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Reduction of Some Classical Non-Holonomic Systems with Symmetry

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Abstract

Two types of nonholonomic systems with symmetry are treated: (i) the configuration space is a total space of a G-principal bundle and the constraints are given by a connection; (ii) the configuration space is G itself and the constraints are given by left-invariant forms. The proofs are based on the method of quasicoordinates. In passing, a derivation of the Maurer-Cartan equations for Lie groups is obtained. Simple examples are given to illustrate the algorithmical character of the main results.

Contents

One of the interesting ocurrences of symmetry in mechanics is the rolling of a solid body without slipping along a two dimensional surface (possibly of a complex profile). The results of this process are studied by the mechanics of nonholonomic systems .... Recently, deep and interesting connections of this subject with Lie groups were discovered ....
A. T. FOMENKO, Visual and hidden Symmetry in geometry, in *Comp. Math. Appl.*, **17**, 1989.

1. Introduction

The question of reducing ordinary or partial differential equations which are invariant under the action of a Lie group has attracted considerable attention in recent years. To *reduce* means to obtain equations with fewer coordinates or, when possible, to obtain a globally defined differential operator on a quotient manifold. For a comprehensive introduction to the subject, see the book by OLVER [O].

The concern here is the reduction of *classical non-holonomic* Lagrangian systems with symmetry. Two cases, which form the core of the paper, are treated in detail: (i) the configuration space is a Lie group G with left-invariant metric and (non-holonomic rigid body) constraints; (ii) the configuration space is a principal bundle $G \to P \to M$ and the constraints distribution defines a *connection* (non-Abelian Čaplygin systems). The setting for more general situations is given in the last sections (§§ 7, 8, 9).

The theory of non-holonomic systems has been much developed in the Soviet Union: My basic sources are the book by NEIMARK & FUFAEV [NF], and the articles by VERSHIK [V] and VERSHIK & FADEEV [VF]. My basic motivation was to generalize the theory to the case of a non-Abelian symmetry group.

Čaplygin systems. ([NF, Chapter III, § 3]). Consider the Lagrange-d'Alembert equations of mixed type [1]

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = Q_k(q, \dot{q}) - \sum_{j=1}^{r} \mu_j a_{jk}, \quad 1 \leqq k \leqq n, \tag{1.1}$$

$$\sum_{k=1}^{n} a_{jk}(q)\, \dot{q}_k = 0, \quad 1 \leqq j \leqq r.$$

Suppose that the Lagrangian and external force do not depend on the coordinates $q_{m+1}, \ldots, q_n$, and that the constraints can be written in the form

$$\dot{q}_K = \sum_{i=1}^{m} b_{K,i}\dot{q}_i, \quad m+1 \leqq K \leqq n \tag{1.2}$$

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[1] Recent influential books on Analytical Mechanics (such as those of ABRAHAM & MARSDEN [AM] or ARNOLD [A]) focus their approach to the foundations on the passage from Newton's laws with potential forces to the Lagrangian and Hamiltonian formalisms. SOMMERFELD, however, considers d'Alembert's Principle as more fundamental [So, §§ 8, 10, 12]: "the principle [of virtual work] was already sketched by GALILEO ... It achieved its dominating position ... with "Mécanique Analytique" of LAGRANGE ... We regard it practically as definition of a mechanical system".

Or, in LAGRANGE's own words [L]: "Si l'on imprime à plusieurs corps des mouvements qu'ils soient forcés de changer à cause de leur action mutuelle, il est clair qu'on peut regarder ces mouvements comme composés de ceux que les corps peudront reélment, et d'autres mouvements qui sont détruits; d'ou il suit que ces derniers doivrent être tels, que les corps animés de ces seuls mouvements se fassent équilibre".

where the functions $b_{K,i}$ do not depend on the last $r = n - m$ coordinates. We interpret this as invariance under the Abelian group R^{n-m}. ČAPLYGIN showed around 1895 that the system reduces to a second-order ordinary differential equation for the variables q_i $(1 \leqq i \leqq m)$, with the multipliers eliminated

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{q}_i} - \frac{\partial L^*}{\partial q_i} = Q_i^* + \sum_K \left(\frac{\partial L}{\partial \dot{q}_K}\right)^* \left[\sum_{j=1}^{m} \left(\frac{\partial b_{K,i}}{\partial q_j} - \frac{\partial b_{K,j}}{\partial q_i}\right) \dot{q}_j\right]. \tag{1.3}$$

The symbol L^* means that the $\dot{q}_K$ are replaced by (1.2) in L *before* the Euler-Lagrange differential is computed. On the other hand, in $(\partial L/\partial \dot{q}_K)^*$ the substitution is effected *after* the differentiation.

As an exercise, the reader may work out the reduced equations for an arrow or javelin: they rotate about their centers of mass so that they always stay tangent to the paths described by their centers of mass.

Non-Abelian Čaplygin systems. The configuration space R^n is replaced by a manifold P^n acted on by a Lie group G. Assume $G \to P \to M$ is a principal bundle, $M = P/G$, $\dim G = r$, $\dim M = m = n - r$. The Lagrangian is supposed to be G-equivariant. For natural systems $L = T - V$ this means that T is a metric on P on which G acts by isometries, and V is an equivariant potential which can be considered a function on M. The constraints are defined by a smooth distribution D_p of m-dimensional subspaces of T_pP which project isomorphically over $T_{\pi(p)}M$ and such that $D_{g.p} = gD_p$ (i.e. the constraints define a connection on the bundle; see § 2 for background). It is geometrically obvious that the Lagrange-d'Alembert constrained system on TP must project to a second-order ordinary differential equation on M. I was able to find a local expression in suitable coordinates, analogous to (1.3), and characterize it intrinsically. These results were announced in [Ko]. M. DE LEON & P. RODRIGUES kindly included the result in their book [LR]; a different proof is presented there.

Rigid body with constraints. Consider a Lie group G together with a left-invariant Riemannian metric. The Euler equations [A, AM] for this generalized rigid body give a vector field in the Lie algebra g. Left-invariant constraints on TP are defined by a subspace D of g. An educated and correct guess is that the Lagrange-d'Alembert constrained system on G reduces to a vector field on D which is just the orthogonal projection of the unconstrained Euler vector field.

Outline. for the convenience of non-experts, I review in § 2 some basic results about connections on a principal bundle, which will be used subsequently. The main results are stated in § 3, in *local* as well as *intrinsic* versions. Some standard examples from the literature are revisited in § 4. The algorithmically minded reader should compare them with the derivations of the equations of motion in the references [H, NF].

Concerning the proofs, I use "quasicoordinates", a method strongly advocated by HAMEL [H], which is reviewed in §§ 5.1 and 5.2, and applied to unconstrained systems in § 5.3 and § 5.4. In the holonomic case, this approach suffers from an inherent syndrome, its local character. A comparison is made with the intrinsic method of MARSDEN & WEINSTEIN [MW]. Nonetheless, as a surprising byproduct of HAMEL's approach, when revisiting ARNOLD's generalized rigid bodies [A], I found another proof of the Maurer-Cartan structure equations for a Lie group. (This is actually an "overkill": for two standard proofs, as well as applications, of this basic fact in Lie group theory, see the textbook by SATTINGER & WEAVER [SW].)

In § 6, I prove the theorems stated in § 2, and outline in § 7 the equations for equivariant Lagrangians with constraints, these being group-invariant but failing to define a connection. In contradistinction to HAMEL's approach for nonholonomic systems, in the particular case of *natural* Lagrangians there is a more "geometrical", or "intrinsic" viewpoint, which can be traced back to E. CARTAN [Ca] and more recently revived by VERSHIK [V]. The equations of motion represent the geodesics of a non-Riemannian affine connection. Interesting enough, HAMEL's method yields a *different* affine connection, with the same geodesics. This is shown in § 8. In the final comments (§ 9) I also describe some recent results of which I became aware after the paper was submitted.

Remarks. (i) I use the word *reduction*, but in the context of § 7 the word *factorization* is perhaps more appropriate because the system does not need to project completely into a quotient manifold. In fact, the word "reduction" has by now a standard meaning in Hamiltonian mechanics (namely the Marsden-Weinstein method [MW]). (ii) The methods of HAMEL and of MARSDEN & WEINSTEIN have complementary drawbacks: the latter was not designed to deal with non-holonomic constraints, but on the other hand, HAMEL's approach does not easily recognize the integrals of motion for holonomic systems with symmetry. (iii) One promising source of applications is robotics. For instance, in Space Engineering one frequently encounters large coupled systems of articulated rigid bodies (with or without constraints) which are invariant under the group of rigid motions SO(3). Underwater systems may have the symmetry group $SO(3) \sim \times R^3$. Furthermore, the non-Abelian Čaplygin setting can be thought of as a "toy model" (in harmony with HERTZ's ideas [He] [2] of "concealed forces") for a spinless particle on a Yang-Mills field [Ko, Bu].

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[2] HERTZ [He] wanted to construct the Foundations of Mechanics disposing entirely with the notion of force, replacing it by equivalent velocity constraints. His basic principle, which yields precisely the Lagrange-d'Alembert equations, states that the geometric curvature of the path is always a minimum, subjected to the constraints. In his review of HERTZ's book, POINCARÉ [P] says: "Ah bien, d'après HERTZ, toutes les fois que nous imaginons une force, nous sommes dupes d'une illusion". And he proposes the following problem: "Peut-on imaginer un système articulé que imite un système de forces, définie par une loi quelconque où en approchant autant qu'on voudra?". Next, POINCARÉ discusses attempts of KELVIN and MAXWELL to give a material realization for the properties of the "ether". The following provocative statement could perhaps be applied to the various theories being tried now in Particle

2. Review of some facts about connections

Let $\sigma : U_M \subset M \to P$ be a local section of the principal G-bundle with projection $\mathrm{pr}: P \to M^m$, and $f: U \subset R^m \to U_M$ a parametrization in M. Let $\{e_K\}_{K=1,\dots,r}$ be a basis of the Lie algebra g, $\exp: g \to G$ the exponential mapping, and $e^X = \exp(\Sigma x_K e_K)$.

With these data one associates the *gauge parameters* (q, X) for P and $(q, \dot{q}, X, \dot{X})$ for TP:

$$F: U \times g \to P, \quad F(q, X) = \exp(X) \cdot s(q), \quad s = \sigma \circ f,$$

$$DF: U \times R^m \times g \times g \to T(\mathrm{pr}^{-1}(U_M)), \tag{2.1}$$

$$(q, \dot{q}, X, \dot{X}) \to u_p = (e^X) * s'(q)\,\dot{q} + (s(q)) * D_X e^X \cdot \dot{X}.$$

Here the customary notations $g: P \to P$, $g(p) = g \cdot p$, $p: G \to P$, $p(g) = g \cdot p$, $L_g: G \to G$, $L_g(h) = gh$ are used. With an element Y in the Lie algebra g, one associates the *vertical* vector field $V_p(Y) = D_e p \cdot Y$.

However, I find that it is more convenient to consider instead the *gauge quasicoordinates* $(q, \dot{q}, g, \dot{\pi})$ on TP:

$$TF: U \times R^m \times G \times g \to T(\mathrm{pr}^{-1}(U_M)), \tag{2.2}$$

$$(q, \dot{q}, g, \dot{\pi}) \to u_p = g * [s'(q)\,\dot{q} + V_{s(q)}\,\dot{\pi}].$$

where the dot over the symbol $\dot{\pi} \in g$ is used only for "historical reasons", to indicate that $\dot{\pi}$ is an ingredient of a velocity vector. The quasicoordinate π has for us a "metaphysical character", that is, it will never appear in our formulas. We do not define it as a mathematical object.

A *connection* on P is a distribution $\{D_p \subset T_p P\}$ of m-dimensional "horizontal" subspaces which project isomorphically over $T_{\pi(p)}M$ and are also invariant under the G-action: $g * D_p = D_{gp}$. Thus any vector v_p decomposes into its horizontal, Hor (v_p), and vertical, Vert (v_p), components. The *connection 1-form* φ is the Lie algebra-valued 1-form in P such that

$$\mathrm{Vert}\,(v_p) = V_p(\varphi(v_p)). \tag{2.3}$$

The connection can also be thought of in terms of *horizontal lift operators* $h_p: T_{\pi(p)}M \approx D_p$. Along the section s, the connection is represented by a function b from U to Hom (R^m, g) such that $h_{s(q)}\dot{q} = D_q s \cdot \dot{q} + V_{s(q)}\, b(q) \cdot \dot{q}$ is a horizontal vector. The following result is elementary.

Proposition 2.1 (Connection 1-form).

(i) $$g * V_p(Y) = V_{gp}[\mathrm{Ad}_g(Y)] \quad \left(\mathrm{Ad}_g Y = \frac{d}{dt}\bigg|_{t=0} g e^{tY} g^{-1}\right) \tag{2.4a}$$

Physics: "Les savants anglais ... ne sont pas effrayés par la complication de ces modèles où l'on a multiplié des triangles, des bielles, des coulisses, comme dans un atelier de mécanicien".

