscr{D}f_2(r, x)|\,dx}{\int_{B_R} (\mathscr{D}f_2(r, x))^2\,dx} \to 0 \quad \text{as } r \to 0. \tag{3.23}$$

Let us denote $h(r, x) = f_2(r, x) - 1$. Then

$$\int_{B_R} h\,dx = 0,$$

and hence

$$\int_{S_R} \frac{\partial(\mathscr{D}h)}{\partial n}\,ds = 0.$$

Since $\mathscr{D}h$ is a radial function, we conclude that

$$\mathscr{D}h = |\nabla \mathscr{D}h| = 0 \quad \text{on } S_R.$$

By (3.21), we have

$$\frac{\int_{B_R} h^2\,dx}{\int_{B_R} (\mathscr{D}h)^2\,dx} \geqq \lambda(B_R).$$

Furthermore $h^2 = f_2^2 - 2f_2 + 1$ and so taking into account (3.23) we get

$$\liminf_{r \to 0} \frac{\int_{B_R} f_2^2\,dx}{\int_{B_R} (\mathscr{D}f_2)^2\,dx} \geqq \lambda(B). \tag{3.24}$$

This inequality and (3.22) complete the proof of the lemma.

Let $f \in L^2(G)$, $G \subset \mathbb{R}^2$ be a bounded domain. Let us assume that $\mathscr{D}f > 0$ in G. Then

$$\int_G (\mathscr{D}f)^2\,dx \leqq \int_{G^*} (\mathscr{D}f^*)^2\,dx, \tag{3.25}$$

with equality only in the case when $f \equiv f^*$ (cf. [9, 13, 14]).

Let u_1 be a ground state of problem (1.1), and assume that $\operatorname{meas}\Omega = \pi$. Let

$$\Omega^+ := \{x \in \Omega, u_1(x) > 0\}, \quad \Omega^- := \{x \in \Omega, u_1(x) < 0\}, \quad \Delta u_1 := f. \tag{3.26}$$

Let f_+ be the restriction of f to Ω^+ and f_- be the restriction of $(-f)$ to Ω^-. From (3.11) it follows that

$$\lambda(\Omega) = \frac{\int f^2\,dx}{\int u_1^2\,dx} \geqq \frac{\int f_+^{*2}\,dx + \int f_-^{*2}\,dx}{\int (\mathscr{D}f_+^*)^2\,dx + \int (\mathscr{D}f_-^*)^2\,dx}$$

with equality only in the cases when either Ω is a disk or Ω is a union of two disjoint disks. Without loss of generality we may exclude the case of two disks from our consideration. Since $\partial u_1/\partial n = 0$ on $\partial\Omega$, Green's formula implies

$$\int_\Omega f\,dx = 0$$

and hence

$$\int_{\Omega^+} f_+\,dx = \int_{\Omega^-} f_-\,dx,$$

$$\int_{B_r} f_+^*\,dx = \int_{B_R} f_-^*\,dx.$$

Thus by Lemmas 3.19 and 3.20 we have

$$\lambda(\Omega) \geqq \lambda(B)$$

with equality only in the case when Ω is a disk. The theorem is proved.

Acknowledgments. I thank C. BANDLE, T. HOFFMANN-OSTENHOF, V. KONDRATIEV, E. LANDIS and especially G. TALENTI for fruitful discussions. This work was supported by Erwin Schrödinger International Institute of Mathematical Physics, Wien, Austria.

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Earth Physics Institute
Moscow, Russia

(Accepted March 21, 1994)