(ii) *For* $p = g \cdot s(q)$ *the vertical component of* u_p *is* $V_p(\varphi(u_p))$, *where*

$$\varphi(u_p) = \mathrm{Ad}_g(\dot{\pi} - b(q)\,\dot{q}). \tag{2.4b}$$

(iii) *The pull-back of* φ *to* $U \times g$ *via* F *is given by*

$$(F^*)_{(q,X)}(\dot{q},\dot{X}) = \frac{d}{dt}\bigg|_{t=0} e^{\dot{X}+t\dot{X}} e^{-X} - \mathrm{Ad}_{\exp(X)}\, b(q)\,\dot{q}. \tag{2.4c}$$

In Yang-Mills theory the components of φ are called *gauge potentials.*

The *curvature* of the connection is a 2-form Ω on P with values on the Lie algebra, given by $\Omega = d\varphi \circ \mathrm{Hor}$. It is *horizontal* (i.e., it vanishes when at least one of the vectors is vertical) and it is *Lie-algebra equivariant*:

$$(L_g^*\Omega)(u_p, v_p) = \mathrm{Ad}_g\, \Omega(u_p, v_p) \tag{2.5}$$

The *local curvature on the gauge* σ is the pull-back $\Omega^\sigma = \sigma^*\Omega$ to M^m. Physically its components form the *gauge field strengths.* Their explicit expressions are well known (see, e.g., [CMD, page 375]):

Proposition 2.2 (Local curvatures).

(i) *Let* b *be represented by its matrix* (b_{Kj}) *with respect to a basis* e_K, $1 \leqq K \leqq r$ *of the Lie algebra. Then*

$$\Omega^s(q)(\dot{q},\dot{Q}) = \sum_{L=1}^{r} \sum_{j,k=1}^{m} \left(\frac{\partial b_{Lj}}{\partial q_k} - \frac{\partial b_{Lk}}{\partial q_j}\right) \dot{q}_j \dot{Q}_k e_L - [b(q)\,\dot{q}, b(q)\,\dot{Q}]. \tag{2.6a}$$

(ii) *If* τ *is another gauge with transition function* $\tau(m) = h(m)\,\sigma(m)$, *then*

$$\Omega^\tau = \mathrm{Ad}_{h(m)}\, \Omega^\sigma \quad (h: U_M \to G). \tag{2.6b}$$

3. Main results

To begin, a brief summary of some basic concepts from the "geometry and physics" of the second tangent bundle is given. For details, see [VF] or [G].

Let Q be a manifold, and let $\pi^1: TQ \to Q$ and $\pi^2: T(TQ) \to TQ$ respectively be the first and second tangent bundles. A vector field X on TQ is called *special* if $d\pi^1 X_{q,v} = v_q$. Given a Lagrangian $L: TQ \to R$, the *Euler-Lagrange differential* is a mapping EL from special fields to 1-forms on Q, which in coordinates has the familiar expression (see [SM] for a direct proof of its covariance under change of coordinates)

$$EL(q,\dot{q},\ddot{q})\,\frac{\partial}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}. \tag{3.1}$$

Thus $EL(X_{q,v};L) \in T_q^*Q$ is an intrinsic object. Instrumental for several constructions is the *fiber derivative* or *Legendre transformation*

$$FL_q = \mathrm{Leg}_q = d(L/T_qQ) : T_qQ \to T_q^*Q. \tag{3.2}$$

A *force field* is a mapping $F: TQ \to T^*Q$ which commutes with the projections (but it can be nonlinear on the fibers). The reason why the target space is T^*Q is that, physically, the force is an agent which does "work" on "virtual displacements" $v \in TQ$. When $F = dV$, V being a function on Q, we say that it derives from the *potential* V. In this case the domain can be considered Q instead of TQ.

For unconstrained, force-free systems, the Euler-Lagrange differential vanishes along the special field which gives the equations of motion: $EL(X_{q,v};L) = 0$. If there are external forces, d'Alembert's Principle states that *on the special vector field which gives the actual motion, the Euler-Lagrange differential equals the external force.* If in addition there are constraints, *the restricted Euler-Lagrange differential* Restr(EL) *equals the restricted external force* Restr(F) *along the constrained special field.* In other words, the constraint distribution $D_q \subset T_qM$ imposes the conditions:

(i) $\quad d\pi X_{q,v} = v_q \in D_q$ (constrained special field), (3.3)

(ii) $\quad \mathrm{Restr}(EL) = j^* \circ EL, \quad \mathrm{Restr}(F) = j^* \circ F$ (restricted 1-forms),

where $j_q^* : T_q^*M \to D_q^*$ is the dual of $j_q : D_q \to T_qM$.

3.1. Non-Abelian Čaplygin systems

Let $\pi : P^n \to M^m$ be a principal G-bundle and consider a G-invariant Lagrangian L on P, subjected to a G-invariant external force $F: TP \to T^*P$. This means that for the associated actions of G in TP and T^*P, we have $F_{g(p,v)} = gF_{(p,v)}$. These actions are defined by $g(p,v) = (gp, g * v)$ for $v_p \in T_pP$ and $g \cdot (p,w) = (gp, (g^{-1})^* w_p) \in T_{gp}^*P$, for $w_p \in T_p^*P$. Recall that the *momentum mapping* is given by $J: T^*P \to g^*$, $J^X(w_p) = w_p(V_p(X))$.

It is assumed that the classical linear constraints are given by a connection D on P so that only horizontal paths are allowed.

First, I shall state the result in *local* form for *arbitrary* Lagrangians, and then immediately give the *intrinsic* interpretation of its ingredients. Finally, the global version for *natural* Lagrangians will be explicitly given, in order to compare it with another approach in § 8.

Consider gauge coordinates as in § 2, and recall that the constraints are represented by $\dot{\pi} = b(q) \cdot \dot{q}$, where $b(q)$ is a $r \times m$ matrix relative to the standard basis in R^m and a basis $\{e_K\}_{K=1,\dots,r}$ of g, $[e_K, e_L] = \Sigma c_{KL}^l e_l$.

Theorem 3.1 (Local expression for the projected equations).

(i) *Relative to these "quasicoordinates",* (see (2.2))

$$L = L^s(q, \dot{q}, \dot{\pi}) = L(s'(q)\, \dot{q} + V_{s(q)}\, \dot{\pi}), \quad F = F^s = \Sigma Q_i(q, \dot{q}, \dot{\pi})\, dq_i. \tag{3.4a}$$

(ii) *The equations of motion are explicitly given by*

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot q_i} - \frac{\partial L^*}{\partial q_i}$$

$$= \sum_{K=1}^{r} \left(\frac{\partial L}{\partial \dot\pi_K}\right)^* \sum_{j=1}^{m} \left(\frac{\partial b_{Ki}}{\partial q_j} - \frac{\partial b_{Kj}}{\partial q_i} + \sum_{U,V=1}^{r} b_{Ui} b_{Vj} c_{UV}^K\right) \dot q_j + Q_i^*(q, \dot q). \quad (3.4\text{b})$$

*Here the symbol * means that $\dot\pi$ is replaced by $b(q)\,\dot q$ wherever it occurs. (This formula generalizes Čaplygin's* [NF] *by the appearance of terms involving the structure constants of the Lie algebra.)*

(iii) *If a solution $q(t)$ is known, the full dynamics in P is recovered by integrating a linear ordinary differential equation with time-dependent coefficients:*

$$\frac{dg}{dt} = (L_g)_* \cdot b\big(q(t)\big)\,\dot q. \quad (3.4\text{c})$$

Remarks. a) It is easy to see that (3.4b) is the coordinate expression of a *globally defined* second-order equation on M. Indeed, the left side is the Euler-Lagrange differential of $L^*(u_m) = L(v_p)$ where u_p is any horizontal lift of u_m. Analogously Q_i^* are the covariant components of $F^*(u_m) = F(v_p)$. The "nonholonomic force" components, in the basis dq_i $(1 \leqq i \leqq m)$, correspond to the components of the 1-form on $U \subset R^m$

$$\left\langle \left(\frac{\partial L^s}{\partial \dot\pi}\right)^*, \Omega^s(\cdot, \dot q) \right\rangle \quad (3.5)$$

where $(\partial L^s/\partial \dot\pi)^* \in g^*$ and $\Omega^s(\cdot, \dot q) \in g$.

If another section τ is considered, with transition function $h: U_M \to G$, $\tau(m) = h(m)\,\sigma(m)$, then by a direct calculation one can check that

$$\left(\frac{\partial L^s}{\partial \dot\pi}\right)^* = \mathrm{Ad}^*_{h(s(q))} \left(\frac{\partial L^\tau}{\partial \dot\pi}\right)^*, \qquad \Omega^\tau = \mathrm{Ad}_{h(s(q))} \Omega^s \quad (3.6)$$

where $s = \sigma$ if $t = \tau \circ f$ and one concludes that (3.4b) is indeed the coordinate expression of a *globally defined* ordinary differential equation on M.

(b) A bit of invariant formulation can help one to grasp the geometric meaning of (3.5). Define $L^\#$ by

$$L^\#(q, \dot q, \dot\pi) = L^s(q, \dot q, \dot\pi + b(q)\,\dot q) = L(h_{s(q)}\dot q + V_{s(q)}\dot\pi) \quad (3.4\text{a})'$$

so that $L^*(q, \dot q) = L^\#(q, \dot q, 0) = L(h_{s(q)}\dot q)$ makes explicit, in coordinates, the *projected* Lagrangian in TM, via the connection. Moreover, $(\partial L^s/\partial \dot\pi)^* = \left.\frac{\partial L^\#}{\partial \dot\pi}\right|_{\dot\pi = 0}$ and then

$$\left\langle \left(\frac{\partial L^s}{\partial \dot\pi}\right)^*, \Omega^s(\dot q, \cdot) \right\rangle = \left.\frac{d}{dt}\right|_{t=0} L\big(h_{s(q)}(\dot q) + tV_{s(q)}\,[\Omega^s(\dot q, \cdot)]\big)$$

$$= FL_{s(q)}\big(h_{s(q)}(\dot q)\big) \cdot V_{s(q)}\,[\Omega^s(\dot q, \cdot)]. \quad (3.5)'$$

where FL is the fiber derivative. This can be further interpreted as follows. Consider the sequence of mappings

$$T_mM \xrightarrow{h} D_p \xrightarrow{i} T_pP \xrightarrow{FL} T_p^*P \xrightarrow{J} g^*,$$

$$T_mM \xrightarrow{\Omega} g \otimes (T_mM)^*, \quad \Omega(v_m) = \Omega^p\,(h_p(v_m), h_p(\cdot)) \tag{3.6}'$$

where $\pi(p) = m$. The first sequence is Ad*-ambiguous, while the second is Ad-ambiguous, where Ad and Ad* are the adjoint and coadjoint actions of G in g and g^*. These ambiguities cancel out when the pairing (3.5) is taken. This is instrinsically given by bracketing the two sequences (3.6)′.

c) The *nonholonomic force* is the first term in the right-hand side of (3.4b). If there is no external force F, the reduced nonholonomic system has a common feature with the constrained Lagrangian L^* on TM: both preserve the total energy $H = \dot{q}(\partial L^*/\partial\dot{q}) - L^*$. In other words, the nonholonomic force does no work along actual trajectories, so it has a *gyroscopic character*.

I proceed to give a geometric characterization of the nonholonomic force (3.5) in the case of natural systems:

Natural Čaplygin systems

The Lagrangian on TP is of the form $L = T - V$ where (i) $V(gp) = V(p)$ for all $g \in G$, $p \in P$ and (ii) T is the quadratic form of a Riemannian metric on P, for which G acts by isometries. To this kinetic energy T one may add a term linear on the velocities, in other words, a G-invariant 1-form μ on P (for instance, if the configuration space is not inertial). A *non-Abelian natural Čaplygin system* consists therefore of the quadruple

$$(G \to P \to M, \quad L = T + \mu - V, \quad D, \quad F).$$

Local data. Given gauge parameters $F(q,X) = e^X s(q)$, $X = \Sigma X_K e_K$, the metric coefficients of T are given by

$$g_{ij} = g_{ij}(q) = \left\langle \frac{\partial}{\partial q_i}, \frac{\partial}{\partial q_j} \right\rangle \quad \text{(these do not depend on } X\text{)},$$

$$g_{KL} = g_{KL}(q,X) = \left\langle \frac{\partial}{\partial X_K}, \frac{\partial}{\partial X_L} \right\rangle \quad \text{(these depend on } X\text{)}, \tag{3.7}$$

$$g_{iK} = g_{iK}(q,X) = \left\langle \frac{\partial}{\partial q_i}, \frac{\partial}{\partial X_K} \right\rangle \quad \text{(these depend on } X\text{)}.$$

Here $\partial/\partial q_i = (e^X)_* \, \partial s/\partial q_i$, $\partial/\partial X_K = V_{\exp(X)\cdot s(q)} \quad [(R_{\exp(-X)})_* \, (D_0 e^X) \cdot e_K]$, where the indices $i,j,\ldots$ range over $1,\ldots,m$ and the $K,L,\ldots$ over $1,\ldots,r$. Thus in spite of G-invariance, when G is non-Abelian, g_{KL} and g_{iK} depend explicitly on the position X along the fiber.

This is why it is really more convenient to use the gauge quasicoordinates $(q, \dot{q}, g, \dot{\pi})$, $\dot{\pi} = \Sigma \dot{\pi}_K e_K$, introduced in the previous section. Now the metric is given by *equivariant* local data (in an abuse of notation):

$$g_{ij}(q) = \left\langle \frac{\partial}{\partial q_i}, \frac{\partial}{\partial q_j} \right\rangle, \quad g_{KL}(q) = \left\langle \frac{\partial}{\partial \pi_K}, \frac{\partial}{\partial \pi_L} \right\rangle, \quad g_{iK}(q) = \left\langle \frac{\partial}{\partial q_i}, \frac{\partial}{\partial \pi_K} \right\rangle \tag{3.8}$$

where $\partial/\partial q_i = g_* \partial s/\partial q_i$, $\partial/\partial \pi_K = g_* V_{s(q)}(e_K)$. In old language, the values $\dot{\pi}_K$ would be called *quasivelocities.*

In summary, the kinetic energy may be written as $T = T(q, \dot{q}, \dot{\pi})$ and the potential energy depends only on q, so that $V = V(q)$.

Furthermore, I must also put into play the connection, defined by the distribution of horizontal subspaces $\{D_p\}_{p \in P}$. A vector u_p at $p = s(q)$ with quasicoordinates $(q, \dot{q}, g = e, \dot{\pi})$ decomposes into its horizontal and vertical components:

$$\text{Hor}\,(u_p) = D_q s \cdot \dot{q} + D_e s(q)\, b(q) \cdot \dot{q}, \qquad \text{Vert}\,(u_p) = V_{s(q)}[\dot{\pi} - b(q) \cdot \dot{q}]. \tag{3.9}$$

It follows that $T(u_p) = T(q, \dot{q}, \dot{\pi})$ may be expressed alternatively as

$$T(q, \dot{q}, \dot{\pi}) = T_q^*(\dot{q}) + T_{s(q)}^G(\dot{\pi} - b(q)\, \dot{q}) + \langle \text{Hor}(u_p), \text{Vert}(u_p) \rangle. \tag{3.10}$$

Some globally defined geometric objects. The *projected metric* T^* is the metric on the base manifold M given by $T^*(v_m) = T(h_p(v_m))$ where p is a point on the fiber over m and h_p is the horizontal lift operator. Its expression in coordinates is

$$2T_q^*(q) = \Sigma\, g_{ij}^*(q)\, q_i q_j, \quad g_{ij}^* = \left\langle h\left(\frac{\partial}{\partial q_i}\right), h\left(\frac{\partial}{\partial q_j}\right) \right\rangle. \tag{3.11}$$

Fix a point $p \in P$. The *p-associated metric* in G is the left-invariant metric whose quadratic form on the Lie algebra is given by

$$T_p^G(Y) = T[V_p(Y)]. \tag{3.12}$$

Caveat. Although the fiber $G \cdot p$ has, via restriction, a natural left-invariant Riemannian metric, there is no naturally associated metric on G. The metric T^G depends *inherently* on the choice of the base point p. The ambiguity is given by the adjoint representation of G on the quadratic forms.

There is one more projected object, when the Lagrangian contains a linear term. The *projected 1-form* is the 1-form on the base manifold defined by

$$\mu_m(v_m) = \mu_p(h_p(v_m)). \tag{3.13}$$

The geometric content of Theorem 3.1′ below is that a force-free, constrained natural Čaplygin system on P projects to an *unconstrained* system on M, but with an interesting external force. To describe this external force intrinsically, we use the following definitions:

The *metric-connection* tensor is the real-valued, (3,0)-horizontal tensor field K on P given by

$$K_p(u_p, v_p, w_p) = \langle \text{Hor}_p(u), V_p[\Omega_p(v, w)] \rangle \tag{3.14a}$$

where Ω is the curvature of the connection. The *μ-connection* tensor is the real-valued, (2,0)-horizontal tensor field J on P given by

$$J_p(v_p, w_p) = \mu_p\{V_p[\Omega_p(v, w)]\}. \tag{3.14b}$$

It readily follows from the Lie algebra equivariance of the connection and curvature forms ((2.4) and (2.5)), that K and J are antisymmetric in v, w and are equivariant over the reals:

$$K_{gp}(g * u_p, g * v_p, g * w_p) = K_p(u_p, v_p, w_p), \tag{3.15a}$$

$$J_{gp}(g * v_p, g * w_p) = J_p(v_p, w_p). \tag{3.15b}$$

Consequently K and J can be considered as tensors on the base M:

$$K_m(u_m, v_m, w_m) = K_p(h_p(u_m), h_p(v_m), h_p(w_m)), \tag{3.15a)$'$}$$

$$J_m(v_m, w_m) = J_p(h_p(v_m), h_p(w_m)) \tag{3.15b)$'$}$$

where p is any point of the fiber over m.

Reduction of natural Čaplygin systems: invariant formulation

Theorem 3.1′. *The equations of motion for a non-Abelian natural Čaplygin system project to a special second-order vector field X on the base manifold M satisfying*

$$EL(X_{m,v}; L^*) + K_m(v_m, v_m, \cdot) + J_m(v_m, \cdot) = F^*_{m,v}(\cdot),$$

where the projected external force is defined as

$$F^*_{m,v}(\cdot) = F_{p,h(v)}(h_p(\cdot)).$$

Remarks. (i) If the vertical subspace $V_p[\Omega]$ corresponding to the span of the curvature is orthogonal to the horizontal subspace D_p, then the tensor K_p vanishes. In particular, this occurs when the connection flat is (holonomic constraints) or when D_p is orthogonal to $V_p[g]$.
(ii) A geometric interpretation of the left-hand side of the first equation of Theorem 3.1′ in terms of the concept of an *affine connection* will be given in § 8.

3.2. Rigid body with constraints

To begin, I review ARNOLD's generalization [A, Appendix 2] of the classical Euler equations for rigid bodies, in the setting of a left-invariant metric on a Lie group G. The procedure can be interpreted in terms of Poisson geometry [W], and holds for any G-invariant Hamiltonian on T^*G. Recall that G acts on T^*G via $g \cdot p_h = (L_g)^{-1*} p_h$. Pulling back the canonical 2-form ω in T^*G to $G \times g^*$ via left translations $p_g \to (g, L_g^* p_g)$ one gets

$$\omega_{(g,\mu)}((X_1, z_1), (X_2, z_2)) = z_1(L_g^{-1}) * X_2 - z_2(L_g^{-1}) * X_1 - \mu(L_g^{-1}) * [X_1, X_2]. \tag{3.16}$$

The following well-known result then follows immediately [A, AM]:

Lemma. *The Hamiltonian vector field associated with a G-invariant function* $H: T^*G \to R$, *written in coordinates* $G \times g^*$, *is given by*

$$\frac{dg}{dt} = L_g * \frac{\delta H}{\delta \mu}, \quad \frac{d\mu}{dt} = \left\{ \frac{\delta H}{\delta \mu}, \mu \right\}. \tag{3.17}$$

Here $\delta H/\delta \mu$ *is the* Lie algebra gradient $dH(\mu) \cdot = \langle \cdot, \delta H/\delta \mu \rangle$ *and* $\{ , \}: g \times g^* \to g^*$ *denotes the bilinear operation* $\{X, \mu\} Y = \mu[X, Y]$.

Example (*Arnold's generalized rigid body*). Suppose that the Lagrangian L is defined by a non-degenerate quadratic form (not necessarily positive-definite) on the Lie algebra g, given by $2T = {}^t\Omega Q \Omega$. Here Q is a $n \times n$ matrix ($n = \dim G$) and Ω is a column vector of components relative to a basis $\{e_1, \ldots, e_n\}$ of g.

Let $e_1^*, \ldots, e_n^*$ be the dual basis. The Legendre transformation is

$$\sum \Omega_i e_i \to \sum \mu_i e_i^*, \quad \mu = Q\Omega = \frac{\partial T}{\partial \Omega}, \tag{3.18}$$

and the Hamiltonian is given by

$$T = H = \tfrac{1}{2}\, {}^t\mu\, Q^{-1} \mu. \tag{3.19}$$

The Euler vector field $v_H: d\mu/dt = \{\delta T/\delta \mu, \mu\}$ in g^*, written explicitly in coordinates, is

$$\frac{d\mu_k}{dt} = \sum_{j,l,m} c_{lk}^j Q_{jm} \Omega_m \Omega_l \tag{3.20}$$

where c_{lk}^j are the structure constants $[e_l, e_k] = \Sigma_j c_{lk}^j e_j$.

Equations (3.20) will be used for a "dynamic proof" of the Maurer-Cartan Theorem for Lie groups. See § 5.

A *generalized rigid body with constraints* consists of a pair (T, D) where T is a positive-definite inner product on g and D is a subspace of g:

$$2T = {}^t\Omega Q \Omega, \quad D = \ker e_1^* \cap \ldots \cap \ker e_r^*. \tag{3.21}$$

By left translation, these objects define a metric and a distribution in G.

Theorem 3.2. *The constrained Euler vector field on g is given by the T-orthogonal projection over D of the unconstrained Euler vector field,*

$$\frac{d\Omega}{dt} = \mathrm{Proj}_D\,(\mathrm{Leg}^{-1} v_H), \quad \frac{dg}{dt} = (L_g) * \Omega(t). \tag{3.22a}$$

More explicitly, there is no loss in generality in assuming that $e_{r+1}, \ldots, e_n$ are *orthogonal* with respect to T. Then $\Omega_l = 0$ $(1 \leqq l \leqq r)$ and

$$\dot{\Omega}_l = \sum_{i=1}^{n} \left(\sum_{j,k=r+1}^{n} c_{kl}^i Q_{ij} \Omega_j \Omega_k \right) Q_{ll}^{-1}, \quad r+1 \leqq l \leqq n. \tag{3.22b}$$

The next section is devoted to simple illustrative examples. The proofs of Theorem 1 and 2 will be given in § 6, after preparation in § 5.

4. Examples

Two basic examples in HAMEL's treatise are revisited [H, p. 478 ff.]. The algorithmically oriented reader should compare the derivation of the equations of motion there with the systematic approach advocated here.

4.1. The sleigh of Čaplygin and Carathéodory[3]

The idealized sleigh (Fig. 1 a) is a body having three points of contact with the plane; two of them slide freely but the third, A, behaves as a knife edge subjected to a constraining force R which does not allow transversal velocity. (See also [NF, p. 71]; [So]). More precisely, let xoy be an inertial frame and $\xi A\eta$ a frame moving with the sleigh. Take as generalized coordinates the Carte-

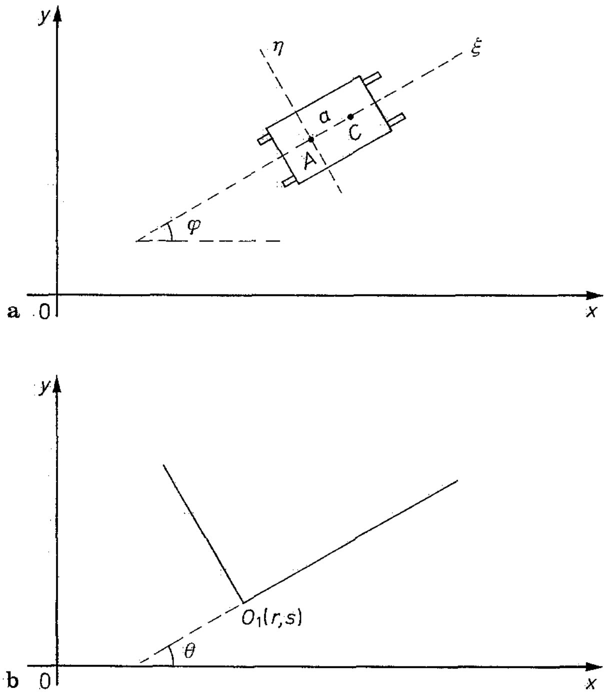

Fig. 1

[3] According to NEIMARK & FUFAEV, "ČAPLYGIN stated and solved in quadratures the problem in 1911 ... CARATHÉODORY published a detailed investigation in 1933 of a special case". SOMMERFELD considers it the simplest nonholonomic system [So, Problem V-3]. OLSSON [Ol] has considered the situation in which the constraint is nonhomogeneous, i.e., when the lateral friction is not sufficient to make $v = 0$.

sian coordinates of the center of mass C of the sleigh and the angle φ between the x- and the ξ-axis. The configuration space is thus $R^2 \times S^1$. Let m be the mass, J the moment of inertia about a vertical axis through C and $a = |AC|$. The reaction force R against the runners is exerted laterally at the point of application A in such a way that the η-component of the velocity is zero. Hence one has the constrained system

$$T = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}J\dot{\varphi}^2, \qquad v = -a\dot{\varphi} + \dot{y}\cos\varphi - \dot{x}\sin\varphi = 0. \tag{4.1}$$

Observe that the "javelin" or "arrow" (see § 1) satisfies (4.1) with $a = 0$. In this case the system is also an "ordinary Čaplygin" system, because the constraint can be written $\dot{x} = \operatorname{cotg}\varphi \cdot \dot{y}$, $L = T - V$, $V = gy$.

The configuration space may be identified with the group G of Euclidian motions of the plane. An element $g \in G$ sending origins and coordinate axes xoy to $x_1 o_1 y_1$ can be written as (r, s, θ) (see Fig. 1b). The group multiplication is given by

$$(r, s, \theta)\,(x, y, \varphi) = (r + x\cos\theta - y\sin\theta,\ \ s + x\sin\theta + y\cos\theta,\ \ \theta + \varphi). \tag{4.2}$$

It is readily seen that the Lagrangian and constraint are left-invariant. In order to apply Theorem 3.2 one must prepare the following data:

Distribution: $D = \operatorname{span}(e_2, e_3) = \ker e_1^*$,

$$e_1 = \frac{\partial}{\partial\varphi},$$

$$e_2 = \cos\varphi\,\frac{\partial}{\partial x} + \sin\varphi\,\frac{\partial}{\partial y},$$

$$e_3 = -\sin\varphi\,\frac{\partial}{\partial x} + \cos\varphi\,\frac{\partial}{\partial y} + \frac{1}{a}\,\frac{\partial}{\partial\varphi},$$

Matrix of T:

$$Q = \begin{pmatrix} J & 0 & J/a \\ 0 & m & 0 \\ J/a & 0 & m + J/a^2 \end{pmatrix}$$

Lie Algebra structure coefficients:

$$[e_2, e_3] = -\frac{1}{a}\,e_3 + \frac{1}{a^2}\,e_1, \qquad [e_3, e_1] = e_2, \qquad [e_2, e_1] = -e_3 + \frac{1}{a}\,e_1.$$

Theorem 3.2 yields, after a short computation:

$$\dot{\Omega}_2 = \frac{\Omega_3^2}{a}, \qquad \dot{\Omega}_3 = -\frac{\Omega_3\Omega_2}{a + (J/ma)}, \qquad \Omega = \Omega_2 e_2 + \Omega_3 e_3 \in D. \tag{4.3}$$

Remark 1. These equations can be solved in terms of elementary functions since they have the energy integral

$$2E = m\Omega_2^2 + \left(\frac{m+J}{a^2}\right)\Omega_3^2. \tag{4.4}$$

This can be verified also from the explicit formulas for the Ω_i:

$$(\dot{x}, \dot{y}, \dot{\varphi})^{\mathrm{t}} = P(\Omega_1, \Omega_2, \Omega_3)^{\mathrm{t}}, \quad (\Omega_1, \Omega_2, \Omega_3)^{\mathrm{t}} = P^{-1}(\dot{x}, \dot{y}, \dot{\varphi})^{\mathrm{t}}$$

$$P^{-1} = \begin{pmatrix} \sin\varphi/a & -\cos\varphi/a & 1 \\ \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix}, \qquad P = \begin{pmatrix} 0 & 1/a & 1 \\ \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \end{pmatrix}. \tag{4.5}$$

Using (4.5) one can find the motions $(x(t), y(t), \varphi(t))$ in terms of elliptic functions. For details, see the original references.

Remark 2. It is perhaps instructive to rewrite (4.3) in terms of more easily recognizable physical quantities. If (u, v) are the (ξ, η)-components of the velocity at A, then

$$(u, v) = e^{-i\varphi}(\dot{x} + i\dot{y} - ai\dot{\varphi}e^{-i\varphi}) = (\cos\varphi\dot{x} + \sin\varphi\dot{y}, -a\dot{\varphi} + \cos\varphi\dot{y} - \sin\varphi\dot{x})$$

and a comparison with (4.5) yields $u = \Omega_2$, $v = -a\Omega_1 = 0$, so $\omega = \dot{\varphi} = \Omega_3/a$,

$$\dot{u} = a\omega^2, \quad \dot{\omega} = \frac{-mau\,\omega}{J + ma^2}, \tag{4.6}$$

which are the equations found by Čaplygin and Carathéodory.

4.2. *The "two-wheeled carriage"*

The system depicted in Fig. 2 has configuration space $P = G \times T^2$ where again $G = \{(x, y, \varphi)\}$ is the group of rigid motions in the plane and T^2 is the torus $S^1 \times S^1 = \{(q_1, q_2)\}$. (See [NF, p. 103].) Let $2r$ be the lateral length, a the radius of the wheels, C_0 the center of mass, situated at distance l from point (x, y). Imposing the constraints of no lateral sliding and no sliding on both wheels, one gets the distribution of 2-dimensional subspaces

$$\begin{aligned} \dot{x}\sin\varphi & - \dot{y}\cos\varphi & & & = 0, \\ \dot{x}\cos\varphi & + \dot{y}\sin\varphi & + r\dot{\varphi} & + a\dot{q}_1 & = 0, \\ \dot{x}\cos\varphi & + \dot{y}\sin\varphi & - r\dot{\varphi} & + a\dot{q}_2 & = 0. \end{aligned} \tag{4.7}$$

It is a simple matter to verify that these constraints define a connection on the bundle $G \to G \times T^2 \to T^2$.

Let m_0 be the mass of the body without wheels and k_0 be the radius of gyration about the vertical through (x, y). The kinetic energy of the body of the carriage is given by

$$T_b = \tfrac{1}{2} m_0[\dot{x}^2 + \dot{y}^2 + 2l\dot{\varphi}(\dot{y}\cos\varphi - \dot{x}\sin\varphi) + k_0^2\dot{\varphi}^2].$$

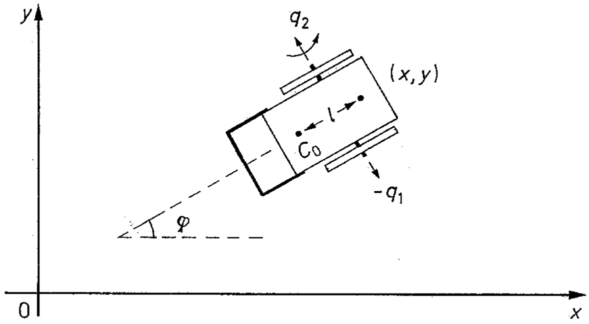

Fig. 2

Let m_1 be the mass of a wheel, C its axial moments of inertia and A its moment of inertia about a diameter. Then

$$T_{\text{right, left}} = \tfrac{1}{2} m_1[\dot{x}^2 + \dot{y}^2 \pm 2r\dot{\varphi}(\dot{x}\cos\varphi + \dot{y}\sin\varphi) + r^2\dot{\varphi}^2] + \tfrac{1}{2}A\dot{\varphi}^2 + \tfrac{1}{2}C\dot{q}_{1,2}^2$$

so that the total energy is

$$T = \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2) + m_0 l\dot{\varphi}(\dot{y}\cos\varphi - \dot{x}\sin\varphi) + \tfrac{1}{2}J\dot{\varphi}^2 + \tfrac{1}{2}C(\dot{q}_1^2 + \dot{q}_2^2), \qquad (4.8\text{a})$$

$$m = m_0 + 2m_1, \quad J = m_0 k_0^2 + 2m_1 r^2 + 2A. \qquad (4.8\text{b})$$

Equations (4.7), (4.8) define a non-Abelian Čaplygin system. The relevant data are as follows:

Connection: $(\dot{x}, \dot{y}, \dot{\varphi})^{\mathrm{t}} = b(q_1, q_2)\,(\dot{q}_1, \dot{q}_2)^{\mathrm{t}}$ (section $s: x = y = \varphi = 0$ for all q),

$$b(q_1, q_2) = \begin{pmatrix} -a/2 & -a/2 \\ 0 & 0 \\ -a/2r & a/2r \end{pmatrix},$$

Structure constants: Extend $e_1 = (\partial/\partial x)_{\mathrm{id}}$, $e_2 = (\partial/\partial y)_{\mathrm{id}}$, $e_3 = (\partial/\partial\varphi)_{\mathrm{id}}$ as left-invariant vector fields. Then

$$c_{1,2}^i = 0, \quad c_{13}^{1,2,3} = 0, -1, 0, \quad c_{23}^{1,2,3} = 1, 0, 0$$

The Lagrangian in quasicoordinates: Along the section s

$$T = \tfrac{1}{2} m(\dot{\pi}_1^2 + \dot{\pi}_2^2) + m_0 l\dot{\pi}_2\dot{\pi}_3 + \tfrac{1}{2}J\dot{\pi}_3^2 + \tfrac{1}{2}C(\dot{q}_1^2 + \dot{q}_2^2)$$

and therefore, by group invariance, this holds everywhere.

The projected metric is given by

$$T^* = \tfrac{1}{8}\, ma^2(\dot q_1^2 + \dot q_2^2) + \frac{Ja^2}{8r^2}(\dot q_2 - \dot q_1)^2 + \tfrac{1}{2}\, C(\dot q_1^2 + \dot q_2^2). \tag{4.9}$$

Applying Theorem 1 one gets, after a straightforward calculation, that

$$\begin{aligned} \frac{ma^2}{4}(\ddot q_1 + \ddot q_2) + \frac{Ja^2}{4r^2}(\ddot q_1 - \ddot q_2) + C\ddot q_1 &= F_1, \\ \frac{ma^2}{4}(\ddot q_1 + \ddot q_2) - \frac{Ja^2}{4r^2}(\ddot q_1 - \ddot q_2) + C\ddot q_2 &= F_2 \end{aligned} \tag{4.10a}$$

where the nonholonomic force is given by

$$F_1 = (m_0 l a^3 l 4r^2)\, \dot q_2(\dot q_2 - \dot q_1), \quad F_2 = -(m_0 l a^3 l 4r^2)\, \dot q_1(\dot q_2 - \dot q_1). \tag{4.10b}$$

As a check, one can verify that T^* is a conserved quantity.

5. Hamel's approach to mechanics

5.1. Quasicoordinates, transpositional coefficients

Let $A = (a_{ij}(q))$, $B = A^{-1} = (b_{ij}(q))$ be $n \times n$ invertible matrices of functions of $q \in U$ in R^n. Consider a *Cartan moving frame* [4] on U:

$$X_j = \frac{\partial}{\partial \pi_j} = \sum_{i=1}^{n} b_{ij} \frac{\partial}{\partial q_i}, \quad 1 \leqq j \leqq n, \tag{5.1}$$

$$\Sigma\, \dot\pi_j X_j = \Sigma\, \dot q_i \frac{\partial}{\partial q_i}, \quad \dot\pi = A(q)\, \dot q. \tag{5.2}$$

Following the classical terminology, the numbers $\dot\pi_j$ are called *quasivelocities*. Recall that the symbols π_j are called *quasicoordinates*, but unless the fields X_j commute, they do not have a mathematical meaning. The motivation for the notation $\partial/\partial\pi_j$ is as follows: If π_j were true coordinates,

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[4] Not surprisingly, Cartan advocated the use of his "moving-frame method" in the study of nonholonomic systems [Ca]. For natural Lagrangians, he observes that d'Alembert's Principle expresses the following: "la différence géométrique entre l'accélération du point et la force est un vecteur normal à l'élément plan à m dimensions défini par les [contraintes]". When the constraints are nonholonomic, one is obliged to make use of all phase space. However: "il est possible en général de modifier cet espace sans que les propriétés mécaniques du système soient altérées". In Sections 4–6, Cartan uses his Lie group methods to find canonical expressions for the differential forms defining the constraints. In sections 7–10 he interprets the dynamics in terms of two connections, one of which depends on the metric in the directions of the constraints, the other along the normal directions.

then

$$\frac{\partial f}{\partial \pi_j} = \sum_i \frac{\partial f}{\partial q_i} \frac{\partial q_i}{\partial \pi_j} = \sum_i \frac{\partial f}{\partial q_i} b_{ij} = X_j(f).$$

Historically, the mathematical meaning of HAMEL's *transpositional symbols* $\gamma^i_{kj} = -\gamma^i_{jk}$, defined by

$$\gamma^i_{kj} = \sum_{s,l=1}^{n} b_{sk} b_{lj} \left(\frac{\partial a_{is}}{\partial q_l} - \frac{\partial a_{il}}{\partial q_s} \right), \quad 1 \leqq i, j, k \leqq n, \tag{5.3}$$

was the object of some controversy [NF, § III 5, 6]. The following simple fact, which seems not to have been observed in the classical works, gives a standard differential geometric interpretation to them. Let

$$\theta_i = \text{``}d\pi_i\text{''} = \sum_{j=1}^{n} a_{ij}\, dq_j \tag{5.4}$$

be the dual 1-forms of the moving frame. (The quotation marks are intended to remind us that in general they are not closed.) A straighforward calculation (see, e.g., [Sp, vol. 2, p. 7–9]) yields:

Proposition 5.1. (i) *The transpositional symbols are precisely the structure coefficients for the moving frame, i.e.,*

$$d\theta_i = \sum_{j<k} \gamma^i_{jk}\, \theta_k\, \theta_j. \tag{5.5}$$

(ii) *If*

$$a^i_{kj} = \tfrac{1}{2} (\gamma^i_{jk} + \gamma^j_{ki} - \gamma^k_{ij}), \quad \omega_{ij} = \sum_k a^i_{jk}\, \theta_k, \tag{5.6a}$$

then

$$\omega_{ij} = -\omega_{ji}, \quad d\theta_i = \sum_k \omega_{ki}\, \theta_k. \tag{5.6b}$$

(iii) *The above equations uniquely define the connection forms ω_{ik} for the moving frame. If this frame is declared to be an orthonormal basis for a Riemannian metric, then the associated Levi-Civita connection is*

$$\nabla \cdot X_j = \sum_i \omega_{ij}(\cdot)\, X_i, \tag{5.7a}$$

whose curvature is given by

$$R(X_k, X_l)\, X_j = \sum_i \Omega_{ij}(X_k, X_l)\, X_i, \quad \Omega_{ij} = d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj}. \tag{5.7b}$$

5.2. Hamel's equations

Recall, from the introduction, that as the fundamental law of mechanics we take equations (1.1) in the form of d'Alembert's Principle:

$$\sum_k \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} - Q_k \right) \delta q_k = 0. \tag{5.8}$$

Here $L(q, \dot{q}, t)$ is the Lagrangian (which can be an arbitrary function, not necessarily representing a natural mechanical system), Q_k the covariant components of the generalized external force, and δq_k the admissible variations, compatible with the constraints. After some tricky manipulations (see [NF, § III.5]) the transformation to quasicoordinates

$$L^*(q, \dot{\pi}, t) = L(q, B(q)\,\dot{\pi}, t) \tag{5.9}$$

yields

Proposition 5.2. *D'Alembert's Principle, written in Hamel's quasicoordinates, yields the equations*

$$\sum_{k=1}^{n} \left[\frac{d}{dt} \frac{\partial L^*}{\partial \dot{\pi}_k} - \frac{\partial L^*}{\partial \pi_k} + \sum_{i,j=1}^{n} \gamma_{kj}^{i} \frac{\partial L^*}{\partial \dot{\pi}_i} \dot{\pi}_j - R_k \right] \theta_k = 0 \tag{5.10}$$

where $R_k = \Sigma_s Q_s b_{sk}$ and $\partial/\partial\pi_k$ denotes the directional derivative along X_k.

Corollary 5.3. *Suppose that r constraints are present. Let the moving frame $\{X_i\}$ be chosen so that the first m vectors satisfy the constraints. In other words, the last r quasivelocities vanish: $\dot{\pi}_K = 0$ $(m+1 \leqq K \leqq n,\ r = n-m)$. Then (5.10) implies that*

$$\frac{d}{dt} \frac{\partial L^*}{\partial \dot{\pi}_k} - \frac{\partial L^*}{\partial \pi_k} + \sum_{i=1}^{n} \frac{\partial L^*}{\partial \dot{\pi}_i} \sum_{j=1}^{n} \gamma_{kj}^{i} \dot{\pi}_j = R_k, \quad 1 \leqq k \leqq m, \tag{5.11 a}$$

which are supplemented with

$$(\dot{q}_1, \ldots, \dot{q}_n)^{\mathrm{t}} = B(\dot{\pi}_1, \ldots, \dot{\pi}_m)^{\mathrm{t}}. \tag{5.11 b}$$

These $m+n$ equations for the unknowns $q_1, \ldots, q_n$, $\dot{\pi}_1, \ldots, \dot{\pi}_m$ *automatically* dispense with the constraint reactions.

5.3. Reduction of unconstrained Lagrangians with group symmetry

Let $G^r \to P^n \to M^m$ be a principal bundle, and (σ, f) a gauge coordinate system. Introduce the quasivelocities $(\dot{\pi}, \dot{q})$ on P via

$$\dot{g} = (L_g) * \dot{\pi}. \tag{5.12 a}$$

Recall that equations (3.8) describe the case of G-invariant natural systems. More generally, a G-invariant Lagrangian L on TP satisfies $\partial L/\partial\pi = 0$, or in gauge quasicoordinates $(q, \dot{q}, g, \dot{\pi})$, one has (cf. (3.4 a))

$$L = L^s(q, \dot{q}, \dot{\pi}). \tag{5.12 b}$$

Proposition 5.4. *Hamel's equations for G-invariant unconstrained Lagrangians are given by*

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\pi}_k} + \sum_{i,j=l}^{r} \gamma_{kj}^{i} \frac{\partial L}{\partial \dot{\pi}_i} \dot{\pi}_j = 0, \tag{5.13 a}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \tag{5.13 b}$$

where $\gamma_{kj}^{i} = c_{kj}^{i}$ *are the structure constants of the group,* $[e_k, e_j] = \Sigma_i c_{kj}^{i} e_i$.

Proof. Consider the exponential mapping $\exp: g \to G$, $X = \Sigma x_i e_i \mapsto e^X$. Thus (X, q) are true coordinates via equations (2.1), $F(X, q) = e^X s(q)$, and (5.12a) yields

$$\dot{\pi} = (L_{\exp(-X)}) * (D_X e^X) \dot{X} =: T(X) \cdot \dot{X}. \tag{5.14 a}$$

The operator A in (5.1) is given here by $(\pi, q)^t = A(X, q)^t$,

$$A = \begin{pmatrix} T(X) & 0 \\ 0 & I \end{pmatrix}, \quad B = A^{-1} = \begin{pmatrix} T(X)^{-1} & 0 \\ 0 & I \end{pmatrix}. \tag{5.14 b}$$

Geometrically, this corresponds to the moving frame obtained by left translating the vectors $\partial/\partial q_i$ and $V(e_K)$ at $s(q)$. With regard to the transpositional symbols for this frame, it is readily seen from (5.3) that

(i) the sums run only for l, $s = 1, \ldots, r$;
(ii) the nontrivial γ_{jk}^{i} are those with $i, j, k = 1, \ldots, r$ (the others vanish):

$$\gamma_{kj}^{i} = \sum_{s,l=1}^{r} T(X)_{sk}^{-1}\, T(X)_{lj}^{-1} \left(\frac{\partial}{\partial x_l} T_{is} - \frac{\partial}{\partial x_s} T_{il}\right). \tag{5.15}$$

Since the left-invariant vector fields e_K on G are being identified with the vertical fields $g * V_{s(q)}(e_K)$, it is clear that these agree with the structure coefficients of the frame $\{e_K\}$ on G. The Maurer-Cartan equations [SW] for Lie groups, for which we shall give an independent proof shortly (Proposition 5.6), can be written

$$\gamma_{kj}^{i}(X) = c_{kj}^{i}. \tag{5.16}$$

The result now follows immediately from Proposition 5.2.

Remarks. (i) If G is Abelian, then $\pi_k = X_k$ are bona fide "cyclic" coordinates. Here the $c_{kj}^{i} = 0$, and equations (5.13 a) give the well-known conserved momenta $l_k = \partial L / \partial \dot{\pi}_k$. Solving for the $\dot{\pi}_k$ in terms of the l_k and inserting the result into (5.13 b), *after* the Euler-Lagrange differentiation, one gets reduced equations in the base manifold. These are second-order ordinary differential equations for q and depend on l as parameters. However, these equations do not appear with a nice structure and seem to have only a local character. The Hamiltonian formalism appears more convenient. The local expression

$H = H(q,p,l)$ does not depend on the choice of section, so H becomes a global Hamiltonian on T^*M depending on the momenta l_k. The only possible complication appears if the bundle is non-trivial.

(ii) If G is non-Abelian, equations (5.13 a) also represent conserved quantities, as we show next. Although (5.13 b) is unavoidably of local character, these equations retain the familiar flavor of an Euler-Lagrange differential with respect to q, and (5.13 a) resembles the equations of a generalized rigid body.

Proposition 5.5 (Noether's Theorem). *Let* $\exp(X(t)) = g(t)$ *be the component of the flow along G, relative to the gauge coordinates* (σ, f). *Then equations* (5.13) *admit the conserved quantities*

$$l_K = \frac{\partial L}{\partial \dot{\pi}} (\mathrm{Ad}_{\exp(-X(t))}\, e_K). \tag{5.17}$$

Proof. One may assume $X(0) = 0$ and $\dot{X}(0) = \dot{\pi}$. Then

$$\frac{dl_K}{dt} = \frac{d}{dt}\bigg|_{t=0} \frac{\partial L}{\partial \dot{\pi}} \cdot e_K + \frac{\partial L}{\partial \dot{\pi}} \cdot \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}_{\exp(-X)} \cdot e_K$$

$$= \frac{d}{dt}\bigg|_{t=0} \frac{\partial L}{\partial \dot{\pi}_K} + \frac{\partial L}{\partial \dot{\pi}} [e_K, \dot{\pi}]$$

$$= \frac{d}{dt}\bigg|_{t=0} \frac{\partial L}{\partial \dot{\pi}_K} + \frac{\partial L}{\partial \dot{\pi}} \left(\sum_{i,j} c^i_{kj}\, \dot{\pi}_j e_i \right) = 0 \quad \text{(by (5.13 a))}.$$

Notice that the explicit presence of $X(t)$ in (5.17) renders it impossible to reduce Hamel's equations (5.13) further via the first integrals $\mu \in g^*$, unless one assumes μ to be fixed under the coadjoint action of G in g^*. This will be discussed in the following section 5.4.

I now show that the Maurer-Cartan equations (5.15), (5.16) can also be derived from "purely mechanical arguments". First recall

Example: *unconstrained rigid body*. Consider the setting of § 3.2, but seen now from HAMEL's viewpoint. One has a trivial principal bundle $P = G$ and M is a point. Equation (5.13 b) disappears, and (5.13 a) for a quadratic $H = T = \frac{1}{2}\,\Omega\, {}^t Q \Omega$ becomes

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\pi}_k} = -\sum_{i,j} \gamma^i_{kj} \left(\frac{\partial T}{\partial \dot{\pi}_i}\right) \dot{\pi}_j$$

where, for the moment, we *pretend* to be unaware of the Maurer-Cartan Theorem. However, since $\mu_k = \partial T / \partial \dot{\pi}_K$, one gets

$$\dot{\mu}_k = -\sum_{i,j,m} \gamma^i_{kj}\, Q_{im} \dot{\pi}_m \dot{\pi}_j.$$

Comparing this equation with (3.20) and doing a simple relabelling of indices, one indeed obtains a "symplectic" proof of Maurer-Cartan equations: $\gamma^i_{kj}(X) = c^i_{kj}$. Perhaps it is worth stating this result explicitly:

Proposition 5.6 (Maurer-Cartan Theorem). *Let $\{\theta_i\}_{i=1,\dots,n}$ be left-invariant 1-forms on a Lie group G such that $\theta_i(\mathrm{id}) = e_i^*$ where $\{e_i\}_{i=1,\dots,n}$ is a basis of g. Then*

$$d\theta_i = \sum_{j<k} \gamma^i_{jk}\theta_k\theta_j \tag{5.18a}$$

where the γ^i_{jk} defined by

$$\gamma^i_{kj} = \sum_{s,l=1}^{r} T(X)^{-1}_{sk}\, T(X)^{-1}_{lj}\left(\frac{\partial}{\partial x_l}T_{is} - \frac{\partial}{\partial x_s}T_{il}\right),$$

$$T(X) = (L_{\exp(-X)}) * (D_X e^X) \tag{5.18b}$$

are actually constants, equal to the structure coefficients c^i_{jk}. In particular,

$$c^i_{jk} = \left[\frac{\partial T_{ik}}{\partial X_j} - \frac{\partial T_{ij}}{\partial X_k}\right]\bigg|_{X=0}. \tag{5.18c}$$

5.4. Marsden-Weinstein reduction: Expression in coordinates

In the more abstract context of G-invariant *Hamiltonian* systems on a symplectic manifold X^{2n}, for which there is a G-action possessing a momentum mapping, there is a widely used reduction procedure known as the Marsden-Weinstein method ([MW], [A, Appendix 5]).

The special case of the Marsden-Weinstein reduction where $X = T^*M$ and M is a principal G-bundle with base N was analysed by KUMMER [Ku]. For completeness, I outline without proofs the main features, which use a connection on M in order to obtain a *globally* defined Hamiltonian in T^*N.

Momentum mapping $J: T^*M \to g^*$, $J(p_m)(\cdot) = p_m(V_m(\cdot))$.
(i) $J_Y = J(\cdot)(Y)$ is the Hamiltonian function of the Hamiltonian vector field

$$v(m,p) = \frac{d}{dt}\bigg|_{t=0} e^{tY}p_m \in T_{(m,p)}T^*M.$$

(ii) J is G-equivariant, $J(g\cdot p_m) = \mathrm{Ad}^*_g J(p_m)$.
(iii) J is an integral of motion for any G-invariant Hamiltonian on T^*M.

Reduced phase space and Hamiltonian [MW]. Given a G-invariant Hamiltonian $H: T^*M \to R$, and an element $\mu \in g^*$, the standard symplectic form on T^*M yields, by projection, a symplectic structure on the reduced phase space $F_\mu = J^{-1}(\mu)/G_\mu$, where G_μ is the isotropy subgroup of μ in the coadjoint action. By restriction and G-invariance, H projects to a Hamiltonian function on F_μ.

Kummer's model for the reduced phase space: $F_\mu \approx T^*N$. *Assume* that μ is G-equivariant, so that $G_\mu = G$. (If it is not, replace the action of G on M by the G_μ-action.) In other words, $\mu(\mathrm{Ad}_g(\cdot)) = \mu(\cdot)$ for all $g \in G$. Observe that in terms of a basis $\{e_i\}$ of g with $\mu = e_n^*$, this means that

$$c_{ij}^n = 0, \qquad \text{for all } i,j.$$

$$\left(\text{The proof is immediate: } 0 = \frac{d}{dt}\bigg|_{t=0} \mu(\mathrm{Ad}_{\exp(tX)}(\cdot)) = \mu[X,\cdot] \text{ for all } X \in g.\right)$$

The idea is to use any connection D on the bundle $\pi: M \to N$ to define a mapping $i_{D,\mu}: T^*N \approx F_\mu$ as follows: Given $p_n \in T^*N$, the G-equivalence class $i_{D,\mu}(p_n)$ is the set of all elements $P_m \in T^*M$ (m is any point in the fiber over n)

$$P_m(\cdot) = p_n(d\pi(\cdot)) + \mu(\varphi(\cdot)) \tag{5.19}$$

where φ is the connection 1-form. (That μ is fixed under the coadjoint action is crucial in order that $P_m \in J^{-1}(\mu)$.) KUMMER showed that the symplectic form on F_μ pulls back to the standard symplectic form on T^*N *plus a "magnetic term"*. This magnetic term is the pull-back, via the natural projection $T^*N \to N$, of a connection-related 2-form on N, which is precisely the μ-component of its curvature. Again, the fact that μ is fixed under the coadjoint action allows us to consider this μ-component as a 2-form on N.

Expression in coordinates. Let (q, X) be gauge coordinates on M, (p, q) natural coordinates on T^*N, and (q, X, P, Π) natural coordinates on T^*M. "Natural" means that

$$p_q\left(\dot{q}\frac{\partial}{\partial q}\right) = p \cdot \dot{q}$$

$$(P_q + \Pi_X)\left((e^X) * s'(q)\,\dot{q} + s(q)^* \cdot D_X e^X \dot{X}\right) = P \cdot \dot{q} + \Pi \cdot \dot{X}. \tag{5.20}$$

The element $i_{D,\mu}(p_q)$ is represented at $F(q, X) = e^X s(q)$ by the linear functional

$$(\dot{q}, \dot{X}) \to p \cdot \dot{q} + \mu\left((L_{\exp(-X)}) * D_X e^X\right) \dot{X} - b(q)\,\dot{q}$$

so that

$$P = p - b(q)^* \mu,$$

$$\Pi = [(DL_{\exp(-X)})\,(D_X e^X)]^* \mu = T(X)^* \mu \tag{5.21}$$

where $*$ denotes the dual linear operators,

$$(L_{\exp(-X)})^*: g^* \to T_{\exp(X)}{}^*G, \qquad (D_X e^X)^*: T_{\exp(X)}{}^*G \to g^*.$$

Given a G-invariant Hamiltonian $H(q, X, P, \Pi)$ in coordinates, the reduced Hamiltonian is given by

$$H_{\mathrm{red}}(p, q; \mu) = H(q, 0, p - b(q)^* \mu, \mu) \tag{5.22}$$

and the reduced symplectic form, which takes into account the magnetic term, is

$$dp \wedge dq + \sum_{i<j} \left(\frac{\partial b_{nj}}{\partial q_i} - \frac{\partial b_{ni}}{\partial q_j}\right) dq_i \wedge dq_j. \tag{5.23}$$

Here $\{e_i\}$ is used as a basis of g with $\mu = e_n^*$, as indicated above. If a solution $p(t), q(t)$ of the reduced system is known, the coordinate $P(t)$ is immediately given by (5.21), while to obtain the coordinate $X(t)$ it is necessary to integrate the following time-dependent ordinary differential equation in g

$$\dot{X} = \frac{\partial H}{\partial \Pi} (q(t), X, P(t), T(X)^* \mu). \tag{5.24}$$

Comparison with Hamel's approach. Let $L: TM \to R$ be a Lagrangian. Its Legendre transformation

$$\text{Leg}: TM \to T^*M, \quad \text{Leg}(v_m)(\cdot) = D_{(m,v)}(L \mid T_m M)(\cdot)$$

composed with the momentum mapping $J: T^*M \to g^*$ yields, as expected, Noether's integrals

$$I: TP \to g^*, \quad I^Y(v_m) = \frac{d}{dt}\bigg|_{t=0} L(v_m + tV_m(Y)).$$

Using gauge coordinates, one has $L = L^s(q, \dot{q}, \dot{\pi})$ where

$$m = g \cdot s(q), \quad g = e^X, \quad v_m = g * [s'(q)\, \dot{q} + V_{s(q)}(\dot{\pi})]$$

$$V_m(\cdot) = g * V_{s(q)}[\text{Ad}_{\exp(-X)}(\cdot)]$$

so that

$$I^Y(v_m) = \frac{\partial L^s}{\partial \dot{\pi}} \cdot \text{Ad}_{\exp(-X)} Y = \text{Ad}_{\exp(-X)}{}^* \frac{\partial L^s}{\partial \dot{\pi}}(Y).$$

If $\mu \in g^*$ is fixed under the coadjoint action, then the invariant set associated with μ is given simply by

$$\frac{\partial L^s}{\partial \dot{\pi}}(q, \dot{q}, \dot{\pi}) = \mu, \tag{5.25}$$

which allows $\dot{\pi}$ to be solved for as a function of $(q, \dot{q}, \mu)$; this solution can be inserted into (5.13 b). One can prove, by brute force, that these "sharply local" reduced equations are equivalent to the globally defined reduced Hamiltonian in KUMMER's construction.

6. Proofs of Theorems 3.1 and 3.2

6.1. Non-Abelian Čaplygin systems

Given the gauge coordinates (σ, f), define the quasivelocities (see (3.4 a)′)

$$\dot{g} = (L_g) * (\dot{\pi} + b(q)\, \dot{q}) \tag{6.1 a}$$

so that a G-invariant Lagrangian L on TP can be written locally as

$$L^*(q,\dot q,\dot\pi) = L^s(q,\dot q,\dot\pi + b(q)\dot q). \tag{6.1b}$$

The constraints are represented simply by $\dot\pi = 0$. The matrix A in (5.1) for which $(\dot q,\dot\pi)^t = A(\dot q,\dot X)^t$ is given by

$$A = \begin{pmatrix} I & 0 \\ -b(q) & T(X) \end{pmatrix}, \quad B = A^{-1} = \begin{pmatrix} I & 0 \\ T(X)^{-1}b(q) & T(X)^{-1} \end{pmatrix}. \tag{6.2}$$

Proof of Theorem 3.1. By group invariance, the transpositional coefficients can be calculated at $X = 0$. One verifies readily that $\gamma^i_{kj} = 0$ $(1 \leqq i \leqq m)$ and

$$\gamma^{m+l}_{kj} = \frac{\partial b_{lj}}{\partial q_k} - \frac{\partial b_{lk}}{\partial q_j} + \sum_{S,L=1}^{r} b_{Sk} b_{Lj} c^l_{SL}$$

where one uses the Maurer-Cartan formulas (5.18). The result now follows immediately from Proposition 5.2.

Proof of Theorem 3.1′. It remains to give an intrinsic characterization of the nonholonomic terms when the Lagrangian is natural. For a quadratic form T, recall that

$$T^s(q,\dot q,\dot\pi) = T^*_M(q,\dot q) + T^G(\dot\pi - b(q)\dot q) + \left(h_{s(q)}\frac{\partial}{\partial q}, s(q)_*(\dot\pi - b(q)\dot q)\right)$$

so that

$$\left.\frac{\partial T^s}{\partial\dot\pi}\right|_{\dot\pi = b(q)\dot q}(\cdot) = \left(h_{s(q)}\frac{\partial}{\partial q}, s(q)^*\cdot\right) \tag{6.3}$$

and this gives rise to the tensor K. Similarly, if L contains a linear term μ in the velocities, it yields the tensor J.

6.2. Generalized rigid bodies with left-invariant constraints

Identify $T^*G \approx G\times g^*$ by left transport, as in the Lemma from § 3.2. Given any 1-form ψ on G (not necessarily left-invariant), consider the vector field X_ψ on $G\times g^*$ defined by

$$\pi^*\psi = \omega(\cdot, X_\psi) \tag{6.4}$$

where $\pi: G\times g^* \to G$ is the projection on the first factor.

It is readily seen that $X_\psi = (0, -L^*_g\psi(g))$ so that X is vertical (tangent to the fibers) and furthermore constant on each fiber. When ψ is left-invariant $X_\psi = (0, -\psi_{\mathrm{id}})$, it is also constant with respect to $g \in G$, i.e., it is the same in all fibers.

Given a left-invariant Hamiltonian H on T^*G subjected to left-invariant constraints $D: \psi^k = 0$, $1 \leqq k \leqq r$, consider the Lagrange-d'Alembert equations in Hamiltonian form (see e.g., [We]) and pass subsequently to the coordinates $G \times g^*$. The result is (compare with Lemma 3.2):

Lemma. *The constrained vector field (in $G \times g^*$) is given by*

$$\dot{g} = (L_g) * \frac{\delta H}{\delta \mu}, \quad \dot{\mu} = \left\{ \frac{\delta H}{\delta \mu}, \mu \right\} - \sum_{k=1}^{r} \lambda_k(\mu)\, \psi_{\mathrm{id}}^k. \tag{6.5}$$

Proof of Theorem 3.2. It remains only to eliminate the multipliers in the particular case where $2H = {}^t\mu\, Q^{-1} \mu$, and Q is positive-definite. Observe that the inverse Legendre transformation $\mathrm{Leg}^{-1}: g^* \to g$ maps the span of the ψ_{id}^k into the T-orthogonal complement of $D = \bigcap_k \ker \psi_{\mathrm{id}}^k$. Thus the constrained vector field is precisely the T-orthogonal projection of $\mathrm{Leg}^{-1}(v_H)$ over D.

7. Factorization of more general classical nonholonomic systems

Again, consider a G-equivariant Lagrangian in the total space P of a principal bundle $G^r \to P^n \to M^m$, subjected to classical constraints. It is natural to ask what happens when the number of constraints is different from $r = \dim G$. Let d be the dimension of the subspaces D_p in T_pP of admissible velocities. We have worked out the case $d = m$, $D_p \approx T_{\pi(p)}M$. In the case $d < m$, $D_p \cap V_p = \{0\}$, one expects to obtain a reduced system in TM with $m - d$ constraints. When $d > m$, the system factorizes into two (coupled) components, one being a special field on TM, the other an operator on subspaces of g.

I outline the main features of the latter. A *pseudoconnection* (lacking a better name) is a distribution $\{D_p\}$ of d-dimensional subspaces of T_pP such that (i) $g * D_p = D_{gp}$; (ii) $K_p = D_p \cap V_p$ has the same dimension $d - m$ for all $p \in P$.

Assume that the Lagrangian L on TP is natural, $L = T - V$. Using the metric one can define the decomposition $TP = N \oplus K \oplus S$ where S_p is the orthogonal complement of K_p in V_p and N_p is the orthogonal complement of K_p in D_p. Since $\dim S_p = r - (d - m)$, it follows that $\dim N_p = m$ and hence that N defines a true connection on the principal bundle.

Fix a gauge system (σ, f). At each $q \in U$, the metric $T \mid V_{s(q)}$ is transferred to g, and there is an associated splitting $K_q \oplus S_q$ of g, varying smoothly with q. Construct a smooth family of basis vectors $k_i(q)$, $1 \leqq i \leqq d - m$, for K_q and $s_j(q)$, $1 \leqq j \leqq r - (d - m)$, for S_q.

Let X be coordinates relative to a fixed basis e_k, $1 \leqq k \leqq r$ for the Lie algebra, and $S(q)$ the matrix transforming the X-coordinates to the coordinates (μ, β) on the basis $\{k_i, s_j\}$,

$$\sum \dot{\pi}_k e_k = \sum \dot{\mu}_i k_i(q) + \sum \dot{\beta}_j s_j(q). \tag{7.1}$$

Thus the quasivelocities for this setting are

$$(\dot{q}, \dot{\mu}, \dot{\beta})^{\mathrm{t}} = A(q, X) \cdot (\dot{q}, \dot{X})^{\mathrm{t}},$$

$$A(q,X) = \begin{pmatrix} I & O \\ 0 & S(q) \end{pmatrix} \begin{pmatrix} I & O \\ -b(q) & T(X) \end{pmatrix}, \tag{7.2}$$

and the constraints are given by $\dot{\beta} = 0$. The Lagrangian is of the form

$$L(q, \dot{q}, \dot{\mu}, \dot{\beta}) = L^s\left(q, \dot{q}, \dot{\pi}(q, \dot{\mu}, \dot{\beta}) + b(q)\,\dot{q}\right) \tag{7.3}$$

and explicit expressions for the transpositional coefficients can be obtained, using the definition (5.3) with A given by (7.3). The structure of Hamel's equations is as follows:

$$\begin{aligned}
0 = {} & \frac{d}{dt}\frac{\partial L^*}{\partial \dot{q}_k} - \frac{\partial L^*}{\partial q_k} + \sum_{i=1}^{m} \frac{\partial L^*}{\partial \dot{q}_i}\left(\sum_{j=1}^{m} \gamma^i_{kj}\,\dot{q}_j + \sum_{j=1}^{d-m} \gamma^i_{k,m+p}\,\dot{\mu}_p\right) \\
& + \sum_{i=1}^{d-m} \frac{\partial L^*}{\partial \dot{\mu}_i}\left(\sum_{j=1}^{m} \gamma^{i+m}_{kj}\,\dot{q}_j + \sum_{p=1}^{d-m} \gamma^{i+m}_{k,m+p}\,\dot{\mu}_p\right) \\
& + \sum_{i=1}^{r-(d-m)} \frac{\partial L^*}{\partial \dot{\beta}_i}\left(\sum_{j=1}^{m} \gamma^{d+i}_{kj}\,\dot{q}_j + \sum_{p=1}^{d-m} \gamma^{d+i}_{k,m+p}\,\dot{\mu}_p\right), \\
0 = {} & \frac{d}{dt}\frac{\partial L^*}{\partial \dot{\mu}_k} + \sum_{i=1}^{m} \frac{\partial L^*}{\partial \dot{q}_i}\left(\sum_{j=1}^{m} \gamma^i_{m+k,j}\,\dot{q}_j + \sum_{p=1}^{d-m} \gamma^i_{m+k,m+p}\,\dot{\mu}_p\right) \\
& + \sum_{i=1}^{d-m} \frac{\partial L^*}{\partial \dot{\mu}_i}\left(\sum_{j=1}^{m} \gamma^{i+m}_{m+k,j}\,\dot{q}_j + \sum_{p=1}^{d-m} \gamma^{i+m}_{m+k,m+p}\,\dot{\mu}_p\right) \\
& + \sum_{i=1}^{r-(d-m)} \frac{\partial L^*}{\partial \dot{\beta}_i}\left(\sum_{j=1}^{m} \gamma^{d+i}_{m+k,j}\,\dot{q}_j + \sum_{p=1}^{d-m} \gamma^{d+i}_{m+k,m+p}\,\dot{\pi}_p\right).
\end{aligned} \tag{7.4}$$

8. Natural Čaplygin systems: affine connections

Riemannian Geometry and Mechanics meet in the study of natural unconstrained Lagrangians $L = T - V$ on a tangent bundle TQ. The following results, for instance, are well known:

(i) The paths with total energy h agree (up to reparametrizations) with the geodesics of the *Jacobi metric* $T_J = (h - V)\,T$. In particular, if the energy surface is compact, and the sectional curvatures negative, the system is ergodic ([A, Appendix 1]).

(ii) The equations of motion can be put in the form

$$\nabla_{\dot{X}} \dot{X} = - \sum_{i,j} g^{ij}(q)\,\frac{\partial V}{\partial q_j}\,\frac{\partial}{\partial q_i} \tag{8.1 a}$$

where the left-hand side is the *acceleration* according the *Riemannian connection* associated with the metric T, and the right-hand side is the contravariant expression of the force due to the potential V. In view of (i), the trajectories agree with the geodesics of T_J, given by

$$\nabla^J_{\dot{X}} \dot{X} = 0. \tag{8.1 b}$$

One can give a proof of (i) without using the Principle of Least Action, as usually done (e.g., in [A, § 45D]): In local coordinates $2T = \dot{q}\, G(q)\, \dot{q}$, $V = V(q)$. Equivalently, the Lagrangian L corresponds to the Hamiltonian on T^*Q:

$$H(p, q) = \tfrac{1}{2} p^t G(q)^{-1} p + V(q), \tag{8.2}$$

and the Legendre transform of the Jacobi-metric is

$$H^J(p, q) = \tfrac{1}{2} p^t G(q)^{-1} (h - V(q))^{-1} p. \tag{8.3}$$

The result of JACOBI follows from the simple observation, via a straightforward calculation, that for solutions of (8.2) with $H = h$,

$$\operatorname{grad} H^J = (h - V)^{-1} \operatorname{grad} H. \tag{8.4}$$

Lagrangians which consist only of a kinetic energy term are called *inertial.* JACOBI's result means that by changing the metric to a conformal one, with conformality factor $h - V$, all natural systems may be assumed to be inertial. The Jacobi metric is useful for constrained systems as well, an observation which seems not to have been sufficiently explored in the theory of nonholonomic systems:

Proposition 8.1. *The trajectories with energy h of the nonholonomic Lagrangian system $L = T - V$ with classical constraints given by the distribution D_q coincide with those of T_J with the same constraints.*

Proof. Write the d'Alembert-Lagrange equations in Hamiltonian form:

$$L : \dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j} + \Sigma \mu_i a_{ij}, \quad \Sigma a_{ij} \frac{\partial H}{\partial p_j} = 0,$$

$$T_J : \dot{q}_j = \frac{\partial H^J}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H^J}{\partial q_j} + \Sigma \lambda_i a_{ij}, \quad \Sigma a_{ij} \frac{\partial H^J}{\partial p_j} = 0.$$

In view of (8.4) the vector fields are proportional: The uniquely defined multipliers for the second set satisfy $\lambda = \mu(h - V)$.

Concerning the nonholonomic counterpart of (ii), VERSHIK & FADDEEV [VF] stressed the role of *projected affine connections*, which are the analogues of Levi-Civita connections on linear tangent subbundles D of TP. Define

$$\nabla^D_X Y = \mathrm{pr}_D \nabla_X Y \tag{8.5}$$

where X, Y are vector fields satisfying the constraints, pr_D is the T-orthogonal projection over D, and $\nabla_X Y$ the Riemannian connection of T.

Caveat: Strictly speaking, ∇^D should not be called an affine connection, because unless it is integrable, D is not a full tangent bundle.

Proposition 8.2 ([VF]). *For inertial constrained Lagrangians $L = T$, $v_p \in D_p$, the solutions $\gamma(t)$ of the Lagrange-d'Alembert equations satisfy*

$$\nabla^D_{\dot\gamma(t)} \dot\gamma(t) = 0. \tag{8.6}$$

In view of Proposition 8.1, the projected connection can be always used in studying natural nonholonomic systems: One can use $T = T_J$ to assume, without loss of generality, that the system is inertial. Proposition 8.2 follows immediatly from the *geometric interpretation* of the Lagrange-d'Alembert equations given by CARTAN ([Ca])[4]:

Lemma. *Consider a mechanical system with n-degress of freedom having kinetic energy T (in coordinates $2T = \dot q^{\mathrm t} G(q)\,\dot q$), subjected to r linear constraints D (in coordinates $A(q)\,\dot q = 0$, A an $r\times n$ matrix of rank r) and an external force F (whose work $\Sigma P_i dq_i$ for admissible displacements is known). Then the Lagrange-d'Alembert equations express the fact that the difference between the acceleration (according to the metric T) and the external force, i.e., the reaction to the constraints, at any point q of the configuration space, is normal to the plane D_q.*

Proof. The Lagrange-d'Alembert equations can be written vectorially as

$$\frac{d}{dt} G(q)\,\dot q - \frac{\partial T}{\partial q} = P + {}^{\mathrm t}\!A\mu.$$

Multiply this equality by G^{-1}. In the old differential-geometric language, this means taking the contravariant components of the acceleration and of the forces (external and constraint reaction), in the right-hand side. The constraint-reaction force is therefore

$$N = \Sigma\, G^{-1} A^{\mathrm t} \mu_i \frac{\partial}{\partial q_i}\,, \tag{8.7}$$

and it is orthogonal to D: If $v = \Sigma\, \dot q_i(\partial/\partial q_i) \in D_q$, then $A\dot q = 0$ and

$$(v,N) = {}^{\mathrm t}\dot q\, G(G^{-1}A^{\mathrm t}\mu) = (A\dot q)^{\mathrm t}\mu = 0.$$

Consider now the basic setting, a G-principal bundle $\pi : P \to M$, with a connection D and G-invariant natural Lagrangian $L = T - V$. In view of Proposition 8.1, there is no loss in generality by assuming that L is purely inertial (replace T by T_J if there is a potential V).

Theorem 8.3. *The solutions of the reduced system on TM are the geodesics of the affine connection on M defined by*

$$\nabla^{M,D}_X Y =: \pi_*\left(\mathrm{pr}_D \nabla_{h(X)} h(Y)\right) = \pi_*(\nabla^D_{h(X)} h(Y)) \tag{8.8}$$

where h is the horizontal lift operator.

Proof. It is easily verified that $\nabla^{M,D}$ is well defined (i.e., it does not depend on where the lifts are done in the fibers) and satisfies the axioms of an affine connection. The result follows immediately from Proposition 8.2.

Since $\nabla^{M,D}$ is a true connection on the tangent bundle of M, it is possible to study its differential-geometric entities: torsion, curvature, holonomy groups, validity of a result of HOPF & RINOW, and so on. These developments will not be pursued here, in part because of the observation that HAMEL's approach yields a *different* affine connection on TM (but with the same geodesics, of course), as we now show.

Recall a standard operation in differential geometry. If Q is a Riemannian manifold with metric T, operators R_T of "raising" and L_T of "lowering" indices are defined as follows. If $X = \Sigma a^i(\partial/\partial q_i)$ is a vector field, then the associated 1-form $w_X = \Sigma a_i dq^i$ is defined by lowering indices, $a_i = \Sigma_j g_{ij} a^j$, and conversely, $a^i = \Sigma_j g^{ij} a_j$. Intrinsically $w_X(Y) = \langle X, Y \rangle$. This construction can be naturally extended to arbitrary tensor fields, transforming any contravariant slot (vector field) into a covariant one (1-form), or vice-versa. Also recall that a covariant slot can be removed if one makes the tensor field be vector-valued: For instance, if $B(X, Y, w)$ is a (2,1)-tensor, then it is identified with a vector field-valued $B(X, Y)$ via $B(X, Y, w) = w(B(X, Y))$. The latter operation does not depend on the metric.

For instance, the (3,0)-tensor field $K(X, Y, Z)$ defined in § 3 via the operation of raising the indices of the last slot, becomes the (2,1)-tensor

$$B(X, Y, w) = K(X, Y, R_T w). \tag{8.9}$$

Proposition 8.4. *Let* $(G \to P \to M,\ L = T,\ D)$ *be a natural Čaplygin system, with an inertial Lagrangian. Then the solutions of the reduced equations on* M*, obtained by Hamel's method, are the geodesics of both of the following affine connections*:

$$\nabla_1^{M,H}{}_X Y = \nabla_X^* Y + B(X, Y), \qquad \nabla_2^{M,H}{}_X Y = \nabla_X^* Y + B(Y, X) \tag{8.10}$$

where $\nabla_X^* Y$ *is the Riemannian connection of the projected metric* T^* *and* $B(X, Y)$ *is the* (2,1)*-tensor field corresponding to the tensor* K *on* M, *via* R_{T^*}.

Proof. Going back to Proposition 3.1, one writes the equations of motion in terms of contravariant components. If $\dot{\gamma} = \Sigma \dot{q}_i(\partial/\partial q_i)$, then

$$\nabla_{\dot{\gamma}}^* \dot{\gamma} + \sum_{i,k} g^{*ki} \left\langle h_{s(q)}(q), V\left(\Omega^s\left(q, \frac{\partial}{\partial q_i}\right)\right)\right\rangle \frac{\partial}{\partial q_k} = 0$$

so that in local coordinates, the second term is $B(\dot{q}, \dot{q})$ with

$$B\left(\Sigma a_i \frac{\partial}{\partial q_i}, \Sigma b_j \frac{\partial}{\partial q_j}\right) = \sum_{k,l} g^{*kl} \left\langle h(a), V\left(\Omega^s\left(b, \frac{\partial}{\partial q_l}\right)\right)\right\rangle \frac{\partial}{\partial q_k}. \tag{8.11}$$

This has a global meaning: it is the raising of the third-slot of the tensor K.

Remark. I have used the fact that the set of connections is an affine space, modelled over the infinite-dimensional vector space of (2,1)-tensors. Clearly either one of $B_1(X,Y) = B(X,Y)$ or $B_2(X,Y) = B(Y,X)$ can be used: $\nabla_1^{M,H}$ and $\nabla_2^{M,H}$ have the same geodesics, since $B_1 - B_2$ is an antisymmetric tensor (see Hicks [Hi] for these properties of affine connections).

In view of this remark, one may pose a "philosophical" question: which one of these connections has a better link with the dynamics of the nonholonomic system? In doubt, one can average them, and an educated guess is that this average would give $\nabla^{M,D}$. It turns out that this guess is not quite correct:

Proposition 8.5. *The projected connection in M is explicitly given by*

$$\nabla_Y^{M,D} Z = \nabla_Y^* Z + \tfrac{1}{2}[B(Y,Z) + B(Z,Y)] + \tfrac{1}{2}\, C(Y,Z) \tag{8.12}$$

where $C(Y,Z)$ is the tensor obtained by raising the first slot of $K(X,Y,Z)$.

Remark. Since C is an antisymmetric tensor, it does not enter in the differential equations for the geodesics of the projected connection.

Proof (by a direct computation). Given coordinates q on M, it is necessary to obtain the Christoffel symbols of $\nabla^{M,D}$:

$$\nabla_{\partial/\partial q_k}^{M,D} \frac{\partial}{\partial q_j} = \sum_i \Gamma_{jk}^i \frac{\partial}{\partial q_i}. \tag{8.13}$$

To make calculations simpler, one may consider a section $s: R^m \to P$ such that the image of $s'(q_0)$ is the horizontal subspace at q_0 $(b(q_0) = 0)$ and also that the vectors $s'(q_0) \cdot \partial/\partial q_i$ are orthonormal. Thus, using the definition of the projected connection, we obtain

$$\Gamma_{jk}^i(q_0) = \left\langle \nabla_{h(\partial/\partial q_k)} h\left(\frac{\partial}{\partial q_j}\right), h\left(\frac{\partial}{\partial q_i}\right)(q_0) \right\rangle$$

where $\langle\, ,\rangle$ represents the metric of P.

One now needs a formula which comes from a well-known trick in differential geometry. Let X, Y, Z be vector fields on a Riemannian manifold, not necessarily commuting. If ∇ is the associated Riemannian connection, then

$$2\langle \nabla_X Y, Z\rangle = X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle - \langle [Y,Z],X\rangle - \langle [X,Z],Y\rangle - \langle [Y,X],Z\rangle. \tag{8.14}$$

Let $X_i = h(\partial/\partial q_i)$, and observe that the $g_{ij}^* = \langle X_i, X_j\rangle$ give the coefficients of the projected metric. Now $g_{ij}^*(q_0) = \delta_{ij}$. For $X = X_i$, $Y = X_j$, $Z = X_k$, the first three terms of (8.14) give the familiar expression

$$\frac{\partial}{\partial q_i} g_{jk}^* + \frac{\partial}{\partial q_j} g_{ik}^* - \frac{\partial}{\partial q_k} g_{ij}^*.$$

The last three terms can be identified (after a careful, but straightforward inspection) with the tensors B_1, B_2 and C computed at q_0, by using

$$\langle [X_i, X_k], X_j \rangle (q_0) = \sum_L g^*_{Lj}(q_0) \left(\frac{\partial b_{Lk}}{\partial q_i} - \frac{\partial b_{Li}}{\partial q_k} \right) \Bigg|_{q=q_0}, \tag{8.15}$$

where $g^*_{Lj} = \langle X_j, V_{s(q)} e_L \rangle$.

To prove (8.15), it is enough to show that

$$[X_i, X_k] \, (s(q_0)) = \sum_L \left(\frac{\partial b_{Lk}}{\partial q_i} - \frac{\partial b_{Li}}{\partial q_k} \right) \Bigg|_{q=q_0} \cdot V_{s(q_0)} e_L,$$

which follows from the explicit formula for the Lie-brackets in (q, X)-coordinates, taking advantage of the convenient assumption that $b(q_0) = 0$.

9. Final comments

9.1. *Other types of constrained systems with symmetry*

In this paper I have treated only the *standard* situation of Lagrangians $L : TP \to R$ with *classical* constraints, i.e., given by a distribution of linear subspaces $D_p \subset T_p P$. Several authors have considered other types of mathematical problems involving linear or nonlinear constraints, such as:

(i) *Lagrangians with nonlinear constraints on P.* Gauss' Principle of Least Constraint can be invoked in order to derive the equations of motion. A physical example was given by HAMEL (see [NF, § IV.2]). NEIMARK & FUFAEV notice that this example arises from another with standard constraints, taking limits for certain parameters. However, the limits of the solutions of the "standard" problem do not agree with the solutions of the limit problem.
(ii) *Lagrangians with linear constraints on TP.* VERSHIK & FADDEEV [VF] extend d'Alembert's Principle to include *admissible codistributions* represented in local coordinates by $\beta = \{\Sigma_j (a^j_i \, dq^i + b^j_i \, dv^i)\}_{j=1,\dots,p}$ with rank $b = p$. The constraints are given by $\ker \beta$.
(iii) *Hamiltonians with constraints.* WEBER [We] considers a symplectic manifold (M^{2n}, ω), a Hamiltonian $H : M \to R$, and a linearly independent system $\{\beta_i\}_{i=1,\dots,m}$, $m < n$, of differential forms on M, and extends d'Alembert's principle as

$$\omega(X, \cdot) = dH + \Sigma \mu_i \beta_i, \quad \beta_i(X) = 0 \quad (1 \leqq i \leqq m). \tag{9.1}$$

(iv) *Dirac theory of degenerate Lagrangians.* In some circumstances, Dirac's theory of constraints has a link with reduction schemes. This is discussed in the paper by CANTRIJN et al. [CCCI]. On the other hand, for degenerate

Lagrangians $L: TP^n \to R$, of rank $n - m$, it is frequently possible to define an equivalent Hamiltonian on T^*P, satisfying m relations $G_i(p, q) = 0$. The differentials $\beta_i = dG_i$ are called generalized holonomic constraints by WEBER. It is asserted in [We, § 3] that Dirac's canonical equations coincide with (9.1), but it seems that this does not hold in general.

(v) *Carathéodory metrics and control problems.* A distribution D on a manifold Q may be called *fat* if the algebra of commutators of a local basis of vector fields for the distribution span T_qQ at any point $q \in Q$. If Q is given a metric g, the Carnot-Carathéodory distance $d(q_1, q_2)$ is the infimum of the q-distances along horizontal curves joining these points. This is related to the Lagrange problem of Calculus of Variations and some variational problems in control theory (see BROCKETT [B]). PANSU [Pa] and MITCHELL [Mi] have studied the Hausdorff dimension of these singular metrics.

In all these contexts the ingredients can be assumed *equivariant* under the action of a Lie group. A research program is to obtain reduction procedures, even in local form, for the above settings.

9.2. Some recent results

After the submission of this paper, the translation of a treatise by ARNOLD, KOZLOV & NEISHTADT [AKN] and several interesting articles have appeared. I add some brief comments about these new developments.

(i) *Vakonomic Mechanics and Lagrange variational problems with symmetry.* Unfortunately, the same word "nonholonomic" has been used historically with two different mathematical meanings. Perhaps to remedy this, KOZLOV has introduced the term "vakonomic" [AKN, § I.4], meaning mechanics of the variational axiomatic kind. The relationship with the Dirac formalism of constraints is clarified in [AKN, § I.5]. R. MONTGOMERY [Mo] has started a program to study shortest loops with a given holonomy. He shows that the solutions are trajectories in Yang-Mills potentials. Applications range from quantum mechanics to mechanical engineering. The old problem of physical realization of constraints was revisited by KOZLOV [AKN, § I.6], [Koz]: a unified framework for the two different equations of motion is obtained. From the analytical viewpoint (topic (v) above) a very nice survey by GERSHKOVICH & VERSHIK [GV] was published, with recent results included. See also [VG] and [BG] for interesting examples.

(ii) *The notion of integrability for nonholonomic systems.* In a first version of this paper, the question whether there is a suitable generalization of the concept of Liouville integrability for classical nonholonomic systems of generalized Čaplygin type was posed. The reduced system in TM (or T^*M under the L^*-Legendre transformation) is not Hamiltonian in general, but it can be conjectured that there is an invariant measure. This is indeed the case in several examples: while preparing the revised version of this paper, the recent work by VESELOV & VESELOVA [VV] was encountered. There, the authors show that the rigid-body with left-invariant constraints, when written in a suitably extended phase space (including the constraint reaction), has an invariant measure. In the case of $G = SO(3)$ with one constraint, there is even a relation with the celebrated Neumann problem of a point moving on the sphere with quadratic potential. A theory of integrable nonholonomic systems is presented in [AKN, § IV.4]: the motion takes place along tori, along their Kronecker curves, but nonuniformly. A perturbation theory for systems with *almost holonomic* constraints is presented by Tatarinov [Ta].

(iii) *Reduced nonholonomic systems.* Stanchenko [St] gives some further examples of invariant measures. That paper studies the *Abelian* Čaplygin nonholonomic systems, from the outset, in terms of differential forms. The reduced equations are of the form $\Omega_{\check{C}\mathrm{apl}}(\cdot, X_{\mathrm{red}}) = dH_{\mathrm{red}}$, but the 2-form $\Omega_{\check{C}\mathrm{apl}}$ need not be closed, so the system in general is not Hamiltonian. It would be interesting to generalize these results for the non-Abelian case. [AKN, § III.1.2] consider symmetries in nonholonomic mechanics, but in a different setting: they assume that the vertical fields satisfy the constraints. In their case the momentum is conserved, while it is not conserved in the setting considered here.

(iv) *Some applications to Mechanical Engineering.* Jankowski & Maryniak [JM] study the helicopter as a controlled system with nonholonomic constraints. The equations of motion are of d'Alembert type, but the constraints are not physically realized by friction forces as usual. Rather, they are consequences of the control laws. This approach was first advocated by H. Beghin [Be, Ap]. Control laws superimposed on classical nonholonomic systems were considered by Bloch & McClamroch [BM]. Kane [Ka] has developed the computer algebra program AUTOLEV to derive the equations of motion, based directly on d'Alembert's priciple. Fewer algebraic manipulations are done than in the Euler-Lagrange approach. The program can handle quasicoordinates as well, so it works also for nonholonomic systems. It would be interesting to extend it to systems with Lie group symmetries.

(v) *Quantum mechanical nonholonomic systems.* R. J. Eden [E] considered the modifications in the Heiseberg formalism necessary to take into account the classical nonholonomic constraints. Recently, P. Pitanga [Pi] revisited this issue, also presenting some concrete examples. Interestingly, the nonholonomic constraints are manifested as nonintegrable phase factors in the wavefunction.

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Caminante, no hay camino.
Se hace camino al andar.

References

[A] Arnold, V. I., *Mathematical Methods of Classical Mechanics*, Springer, 1978.

[AKN] Arnold, V. I., Kozlov, V. V., & Neishtadt, A. I., *Dynamical Systems III*, Springer, 1989.

[AM] Abraham, R. & Marsden, J., *Foundations of Mechanics*, Benjamin, 1978.

[AP] Appel, P., *Traité de Mécanique Rationnelle*, Gauthier-Villars, 1953.

[B] Brockett, R., Control theory and Singular Riemannian Geometry, in *New Directions in Applied Mathematics*, P. J. Hilton & G. S. Young, eds., Springer, 1982.

[BG] Baranov, Yu. S. & Gliklikh, Yu. E., A mechanical connection on the group of volume preserving diffeomorphisms, *Func. Anal. Appl.* **22**, 133–135, 1988.

[Be] Beghin, H., Etude théorique des compas gyrostatiques, Anschutz et Sperry, Imprimerie Nationale, Paris, 1921.

[BM] Bloch, A. M. & McClamroch, N., Control of mechanical systems with classical nonholonomic constraints, preprint, Ohio State University.

[Bu] Burns, J. A., Ball rolling on a turntable: analog for charged particle dynamics, *Amer. J. Phys.* **49**, 5658, 1981.

[Ca] Cartan, E., Sur la représentation géométrique des systèmes matériels non holonomes, *Proc. Int. Cong. Math.*, vol. 4, Bologna, 1928, 253–261.

[CCCI] Cantrijn, F., Carinena, J. F., Crampin, M. & Ibort, L. A., Reduction of degenerate Lagrangian systems, *J. Geom. Phys.* **3**, 353–400, 1986.

[CMD] Choquet-Bruhat, Y., DeWitt Morette, C. & Dillard-Bleick, M., *Analysis, Manifolds and Physics*, North-Holland, 1987.

[E] Eden, R. J., The quantum mechanics of nonholonomic systems, *Proc. Roy. Soc. London* **A 205**, 583–595, 1951.

[G] Godbillon, C., *Géométrie Différentielle et Mécanique Analytique*, Hermann, 1969.

[GV] Gershkovich, V. & Vershik, A., Nonholonomic manifolds and nilpotent analysis, *J. Geom. Phys.* **5**, 407–452, 1988.

[H] Hamel, G., *Theoretische Mechanik*, Springer, 1949.

[He] Hertz, H. R., *The Principles of Mechanics*, Dover, 1956.

[Hi] Hicks, N., *Notes on Differential Geometry*, Van Nostrand, 1965.

[JM] Jankowski, K. & Maryniak, J., Le système commandé en tant que système a liaisons non holonomes, *J. Méc. Th. Appl.*, **7**, 157–173, 1988.

[Ka] Kane, T., AUTOLEV software, Stanford University.

[Ko] Koiller, J., Nonholonomic systems with symmetry (and a model for a classical particle on a Yang-Mills field), *Proc. XV Int. Colloq. on Group Theoretical Methods in Physics*, R. Gilmore, ed., World Scientific, 1987.

[Koz] Kozlov, V. V., Realization of nonintegrable constraints in classical mechanics, *Sov. Phys. Dokl.* **28**, 735–737, 1983.

[Ku] Kummer, M., On the construction of the reduced phase space of a Hamiltonian systems with symmetry, *Indiana Univ. Math. J.* **30**, 281–291, 1981.

[L] Lagrange, J. L., *Mécanique Analytique*, *Oeuvres*, v. 11, Gauthier-Villars, 1888.

[LR] de Leon, M. & Rodrigues, P. R., *Methods of Differential Geometry in Analytical Mechanics*, North-Holland Math. Studies **158**, 1989.

[Mi] Mitchell, J., A local study of Carnot-Carathéodory metrics, Thesis, State Univ. New York, Stony Brook, 1982.

[Mo] Montgomery, R., Isoholonomic problems and some applications, *Commun. Math. Phys.* **128**, 565–592, 1990.

[MW] Marsden, J. & Weinstein, A., Reduction of symplectic manifolds with symmetry, *Rep. Math. Phys.* **5**, 121–130, 1974.

[NF] NEIMARK, YU. & FUFAEV, N. A., *Dynamics of Nonholonomic Systems.* Amer. Math. Soc. Translations 33, 1972.

[O] OLVER, P. J., *Applications of Lie groups to Differential Equations*, Springer, 1986.

[Ol] OLSSON, R., Some remarks on a paper by C. Carathéodory, *Proc. Int. Cong. Math.*, Amsterdam, 1954, p. 349.

[Pa] PANSU, P., Géométrie du groupe d'Heisenberg, Thèse, Paris VII, 1982.

[P] POINCARÉ, H., Les idées de Hertz sur la mécanique, *Oeuvres* VII, 231–250, Gauthier-Villars, 1952.

[Pi] PITANGA, P., Projector constrained quantum dynamics, *Nuovo Cimento* **101** A, 345–351, 1989 (see also *Nuovo Cimento* **103** A, 1529–1533, 1990).

[SM] SIEGEL, C. & MOSER, J., *Lectures on Celestial Mechanics*, Springer, 1971.

[So] SOMMERFELD, A., *Mechanics*, Academic Press, 1952.

[Sp] SPIVAK, M., *A Comprehensive Introduction to Differential Geometry*, vol. 2., Publish or Perish, 1970.

[SW] SATTINGER, D. H. & WEAVER, O. L., *Lie groups and algebras with applications to Physics, Geometry and Mechanics*, Springer, 1986.

[St] STANCHENKO, S. V., Nonholonomic Chaplygin systems, *Prikl. Mat. Mekh.* **53**, 11–17, 1989.

[Ta] TATARINOV, YA. V., Consequences of a nonintegrable perturbation of the integrable constraints: model problems of low dimensionality, *Prikl. Mat. Mech.* **51**, 579–586, 1987.

[V] VERSHIK, A. M., Classical and non-classical dynamics with constraints, in Springer Lect. Notes Math. 1108, ed. YU. G. BORISOVICH & XU. E. GLIKLIKH, 1984.

[VF] VERSHIK, A. M. & FADEEV, L. D., Lagrangian mechanics in invariant form, *Selecta Math. Sov.* **1**, 339–350, 1981.

[VG] VERSHIK, A. M. & GERSHKOVICH, V. YA., Geodesic flows on SL(2,R) with nonholonomic restrictions, *Sem. Inst. Steklov* **155**, 7–17, 1986.

[VV] VESELOV, A. & VESELOVA, L., Currents on Lie groups with nonholonomic connection and integrable nonhamiltonian systems, *Funct. Anal. Appl.* **20**, 308–309, 1987.

[W] WEINSTEIN, A., The local structure of Poisson manifolds, *J. Diff. Geom.* **18**, 523–557, 1983.

[We] WEBER, R. W., Hamiltonian systems with constraints and their meaning in mechanics, *Arch. Rational Mech. Anal.* **91**, 309–335, 1986.

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